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Autometrized Griss Algebras II

By N. PRABHAKARA RAO

This is a continuation of [5] wherein we have introduced the notion of a Griss algebra and made a preliminary study of a Griss algebra autometrized by the "symmetric difference". In this paper, we study about congruence relations (algebraic and metric), construction of quotient algebras, direct limt of Griss algebras and about the uniformities on Griss algebras through the metric.

1. As any Griss algebra is a universal algebra (even though it may not in general "equationally defined"), there is already the notion of a congruence relation (which we call "algebraic congruence relation" in this paper) as an equivalence relation having substitution property with respect to all the operations. Specifically, here, in the case of a Griss algebra an equivalence relation θ is an algebraic congruence relation if and only if

(1.1) $a \equiv b(\theta)$ and $c \equiv d(\theta)$ imply $a \cup c \equiv b \cup d(\theta)$,

(1.2) $a \equiv b(\theta)$ and $c \equiv d(\theta)$ imply $a - c \equiv b - d(\theta)$.

For universal algebras which are equationally definable, the quotient sets (obtained through congruence relations) can be made into again algebras of the same "species" satisfying the same set of equations. But here by means of an example (see Example 1 below), we show that it is not possible

to make the quotient of a Griss algebra by an algebraic congruence relation into a Griss algebra. However we obtain a necessary and sufficient condition on an algebraic congruence relation θ of a Griss algebra G to make G/θ into a Griss algebra (see Theorem 1 below).

As we have mentioned in our earlier paper [5] several Griss algebras are "normal autometrized algebras" (in the sense of [8]). For the sake of completeness we give here the definition:

An autometrized algebra $A = (A, +, \leq, *)$ is called *normal* if and only if

(1) $a \leq a * 0$ for any a in A,

(2) $(a + b) * (c + d) \leq a * c + b * d$ for any a, b, c, d in A,

(3) $(a * b) * (c * d) \leq a * c + b * d$ for any a, b, c, d in A,

(4) $a \leq b$ implies there exists an $x \geq 0$ such that $a + x = b$.

In [8] we have introduced the notion of a congruence relation (called "metric congruence relation" in this paper) in a normal autometrized algebra as an equivalence relation θ satisfying (1.1) and

(1.3) $a \equiv b(\theta)$ and $c \equiv d(\theta)$ imply $a * c = b * d(\theta)$

(1.4) $a \equiv b(\theta)$ and $x * y \leq a * b$ imply $x \equiv y(\theta)$,

and we have obtained a necessary and sufficient condition on such a congruence θ to make A/θ into a normal autometized algebra. Now consequently in the case of a normal autometrized Griss algebra, we show that the two possible ways of constructing a quotient (i.e. through algebraic and metric congruences) lead to the same Griss algebra (see Theroem 4 below).

If θ is any equivalence relation on a set G, as usual G/θ denotes the quotient set (i.e. the set of all θ-equivalence classes) and for any x in G, $\bar{x}$ denotes the equivalence class containing x.

Now we prove the following theorem:

Theorem 1. *Let* θ *be an algebraic congruence relation on a Griss algebra* G. *For any* $\bar{a}, \bar{b}$ *in* G/θ *define* $\bar{a} - \bar{b} = \overline{a - b}$ *and* $\bar{a} \leq \bar{b}$ *if and only if* $\overline{a - b} = \bar{0}$. *Then* G/$\theta$ *is a Griss algebra if and only if* θ *has the following property*

(A) $a - b \equiv 0(\theta)$ *and* $b - a \equiv 0(\theta)$ *imply* $a \equiv b(\theta)$ *for any* a, b *in* G.

Proof. Suppose θ has the property (A). We shall show G/θ is a Griss algebra.

Let $\bar{a} = \bar{a}_1$ and $\bar{b} = \bar{b}_1$. Then $a \equiv a_1(\theta)$ and $b \equiv b_1(\theta)$. Hence $a - b \equiv a_1 - b_1(\theta)$ (by 1.2), so that $\overline{a - b} = \overline{a_1 - b_1}$. Hence $-$ is well defined. Let $\bar{a} \leq \bar{b}$ and $\bar{a} = \bar{a}_1$, $\bar{b} = \bar{b}_1$. Then $\overline{a - b} = \bar{0}$ and hence we have $\overline{a_1 - b_1} = \bar{a}_1 - \bar{b}_1 = \bar{a} - \bar{b} = \overline{a - b} = \bar{0}$ which implies that $\bar{a}_1 \leq \bar{b}_1$. Hence $\leq$ is well defined.

Since $0 - a = 0$ for every a in G, we have $\overline{0 - a} = \bar{0}$ which implies that $\bar{0} \leq \bar{a}$. Hence $\bar{0}$ is the least element of G/θ. Also, if $x \leq y$ then $x - y = 0$ which implies that $\overline{x - y} = \bar{0}$. Hence if $x \leq y$ then $\bar{x} \leq \bar{y}$.

