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(Citation)
Mathematics Seminar Notes,5(1):109-116

(Issue Date)
1977

(Resource Type)
journal article

(Version)
Version of Record

(JaLCDOI)
https://doi.org/10.24546/0100492017

(URL)
https://hdl.handle.net/20.500.14094/0100492017

MATHEMATICS SEMINAR NOTES
Vol. 5 (1977) April 1977

On a Homotopy Version of 4-Dimensional Whitney's Lemma

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1. Whitney's lemma ([W]) states that intersection points of smooth n-submanifolds of a simply connected 2 -manifold can be eliminated if the intersection number of the two submanifolds is equal to zero and $2n \geqq 6$. However, this lemma fails for $2n = 4$. This was first pointed out by Kervare and Milnor (K&M]) in the smooth category. And Matsumoto [M] gave an example (Example 1) in the PL category which shows that it is not always possible to represent two homology classes ξ_1, ξ_2 with the intersection number $\xi_1 \cdot \xi_2 = 0$ by disjoint PL embedded 2-spheres. Furthermore he also gave an example (Example 2) in which one can not represent a homology class ξ with $\xi \cdot [S_i] = 0$ ($\{S_i\}$ being a finite set of embedded 2-spheres) by a continuous map of a 2-sphere whose image is disjoint of these 2-spheres $\{S_i\}$. In this paper we shall consider the above Example 2 and prove that if 2-spheres $\{S_i\}$ of the above may be deformed by a homotopical deformation, we can eliminate the intersection.

It is induced from the following theorem.

Theorem 1. *Let* M^4 *be a closed* 1*-connected* 4*-manifold and* P^2, Q^2 *be closed surfaces. If* $f: P^2 \to M^4$ *and* $g: Q^2 \to M^4$ *are continuous maps with*

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an intersection number $I(f(P), g(Q)) = 0$, *there are continuous maps* $f':P^2 \to M^4$, $g:Q^2 \to M^4$ *such that* $f' \simeq f$, $g' \simeq g$ *(homotopic) and* $f'(P) \cap g'(Q) = \phi$.

Furthermore, for $P^2 \cong S^2$, a 2-sphere, we obtain a necessary and sufficient condition that the intersection $f(S^2) \cap g(Q^2)$ can be eliminated by a homotopical deformation for only $f(S^2)$.

Theorem 2. *Let* M^4 *be a closed* 1*-connected* 4*-manifold and* $f:S^2 \to M^4$ *be continuous maps with* $I(f(S^2), g(Q^2)) = 0$. *Then there is a map* $f_1:S^2 \to M^4$ *such that* $f_1 \simeq f$ *and* $f_1(S^2) \cap g(Q^2) = \phi$ *if and only if* $< f'(P') > \varepsilon\, Im(\phi)$ *where* $\phi:\pi_2(M^4 - g(Q^2)) \to H_2(M^4 - g(Q^2))$ *is Hurewicz homomorphism and for* $< f'(P') >$ *see Definition* 1 *(section* 2*).*

Throughout the paper we work in the piecewise linear category ([Z]).

Let M^4, P^2 be closed 4- and 2-manifolds and $f:P^2 \to M^4$ be a map.

Then $S(f)$ denotes a closure of a set of selfintersection points of f; $S(f) = Cl\{x \varepsilon P \mid f^{-1}f(x) = \{x\}\}$, and let $S_r(f) = \{x \varepsilon S(f) \mid \# f^{-1}f(x) \leqq r\}$ where $\# X$ is a number of elements of a set X. Since we may assume f is in general position with respect to itself (up to ε-homotopy) i.e. $dim\, S_r(f) \leqq r dim\, P - (r-1) dim\, M$, $S(f)$ is a finite set. $\kappa(f)$ denotes a set of locally knotted points of f; $\kappa(f) = \{x \varepsilon P \mid f$ *is locally knotted at* $x\}$. Then we may assume $dim\, \kappa(f) \leqq 0$ and $\kappa(f) \cap S(f) = \phi$ [N]. $\simeq$, $\sim$ mean homotopic and homologous respectively. Let X be a subpolyhedron of a polyhedron Y, then $U(X, Y)$ means a regular neighborhood of X in Y if otherwise is stated.

