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Chilana, Ajit Kaur

Kiran, Usha

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Algebra of Sequences in Locally Convex Algebras

By Ajit Kaur CHILANA and Usha KIRAN^{*}

0. Introduction

Pietsch [6] and Rosier [7] combine a perfect sequence space Λ of scalars and a locally convex space E and topologize them to make them into locally convex spaces. In this paper, we take E to be a locally convex algebra and study the conditions on Λ and E which make the resulting space a locally convex algebra with properties such as A -convexity, local multiplicative convexity and having jointly continuous multiplication.

1. Notation and Terminology

The basic terminology will be as in [4]. A complex (real) algebra E with a locally convex linear topology for which multiplication is separately continuous will be called a *locally convex algebra*. E is called *locally multiplicatively convex*, in short, *lmc*, if there exists a basis of idempotent convex neighborhoods of zero [5]. Following [3], an absolutely convex set B in E is called *right (left) A-convex* if it absorbs Bx (xB) for every x in E , it will be called *A-convex* if it is both right and left A -convex. A locally convex algebra is called (*right, left*) A -

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convex if there exists a basis of (right, left) A -convex neighborhoods of zero. Following [2], a subset U of E is called *right (left) boundedly multiplicatively convex*, in short, *bmc*, if for every bounded set B there exists a scalar $\rho > 0$, depending upon B and U , such that $UB \subset \rho U$ ($BU \subset \rho U$). It will be called *bmc* if it is both right and left *bmc*. A locally convex algebra is called *(right, left) locally boundedly multiplicatively convex*, in short *lbmc*, if there exists a basis of (right, left) *bmc* neighborhoods of zero. Multiplication in a locally convex algebra will be called *right (left) hypocontinuous* if given a neighborhood U of zero and a bounded set B there exists a neighborhood V of zero satisfying $VB \subset U$ ($BV \subset U$). We say that multiplication is *hypocontinuous* if it is both right and left hypocontinuous. Multiplication is *jointly continuous* if for each neighborhood U of zero there exists a neighborhood V of zero such that $V^2 \subset U$. Let $\mathcal{U}(E)$ be a basis of absolutely convex closed neighborhoods of zero which will be taken to be A -convex in case E is A -convex and similarly for other conditions on E . Let $\mathcal{P}(E) = \{p_U : U \in \mathcal{U}(E)\}$ be the corresponding collection of seminorms. Let λ be a normal sequence space such that $\mathcal{P} \subset \lambda \subset m$, where $\mathcal{P}$ is the space of all complex (real) sequences with only a finite number of nonzero co-ordinates and m is the space of all bounded complex (real) sequences. Let λ^* denote the α -dual of λ . Let $\lambda\{E\}$ be the space of all E -valued sequences $x = (x_n)$ such that the scalar sequence $(p_U(x_n))$ is in λ for each U in $\mathcal{U}(E)$. Let $\mathbb{N}$ denote the set of natural numbers.

2. λ -hypocontinuity and λ -local multiplicative convexity in algebras

Definition 2.1. A subset U of E is called *right (left) λ -multiplicatively convex*, in short, *right (left) λ -mc*, if for every $x = (x_n)$ in

$\lambda\{E\}$ there exists a scalar $a > 0$, depending upon x and U such that for each n in $\mathbb{N}$, $Ux_n \subset aU$ ($x_n U \subset aU$). It will be called λ -mc if it is both right and left λ -mc.

Definition 2.2. A locally convex algebra is called *(right, left) λ -locally multiplicatively convex*, in short, *(right, left) λ -lmc* if there is a basis of closed absolutely convex (right, left) λ -mc neighborhoods of zero.

Definition 2.3. Multiplication in a locally convex algebra E is said to be *right (left) λ -hypocontinuous* if given a neighborhood U of zero and $x = (x_n)$ in $\lambda\{E\}$ there exists a neighborhood V of zero satisfying $Vx_n \subset U$ ($x_n V \subset U$) for each $n \in \mathbb{N}$. We say that multiplication is λ -*hypocontinuous* if it is both right and left λ -hypocontinuous.

Clearly, if μ is any other normal sequence space such that $\lambda \subset \mu$ then E is μ -lmc implies that E is λ -lmc which further implies that E is A -convex. Also E is μ -lmc implies that E has μ -hypocontinuous multiplication which in turn implies that E has λ -hypocontinuous multiplication. We investigate the possibility of implications in the reverse direction.