$\bar{a} \leq \bar{a}$ is obvious. Let $\bar{a} \leq \bar{b}$ and $\bar{b} \leq \bar{a}$. Then $a - b \equiv 0(\theta)$ and $b - a \equiv 0(\theta)$. Hence $a \equiv b(\theta)$ by (A). Hence $\bar{a} = \bar{b}$. Let $\bar{a} \leq \bar{b}$ and $\bar{b} \leq \bar{c}$. Then $a - b \equiv 0(\theta)$ and $b - c \equiv 0(\theta)$. Hence $(a - b) \cup (b - c) \equiv 0(\theta)$ which implies that $\overline{(a - b) \cup (b - c)} = \bar{0}$. Since $a - c \leq (a - b) \cup (b - c)$, we have $\overline{a - c} \leq \overline{(a - b) \cup (b - c)} = \bar{0}$. Hence $a - c = \bar{0}$ which implies that $\bar{a} \leq \bar{c}$. Hence $\leq$ is a partial ordering on G/θ.

$a \leq a \cup b$ and $b \leq a \cup b$ imply $\overline{a \cup b}$ is an upper bound of $\bar{a}$ and $\bar{b}$. Let $\bar{a} \leq \bar{c}$ and $\bar{b} \leq \bar{c}$. Then $a - c \equiv 0(\theta)$ and $b - c \equiv 0(\theta)$ which imply that $(a - c) \cup (b - c) \equiv 0(\theta)$. Now $\overline{(a \cup b) - c} = \overline{(a - c) \cup (b - c)} = \bar{0}$

which implies that $\overline{a \cup b} \leq \bar{c}$. Hence $\overline{a \cup b}$ is the smallest upper bound of $\bar{a}$ and $\bar{b}$. Hence $(G/\theta, \leq)$ is a join semilattice with least element $\bar{0}$.

Also $(\bar{a} \cup \bar{b}) - \bar{c} = \overline{a \cup b} - \bar{c} = \overline{(a \cup b) - c} = \overline{(a - c) \cup (b - c)} = \overline{a - c} \cup \overline{b - c} = (\bar{a} - \bar{c}) \cup (\bar{b} - \bar{c})$. Since $a - c \leq (a - b) \cup (b - c)$ we also have $\bar{a} - \bar{c} = \overline{a - c} \leq \overline{(a - b) \cup (b - c)} = (\bar{a} - \bar{b}) \cup (\bar{b} - \bar{c})$. If $\bar{a} - \bar{b} = \bar{0}$ then $\overline{a - b} = \bar{0}$ which implies that $\bar{a} \leq \bar{b}$. And $\bar{a} - \bar{a} = a - a = \bar{0}$. Hence G/θ is a Griss algebra.

Suppose G/θ is a Griss algebra. Let $a - b \equiv 0(\theta)$ and $b - a \equiv 0(\theta)$. Then $\overline{a - b} = \bar{0}$ and $\overline{b - a} = \bar{0}$ which imply that $\bar{a} \leq \bar{b}$ and $\bar{b} \leq \bar{a}$. Since $\leq$ is a partial ordering on G/θ it follows that $\bar{a} = \bar{b}$. Hence $a \equiv b(\theta)$. This completes the proof of the theorem.

Clearly all algebraic congruence relations in Boolean algebras, Brouwerian algebras and semi-Brouwerian algebras have the property (A). However the following example shows that in a Griss algebra every algebraic congruence relation may not have the property (A).

Example 1. Let G be the set consisting of elements $0, a, b, c, 1$ where $\cup$ and $-$ are defined by the following tables:

$\cup$	0	a	b	c	1
0	0	a	b	c	1
a	a	a	1	a	1
b	b	1	b	b	1
c	c	a	b	c	1
1	1	1	1	1	1

$-$	0	a	b	c	1
0	0	0	0	0	0
a	c	0	c	c	0
b	c	c	0	c	0
c	c	0	0	0	0
1	c	c	c	c	c

Clearly $(G, \cup, ^-, 0)$ is a Griss algebra. Let $\theta = \{(0, 0), (a, a), (b, b), (c, c), (1, 1), (c, 0), (0, c)\}$. It is routine to verify that θ

is an algebraic congruence relation on G. Now $a - b = c \equiv 0(\theta)$ and $b - a = c \equiv 0(\theta)$. But $a \not\equiv b(\theta)$.

In the remainder of this section G stands for a normal autometrized Griss algebra. The following theorem shows that in such an algebra, algebraic congruence relations having the property (A) and metric congruence relations are the same.

Theorem 2. *Algebraic congruence relations having the property* (A) *and metric congruence relations are one and the same in* G.

