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Characterizations of Certain Elements in Semigroups

By Sándor LAJOS

Let S be a semigroup. An element a of S is called *regular* if there is an element x in S such that

(1) $$axa = a.$$

The element a is said to be *intraregular* if there exist two elements x, y in S such that

(2) $$xa^2y = a.$$

The element a of S is called *completely regular* if it is regular and the condition

(3) $$ax = xa$$

holds for the generalized inverse x of a.

It is known that an element a of a semigroup S is regular if and only if the condition

(4) $$L(a) \cap R(a) = R(a)L(a)$$

holds (see [3]). An element a of S is intraregular if and only if the inclusion

(5) $$L(a) \cap R(a) \subset L(a)R(a)$$

holds. An element a of a semigroup S is completely regular if and only if there exists an idempotent element e in S such that

(6) $$L(a) \cap R(a) = R(e)L(e).$$

In this short note some further ideal-theoretic characterizations will be given for regular (intraregular, completely regular) element of a semigroup S.

Theorem 1. *An element a of a semigroup S is regular if and only if the condition*

$$B(a) \cap Q(a) = R(a)L(a)$$

holds.

Proof. First, assume that S is a semigroup and a is a regular element of S. Then (1) implies that $B(a) = Q(a) = aSa$, $R(a) = aS$, and $L(a) = Sa$. Hence it follows that

$$aS \cap Sa = aSa = (aS)(Sa).$$

Conversely, if we have (7), then

$$a \in B(a) \cap Q(a) = R(a)L(a) = (aS^1)(S^1a),$$

whence it follows that a is regular.

Theorem 2. *An element a of a semigroup S is intraregular if and only if the inclusion*

$$B(a) \cap Q(a) \subset L(a)R(a) \tag{8}$$

holds.

Proof. Let $a \in S$ be an intraregular element, i.e., (2) holds true. If $b \in B(a) \cap Q(a)$, then we have

$$b = aza = aw \qquad (z, w \in S^1).$$

Hence

$$b = xa^2yza = (xa)(ayza) \in L(a)R(a),$$

that is, the inclusion (8) holds.

Conversely, assume (8). Then we have

$$a \in B(a) \cap Q(a) \subset L(a)R(a) = (S^1a)(aS^1),$$

whence it follows at once that a is intraregular.

Theorem 3. *An element a of a semigroup S is both regular and intraregular if and only if the condition*[*)]

(9) $$L(a) \cap R(a) = R(a)I(a^2)L(a)$$

holds, where $I(a^2)$ denotes the principal interior ideal of S generated by the element a^2 of S.

Proof. First, assume that $a \in S$ is both regular and intraregular, i.e., there are elements x, y in S such that $a = axa^2ya$. Suppose that $b \in L(a) \cap R(a)$. Then we have $L(a) \cap R(a) = aS \cap Sa = aS^2a$, whence $b = aza$ ($z \in S^2$). Finally, we conclude that

$$b = (axa^2ya)za = (ax)a(ay)a(za) \in R(a)I(a^2)L(a),$$

and the inclusion (9) holds true.

Conversely, assume (9). Then we have

$$a \in L(a) \cap R(a) = R(a)I(a^2)L(a) =$$
$$= (aS^1)(a^2 \cup Sa^2S)(S^1a),$$

whence it follows that a is both regular and intraregular.

Theorem 4. *An element a of a semigroup S is completely regular if and only if the principal bi-ideal $B(a)$ is a monoid.*

Proof. If $a \in S$ is a completely regular element, then $B(a) = R(e)L(e)$

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*) A subsemigroup A of a semigroup S is said to be *interior ideal* of S if $SAS \subset A$. $I(a)$ denotes the principal interior ideal of S generated by the element a of S.

(this follows from (6)), and e is the two-sided identity element of $B(a)$.

Conversely, suppose that $B(a)$ is a monoid and e is the identity of $B(a)$. Then $e = a$, or $e \in aS^1a$. If $e = axa$, where $x \in S$, then it follows that

$$a = a^2xa = axa^2,$$

whence

$$a = a^2xaxa^2,$$

and thus a is completely regular (see [5]).

Corollary. *Let M_n be the multiplicative semigroup of all $n \times n$ matrices over an arbitrary field. An element A of M_n has a group inverse X if and only if the set AM_nA is a multiplicative monoid.*

$X \in M_n$ is a *group inverse* of A if $AXA = A$, $XAX = X$, and $AX = XA$ (see [1]).

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