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Orlicz's Property for Volterra Equations

By Shin-ichi NAKAGIRI^{*}

1. Introduction

This paper is concerned with a study of uniqueness of solutions of a non-linear integral equation of Volterra-type

$$(1) \quad x(t) = f(t) + \int_0^t g(t, s, x(s)) ds.$$

Here t belongs to the interval $R^+ = [0, \infty)$, and x , f and g have values in R^n . Let C be the set of continuous forcing functions f and let G be the set of kernel functions g .

Our main purpose in this paper is to show that the uniqueness of solutions for the integral equation (1) with (f, g) in $C \times G$ is a generic property. In other words, the set of all $(f, g) \in C \times G$ for which the equation (1) fails to have a unique solution is of the Baire's first category (or meager) in $C \times G$. This property, called Orlicz's property, has been proved by Orlicz [8] for ordinary differential equations and then by Alexiewicz and Orlicz [1] for hyperbolic partial differential equations. However the treatment of this problem for the case of Volterra integral equations seems to be new. Moreover by using same methods in this paper,

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we can discuss the analogous problem for Volterra integro-differential equations of type

$$(2) \quad \begin{cases} x'(t) = f(t, x(t)) + \int_0^t g(t, s, x(s))ds, \\ x(0) = x_0. \end{cases}$$

For related arguments of the generic property of existence and uniqueness of functional-differential equations in infinite dimensional spaces, we refer [2] and [3].

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2. Preliminaries

Let R^n be the Euclidean n -space with the usual norm $|\cdot|$. We denote by C the space of continuous functions $f: R^+ \rightarrow R^n$ endowed with the topology of uniform convergence on compact intervals. As is well-known, this topology is metrizable, that is to say, the topology is determined by a metric d on C , and C is complete with respect to d .

Put $\Delta_k = \{(t, s) \in R^+ \times R^+ : s \leq t \leq k\}$, $\Delta = \bigcup_{k=1}^{\infty} \Delta_k$, and $D = \Delta \times R^n$. Let G denote the space of all functions $g: D \rightarrow R^n$ which are bounded and uniformly continuous on $\Delta_k \times R^n$ for each k , endowed with the topology of uniform convergence on $\{\Delta_k \times R^n : k = 1, 2, \dots\}$. The topological space G is a complete metric space with the metric d_G given by

$$d_G(g, h) = \sum_{k=1}^{\infty} 2^{-k} \cdot \min\{1, \max\{|g(t, s, x) - h(t, s, x)| : (t, s, x) \in \Delta_k \times R^n\}\}$$

for all $g, h \in G$. The product space $X = C \times G$ is also a complete metric space with the metric $d_X = \max\{d, d_G\}$.

By a (right) *maximally defined solution* of the equation (1) for (f, g) , we mean a continuous function $x: [0, \alpha(x)) \rightarrow R^n$ such that (1) is satisfied

on $[0, \alpha(x))$ and $\alpha(x)$ is either ∞ or a finite number such that $|x(t_j)| \uparrow \infty$ for some sequence $t_j \uparrow \alpha(x)$. In our previous papers [5], [6] and [7], we have obtained the existence of maximally defined continuous solution, the Kneser's property of solutions and continuous dependence of solutions with respect to (f, g) for some wider class of (f, g) than X . It is evident from the definition of G that $\alpha(x) = \infty$ for any solution x of (1) for $(f, g) \in X$. Hence all solutions considered in this paper are the members of C . It is easy to see that the metric topology on G is a jointly continuous topology, and so the following lemma follows from a result in [7].

Lemma 1. *Suppose that a sequence $\{(f_n, g_n)\}$ in X converges to $(f, g) \in X$. Let $x_n(\cdot)$ be a solution of (1) for (f_n, g_n) . Then there exist a solution $x(\cdot)$ in C and a subsequence $\{x_m(\cdot)\}$ of $\{x_n(\cdot)\}$ such that $x_m(\cdot)$ converges to $x(\cdot)$ in C .*

3. Uniqueness and Denseness

We denote by H the set of all $(f, g) \in X$ such that the solution of (1) for (f, g) is unique. Then we have the following

Theorem 1. *The subspace H is dense in X .*

To prove this theorem we need some preparations. A function $g \in G$ is said to be G -Lipschitzian if for each k there exists a constant $L(k)$ such that

$$|g(t, s, x) - g(t, s, y)| \leq L(k)|x - y|$$

for all $(t, s, x), (t, s, y) \in \Delta_k \times R^n$. Let L denote the set of all G -Lipschitzian functions in G . The method of smooth approximations estab-

lishes the following

Lemma 2. *The set L is dense in G .*

Proof. Let $\varphi(x)$ be an infinitely differentiable function with compact support in R^n satisfying $\int_{R^n} \varphi(x) dx = 1$. For each $g \in G$ and $\varepsilon > 0$, we define ε -smooth approximation g_ε about spatial variable x by

$$g_\varepsilon(t, s, x) = \int_{R^n} g(t, s, z) \varphi\left(\frac{x-z}{\varepsilon}\right) \frac{dz}{\varepsilon^n}.$$

