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(1, 2)-Ideal Characterizations of Unions of Groups

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Let S be a semigroup. A subsemigroup A of S is called (1, 2)-*ideal* of S if $ASA^2 \subset A$ (cf. [1]). S is said to be *completely regular* if it is the union of disjoint groups (see [8], [9]). It is known that S is completely regular if and only if $a \in aSa^2$ for every element a of S.

In the present note, some characterizations of completely regular semigroups will be proved in terms of (1, 2)-ideals.

Theorem 1. *A semigroup S is the union of groups if and only if every principal* (1, 2)*-ideal of S can be generated by an idempotent element of S.*

Proof. Let S be a completely regular semigroup, $a \in S$. Then there is an element x in S such that

$$axa = a \quad \text{and} \quad ax = xa.$$

Let $ax = e$. Then $e^2 = e$, and we show that $\{a\}_{(1,2)} = \{e\}_{(1,2)}$. First, we have

$$\{a\}_{(1,2)} = \{a, a^2, a^3\} \cup aSa^2 = aSa^2 =$$
$$= (axa)S(axa) \subset (ax)S(xa) = eSe = \{e\}_{(1,2)}.$$

On the other hand,

$$\{e\}_{(1,2)} = eSe = (ax)S(ax)^2 \subset aSa^2 = \{a\}_{(1,2)}.$$

Hence it follows that $\{a\}_{(1,\,2)} = \{e\}_{(1,\,2)}$, where $e^2 = e \in S$.

Conversely, if $\{a\}_{(1,\,2)} = \{e\}_{(1,\,2)}$ with $e = e^2$, then $a = ese$, $s \in S$, whence it follows that $ea = ae = a$. On the other hand, $e = a$, $e = a^2$, $e = a^3$ or $e = ata^2$, where $t \in S$. Hence $a = a^2$, $a = a^3$, $a = a^4$ or $a = a^2ta^2$, that is, in each cases, $a \in aSa^2$. This holds for every element a of S, and therefore S is completely regular.

The following criterion can be proved similarly.

Theorem 2. *A semigroup S is completely regular if and only if for every element a of S there exists an idempotent element e in S such that*

$$\{a\}_{(1,\,2)} = R(e)L(e).$$

Theorem 3. *A semigroup S is completely regular if and only if every principal (1, 2)-ideal of S is a monoid.*

Proof. The necessity follows at once from Theorem 1, because $\{e\}_{(1,\,2)} = eSe$, and e is the two-sided identity of $\{a\}_{(1,\,2)}$.

Suppose that $\{a\}_{(1,\,2)}$ is a monoid with identity element e. Then $ae = ea = a$ and $e = a$, $e = a^2$, $e = a^3$ or $e = asa^2$ with $s \in S$. Hence it follows that $a \in aSa^2$ in each cases. Thus S is completely regular.

Theorem 4. *For a semigroup S the following conditions are equivalent:*

(A) *S is completely regular.*

(B) *$A \cap B \subseteq AB$ for every couple A, B of (1, 2)-ideals of S.*

(C) *$\{a\}_{(1,\,2)} \cap \{b\}_{(1,\,2)} \subseteq \{a\}_{(1,\,2)}\{b\}_{(1,\,2)}$ for every couple a, $b \in S$.*

(D) $\{a\}_{(1,\,2)}$ *is globally idempotent for every element* a *of* S.

(E) *Every* (1, 2)*-ideal of* S *is globally idempotent.*

Proof. To prove (A) implies (B), assume that S is a completely regular semigroup, A, B are (1, 2)-ideals of S, and $a \in A \cap B$. Then there is an element x of S such that $a = a^2xa^2 = a(axa^2) \in AB$, because of $axa^2 \in BSB^2 \subseteq B$.

Evidently, (B) implies (C) and (E), moreover (C) implies (D), and (E) implies (D). We have to show that (D) implies (A).

If $\{a\}_{(1,\,2)}$ is globally idempotent, it follows that

$$a \in \{a\}_{(1,\,2)} = (\{a, a^2, a^3\} \cup aSa^2)^2$$
$$\subseteq \{a^2, a^3\} \cup aSa^2,$$

whence it follows that $a \in aSa^2$ in each cases. This holds for every element a of S, i.e., S is completely regular. This completes the proof.

For earlier characterizations of semigroups that are unions of groups, see [2]-[7].

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