2. We shall now proceed to establish our results.

Proof of Theorem 1. We may assume that f, g are in general position themselves and f is in general position with respect to g (up to ε-homotopy) i.e. $S(f)$, $S(g)$ and $f(P^2)$ $g(S^2)$ are all finite points sets. Since $I(f(P^2), g(Q^2)) = 0$, $f(P^2) \cap g(Q^2)$ is a set of even points $\{a_1, \cdots, a_p, b_1, \cdots, b_p\}$ such that the intersection numbers of $f(P)$ and $g(Q)$ at a_i, b_i are $+1$, -1 respectively i.e. $I(a_i) = +1$, $I(b_i) = -1$. Furthermore we may assume $(f(P) \cap g(Q)) \cap f(S(f) \cup \kappa(f)) = \phi$ and $(f(P) \cap g(Q)) \cap g(S(g) \cup \kappa(g)) = \phi$. Let α_i be a simple arc on $f(P)$ joining a_i and b_i and let β_i be a simple arc on $g(Q)$ joining a_i and b_i such that $\alpha_i \cap f(S(f)) = \phi$ and $\beta_i \cap g(S(g)) = \phi$. And let $\gamma_i = \alpha_i \cup \beta_i$, then γ_i is a simple closed curve in M^4. Since M^4 is 1-connected, there is a continuous map $h: D^2 \to M^4$ such that $h(\partial D^2) = \gamma_i$ and $h|\partial D^2$ is an embedding where D^2 is a 2-ball. Since we may assume h be in general position with respect to itself up to ε-homotopy, we may assume $h^{-1}(h(D^2) \cap f(P)) \cap (S(h) \cup \kappa(h)) = \phi = h^{-1}(h(D^2) \cap g(Q)) \cap (S(h) \cup \kappa(h))$. We take simple arcs A_i, B_i on D^2 joining $h^{-1}(a_i)$ and $h^{-1}(b_i)$ satisfying the following conditions:

(1) $A_i \cap B_i = \{h^{-1}(a_i), h^{-1}(b_i)\} = \partial A_i = \partial B_i$,

(2) a region surrounded by $h^{-1}(\alpha_i)$ and A_i contains all $S(h)$ and $h^{-1}(h(D^2) \cap f(P))$,

(3) a region surrounded by $h^{-1}(\beta_i)$ and B_i contains all $h^{-1}(h(D^2) \cap g(Q))$ (Fig. 1).

Let $\widetilde{D}^2$ be a 2-ball in D^2 surrounded by A_i and B_i. Using $h(D^2)$, we can construct a map $F: P \times I \to M^4$ such that $F_0 = f$, $F_1(f^{-1}(\alpha_i)) = h(A_i)$ and $F_t|P\text{-}U(f^{-1}(\alpha_i), P) = f|P\text{-}U(f^{-1}(\alpha_i), P)$ where $U(f^{-1}(\alpha_i), P)$

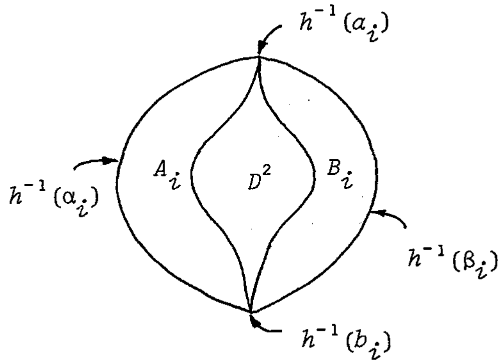

Figure 1.