Since g is bounded and continuous on each $\Delta_k \times R^n$, the function g_ε is also bounded and continuous on $\Delta_k \times R^n$ and satisfies the following inequality:

$$\begin{aligned} |g_\varepsilon(t, s, x) - g_\varepsilon(t, s, y)| &\leq \int_{R^n} |g(t, s, z)| \left| \varphi\left(\frac{x-z}{\varepsilon}\right) - \varphi\left(\frac{y-z}{\varepsilon}\right) \right| \frac{dz}{\varepsilon^n} \\ &\leq \max\{|g(t, s, x)| : (t, s, x) \in \Delta_k \times R^n\} \cdot \int_{R^n} \left\{ \sup_{\xi \in \text{supp } \varphi} \left| \frac{\partial \varphi}{\partial x_i}(\xi) \right| \right\} \frac{|x-y|}{\varepsilon^{n+1}} dz \\ &\leq L(k, \varepsilon) |x-y| \end{aligned}$$

for all $(t, s, x), (t, s, y) \in \Delta_k \times R^n$, where $\text{supp } \varphi$ denotes the support of φ . Hence $g_\varepsilon \in L$. In the equality

$$\begin{aligned} g_\varepsilon(t, s, x) &= g(t, s, x) \int_{R^n} \varphi\left(\frac{x-y}{\varepsilon}\right) \frac{dy}{\varepsilon^n} \\ &\quad - \int_{R^n} [g(t, s, x) - g(t, s, y)] \varphi\left(\frac{x-y}{\varepsilon}\right) \frac{dy}{\varepsilon^n}, \end{aligned}$$

the integral of the first term is equal to 1 by the assumption, and the absolute value of the second term is bounded by

$$\sup_{|x-y| \leq \varepsilon r} |g(t, s, x) - g(t, s, y)| \int_{R^n} \left| \varphi\left(\frac{x-y}{\varepsilon}\right) \right| \frac{dy}{\varepsilon^n}$$

where r is a number such that $\text{supp } \varphi \subset \{x : |x| < r\}$. Here the second factor is equal to the constant $\int |\varphi(y)| dy$, and from the uniform continuity of $g \in G$ in x , the first factor approaches to 0 uniformly on $\Delta_k \times R^n$,

$k = 1, 2, \dots$, as $\varepsilon \rightarrow 0$. This shows $g_\varepsilon \rightarrow g$ in G , and hence L is dense in G .

Proof of Theorem 1. Since for any $(f, g) \in C \times L$ the solution of (1) for (f, g) is unique (see for example, Miller [4]), it is clear that $C \times L$ is contained in H . Hence by Lemma 2, H is dense in X .

4. Orlicz's Property for (1)

To characterize the uniqueness of solutions, we define a number $K(f, g)$ which clarifies the similar complicated number introduced by Lasota and Yorke [3]. That is, this number is defined to be the $\limsup$ (when $(f_1, g_1) \rightarrow (f, g)$ and $(f_2, g_2) \rightarrow (f, g)$) of the distance between any solution $x_1(\cdot)$ of (1) for (f_1, g_1) and any solution $x_2(\cdot)$ of (1) for (f_2, g_2) in the space C . Let $S(f, g) \subset C$ be the set of all solutions of (1) for (f, g) , and put $S(X) = \bigcup \{S(f, g) : (f, g) \in X\}$. Let (f, g) be an arbitrary fixed element of X . For each $\delta > 0$, define

$$V_\delta(f, g) = \{(f', g') \in X : d_X((f', g'), (f, g)) \leq \delta\} \subset S(X),$$

where d_X denotes the metric of X . Moreover, put

$$W_\delta(S(f, g)) = \bigcup \{S(f', g') : (f', g') \in V_\delta(f, g)\},$$

$$K_\delta(f, g) = \sup \{d(x_1(\cdot), x_2(\cdot)) : x_1(\cdot), x_2(\cdot) \in W_\delta(S(f, g))\}.$$

Finally we define

$$K(f, g) = \limsup_{\delta \rightarrow 0} K_\delta(f, g).$$

It is evident that

$$K(f, g) \geq d(S(f, g)) = \max \{d(f, g) : x(\cdot), y(\cdot) \in S(f, g)\}.$$

Here we note that $d(S(f, g))$ can be defined, because $S(f, g)$ is compact in C (see [6]).

The following lemma is essential for the proof of our main theorem.

Lemma 3. The function $K: X \rightarrow R^+$ is upper semi-continuous. Moreover, $K(f, g) = 0$ if and only if $(f, g) \in H$.