Theorem 2.4. *Suppose that E has a continuous norm, then E has φ -hypocontinuous multiplication. Further, if E is A -convex, then it is φ -lmc.*

Proof. Let p be a continuous norm on E . Let $(x_n) \in \varphi\{E\}$. Then $p(x_n) = 0$ and therefore $x_n = 0$ for all but a finite number of n 's. The rest of the proof follows easily.

Theorem 2.5. *Let $\lambda \neq \varnothing$ be a normal sequence space of bounded sequences and E be a metrizable locally convex algebra. Then (right, left) λ -hypocontinuity is equivalent to (right, left) hypocontinuity.*

Proof. Clearly (right, left) hypocontinuity in E implies (right, left) λ -hypocontinuity. Let E have right λ -hypocontinuous multiplication. Then there exists an increasing sequence $\mathcal{P} = \{p_n : n \in \mathbb{N}\}$ of seminorms defining the topology of E with the property that if $y = (y_k) \in \lambda\{E\}$ and $n \in \mathbb{N}$ then there exists $p_m \in \mathcal{P}$ such that $p_n(xy_k) \leq p_m(x)$ for all $k \in \mathbb{N}$ and $x \in E$. Let $\alpha = (\alpha_n) \in \lambda \setminus \varnothing$. Since λ is normal we may suppose that $\alpha_n \geq 0$ for all n in $\mathbb{N}$. Let $\beta_n = 1/\alpha_n$ if $\alpha_n > 0$ and $\beta_n = 1$ if $\alpha_n = 0$. Let if possible E not have right hypocontinuous multiplication. Then there exists a seminorm p_n , a bounded sequence (x_m) and a sequence (y_m) in E such that $p_n(x_m, y_m) > m\beta_m p_m(y_m)$. Put $x'_m = 0$ if $\alpha_m = 0$ and $x'_m = x_m$ if $\alpha_m \neq 0$. Then (x'_m) is a bounded sequence. As $(\alpha_m) \in \lambda$ and λ is normal, $(p_j(\alpha_m x'_m)) \in \lambda$ for each $j \in \mathbb{N}$. So $(\alpha_m x'_m) \in \lambda\{E\}$. Since E has λ -hypocontinuity, there exists a seminorm p_j such that $p_n(\alpha_m x'_m y) \leq p_j(y)$ for all $y \in E$. Since $\alpha \notin \varnothing$, there exists $k > j$ such that $\alpha_k \neq 0$. Then $p_j(y) \leq p_k(y)$ for all $y \in E$. So we have $(k/\alpha_k)p_k(y_k) < p_n(x'_k y_k) \leq p_k(y_k)/\alpha_k$. Therefore $k < 1$, which is absurd. Hence we get the result for right hypocontinuity. Similarly we can prove for other cases.

Corollary 2.6. *In a metrizable locally convex algebra hypocontinuity is equivalent to λ_p -hypocontinuity for any $p > 0$.*

Theorem 2.7. *Let $\lambda \neq \varnothing$ be a normal sequence space of bounded sequences. Then a locally convex algebra E is (right, left) lbmc if and only*

if it is (right, left) λ -lmc.

Proof. It is enough to show that every (right, left) λ -mc set is (right, left) bmc. Let A be a right λ -mc set in E . If possible, let A be not right bmc. Then there exists an absolutely convex bounded set B such that AB is not absorbed by A . Let $\alpha = (\alpha_n) \in \lambda \setminus \mathcal{P}$. We may suppose $\alpha_n \geq 0$ for all n . Then for each n with $\alpha_n \neq 0$ there exists $x_n \in A$ and $y_n \in B$ such that $\alpha_n x_n y_n \notin nA$. Put $x_n = 0$ and $y_n = 0$ when $\alpha_n = 0$. Now for each $U \in \mathcal{U}(E)$, $p_U(\alpha_n y_n) = \alpha_n p_U(y_n)$ and as (y_n) is a bounded sequence in E and $(\alpha_n) \in \lambda$ we have that $(p_U(\alpha_n y_n))$ is in λ . So $(\alpha_n y_n) \in \lambda\{E\}$. Since A is right λ -mc, we have a $\rho > 0$ such that for each $n \in \mathbb{N}$, $x_n(\alpha_n y_n) \in \rho A$. Since $\alpha \notin \mathcal{P}$ we can choose an $m > \rho$ with $\alpha_m \neq 0$. Then $x_m y_m \in (m/\alpha_m)A$ which gives a contradiction. Similarly we can prove that a left λ -mc set is left bmc.

Corollary 2.8. An λ_p -lmc algebra is lbmc for each $p > 0$.