is a regular neighborhood of $f^{-1}(\alpha_i)$ in P $mod(f^{-1}(\partial\alpha_i))$ and a map G: $Q \times I \to M^4$ such that $G_0 = g$, $G_1(g^{-1}(\beta_i)) = h(B_i)$ and $G_t|Q-\mathring{U}(g^{-1}(\beta_i), Q) = g|Q-\mathring{U}(g^{-1}(\beta_i), Q)$ where $U(g^{-1}(\beta_i), Q)$ is a regular neighborhood of $g^{-1}(\beta_i)$ in Q $mod(g^{-1}(\partial\beta_i))$. Then $F_1(f^{-1}(\alpha_i)) \cup G_1(g^{-1}(\beta_i))$ bounds a nonsingular 2-ball $h(\widetilde{D}^2)$ and $h(\widetilde{D}^2) \cap (F_1(P) \cup G_1(Q)) = h(\partial\widetilde{D}^2) = F_1(f^{-1}(\alpha_i)) \cup G_1(g^{-1}(\beta_i))$. So there are continuous maps $f_1 : P \to M^4$ and $g_1 : Q \to M^4$ such that $f_1 \simeq f$, $g_1 \simeq g$ and $f_1(P) \cap g_1(Q) = (f(P) \cap g(Q)) - \{a_i, b_i\}$ where the intersections of f_1 and g_1 are more than that of f and g. Doing this operations for all pairs $\{(a_i, b_i);\ i = 1, 2, \cdots, p\}$, we obtain a maps $f' : P \to M^4$ and $g' : Q \to M^4$ such that $f' \simeq f$, $g' \simeq g$ and $f'(P) \cap g'(Q) = \phi$.

From the above proof, we obtain the following.

Corollary. *Let P^2, Q^2 be 2-dimensional compact submanifolds in M^4. If $\pi_1(M-Q) = \{1\}$ and if $I(P, Q) = 0$, there is a continuous map $f : P \to M^4$ such that $f \simeq$ inclusion and $f(P) \cap Q = \phi$. (P can be interchanged Q.)*

Lemma 1. *Let M^4 be a closed 4-manifold and P^2, Q^2 be closed* *surfaces. If* $f: P^2 \to M^4$ $g: Q^2 \to M^4$ *are maps with* $I(f(P), g(Q)) = 0$, *there are a closed surface* P' *and an embedding* $f': P' \to M^4$ *such that* $f' \sim f$ *and* $f'(P') \cap g(Q) = \phi$.

Proof. We may suppose $f(P^2) \cap g(Q^2) = \{a_1, \cdots, a_p, b_1, \cdots, b_p\}$ where $I(a_i) = +1$, $I(b_i) = -1$ (intersection number of $f(P^2)$ and $g(Q^2)$ at a_i, b_i respectively). And $S(f)$ consists of only double points using general position technique. Furthermore we may assume that $f(P^2) \cap g(Q^2)$ and $f(S(f))$ are all locally flat points in M^4. Let α_i ($i = 1, 2, \cdots, p$) be simple arcs on $g(Q^2)$ joining a_i and b_i satisfying the following conditions:

$$\begin{cases} \alpha_i \cap \alpha_j = \phi \ (i \neq j), \\ \alpha_i \cap g(S(g)) = \phi, \\ \alpha_i \cap (f(P^2) \cap g(Q^2)) = \{a_i, b_i\} \\ \alpha_i \cap g(\kappa(g)) = \phi, \end{cases}$$

We can take such α_i since $dim\ Q^2 = 2$. Let $U(\alpha_i) = U(\alpha_i, M^4)$ be a regular neighborhood of α_i in M^4, then it is homeomorphic to $D^1 \times D^3$.

Using $\partial U(\alpha_i)$ we deform $f(P^2)$ to a new manifold $(f(P) - (\mathring{U}(a_i, f(P)) \cup \mathring{U}(b_i, f(P))) \cup_\partial (D^1 \times S^1)$ where $D^1 \times S^1 \subset \partial U(\alpha_i)$, $(D^1 \times S^1) \cap g(Q) = \phi$ and $\partial D^1 \times S^1 = \partial U(a_i, f(P)) \cup \partial U(b_i, f(P))$ (Fig. 2).

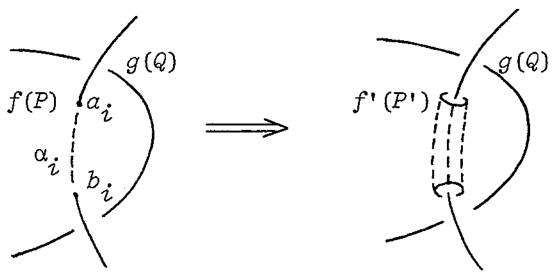

Figure 2.

We do this surgery for all $i = 1, 2, \cdots, p$. Next if we denote $f(S(f)) = \{c_1, \cdots, c_q\}$, pairs $(\partial U(c_j, M^4), \partial U(c_j, f(P)))$ $(j = 1, 2, \cdots, q)$ are elementary links (Fig. 3) since c_j $(j = 1, 2, \cdots, q)$ are all double points of f and since f is locally flat at c_j. We deform $f(P)$ to a new manifold $(f(P) - \mathring{U}(c_j, f(P))) \cup A_j$ where A_j is an annulus embedded in $\partial U(c_j, M^4)$ bounded by $\partial U(c_j, f(P))$ (Fig. 4).

