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Relations Between Some Conjectures in Knot Theory

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We consider the following conjectures:

C.1 : *Every non-trivial strongly invertible knot has Property* P_1.

C.2 : *Every non-trivial strongly invertible knot has Property R.*

C.3 : *If a band is attached to a trivial knot resulting a trivial knot. then the band is isotopic to a half twisted trivial band.*

C.4 : *If a band is attached to a trivial knot resulting a trivial link of 2 components, then the band is isotopic to a trivial band.*

In § 1, we state C.3 and C.4 more precisely.

Concerning Property P_1 and Property R, many mathematicians have studied (for example, see [6] and [7]), but C.1 and C.2 are still open, and of course, so are C.3 and C.4. For C.1 and C.2, we refer the reader to Problem 1.15 and Problem 1.16 in [2], and for C.3 and C.4 to Problem 1.2 in [2] and Section 10 of [8].

In this paper, we shall prove the following results:

Theorem 1. C.1 *and* C.3 *are equivalent.*

Theorem 2. C.2 *and* C.4 *are equivalent.*

For the proof of Theorem, we use Montesinos' technique [4, 5].

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1. Preliminaries

Throughout this paper, we work in the combinatorial category.

By D^n, S^n and R^3, we denote respectively an n-disk, an n-sphere and a 3-space, and by I the unit interval $[0, 1]$. We denote

$$R^3_+ = \{(x, y, z) \in R^3 \ ; \ y \geqq 0\}$$

and

$$R^2 = \{(x, y, z) \in R^3 \ ; \ y = 0\}.$$

We may regard S^3 as a one point compactification $R^3 \cup \{\infty\}$ of R^3.

For a manifold M, ∂M and $int\ M$ are the boundary and the interior of M. In this paper, every submanifold is locally flat.

For manifolds M and N, let M_0 be a submanifold of M. And f is an embedding of M_0 into N, then $M \underset{f}{\cup} N$ is obtained from $M \cup N$ by identifying M_0 and $f(M_0)$.

A *knot* K is an embedded S^1 in S^3 (or R^3), and a *link* L is a disjoint union of knots. A link L (or a knot K) is *trivial* if each component bounds a mutually disjoint 2-disk. A simple arc α in a manifold M, with $\alpha \cap \partial M = \partial\alpha$, is *unknotted* if there exists an simple arc α' in ∂M such that $\alpha \cup \alpha'$ bounds a 2-disk.

A knot K is *strongly invertible* [4] if there exists an orientation preserving involution of S^3 which induces an involution of K with two fixed points. By Waldhausen's results [10], this involution is essentially a rotation about an axis.

Let N be a regular neighbourhood of a knot K in S^3, then we choose a meridian-longitude pair (m, ℓ) for ∂N, where m bounds a disk in N and ℓ is homologous to zero in $S^3 - int\ N$, and m, ℓ are oriented such that their intersection number is +1 on ∂N. Let f be a homeomorphism of ∂N onto itself with $f(m) = m^{\alpha}\ell^{\beta}$, then $(S^3 - int\ N) \underset{f}{\cup} N$ is said to *be obtained from* S^3 *by perfoming* (α, β)*-surgery along* K. Especially, in the case of $\beta = 1$, we call it α*-surgery*.

A knot K, in S^3, has *Property* P_1 if, for any ±1-surgery along K, S^3 is not obtained; and K has *Property R* if, for any 0-surgery along K, $S^1 \times S^2$ is not obtained.

