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A Generalization of Local Dynamical Systems

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The theory of local dynamical systems is motivated by the theory of autonomous ordinary differential equations and this theory was studied extensively by many researchers. In [2], G. R. Sell has shown how to associate in a significant manner a local dynamical system with a nonautonomous ordinary differential equation or a Volterra integral equation, and gave many invariance properties for the limit sets of solutions of equations. However, it has been assumed to define a local dynamical system that the solution of the given equation satisfies some sort of uniqueness condition. On the other hand, E. Roxin [3] gave a type of axiomatically defined generalized dynamical system which based on differential equations without uniqueness and control systems governed by differential equations, but not on integral equations without uniqueness.

In the present note we shall give the axiom of a local generalized dynamical system which admits us to treat Volterra integral equations without uniqueness. This treatment seems to be new. The further theory and applications of our local generalized dynamical systems will appear elsewhere.

Let X be a metric space with metric d_X and 2^X be the set of all

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subsets of X . We define the distance $d_X(a, B)$ between a point a and a set B by $d_X(a, B) = \inf \{d_X(a, b) : b \in B\}$ and the distances $d_X^*(A, B)$ and $d_X(A, B)$ between sets A and B by $d_X^*(A, B) = \sup \{d_X(a, B) : a \in A\}$ and $d_X(A, B) = \max \{d_X^*(A, B), d_X^*(B, A)\}$. Moreover we define an ε -neighboring set of a given set A_0 by $V(A_0, \varepsilon) = \{x \in X : d_X(x, A_0) < \varepsilon\}$. If ψ is any function and A is a set, we write $\psi(A)$ for $\{\psi(x) : x \in A\}$, and analogously for functions with two or more variables.

The set-valued mapping F of a topological space $D(F)$ into 2^X is said to be *continuous* on $D(F)$, if for any $\alpha_0 \in D(F)$ and for every $\varepsilon > 0$ there exists a neighborhood V of α_0 such that $d_X(F(\alpha), F(\alpha_0)) < \varepsilon$ for all $\alpha \in V$; F is said to be *upper semi-continuous* on $D(F)$ if for any $\alpha_0 \in D(F)$ and for every $\varepsilon > 0$ there exists a neighborhood V of α_0 such that $d_X^*(F(\alpha), F(\alpha_0)) < \varepsilon$ for all $\alpha \in V$.

Definition. Let X be a metric space. For each $p \in X$, let there be given a half open interval $I_p = [0, \alpha_p)$, where $\alpha_p > 0$. Let D denote the set $\{(t, p) : p \in X \text{ and } t \in I_p\}$. A set-valued function $\pi : D \rightarrow 2^X$ is a *local generalized semi-flow* (or a *local generalized dynamical system*) on X if the followings hold:

- (1) The set $\pi(t, p)$ is nonempty and closed for all $(t, p) \in D$; in particular $\pi(0, p) = \{p\}$ for each p in X .
- (2) If $t \in I_p$ and $s \in \bigcap \{I_q : q \in \pi(t, p)\}$, then $t + s \in I_p$ and $\pi(t + s, p) = \pi(t, \pi(s, p)) = \bigcup \{\pi(t, q) : q \in \pi(s, p)\}$.
- (3) Each I_p is maximal in the sense that if $I_p = [0, \alpha_p)$ then either $\alpha_p = \infty$ or the closure of the set $\bigcup \{\pi(t, p) : t \in [0, \alpha_p)\}$ is not compact in X .
- (4) D is open in $R^+ \times X$.

(5) $\pi:D \rightarrow 2^X$ is upper semi-continuous.

(6) For each fixed $p \in X$, $\pi(\cdot, p):I_p \rightarrow 2^X$ is continuous.

We note that d_X defines a metric on the class of all nonempty compact subsets of X and a flow defined by an integral equation in this note is a (compact-connected sets)-valued mapping.

Let $\text{Comp}(D)$ be the collection of all compact sets in $D \subset R^+ \times X$. Since X is a metric space, we can give a metric d_D on the space D . Then we have a topology on $\text{Comp}(D)$ by defining a neighborhood basis for each U in $\text{Comp}(D)$. This topology is generated by the neighborhood system $\{W \in \text{Comp}(D) : W \subset V_D(U, \epsilon)\}$ for any $\epsilon > 0$, where $V_D(U, \epsilon)$ is the ϵ -neighboring set of U corresponding to the metric d_D .

In what follows, we assume the compactness of $\pi(t, p)$ for all (t, p) in D . If X is locally compact, this assumption is satisfied.

Theorem 1. *The function $\pi:\text{Comp}(D) \rightarrow 2^X$ defined by $\pi(U) = \bigcup\{\pi(t, p) : (t, p) \in U\}$ is upper semi-continuous.*

Theorem 2. *Let M be a compact subset of X . Then $\alpha_M = \inf\{\alpha_p : p \in M\}$ is positive and $\pi(t, M)$ is compact for all $t \in [0, \alpha_M]$. Moreover, the function $\pi(\cdot, M):[0, \alpha_M] \rightarrow 2^X$ defined by $\pi(\cdot, M)(t) = \pi(t, M)$ is continuous.*

Theorem 3. *If $U \in \text{Comp}(D)$ is a connected set, then $\pi(U)$ is compact and connected in X .*

Finally we shall give an example of local generalized semi-flow. Consider the Volterra integral equation of the type

$$P(f, g): \quad x(t) = f(t) + \int_0^t g(t, s, x(s))ds.$$

Under some conditions on f and g (for details, see [1]), we have an inextendable solution $x(t)$ of $P(f, g)$ defined on $[0, \alpha(f, g, x))$. Let $S(f, g)$ be the solution family of $P(f, g)$ and

$$\alpha(f, g) = \inf \{ \alpha(f, g, x) : x \in S(f, g) \}.$$

Let $C \times G$ be a base space (for the definition, see [1]).

For any $(t, f, g) \in D = \{(t, f, g) : f \in C, g \in G \text{ and } t \in [0, \alpha(f, g))\}$, we define the set $T(f, g, t) \in 2^C$ by

$$T(f, g, t) = \{f(t + \cdot) + \int_0^t g(t + \cdot, s, \varphi(s))ds : \varphi \in S(f, g)\}$$

and the τ -translation of $g \in G$ by

$$g_\tau(t, s, x) = g(t + \tau, s + \tau, x).$$

Then we have the following theorem:

Theorem 4. *Let $X = C \times G$. Then the mapping $\pi: D \rightarrow 2^X$ defined by $\pi(t, f, g) = (T(t, f, g), g_t)$ determines a local generalized semi-flow.*

References

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