Proof. Suppose θ is an algebraic congruence relation having the property (A). Let $a \equiv b(\theta)$ and $c \equiv d(\theta)$. Then $a - c \equiv b - d(\theta)$ and $c - a \equiv d - b(\theta)$ by (1.2). Hence $(a - c) \cup (c - a) \equiv (b - d) \cup (d - b)(\theta)$ by (1.1). Hence it follows that $a * c \equiv b * d(\theta)$. Let $a \equiv b(\theta)$ and $c * d \leq a * b$. Since $a \equiv b(\theta)$ we have $a * b \equiv 0(\theta)$. Now $\overline{c - d} \leq \overline{c * d} \leq \overline{a * b} = \bar{0}$. Hence $c - d = \bar{0}$ so that $c - d \equiv 0(\theta)$. Similarly we can show that $d - c \equiv 0(\theta)$. Hence $c \equiv d(\theta)$ by the property (A). Hence θ is a metric congruence relation.

Now suppose θ is a metric congruence relation on G. Let S be the metric ideal corresponding to θ (Note that there ix a one to one correspondence between metric congruence relations and metric ideals in a normal autometrized algebra (see Theorem 2 of [8])). Let $a \equiv a_1(\theta)$ and $b \equiv b_1(\theta)$. Then $(a - b) * (a_1 - b_1) = [(a \cup b) * b] * [(a_1 \cup b_1) * b] \leq [(a \cup b) * (a_1 \cup b_1)] \cup (b * b_1)$ (by the third condition of normality). $b \equiv b_1(\theta)$ implies that $b * b_1 \in S$. And $a \equiv a_1(\theta)$, $b_1 \equiv b(\theta)$ imply that $a \cup b \equiv a_1 \cup b_1(\theta)$. Hence it follows that $(a \cup b) * (a_1 \cup b_1) \in S$. So $(a - b) * (a_1 - b_1) \in S$ which implies that $a - b \equiv a_1 - b_1(\theta)$. Hence θ is an algebraic

congruence relation on G. Now let $a - b \equiv 0(\theta)$ and $b - a \equiv 0(\theta)$. Then $a * b \equiv 0(\theta)$ which implies that $a \equiv b(\theta)$. Thus θ is an algebraic congruence relation having the property (A). This completes the proof.

In [8] we have shown (Theorem 3) that if θ is a metric congruence relation on a normal autometrized algebra A then $(A/\theta, +, \leq, *)$ where $+$, $\leq$, $*$ are defined by $\bar{a} + \bar{b} = \overline{a + b}$, $\bar{a} * \bar{b} = \overline{a * b}$ and $\bar{a} \leq \bar{b}$ to mean there exists an $x \geq 0$ such that $a + x \equiv b(\theta)$ is a normal autometrized algebra if and only if θ has the following property:

(B) $a + z_1 \equiv b(\theta)$ and $b + z_2 \equiv a(\theta)$ for some $z_1, z_2 \geq 0$ imply $a \equiv b(\theta)$.

Now if θ is an algebraic congruence relation satisfying the property (A) on G then θ is a metric congruence relation (by Theorem 2) and θ has the property (B) (since $a \equiv b \cup z_2 = b \cup b \cup z_2 = b \cup a \equiv a \cup z_1 \equiv b(\theta)$). Hence by above $(G/\theta, \cup, \leq, *)$ is a normal autometrized algebra where $\bar{a} \cup \bar{b} = \overline{a \cup b}$, $\bar{a} * \bar{b} = \overline{a * b}$ and $\bar{a} \leq \bar{b}$ if and only if there exists an x in G such that $a \cup x \equiv b(\theta)$. The following theorem shows that G/θ is a normal autometrized Griss algebra.

Theorem 3. *For* $\bar{a}, \bar{b}$ *in* G/θ *define* $\bar{a} - \bar{b} = \overline{a - b}$. *Then* G/θ *is a Griss algebra.*

Proof. Clearly $-$ is well defined. We know that $\leq$ is a partial ordering on G/θ. Now we show that $\overline{a \cup b}$ is the smallest upper bound of $\bar{a}$ and $\bar{b}$. Since $a \cup b \equiv a \cup b(\theta)$ we have $\bar{a} \leq \overline{a \cup b}$ and $\bar{b} \leq \overline{a \cup b}$. Let $\bar{a} \leq \bar{c}$ and $\bar{b} \leq \bar{c}$. Then there exist z_1, z_2 such that $a \cup z_1 \equiv c(\theta)$ and $b \cup z_2 \equiv c(\theta)$. Now $c = c \cup c \equiv a \cup z_1 \cup b \cup z_2 = a \cup b \cup z_1 \cup z_2$. Hence $\overline{a \cup b} \leq \bar{c}$. Thus $\overline{a \cup b}$ is the smallest upper bound of $\bar{a}$ and $\bar{b}$. Hence $(G/\theta, \cup)$ is a join semilattice. Clearly $\bar{0}$ is the least element of G/θ. For any