Proposition 0. *Trivial knot has neither Property* P_1 *nor Property R.*

Let M be a 3-dimensional manifold, $\pi : M \to S^3$ be a simplicial map, and L be a link in S^3. Then M is said to be a *covering space of* S^3 *branched over* L if $\pi|_{M-\pi^{-1}(L)} : M - \pi^{-1}(L) \to S^3 - L$ is a covering space. We say that M is a *2-fold covering space of* S^3 *branched over* L if the associated covering space is 2-fold. By Waldhausen's results [9, 10] imply the following [1] ;

Proposition 1. *Let* M *be a 2-fold covering space of* S^3 *branched over a link* L. *Then* M *is homeomorphic to* $S^1 \times S^2$ *if and only if* L *is a trivial link of 2 components. And* M *is homeomorphic to* S^3 *if and only if* L *is a trivial knot.*

We use the following maps $p, u : R^3 \cup \{\infty\} \to R^3 \cup \{\infty\}$ defined by

(1) $p(\infty) = u(\infty) = \infty$,

(2) $p(r \cos \theta, r \sin \theta, z) = (r \cos 2\theta, r \sin 2\theta, z)$

for $(r \cos \theta, r \sin \theta, z) \in R^3$, and

$$(3)\quad u(x,\ y,\ z) = (-x,\ -y,\ z) \quad \text{for} \quad (x,\ y,\ z) \in R^3.$$

Then u is an orientatoon preserving involution of S^3, where the fixed point set $Fix(u)$ of u is the z-axis $\cup\{\infty\}$; and $p : S^3 \to S^3$ is a 2-fold covering space of S^3 branched over $Fix(u)$.

An *isotopy* of a manifold M is a homeomorphism H of $M \times I$ onto itself such that H is level preserving, i.e., $H(x,\ t) = (h_t(x),\ t)$, where h_t is a homeomorphism of M onto itself for $t \in I$, and $h_0 = i_M$.

An isotopy H of S^3 will be called a *u-isotopy* if, for all $t \in I$, $uh_t = h_tu$. For a subset A in M, if $h_t|_A$ is the identity map for any t, then we say H *fixes* A.

From the definition, we can easily see the following lemma,

Lemma 1. *Let H be a u-isotopy of S^3, then there exists an isotopy $\bar{H}$ of S^3 such that $\bar{h}_1 p = ph_1$.*

Let K be a knot in S^3, and β be an embedding of $D^1 \times I$ into S^3 such that $\beta(D^1 \times I) \cap K = \beta(D^1 \times \partial I)$. Then we define

$$F(K;\ \beta) = (K - \beta(D^1 \times \partial I)) \cup \beta(\partial D^1 \times I)$$

and β is said to be a *band attaching to* K. Two bands β_1 and β_2, attaching to a knot K, are *equivalent* if there exists an isotopy H of S^3 such that $h_1(K) = K$ and $h_1(\beta_1(D^1 \times I)) = \beta_2(D^1 \times I)$.

Now we restate the conjectures C.3 and C.4 more precisely.

C.3 : *Let K be a trivial knot in S^3, and β_i be a band attaching to K such that $F(K;\ \beta_i)$ is a trivial knot for $i = 1,2$. Then β_1 and β_2 are equivalent.*

C.4 : *In the hypothesis of* C.3, *we assume* $F(K;\ \beta_i)$ *is a trivial link of* 2 *components instead of a trivial knot. Then* β_1 *and* β_2 *are equivalent.*

2. A property on trivial knots

Let K be a trivial knot in S^3 such that $u(K) = K$ and $K \cap Fix(u) = \{P_1,\ P_2\}$, two points.

Because K is a trivial knot, the following is easily seen [3],

Lemma 2. *There exists a* 2*-disk* D *in* S^3 *such that* $\partial D = K$ *and* $D \cap u(D)$ *is a set of simple closed curves.*

Moreover we can prove the following by well known cut-and-paste technique:

Lemma 3. *There exists a* 2*-disk* D *in* S^3 *such that* $\partial D = K$ *and* $D \cap u(D) = K$.

Now we will prove the following lemma :

Lemma 4. *Let* S *be a* 2*-sphere in* S^3 *with* $u(S) = S$, *then there exists a u-isotopy* H *of* S^3 *and a simple closed curve* k *in* $S \cap R^2$ *such that* H *fixes* k *and* $h_1(S) \cap R^2 = k$.