Proof. To prove the upper semi-continuity, suppose the contrary. Then $\limsup_{n \rightarrow \infty} K(f_n, g_n) > K(f, g)$ for some sequence $\{(f_n, g_n)\}$ converging to $(f, g) \in X$. Hence there exist an $\varepsilon > 0$ and a subsequence $\{(f_m, g_m)\}$ of $\{(f_n, g_n)\}$ such that $K(f_m, g_m) > K(f, g) - \varepsilon$. From the definition of $K(f_m, g_m)$, there exist solutions $x_{1,m}$ and $x_{2,m}$ of (1) for $(f_{1,m}, g_{1,m})$ and $(f_{2,m}, g_{2,m})$, respectively, such that $d(x_{1,m}, x_{2,m}) > K(f, g) - \varepsilon$. Since $(f_{1,m}, g_{1,m})$ and $(f_{2,m}, g_{2,m})$ can be so chosen that $(f_{1,m}, g_{1,m}) \rightarrow (f, g)$ and $(f_{2,m}, g_{2,m}) \rightarrow (f, g)$ as $m \rightarrow \infty$, we can find, by Lemma 1, solutions $x(\cdot), y(\cdot) \in S(f, g)$ and subsequence $\{\ell\}$ of the sequence $\{m\}$ such that $d(x_{1,\ell}, x) \rightarrow 0$ and $d(x_{2,\ell}, y) \rightarrow 0$ as $\ell \rightarrow \infty$. Hence by continuity of d , we have $K(f, g) - \varepsilon \leq d(x, y) \leq K(f, g)$, a contradiction which proves the upper semicontinuity of K . Now we shall show that $K(f, g) = d(S(f, g))$. Suppose $K(f, g) > d(S(f, g))$. Then the definition of $K(f, g)$ implies there exist an $\varepsilon > 0$ and solution sequences $\{x_{1,n}\}$ and $\{x_{2,n}\}$ such that $x_{1,n} \in S(f_{1,n}, g_{1,n})$, $x_{2,n} \in S(f_{2,n}, g_{2,n})$, $\lim_{n \rightarrow \infty} (f_{1,n}, g_{1,n}) = \lim_{n \rightarrow \infty} (f_{2,n}, g_{2,n}) = (f, g)$ in X , and $d(x_{1,n}, x_{2,n}) > d(S(f, g)) - \varepsilon$. But then, by Lemma 1, we can find two solutions x, y in $S(f, g)$ such that some subsequences of $\{x_{1,n}\}$ and $\{x_{2,n}\}$ converge to x and y in C , respectively. Therefore we have $d(S(f, g)) - \varepsilon \leq d(x, y) \leq d(S(f, g))$, a contradiction. Thus we have $K(f, g) = d(S(f, g))$ which establishes the latter half of the lemma.

Note that Lemma 3 implies the continuity of K at any point of H .

The following theorem extends the theorem of Orlicz [8] to the case of

Volterra integral equations.

Theorem 2. *The set $X - H$ is meager in the space X .*

Proof. Let n be an arbitrary positive integer. Since K is upper semi-continuous, the set $E_n = \{(f, g) \in X : K(f, g) \geq n^{-1}\}$ is closed. To show that E_n is nowhere dense, suppose the contrary. Then by Theorem 1 $H \cap \overset{\circ}{E}_n$ is nonempty, where $\overset{\circ}{E}_n$ denotes the interior of E_n . Let (f_0, g_0) be in $H \cap \overset{\circ}{E}_n$. Then by Lemma 3, we have $K(f_0, g_0) = 0$, which is absurd. Since $X - H = \bigcup_{n=1}^{\infty} E_n$, this completes the proof.

Remark. Each $(f, g) \in H$ has the following property: If $(f_n, g_n) \in X$ converges to $(f, g) \in H$ as $n \rightarrow \infty$ and $x_n(\cdot) \in S(f_n, g_n)$, then $x_n(\cdot)$ converges to $x(\cdot) \in S(f, g)$. Especially $d(S(f, g))$ is continuous at $(f, g) \in H$.

5. Orlicz's Property for (2)

We extend the results in the preceding sections to the equation (2). Since the methods are similar, we omit the proofs of the theorems in this section.

Let F be the space of all functions $f: R^+ \times R^n \rightarrow R^n$ such that f is bounded and uniformly continuous on $[0, k] \times R^n$ for each $k = 1, 2, \dots$. The space F is topologized by the uniform convergence on each subsets $[0, k] \times R^n$, $k = 1, 2, \dots$. F is a complete metric space with this topology.

We consider the problem (2) for each data $(x_0, f, g) \in R^n \times F \times G$. Then the solution family of (2) for (x_0, f, g) is a nonempty compact set in C . Moreover, a continuous dependence theorem which is similar to Lemma 1

is valid for this equation (2). Therefore we have the following

Theorem 3. *Let T be the set of all $(x_0, f, g) \in R^n \times F \times G$ such that the solution of (2) for (x_0, f, g) is unique. Then T is dense in $R^n \times F \times G$.*

Theorem 4. *The complement of T in $R^n \times F \times G$ is meager in $R^n \times F \times G$.*

Moreover, as a stronger version of Theorem 4, we have the following

Theorem 5. *The set of those (f, g) for which the equation (2) does not have unique solution for some $x_0 \in R^n$ is meager in $F \times G$.*

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