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## A Note on States of von Neumann Algebras

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In this note we essentially prove that on a von Neumann algebra (possibly of uncountable cardinality) there exists a family of states having mutually orthogonal supports (projections) converging to the identity operator. The projections thus obtained yield a direct sum decomposition of the von Neumann algebra into subalgebras which can be very useful in the quantum field theory.

In what follows,  $M$  denotes a von Neumann algebra acting on a Hilbert space  $H$ . Let  $\phi$  be a positive linear functional on  $M$  such that  $\|\phi\| = 1$ ; such a  $\phi$  is called a *state* on  $M$ . If  $p$  is the greatest of all projections  $q$  with  $\phi(q) = 0$ , then the projection  $1 - p$  is called the *support* of  $\phi$  (see for example [1; p. 31]). I am grateful to Professor A. Van Daele for many useful discussions.

**Theorem.** *Let  $M$  be a von Neumann algebra. Then there exists a family  $\{\phi_\alpha : \alpha \in \Omega\}$  of normal states whose supports  $e_\alpha$  are mutually orthogonal and  $\sum_{\alpha \in \Omega} e_\alpha = 1$ .*

**Proof.** Let  $\mathcal{J}$  be a collection of all families  $\{\psi_\alpha : \alpha \in \Omega\}$  where  $\psi_\alpha$  are normal states whose supports are mutually orthogonal.  $\mathcal{J}$  can be ordered by inclusion. Let  $\mathcal{J}_0$  be a chain in  $\mathcal{J}$ . Put

$$A = \bigcup_{\beta \in \mathcal{J}_0} \beta.$$

Then every element in  $A$  is an element of some  $\beta$  in  $\mathcal{J}_0$  and therefore is a normal state on  $M$ . Let  $\psi_1$  and  $\psi_2$  be two distinct elements in  $A$ . Then there exist  $\beta_1$  and  $\beta_2$  such that  $\psi_1 \in \beta_1$  and  $\psi_2 \in \beta_2$ . Since  $\mathcal{J}_0$  is a chain, we have  $\beta_1 \subset \beta_2$  or  $\beta_2 \subset \beta_1$ . In the first case  $\psi_1, \psi_2 \in \beta_2$  and hence  $\psi_1$  and  $\psi_2$  have mutually orthogonal supports. Similarly, we have the same result for the second case. It follows that  $A$  is a family of normal states with mutually orthogonal supports. Therefore  $A \in \mathcal{J}$  and is an upper bound for  $\mathcal{J}_0$ . Hence using Zorn's lemma, we obtain a family  $\{\phi_\alpha : \alpha \in \Omega\}$  as a maximal element in  $\mathcal{J}$  with mutually orthogonal supports  $e_\alpha$ . Put

$$e = \sum_{\alpha \in \Omega} e_\alpha.$$

The sum is well-defined because  $e_\alpha$  are mutually orthogonal. If  $e \neq 1$ , then choose a vector  $\xi \neq 0$  in the Hilbert space  $H$  such that  $(1 - e)\xi = \xi$  or in other words  $\xi \in (1 - e)H$ . Put  $\phi(x) = \langle x\xi, \xi \rangle$  for all  $x$  in  $M$ . Then  $\phi$  is a state on  $M$ . Since  $\phi(e) = \langle e\xi, \xi \rangle = 0$ , the support of  $\phi$  is orthogonal to  $e$  and hence to all  $e_\alpha$ . Thus  $\{\phi_\alpha : \alpha \in \Omega\} \cup \{\phi\}$  is again in  $\mathcal{J}$ . This contradicts the maximality, and so  $e = 1$ . This completes the proof of Theorem.

#### Reference

- [1] S. Sakai: C\*-algebras and W\*-algebras, Ergebnisse der Math. und ihrer Grebzgebite, Bd. 60, Springer-Verlag, 1971.

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