Example 2.9. Consider the algebra $\mathcal{C}[0, 1]$ of all continuous complex valued functions on the closed interval $[0, 1]$. A norm p is defined on this algebra by $p(f) = \sup\{|f(x)g(x)| : x \in [0, 1]\}$, where

$$g(x) = \begin{cases} x & \text{for } 0 \leq x \leq 1/2 \\ 1 - x & \text{for } 1/2 \leq x \leq 1. \end{cases}$$

Then $E = (\mathcal{C}[0, 1], p)$ is a normed linear space which is A -convex (Example 2 of [3]). So by applying Theorem 2.4 we get that E is $\mathcal{P}$ -lmc and consequently E has $\mathcal{P}$ -hypocontinuous multiplication. On the other hand, it is proved in Example 2.24 of [2] that E is not lbmc and does not have hypocontinuous multiplication. Again by Theorem 2.5, E does not have λ -hypocontinuous multiplication for any λ with $\mathcal{P} \subset \lambda$ and $\mathcal{P} \neq \lambda$.

Example 2.10. Consider the set $[0, 1]$ with the Lebesgue measure. Let us consider (equivalence classes of) measurable functions defined on $[0, 1]$. Let for $p \geq 1$, $\|f\|_p = \left\{ \int_0^1 |f(x)|^p dx \right\}^{1/p}$. L^ω consists of all functions f for which $\|f\|_1, \|f\|_2, \dots, \|f\|_p, \dots$ are all finite. A base of neighborhoods for 0 in the topology τ_ω on L^ω is given by the sets of the type $\{f: \|f\|_p < \epsilon\}$ where $p \geq 1$ and $\epsilon > 0$ (see [1]). The multiplication in L^ω is jointly continuous and therefore hypocontinuous. So L^ω has λ -hypocontinuous multiplication for all λ . But L^ω is not A-convex and therefore can not even be φ -lmc.

Example 2.11. Let L^∞ denote the subalgebra of essentially bounded measurable functions on $[0, 1]$. Consider L^∞ with the subspace topology τ^* induced on it by the topology τ_ω of L^ω . Then (L^∞, τ^*) is an A-convex algebra which is not lbmc (Example 5.5 of [2]). So by Theorem 2.4, it is φ -lmc and by Theorem 2.7, it is not λ -lmc for any λ such that $\varphi \subset \lambda \subset m$ and $\varphi \neq \lambda$. Also L^∞ has jointly continuous multiplication and therefore, it has λ -hypocontinuous multiplication for all λ .

3. $\Lambda\{E\}$ as a locally convex algebra

Let Λ be a perfect sequence space. A family $\mathcal{M}$ consisting of bounded subset of Λ^* is called a *normal topologizing system* for Λ if $\mathcal{M}$ has the following properties: (i) If $M_1, M_2 \in \mathcal{M}$ then there exists $M \in \mathcal{M}$ such that $M_1 \cup M_2 \subset M$; (ii) If $M \in \mathcal{M}$ and $\rho > 0$ then $\rho M \in \mathcal{M}$; (iii) If $\alpha \in \Lambda^*$ then $\alpha \in M$ for some $M \in \mathcal{M}$; (iv) The normal hull of every set in $\mathcal{M}$ is in $\mathcal{M}$. Rosier [7] has proved that if $\mathcal{M}$ is a normal topologizing system for Λ then the collection of all $M^\circ, M \in \mathcal{M}$ forms a neighborhood base at zero for a locally convex topology on Λ . Also we denote by $\Lambda\{E\}_{\mathcal{M}}$

the space $\Lambda\{E\}$ equipped with the locally convex Hausdorff $\mathcal{M}$ -topology defined by the family of seminorms $\pi_{M,U}(x) = \sup \{ \sum_n |\alpha_n| p_U(x_n) : \alpha \in M \}$, where $M \in \mathcal{M}$ and $U \in \mathcal{U}(E)$.

Theorem 3.1. *If E has Λ -hypocontinuous multiplication then $\Lambda\{E\}_{\mathcal{M}}$ is a locally convex algebra.*

Proof. Let $x \in \Lambda\{E\}$. For each $U \in \mathcal{U}(E)$, there exists a V in $\mathcal{U}(E)$ such that $\max\{p_U(x_n y_n), p_U(y_n x_n)\} \leq p_V(y_n)$ for all $y = (y_n) \in \Lambda\{E\}$ and $n \in \mathbb{N}$. Since Λ is normal, we have that xy and yx are in $\Lambda\{E\}$ for all y in $\Lambda\{E\}$. Also we have $\max\{\pi_{M,U}(xy), \pi_{M,U}(yx)\} \leq \pi_{M,V}(y)$ for all y in $\Lambda\{E\}$. Hence the conclusion follows.