$\bar{a}, \bar{b}, \bar{c}$ in G/θ, $(\bar{a} \cup \bar{b}) - \bar{c} = \overline{a \cup b} - \bar{c} = \overline{(a \cup b) - c} = \overline{(a - c) \cup (b - c)} = \overline{a - c} \cup \overline{b - c} = (\bar{a} - \bar{c}) \cup (\bar{b} - \bar{c})$. Since $\overline{(a - c)} \cup [\overline{(a - b)} \cup \overline{(b - c)}] = \overline{(a - c) \cup (a - b) \cup (b - c)} = \overline{a - c}$, we have

$$\bar{a} - \bar{c} = \overline{a - c} \leq \overline{(a - b) \cup (b - c)} = \overline{a - b} \cup \overline{b - c} = (\bar{a} - \bar{b}) \cup (\bar{b} - \bar{c}).$$

Suppose $\bar{a} - \bar{b} = \bar{0}$. Then $a - b \equiv 0(\theta)$ which implies that $(a - b) * 0 \in S$ where S is the metric ideal corresponding to θ. Since $(a \cup b) * b = a - b \leq (a - b) * 0$ (by the first condition of normality) and since $(a - b) * 0 \in S$ we have $(a \cup b) * b \in S$. Hence $a \cup b \equiv b(\theta)$ which implies that $\bar{a} \cup \bar{b} = \overline{a \cup b} = \bar{b}$. Hence $\bar{a} \leq \bar{b}$. Also $\bar{a} - \bar{a} = \overline{a - a} = \bar{0}$. Hence G/θ is a Griss algebra.

Now combining Theorems 1, 2 and 3 we have the following:

Theorem 4. *Suppose* θ *is an algebraic congruence relation having the property* (A) *on* G. *Then* G/θ *constructed via Theorem* 1 *and Theorem* 3 *are the same.*

2. Imai and Iséki [4] introduced the notion of a homomorphism in a Griss algebra. As any Griss algebra $G = (G, \cup, -, 0)$ is a universal algebra of "species" $(2, 2, 0)$, in the terminology of universal algebra, a mapping f from a Griss algebra G into an algebra H of "species" $(2, 2, 0)$ is called a *homomorphism* if and only if

(2.1) $(a \cup b)f = af \cup bf$ for any $a, b \in G$

(2.2) $(a - b)f = af - bf$ for any $a, b \in G$

(2.3) $0f = 0$.

The definition of Imai and Iséki coincides with this definition if H is also a Griss algebra. We observe (see Example 2 below) that a homomorphic image (in the sense of universal algebra) of a Griss algebra need not

be a Griss algebra. This fact also establishes that in general Griss algebras are not equationally definable. Hence there is need for us to establish the homomorphism theorems for Griss algebras (see Theorems 5, 6, 7 and 8 below).

The class of Griss algebras together with homomorphisms form a category. We show that this category is closed under the formation of direct limits (see Theorem 9 below).

The following example shows that a homomorphic image of a Griss algebra need not be a Griss algebra.

Example 2. Let G_1 be the Griss algebra as in Example 1. Let G_2 be the set consisting of elements $0'$, a', b', $1'$ where $\cup$ is defined by the following table. For any x, y in G_2 define $x - y = 0'$. Let $f:G_1 \to G_2$ be defined by $0f = cf = 0'$, $af = a'$, $bf = b'$, $1f = 1'$. Clearly f is a homomorphism from G_1 onto G_2. Now $a' - b' = 0'$ but $a' \neq b'$. Hence G_2 is not a Griss algebra.

$\cup$	$0'$	a'	b'	$1'$
$0'$	$0'$	a'	b'	$1'$
a'	a'	a'	$1'$	$1'$
b'	b'	$1'$	b'	$1'$
$1'$	$1'$	$1'$	$1'$	$1'$

If f is a homomorphism from a Griss algebra G_1 to a Griss algebra G_2, define $\ker f = \{(x, y) \in G_1 \times G_2 \mid xf = yf\}$. Clearly $\ker f$ is an algebraic congruence relation having the property (A). So we can talk of the quotient $G_1/\ker f$ for any homomorphism f on G_1. We denote elements of $G_1/\ker f$ by $[x]^f$ ($x \in G_1$). Now we prove the following:

Theorem 5. *Let $f:G_1 \to G_2$ be a homomorphism. Then $\pi_f:G_1 \to G_1/\ker f$ defined by $x\pi_f = [x]^f$ is a homomorphism. Further there exists a unique homomorphism $\bar{f}:G_1/\ker f \to G_2$ such that $\pi_f \circ \bar{f} = f$ and $\bar{f}$ is injective.*

If f *is surjective then* $G_1/\ker f$ *is isomorphic to* G_2.

Proof. It is routine to verify that π_f is a homomorphism. Let $[p]^f \in G_1/\ker f$, $p \in G_1$. Then $pf \in G_2$. Define $\bar{f}$ by $[p]^f \bar{f} = pf$. Clearly $\bar{f}$ is a homomorphism such that $\pi_f \circ \bar{f} = f$. Suppose $g: G_1/\ker f \to G_2$ is a homomorphism such that $\pi_f \circ g = f$. Then $[p]^f \bar{f} = pf = p\pi_f \circ g = (p\pi_f)g = [p]^f g$ for any $[p]^f \in G_1/\ker f$. Hence $\bar{f} = g$. If f is surjective, clearly $\bar{f}$ is onto so that $G_1/\ker f$ is isomorphic to G_2.