Proof. We can suppose that $S \cap R^2$ is a set of simple closed curves, and we put $S \cap R^2 = \{a_1, \cdots, a_m, \tilde{a}_1, \cdots, \tilde{a}_m, c_1, \cdots, c_n\}$ such that $\tilde{a}_i = u(a_i)$, $u(a_i) \cap a_j = \emptyset$ and $u(c_i) = c_i$ for all i and j.

Then, in a_i's, there exists an inner most curve on S and R^2, and put a_i one of these curves. Then $\tilde{a}_i$ is also an inner most curve. We choose a 2-disk Δ_i on S such that $\partial\Delta_i = a_i$ and $int\ \Delta_i \cap R^2 = \emptyset$, and let D_i be a 2-disk on R^2 such that $\partial D_i = a_i$. Then there exists a 3-disk V_i in S^3 such that $\partial V_i = D_i \cup \Delta_i$. We can suppose that $D_i \times I$ is embedded in S^3 such that $D_i \times I \cap S = \partial D_i \times I$ and $D_i \times I \cap V_i = D_i \times \{0\} = D_i \times I \cap R^2$. Then $V_i \cup D_i \times I$ is a 3-disk and

$$S' = (S - (\Delta_i \cup \partial D_i \times I)) \cup D_i \times \{1\}$$

is a 2-sphere such that $S' \cap R^2$ doesn't contain a_i.
We put $S'' = p^{-1}(p(S'))$, then S'' is also a 2-sphere such that $u(S'')=S''$ and $S'' \cap R^2 = S \cap R^2 - \{a_i, \tilde{a}_i\}$. By the construction of S'', we get a u-isotopy of S^3 which moves S to S'' and fixes $S \cap R^2 - \{a_i, \tilde{a}_i\}$. Repeating this process, we can move S to S_0 by a u-isotopy of S^3 which fixes $\{c_1, \cdots, c_n\}$, and $S_0 \cap R^2 = \{c_1, \cdots, c_n\}$.

Suppose $n \geqq 2$. For some c_i and c_j $(i \neq j)$, there exists an arc γ on $R^3_+ \cap S_0$ such that γ spans c_i and c_j. We may suppose that c_i bounds a 2-disk B in R^2 such that $B \cap c_j = \emptyset$. (B may intersect with other c_r for $r \neq i, j$.) Let B' be a regular neighbourhood of B in R^2, and we put $c' = \partial B'$. Then $\xi = \gamma \cup u(\gamma)$ is a closed curve on S, c' is a simple closed curve in $S^3 - S$ and has a linking number +1 or -1 with ξ, this means that c' is not homotopic to zero in $S^3 - S$. This is a contradiction. The case $n = 0$ is impossible. Hence $n = 1$. Therefore $k = c_1$ is fixed by the above u-isotopy of S^3. This completes the proof.

Lemma 5. *Let S be a 2-sphere in S^3 such that $u(S) = S$ and $S \cap R^2$ is a simple closed curve in R^2, and K be a trivial knot on S with $u(K) = K$ and $K \cap Fix(u) = \{P_1, P_2\}$. Then there exists a u-isotopy H of*

S^3 *such that* H *fixes* $\{P_1, P_2\}$ *and* $h_1(K) \cap R^3_+$ *is an unknotted arc spanning* P_1 *and* P_2.

Proof. Let $k = K \cap R^2$. Then, by the similar arguments of the proof of Lemma 4, we can eliminate the intersection $k \cap K$ except P_1 and P_2. In fact, this is done by a u-isotopy H of S^3 which fixes $\{P_1, P_2\}$. This completes the proof of Lemma 5.

Combining the above Lemmas 3, 4 and 5, we have the following :

Proposition 2. *Let* K *be a trivial knot in* S^3 *such that* $u(K) = K$ *and* $K \cap fix(u) = \{P_1, P_2\}$. *Then there exists a* u*-isotopy* H *of* S^3 *such that* H *fixes* $\{P_1, P_2\}$ *and* $h_1(K) \cap R^3_+$ *is an unknotted simple arc spanning* P_1 *and* P_2.