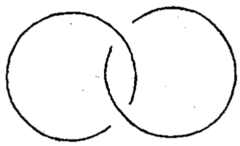

Figure 3.

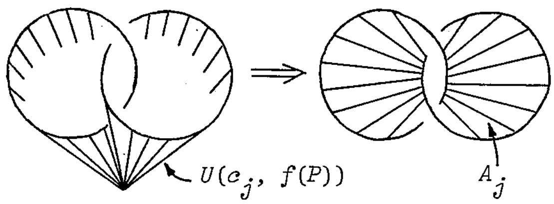

Figure 4.

Doing two kinds of surgeries of the above for all $i = 1, 2, \cdots, p$; $j = 1, 2, \cdots, q$, we obtain a new orientable closed surface P' whose genera is equal to $p + q +$ (*genera of* P) and an embedding $f': P' \to M^4$ such that $f' \sim f$ in M^4 and $f'(P') \cap g(Q) = \phi$.

Definition 1. We define $< f'(P') > \varepsilon H_2(M^4 - g(Q))$ an induced class from $< f(P) > \varepsilon H_2(M^4)$ using lemma 1. Unfortunately this class cannot be uniquely determined from $f(P)$. But if $Q \cong S^2$, 2-sphere and if g is an embedding, it can be uniquely determined by the following lemma.

Lemma 2. *For the above definition, if* $Q \cong S^2$ *and if* $g: S^2 \to M^4$ *is an embedding,* $j_*: H_2(M^4 - g(S^2)) \to H_2(M^4)$ *is injective.*

Proof. In a homology exact sequence of a pair $(M^4, M^4 - g(S^2))$, $H_3(M^4, M^4 - g(S^2)) \cong H_3(U(g(S^2)), \partial U(g(S^2))) \cong H^1(U(g(S^2))) \cong H_1(U(g(S^2))) \cong H_1(g(S^2)) \cong H_1(S^2) = 0$ where $U(g(S^2) = U(g(S^2), M^4)$. So j_* is injective.

Proof of Theorem 2. Necessity of the condition is clear. Now we consider the commutative diagram.

$$\begin{array}{ccc} \pi_2(M^4-g(S^2)) & \xrightarrow{\phi} & H_2(M^4-g(S^2)) \\ {\scriptstyle i_*}\big\downarrow & & \big\downarrow{\scriptstyle j_*} \\ \pi_2(M^4) & \xrightarrow{\psi} & H_2(M^4) \end{array}$$

Since M^4 is 1-connected, ψ is an isomorphism. Let $[f]\varepsilon\pi_2(M^4)$.

Since $I(f(S^2), g(Q)) = 0$, there is an induced class $< f'(P') >\varepsilon H_2(M^4-g(Q))$ from $\psi([f]) = j_*< f'(P') >$. So $< f'(P') >\varepsilon Im(\phi)$ if and only if there is a class $\beta\varepsilon\pi_2(M-g(Q))$ such that $\phi(\beta) = < f'(P') >$. By the commutativity of the diagram, $i_*(\beta) = [f]$ since ψ is an isomorphism i.e. there is a map $f_1:S^2 \to M^4-g(Q)$ such that $[f_1] = \beta$ and $f_1 \simeq f$ in M^4.

It is easy to show the following corollaries.

Corollary 1. *Let* $f:S^2 \to M^4$, $g:Q^2 \to M^4$ *be maps with* $I(f(S^2), g(Q)) = 0$. *If* M^4, $M^4-g(Q)$ *are both* 1*-connected, there is a map* $f_1:S^2 \to M^4$ *such that* $f_1 \simeq f$ *and* $f_1(S^2) \cap g(Q) = \phi$.

Corollary 2. *If the induced class* $< f'(P') > = 0$ *in* $H_2(M^4-g(Q))$, *there is a map* $f_1:S^2 \to M^4$ *such that* $f_1 \simeq f$ *and* $f_1(S^2) \cap g(Q) = \phi$.

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