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On Common Fixed Points of Commuting Mappings

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In his recent paper [3], S. Kasahara has obtained some interesting results by extending the results of Jungck [1, 2] to L-spaces. The aim of this note is to obtain another generalization of the results of [2].

The proof of our theorem requires the following obvious lemma of [2].

Lemma. Let $\{y_n\}$ be a sequence in a complete metric space (X, d) . If there exists $k \in (0, 1)$ such that $d(y_{n+1}, y_n) \leq kd(y_n, y_{n-1})$ for all n , then $\{y_n\}$ converges to a point in X .

Theorem. Let two continuous and commuting mappings S and T from a complete metric space (X, d) into itself satisfy the following conditions:

$$(1) \quad S(X) \subset T(X).$$

$$(2) \quad d(Sx, Sy) \leq a[d(Tx, Sx) + d(Ty, Sy)] + b[d(Tx, Sy) + d(Ty, Sx)] \\ + cd(Tx, Ty)$$

for all x, y in X where a, b, c are nonnegative reals satisfying

$$(3) \quad 0 < 2a + 2b + c < 1.$$

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Then S and T have a unique common fixed point.

Proof. Let $x_0 \in X$ and let x_1 be such that $Tx_1 = Sx_0$. In general, choose x_n so that

$$(4) \quad Tx_n = Sx_{n-1}.$$

We can do this since (1) holds. Therefore by (2),

$$\begin{aligned} d(Tx_{n+1}, Tx_n) &= d(Sx_n, Sx_{n-1}) \\ &\leq a[d(Tx_n, Sx_n) + d(Tx_{n-1}, Sx_{n-1})] \\ &\quad + b[d(Tx_n, Sx_{n-1}) + d(Tx_{n-1}, Sx_n)] + cd(Tx_n, Tx_{n-1}) \\ &\leq (a + b)d(Tx_{n+1}, Tx_n) + (a + b + c)d(Tx_n, Tx_{n-1}) \end{aligned}$$

for all $n \geq 1$. This gives

$$d(Tx_{n+1}, Tx_n) \leq pd(Tx_n, Tx_{n-1}),$$

where $p = (a + b + c)/(1 - a - b)$. Condition (3) makes $p \in (0, 1)$. Therefore the lemma yields that there exists a $t \in X$ such that $Tx_n \rightarrow t$. In view of (4), Sx_n also tends to t . Since both S and T are continuous, we have $S(Tx_n) \rightarrow St$ and $T(Sx_n) \rightarrow Tt$. But S and T commute so that $S(Tx_n) = T(Sx_n)$ for all n . Thus

$$(5) \quad St = Tt$$

and consequently

$$(6) \quad T(Tt) = T(St) = S(Tt) = S(St).$$

Therefore by (2),

$$\begin{aligned} d(St, S(St)) &\leq a[d(Tt, St) + d(T(St), S(St))] \\ &\quad + b[d(Tt, S(St)) + d(T(St), St)] + cd(Tt, T(St)). \end{aligned}$$

Using (5) and (6) we get $(1 - 2b - c)d(St, S(St)) \leq 0$. Therefore $S(St) = St$. So it follows from (6) that St is a common fixed point of S and T .

To show that S and T can have only one common fixed point, let $u =$

$Su = Tu$ and $v = Sv = Tv$. Then by (2),

$$\begin{aligned} d(u, v) &= d(Su, Sv) \\ &\leq a[d(Tu, Su) + d(Tv, Sv)] + b[d(Tu, Sv) + d(Tv, Su)] + cd(Tu, Tv) \\ &= (2b + c)d(u, v). \end{aligned}$$

Hence $u = v$.

Remark 1. In case $a = b = 0$, we get Jungck's theorem [2].

Corollary 1. Let S and T be two continuous and commuting mappings of a complete metric space (X, d) into itself. Suppose that there exists a positive integer n such that

$$\begin{aligned} (7) \quad d(S^n x, S^n y) &\leq a[d(Tx, S^n x) + d(Ty, S^n y)] \\ &\quad + b[d(Tx, S^n y) + d(Ty, S^n x)] + cd(Tx, Ty) \end{aligned}$$

for all x, y in X , and conditions (1) and (3) are satisfied. Then S and T have a unique common fixed point.

Proof. Since S commutes with T , S^n also commutes with T , and $S^n(X) \subset S(X) \subset T(X)$. Thus the theorem pertains to S^n and T , so there exists a unique point $u \in X$ such that $u = Tu = S^n u$. Also $T(Su) = S(Tu) = S(u) = S(S^n u) = S^n(Su)$. This says that Su is a common fixed point of T and S^n . The uniqueness of u implies that $Su = u = Tu$.

Remark 2. In case $a = b = 0$ in Corollary 1, we obtain Corollary 1 of Jungck [2].

As an immediate consequence of our theorem, we have the following

Corollary 2. Let (X, d) be a complete metric space, q a real number with $q > 5$ and n a positive integer. If S is a continuous mapping

of X onto itself such that

$$qd(x, y) \leq d(S^n x, x) + d(S^n y, y) + d(S^n x, y) + d(S^n y, x) + d(S^n x, S^n y)$$

for all x, y in X , then S has a unique fixed point.

In fact, this is obtained by letting $T = S^{n+1}$ and $1/q = a = b = c < 1/5$ in our theorem.

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