Denote by $\mathrm{Hom}(G_1, G_2)$, the set of all homomorphisms from G_1 to G_2.

Theorem 6. *Let* $f \in \mathrm{Hom}(G_1, G_2)$ *and* $g \in \mathrm{Hom}(G_1, G_3)$ *such that* $\ker g \subset \ker f$. *Then there exists* $h \in \mathrm{Hom}(G_1/\ker g, G_1/\ker f)$ *such that* $\pi_g \circ h = i \circ \pi_f$ *where* i *is the identity mapping on* G_1.

Proof. Let $[y]^g \in G_1/\ker g$. Since π_g is surjective there exists an $x \in G_1$ such that $x\pi_g = [y]^g$. Define $h: G_1/\ker g \to G_1/\ker f$ by $[y]^g h = [y]^f$. Let $[y_1]^g = [y_2]^g$. Then $(y_1, y_2) \in \ker g \subset \ker f$ which implies that $[y_1]^f = [y_2]^f$. Hence h is well defined. It is routine to verify that h is a homomorphism. For any $x \in G_1$ we have $x\pi_g \circ h = (x\pi_g)h = [x]^g h = [x]^f = x\pi_f = (xi)\pi_f = xi \circ \pi_f$. Hence $\pi_g \circ h = i \circ \pi_f$.

Theorem 7. *Let* $f \in \mathrm{Hom}(G_1, G_2)$ *and* $g \in \mathrm{Hom}(G_1, G_3)$ *such that* $\ker g \subset \ker f$. *Then there exists* $g^* \in \mathrm{Hom}(G_1/\ker g, G_2)$ *such that* $f = \pi_g \circ g^*$.

Proof. By the above theorem there exists $h \in \mathrm{Hom}(G_1/\ker g, G_1/\ker f)$ such that $\pi_g \circ h = i \circ \pi_f$. Now $g^* = h \circ \bar{f}$ has the required properties.

Theorem 8. *Let* $f \in \mathrm{Hom}(G_1, G_2)$ *and* $g \in \mathrm{Hom}(G_1, G_3)$ *such that* $\ker g \subset \ker f$. *If* g *is surjective then there exists* $\Phi \in \mathrm{Hom}(G_3, G_2)$ *such*

that $f = g \circ \Phi$.

Proof. Since g is surjective, $\bar{g}: G_1/\ker g \to G_3$ is an isomorphism such that $\pi_g \circ \bar{g} = g$. Also $\bar{f}: G_1/\ker f \to G_2$ is a homomorphism such that $\pi_f \circ \bar{f} = f$. By Theorem 6 there exists a homomorphism $h: G_1/\ker g \to G_1/\ker f$ such that $\pi_g \circ h = i \circ \pi_f$. Let $\Phi = (\bar{g})^{-1} \circ h \circ \bar{f}$. It is routine to verify that Φ is a homomorphism from G_3 to G_2. And $g \circ \Phi = \pi_g \circ \bar{g} \circ (\bar{g})^{-1} \circ h \circ \bar{f} = \pi_g \circ h \circ \bar{f} = i \circ \pi_f \circ \bar{f} = i \circ f = f$. This completes the proof.

Following Gratzer [3] by a *direct family of Griss algebras* we mean a triple of the following objects:

(1) A directed partially ordered set $(I, \leq)$;

(2) Griss algebras A_i, $i \in I$;

(3) Homomorphisms f_{ij} of A_i into A_j for all $i \leq j$ such that $f_{ij} f_{jk} = f_{ik}$ if $i \leq j \leq k$ and f_{ii} is the identity mapping for each $i \in I$.

Let A be a direct family of Griss algebras A_i and define $\equiv$ on $\bigcup\{A_i \mid i \in I\}$ by $x \equiv y$ if and only if $x \in A_i$, $y \in A_j$ for some $i, j \in I$ and there exists a $k \in I$ such that $xf_{ik} = yf_{jk}$. Then $\equiv$ is an equivalence relation. Let A_∞ denote the set of all equivalence classes. We call A_∞ the *direct limit* of the direct family.

The following theorem shows that the category of Griss algebra is closed under the formation of direct limits.