Remark 1. In Proposition 2, from the definition of unknotted simple arc, we can choose a u-isotopy H' of S^3 such that h'_1 fixes $\{P_1, P_2\}$ and $h'_1 h_1(K) \subset R^2$.

3. Proof of Theorems 1 and 2

We will prove Theorems 1 and 2 by using Montesinos' technique [4, 5].

Let K be a trivial knot in S^3 and in this section K always means $Fix(u)$, i.e., z-axis $\cup \{\infty\}$. As mentioned in §1, $p : S^3 \to S^3$ is a 2-fold covering space of S^3 branched over K.

First we prove C.1 implies C.3, and C.2 implies C.4.

Let β be an attaching band to K such that $F(K; \beta)$ is a trivial knot or a trivial link of 2 components.

Choose a regular neighbourhood B of β in S^3, then $p^{-1}(B)$ is a regular neighbourhood of the knot $p^{-1}(\alpha)$ where α is a centerline of β defined by $\alpha = \beta(o \times I)$ for $o \in int\ D^1$, and, by the definition of the map p, $p^{-1}(\alpha)$ is a strongly invertible knot in S^3.

Let f be an orientation preserving homeomorphism of ∂B onto itself such that $\partial B \cap K$ is fixed as a set. Then the manifold $(S^3 - int\ B) \underset{f}{\cup} B$ is homeomorphic to S^3, because any homeomorphism of ∂B is extended to the whole 3-disk B.

Let $\tilde{f}$ be an orientation preserving homeomorphism of $\partial(p^{-1}(B))$ onto itself such that $p\tilde{f} = fp$. Then $M = (S^3 - int\ p^{-1}(B)) \underset{\tilde{f}}{\cup} p^{-1}(B)$ is a 2-fold covering space of $(S^3 - int\ B) \underset{f}{\cup} B$ branched over the link

$$(K \cap (S^3 - B)) \underset{f}{\cup} (K \cap B).$$

We can choose the above homeomorphism f of ∂B onto itself such that

$$F(K;\ \beta) = (K \cap (S^3 - B)) \underset{f}{\cup} (K \cap B).$$

Then M is obtained from S^3 by perfoming a surgery along $p^{-1}(\alpha)$. We remark that it is μ-surgery for some integer μ. (Fig.1)

Now we suppose $F(K;\ \beta)$ is a trivial knot, then, by Proposition 1, M is homeomorphic to S^3. Hence we can check that the surgery is ± 1-surgery by easy calculation of the 1-st homology group of M. Then, by hypothesis C.1, $p^{-1}(\alpha)$ is a trivial knot. By Lemma 1, Proposition 2 and Remark 1,we can get an isotopy G of S^3 such that g_1 fixes $\alpha \cap K$ and $g_1(\alpha) \subset R^2$. This implies that, for an attaching band β_i such that $F(K;\ \beta_i)$ is a trivial knot, their centerlines are isotopic keeping K fixed. If β_1 is $(2\nu+1)$ half twisted for $\nu \geqq 0$ (Fig.2), then the complement of $F(K;\ \beta_1)$ has the fundamental group represented by $\langle a,\ b\ ;\ a(ba)^{\nu} = (ba)^{\nu}b \rangle$.