Theorem 3.2. *Let E have Λ -hypocontinuous multiplication. Then*

- (i) *E is Λ -lmc implies that $\Lambda\{E\}_{\mathcal{M}}$ is A -convex.*
- (ii) *If E is lmc and $\sqrt{M} \subset M$ for each M in $\mathcal{M}$ then $\Lambda\{E\}_{\mathcal{M}}$ is lmc, where $\sqrt{M}$ is the normal hull of the set $\{(\sqrt{|\alpha_n|}) : (\alpha_n) \in M\}$.*
- (iii) *If E has jointly continuous multiplication and for every $M \in \mathcal{M}$ there exists $M' \in \mathcal{M}$ such that $\sqrt{M} \subset M'$ then $\Lambda\{E\}_{\mathcal{M}}$ has jointly continuous multiplication.*

Proof. (i) Let E be Λ -lmc and $\mathcal{U}(E)$ a base of closed absolutely convex Λ -mc neighborhoods of 0 in E . Let $x = (x_n) \in \Lambda\{E\}$, $U \in \mathcal{U}(E)$ and $M \in \mathcal{M}$. Since U is Λ -mc, there exists a number $\rho \geq 0$ such that $\max\{p_U(zx_n), p_U(x_n z)\} \leq \rho \cdot p_U(z)$ for $z \in E$ and $n \in \mathbb{N}$. So for $y = (y_n)$ in $\Lambda\{E\}$,

$$\begin{aligned} \pi_{M,U}(xy) &= \sup \{ \sum_n |\alpha_n| p_U(x_n y_n) : \alpha \in M \} \leq \sup \{ \sum_n |\alpha_n| \rho \cdot p_U(y_n) : \alpha \in M \} \\ &= \rho \cdot \pi_{M,U}(y). \end{aligned}$$

Similarly we can prove that $\pi_{M,U}(y_n) \leq \rho \cdot \pi_{M,U}(y)$. Thus $\Lambda\{E\}_{\mathcal{M}}$ is A -

convex

(ii) Let E be lmc and $\mathcal{U}(E)$ be a base of closed absolutely convex idempotent neighborhoods of 0 in E . Let $U \in \mathcal{U}(E)$ and $M \in \mathcal{M}$. Then $(\sqrt{|\alpha_n|}) \in M$ whenever $(\alpha_n) \in M$. Also $p_U(z_1 z_2) \leq p_U(z_1) p_U(z_2)$ for $z_1, z_2 \in E$. So for $x = (x_n), y = (y_n) \in \Lambda\{E\}$ we have

$$\begin{aligned} \pi_{M,U}(xy) &= \sup \{ \sum_n |\alpha_n| p_U(x_n y_n) : \alpha \in M \} \\ &\leq \sup \{ \sum_n (\sqrt{|\alpha_n|} p_U(x_n)) (\sqrt{|\alpha_n|} p_U(y_n)) : \alpha \in M \} \\ &\leq \sup \{ (\sum_n \sqrt{|\alpha_n|} p_U(x_n)) (\sum_n \sqrt{|\alpha_n|} p_U(y_n)) : \alpha \in M \} \\ &\leq \sup \{ \sum_n |\beta_n| p_U(x_n) : \beta \in M \} \sup \{ \sum_n |\beta_n| p_U(y_n) : \beta \in M \} \\ &= \pi_{M,U}(x) \pi_{M,U}(y). \end{aligned}$$

Thus $\pi_{M,U}$ is a submultiplicative seminorm for each $U \in \mathcal{U}(E)$ and $M \in \mathcal{M}$.

Hence $\Lambda\{E\}_{\mathcal{M}}$ is lmc.

(iii) Since E has jointly continuous multiplication, for each U in $\mathcal{U}(E)$ there exists V in $\mathcal{U}(E)$ such that $V^2 \subset U$. If $M, M' \in \mathcal{M}$ are such that $\sqrt{M} \subset M'$ and $U, V \in \mathcal{U}(E)$ are such that $V^2 \subset U$, then we can prove in a manner similar to the one above that

$$\pi_{M,U}(xy) \leq \pi_{M',V}(x) \pi_{M',V}(y)$$

for x, y in $\Lambda\{E\}$. Hence $\Lambda\{E\}_{\mathcal{M}}$ has jointly continuous multiplication.

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Up to September 1977
A. K. Chilana
Department of Mathematics
University of Oregon
Eugene, Oregon 97403
U. S. A.

Department of Mathematics
University of Delhi
Delhi - 110007
India.