Theorem 9. *The direct limit* A_∞ *of a direct family* A *of Griss algebras is a Griss algebra.*

Proof. Let $\bar{x}, \bar{y} \in A_\infty$. Let $x \in A_i$, $y \in A_j$ for some $i, j \in I$. Let k be an upper bound of i and j. Put $x' = xf_{ik}$ and $y' = yf_{jk}$. Then

$x \equiv x'$ and $y \equiv y'$. Define $\bar{x} - \bar{y} = x' - y'$ and $\bar{x} \leq \bar{y}$ if and only if $\overline{x' - y'} = \bar{0}$ (Note that $\bar{0}_i = \bar{0}_j = \cdots$ (= $\bar{0}$ say) where 0_i, 0_j, $\cdots$ are the least elements of A_i, A_j, $\cdots$). It is routine to verify that $-$ is well defined. Let $\bar{x} \leq \bar{y}$ and $\bar{x} = \bar{x}_1$, $\bar{y} = \bar{y}_1$. Then $\overline{x' - y'} = \bar{0}$ and so $\bar{x}_1 - \bar{y}_1 = \bar{x} - \bar{y} = \overline{x' - y'} = \bar{0}$. Hence $\bar{x}_1 \leq \bar{y}_1$. Thus $\leq$ is well defined.

$\bar{x} \leq \bar{x}$ is obvious. Let $\bar{x} \leq \bar{y}$ and $\bar{y} \leq \bar{x}$. Hence $xf_{ik} - yf_{jk} \equiv 0$ and $yf_{jk} - xf_{ik} \equiv 0$ where k is an upper bound of i and j. Hence $(xf_{ik} - yf_{jk})f_{ks} = 0_k f_{ks} = 0_s$ (where s is an upper bound of k, k) and $(yf_{jk} - xf_{ik})f_{kt} = 0_k f_{kt} = 0_t$ (where t is an upper bound of k, k). Hence it follows that $xf_{is} - yf_{js} = 0_s$ and $yf_{jt} - xf_{it} = 0_t$. Let p be an upper bound of s and t. Now $xf_{ip} - yf_{jp} = xf_{is}f_{sp} - yf_{js}f_{sp} = (xf_{is} - yf_{js})f_{sp} = 0_s f_{sp} = 0_p$. Hence $xf_{ip} \leq yf_{ip}$. Similarly we can prove $yf_{ip} \leq xf_{ip}$. Hence $xf_{ip} = yf_{jp}$ which implies that $\bar{x} = \bar{y}$. Let $\bar{x} \leq \bar{y}$ and $\bar{y} \leq \bar{z}$. Let $x \in A_i$, $y \in A_j$, $z \in A_q$. Since $\bar{x} \leq \bar{y}$, $xf_{ik} - yf_{jk} \equiv 0$ where k is an upper bound of i and j. Hence $(xf_{ik} - yf_{jk})f_{ks} = 0_k f_{ks} = 0_s$ where $k \leq s$. Since $\bar{y} \leq \bar{z}$, $yf_{jm} - zf_{qm} \equiv 0$ where m is an upper bound of j and q. Hence $(yf_{jm} - zf_{qm})f_{mt} = 0_m f_{mt} = 0_t$ for some $t \geq m$. To show $\bar{x} \leq \bar{z}$ we have to show $xf_{in} - zf_{qn} \equiv 0$ where n is an upper bound of i and q. Now let p be an upper bound of s, n, t. We have then

$$
\begin{aligned}
(xf_{in} - zf_{qn})f_{np} &= xf_{ip} - zf_{qp} \\
&\leq (xf_{ip} - yf_{jp}) \cup (yf_{jp} - zf_{qp}) \\
&= (xf_{ik}f_{kp} - yf_{jk}f_{kp}) \cup (yf_{jm}f_{mp} - zf_{qm}f_{mp}) \\
&= [(xf_{ik} - yf_{jk})f_{kp}] \cup [(yf_{jm} - zf_{qm})f_{mp}] \\
&= [(xf_{ik} - yf_{jk})f_{ks}f_{sp}] \cup [(yf_{jm} - zf_{qm})f_{mt}f_{tp}] \\
&= 0_s f_{sp} \cup 0_t f_{tp} = 0_p \cup 0_p = 0_p = 0_n f_{np}.
\end{aligned}
$$

Hence $\overline{xf_{in} - zf_{qn}} = \bar{0}$ which implies $\bar{x} \leq \bar{z}$. Therefore $\leq$ is a partial ordering. Clearly $\bar{0}$ is the least element.