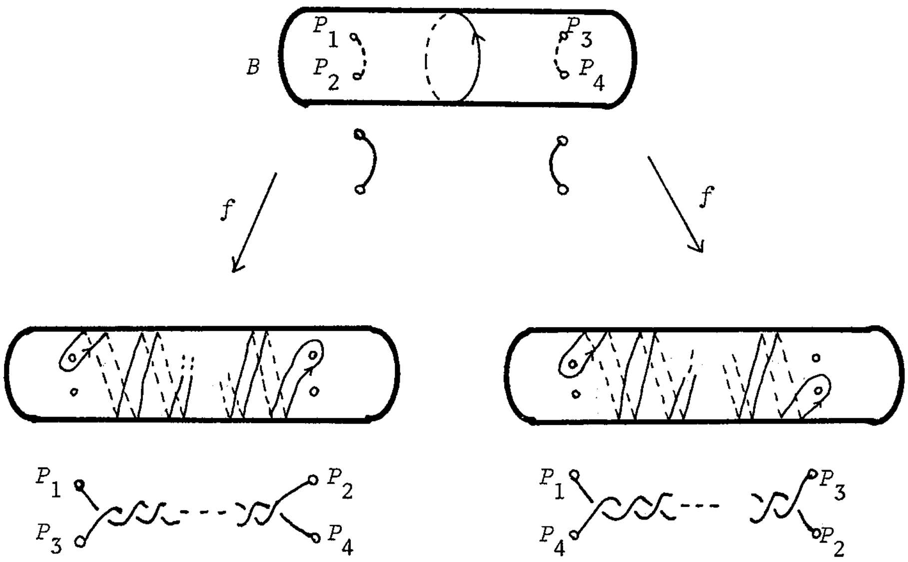


Figure 1

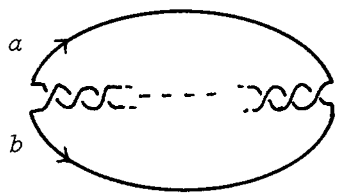


Figure 2

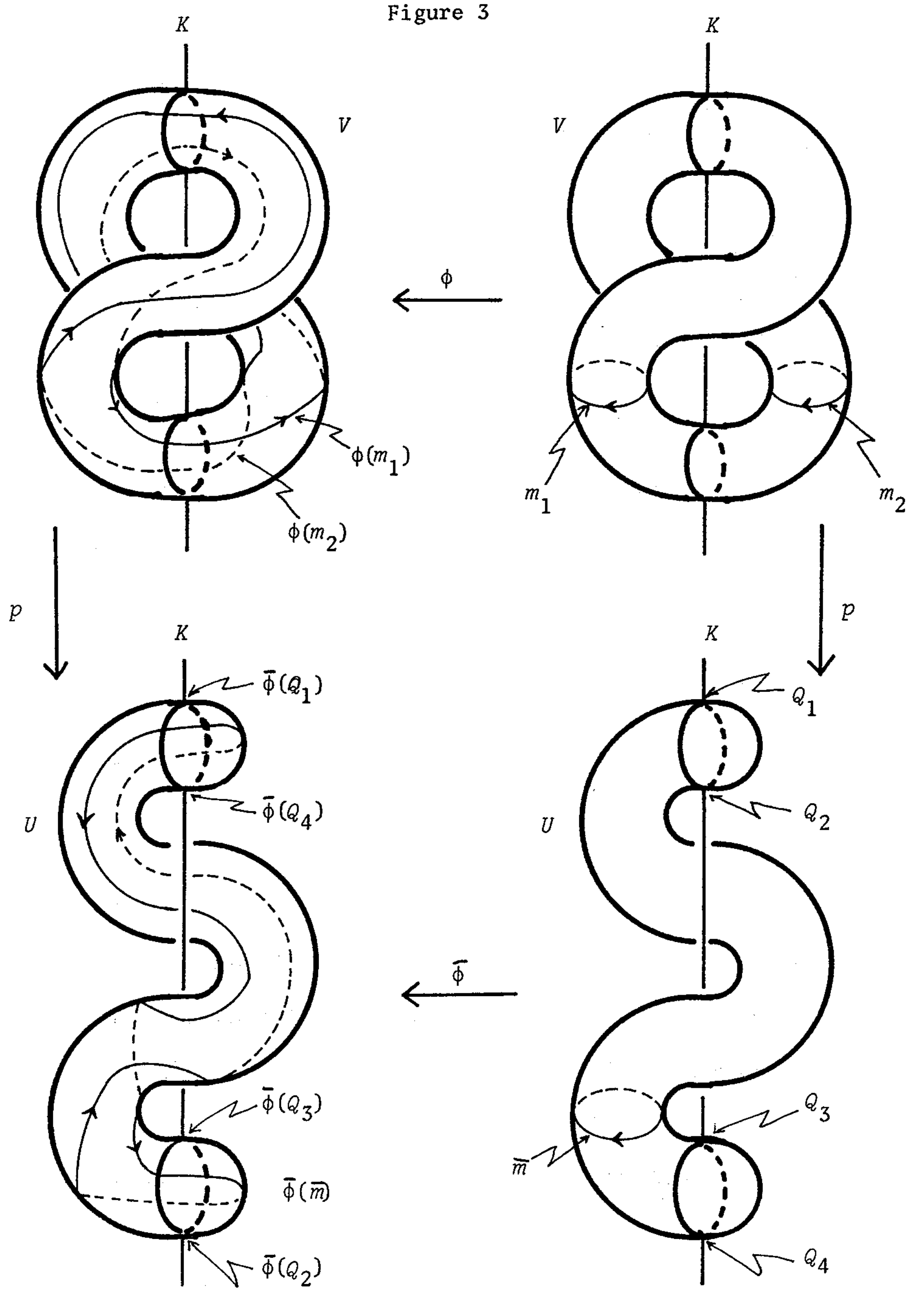
It is well known that, if $\nu \geqq 1$, this group is not isomorphic to the infinite cyclic group, hence $F(K;\ \beta_1)$ is not a trivial knot. Therefore β_1 is a half twisted, i.e., this results the bands β_1 and β_2 are equivalent, Therefore C.3 is implied.