Now we show $(A_\infty, \leq)$ is a join semilattice. Let $\bar{x}, \bar{y} \in A_\infty$. Let $x \in A_i$, $y \in A_j$. Suppose $x' = xf_{ik}$ and $y' = yf_{jk}$ where k is an upper bound of i and j. It is our claim that $\overline{x' \cup y'}$ is the smallest upper bound of $\bar{x}$ and $\bar{y}$. Let t be an upper bound of i and k. Then we have $xf_{it} - [(x' \cup y')f_{kt}] = xf_{it} - [(xf_{ik} \cup yf_{jk})f_{kt}] = xf_{it} - [xf_{it} \cup yf_{jt}] = 0_t$. Hence $xf_{ik} - [(x' \cup y')f_{kt}] \equiv 0$ which implies that $\bar{x} \leq \overline{x' \cup y'}$. Similarly we can prove $\bar{y} \leq \overline{x' \cup y'}$. Let $\bar{z}$ be an upper bound of $\bar{x}$ and $\bar{y}$. Let $z \in A_q$. Let $x'' = xf_{im}$ and $z' = zf_{qm}$ where m is an upper bound of i and q. Suppose $y'' = yf_{jn}$ and $z'' = zf_{qn}$ where n is an upper bound of j and q. Since $\bar{x} \leq \bar{z}$ we have $x'' - z' \equiv 0$ which means that $(x'' - z')f_{mp} = 0_p$ for some $p \geq m$. Since $\bar{y} \leq \bar{z}$ we have $y'' - z'' \equiv 0$ which means that $(y'' - z'')f_{nr} = 0_r$ for some $r \leq n$. Note that $x' \cup y' \in A_k$ and $z \in A_q$. To show $\overline{x' \cup y'} \leq \bar{z}$ we have to show $(x' \cup y')f_{ks} - zf_{qs} \equiv 0$ where s is an upper bound of k and q. Now let t be an upper bound of s, p, r. Then we have

$$
\begin{aligned}
[(x' \cup y')f_{ks} - zf_{qs}]f_{st} &= [(x'f_{ks} \cup y'f_{ks}) - zf_{qs}] \\
&= [(x'f_{ks} \cup y'f_{ks})f_{st}] - zf_{qs}f_{st} \\
&= (x'f_{kt} \cup y'f_{kt}) - zf_{qt} \\
&= (xf_{ik}f_{kt} \cup yf_{jk}f_{kt}) - zf_{qt} \\
&= (xf_{it} \cup yf_{jt}) - zf_{qt} \\
&= (xf_{im}f_{mt} \cup yf_{jn}f_{nt}) - zf_{qt} \\
&= (x''f_{mt} \cup y''f_{nt}) - zf_{qt} \\
&= (x''f_{mt} - zf_{qt}) \cup (y''f_{nt} - zf_{qt}) \\
&= (x''f_{mp}f_{pt} - zf_{qm}f_{mt}) \cup (y''f_{nr}f_{rt} - zf_{qn}f_{nt}) \\
&= (x''f_{mp}f_{pt} - z'f_{mt}) \cup (y''f_{nr}f_{rt} - z''f_{nt}) \\
&= (x''f_{mp}f_{pt} - z'f_{mp}f_{pt}) \cup (y''f_{nr}f_{rt} - z''f_{nr}f_{rt})
\end{aligned}
$$

$$= [(x''f_{mp} - z'f_{mp})f_{pt}] \cup [(y''f_{nr} - z''f_{nr})f_{rt}]$$
$$= 0_p f_{pt} \cup 0_r f_{rt} = 0_t \cup 0_t = 0_t = 0_s f_{st}.$$

Hence $(x' \cup y')f_{ks} - zf_{qs} \equiv 0$ which implies $\overline{x' \cup y'} \leq \bar{z}$. So $\overline{x' \cup y'}$ is the smallest upper bound of $\bar{x}$ and $\bar{y}$.

It is routine to verify that $\bar{x} - \bar{x} = \bar{0}$ and that $\bar{x} - \bar{y} = \bar{0}$ implies $\bar{x} \leq \bar{y}$. Let $\bar{x}, \bar{y}, \bar{z} \in A_\infty$. Suppose $x \in A_i$, $y \in A_j$ and $z \in A_q$. Let $x' = xf_{ik}$ and $y' = yf_{jk}$ where k is an upper bound of i and j. Then

$$(\bar{x} \cup \bar{y}) - \bar{z} = \overline{x' \cup y'} - \bar{z}$$
$$= \overline{(x' \cup y')f_{ks} - zf_{qs}} \text{ (where } s \text{ is an upper bound of } k \text{ and } q)$$
$$= \overline{(x'f_{ks} \cup y'f_{ks}) - zf_{qs}} = \overline{(x'f_{ks} - zf_{qs}) \cup (y'f_{ks} - zf_{qs})}$$
$$= \overline{(xf_{ik}f_{ks} - zf_{qs}) \cup (yf_{jk}f_{ks} - zf_{qs})}$$
$$= \overline{(xf_{is} - zf_{qs}) \cup (yf_{js} - zf_{qs})}$$
$$= \overline{[(xf_{is} - zf_{qs})f_{ss}] \cup [(yf_{js} - zf_{qs})f_{ss}]}$$
$$= \overline{xf_{is} - zf_{qs}} \cup \overline{yf_{js} - zf_{qs}}$$
$$= (\bar{x} - \bar{z}) \cup (\bar{z} - \bar{y}).$$