Next, suppose C.2, then the surgery is 0-surgery. By the similar arguments in the case of C.1, we can conclude C.4.

Now we will prove C.3 implies C.1, and C.4 implies C.2.

Let k be a strongly invertible knot in S^3 such that $u(k) = k$ and

Figure 3


K
K
V
V
φ
φ(m1)
m1
m2
φ(m2)
p
p
K
K
$\bar{\phi}(Q_1)$
Q_1
$\bar{\phi}(Q_4)$
Q_2
U
U
$\bar{\phi}$
$\bar{\phi}(Q_3)$
Q_3
$\bar{m}$
$\bar{\phi}(\bar{m})$
$\bar{\phi}(Q_2)$
Q_4

$k \cap K = \{P_1, P_2\}$. Let V be a regular neighbourhood of k in S^3 such that $u(V) = V$. Choose a homeomorphism ϕ of ∂V onto itself such that $(S^3 - int\ V) \underset{\phi}{\cup} V$ is homeomorphic to S^3 or $S^1 \times S^2$, and that ϕ induces the ± 1- or 0-surgery, moreover we can assume that $\phi u = u\phi$ and $\phi(K \cap \partial V) = K \cap \partial V$. We put $\alpha = p(k)$, then α is a simple arc spanning $p(P_1)$ and $p(P_2)$ such that $int\ \alpha \cap K = \emptyset$. Let $U = p(V)$, then U is a 3-disk. Now there exists an orientation preserving homeomorphism $\bar{\phi}$ of ∂U onto itself such that $p\phi = \bar{\phi}p$ and $\bar{\phi}(K \cap \partial U) = K \cap \partial U$ (see Fig.3). Since ϕ induces ± 1- or 0-surgery, we can find a band β attaching to K such that $F(K;\ \beta) = (K \cap (S^3 - U)) \underset{\bar{\phi}}{\cup} (K \cap U)$.

If we suppose C.3, $F(K;\ \beta)$ is a trivial knot by Proposition 1.

Under the assumption C.4, we get $F(K;\ \beta)$ a trivial link of 2 components by Proposition 1. In both cases, we conclude that k is a trivial knot in S^3. This implies C.1 or C.2.

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