To prove $\bar{x} - \bar{y} \leq (\bar{x} - \bar{z}) \cup (\bar{z} - \bar{y})$, it is sufficient to show that $(\bar{x} - \bar{y}) \cup (\bar{x} - \bar{z}) \cup (\bar{z} - \bar{y}) = (\bar{x} - \bar{z}) \cup (\bar{z} - \bar{y})$. Now let k (resp. n, m, s, t) be an upper bound of i and j (resp. i and q, q and j, m and n, k and s). Then we have

$$(\bar{x} - \bar{y}) \cup (\bar{x} - \bar{z}) \cup (\bar{z} - \bar{y}) = \overline{xf_{ik} - yf_{jk}} \cup \overline{xf_{in} - zf_{qn}} \cup \overline{zf_{qm} - yf_{jm}}$$
$$= \overline{(xf_{ik} - yf_{jk}) \cup (xf_{in} - zf_{qn})f_{ns} \cup (zf_{qm} - yf_{jm})f_{ms}}$$
$$= \overline{(xf_{ik} - yf_{jk})f_{kt} \cup [(xf_{in} - zf_{qn})f_{ns} \cup (zf_{qm} - yf_{jm})f_{ms}]f_{st}}$$
$$= \overline{(xf_{it} - yf_{jt}) \cup [(xf_{in} - zf_{qn}) \cup (zf_{qm} - yf_{jm})f_{mt}]}$$
$$= \overline{(xf_{it} - yf_{jt}) \cup (xf_{it} - zf_{qt}) \cup (zf_{qt} - yf_{jt})}$$
$$= \overline{(xf_{it} - zf_{qt}) \cup (zf_{qt} - yf_{jt})} \quad *)$$

---

*) In fact $xf_{it} - yf_{jt} \leq (xf_{it} - zf_{qt}) \cup (zf_{qt} - yf_{jt})$

$$= \overline{xf_{it} - zf_{qt}} \cup \overline{zf_{qt} - yf_{jt}}$$
$$= (\bar{x} - \bar{z}) \cup (\bar{z} - \bar{y}).$$

Thus $\bar{x} - \bar{y} \leq (\bar{x} - \bar{z}) \cup (\bar{z} - \bar{y})$. Hence A_∞ is a Griss algebra. This completes the proof.

3. In this section, we introduce uniform structures on a Griss algebra via the distance function and extend some of the major results of Rema [6] on "auto-topologies in Boolean algebras".

Following Gratzer [2], by a *dual ideal* P of a Griss algebra G we mean a nonempty subset P of G satisfying the following:

(3.1) $a \in P$ and $a \leq b$ imply $b \in P$.

(3.2) $a \in P$ and $b \in P$ imply existence of an element $d \in P$ such that $d \leq a$ and $d \leq b$.

Now we prove the following theorem:

Theorem 10. *A dual ideal* P *of a Griss algebra* G *is non principal if and only if, for each* $p \in P$ *there exists an element* $q \in P$ *such that* $q < p$.

Proof. Suppose $p \in P$ implies existence of an element $q \in P$ such that $q < p$. If P is a principal dual ideal, say $[a)$, then $a \in P$ implies that $q < a$ for some $q \in P$; but $q \in P$ yields $q \geq a$ which is absurd. So P is non principal.

Suppose P is a non principal dual ideal. Let $p \in P$. Since $P \neq [p)$, there exists an element $r \in P$ such that $r \notin [p)$. Hence $p \not\leq r$. Now $r \in P$ and $p \in P$. Since P is a dual ideal, it follows that there exists an element $q \in P$ such that $q \leq r$ and $q \leq p$. If $q = p$ then $p \leq r$ which is not true. Hence $q \neq p$ and so $q < p$. This completes the proof of the theorem.

Now we state below a few theorems the proofs of which follow almost as on the same lines as in [6].

Theorem 11. *Let* P *be any dual ideal of* G. *Then the subsets* $A_p = \{(x, y) \mid d(x, y) \leq p\}$ $(p \in P)$ *form a base for a uniformity and define a uniform structure on* G.

The uniformities defined above are called *auto-uniformities* and the corresponding topologies are called *auto-topologies*.

Theorem 12. *Let* P *be a non principal dual ideal of* G. *Then the subsets* $U_p = \{(x, y) \mid d(x, y) < p\}$ $(p \in P)$ *form a base for a uniformity and define a uniform structure on* G.

The uniformities defined above are called *strict auto-uniformities* and the corresponding topologies are called *strict auto-topologies*.

Theorem 13. *Let* P *be a non principal dual ideal of* G. *Then the strict auto-uniformity and the auto-uniformity defined by* P *on* G *are equivalent.*

Theorem 14. *Let* P *be any dual ideal of* G, *and* T_P *be the auto-topology defined by* P *on* G. *Then*

(1) *The subsets* $A_p(a) = \{x \mid d(x, a) \leq p\}$ $(a \in G, p \in P)$ *are closed in* (G, T_P).

(2) *The subsets* $A_p(a)$ *are open in* (G, T_P) *and constitute a base of neighborhoods at* a.

From the above theorem, the following is clear.

Theorem 15. *Let* T_P *be the auto-topology defined by* P *on* G. *Then*

(G, T_p) *is zero dimensional.*

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