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On a Composition of Knot Groups II — Algebraic Bridge Index

By Toru MAEDA

The bridge index of (3,1)-knots was introduced by H. Schubert [4]. Let K be a (3,1)-knot. An overpass of a regular projection $p(K)$ of the knot K is a subarc of the diagram $p(K)$ which contains an over-crossing points but no undercrossing points, the number of the maximal-overpasses is called *the bridge number of the projection* $p(K)$. *The bridge index* $b(K)$ *of* K is the least bridge number of all regular projections of K. For the composition $K_1 \# K_2$ of (3,1)-knots K_1 and K_2, H. Schubert proved

$$(*) \qquad b(K_1 \# K_2) = b(K_1) + b(K_2) - 1 \quad .$$

For any locally flat, piecewise linear or smooth (n+2,n)-knots, $n \geqq 1$, A.B. Sosinskii gave a similar result to the above Schubert's equation (*) in [5], Theorem 3.2 . One of the purposes of our paper is to show that the A.B. Sosinskii's statement is false. Before describing this, let us review the A.B. Sosinskii's statement.

Let $K : S^n \to S^{n+2}$ be a locally flat, piecewise linear or smooth knot. For a set of generators of the knot group $\pi_1(S^{n+2}-K(S^n))$ of K, we can choose $\{x, a_1, \ldots, a_s\}$ such that x is a meridian and the linking number of a_i with $K(S^n)$ is zero for each i. The minimal system of generators chosen in this way will be called *a canonical system of generators of the knot group of* K. A.B. Sosinskii's statement is as follows :

" If $\{x, a_1, \ldots, a_s\}$ and $\{y, b_1, \ldots, b_r\}$ are canonical systems for the knot groups of two knots K_1 and K_2 respectively, then the elements z $(= x = y)$, $a_1, \ldots, a_s, b_1, \ldots, b_r$ form a minimal system of generators of the knot group $K_1 \# K_2$. Thus, on adding a knot (whose knot group is not the infinite cyclic group Z), we strictly increase the minimum number of generators of the knot group of the original knot."

In our paper, we will also give the definition of "the algebraic bridge index" (§ 1 and § 2) and give an explanation of "the composition of knot groups" (§ 3) which was introduced in [7],p.108. After these preliminaly arrangements, we will give an example which shows that a similar equation (3.1) for the equation (*) does not hold.

1. Algebraic bridge index

After this, G is always considered to be a finitely presented group which has a Wirtinger presentation (see [3]) and whose first homology group $H_1(G)$ is isomorphic to Z.

Definition 1. x (ε G) is *a meridian element of* G if G has a Wirtinger presentation whose set of generators contains x.

If the second homology group $H_2(G)$ of G is trivial, we can regard G as a knot group, because of the main theorem in M.A. Kervaire [1], p.106 . T. Yajima [6] proved that x is a meridian element of G if and only if

x is a weighted element of G, i.e. x^G, the normal closure of x in G, is equal to G.

We will denote the pair of G and its meridian element x by (G,x).

Definition 2. Let x be a meridian element of G and $P(G,x)$ be any Wirtinger presentation of G whose set of generators contains x. *The algebraic bridge number of* $P(G,x)$ is the number of the generating symbols in $P(G,x)$.

Definition 3. *The algebraic bridge index of* (G,x), denoted by $B(G,x)$, is the least algebraic bridge number for all Wirtinger presentations whose sets of generators contain x.

It is trivial that $B(G,x) = 1$ if and only if G is isomorphic to Z.

Now, we will give a knot group G which has two meridian elements x_1 and y_1 such that $B(G,x_1) \neq B(G,y_1)$, i.e. the algebraic bridge index depends on not only the group G but also its meridian element. Let G be a group which has the following Wirtinger presentation :

$$(1.1) \qquad < x_1,x_2 : x_1(x_2x_1^{-1}x_2^{-1}x_1)x_2^{-1}(x_2x_1^{-1}x_2^{-1}x_1)^{-1} >.$$

G is the knot group of the figure-eight knot. (This group G is not isomorphic to Z, because the factor group of G by $(x_1^2)^G$, the normal closure of x_1^2, is the dihedral group of order 10. Therefore, $B(G,x_1) = 2$. Next, we change the presentation (1.1) to (1.2) by adjoining $a_1 = x_1x_2^{-1}$, and then, by deleting x_2 on Tietze transformations.

$$(1.2) \qquad < x_1,a_1 : x_1a_1^{-1}x_1^{-1}a_1^2x_1^{-1}a_1^{-1}x_1a_1 > .$$

It is trivial that the element $y_1 = a_1^2x_1$ is a weighted element of G.

Therefore, y_1 is a meridian element of G. In fact,

$$\begin{aligned}
G &= < y_1\, a_1 : a_1^{-1}y_1a_1^{-1}y_1^{-1}a_1^4y_1^{-1}a_1^{-1}y_1 > \\
&= < y_1,y_2,y_3,y_4,a_1 : a_1^{-1}y_1a_1^{-1}y_1^{-1}a_1^4y_1^{-1}a_1^{-1}y_1, \\
&\qquad\qquad y_2 = a_1^{-1}y_1a_1,\ y_3 = a_1^{-2}y_1a_1^2,\ y_4 = a_1^2y_1a_1^{-2} > \\
&= < y_1,y_2,y_3,y_4,a_1 : a_1 = y_4y_3^{-1}y_2^{-1}y_1^{-1}, \\
&\qquad\qquad y_2 = a_1^{-1}y_1a_1,\ y_3 = a_1^{-2}y_1a_1^2,\ y_4 = a_1^2y_1a_1^{-2} > \\
&= < y_1,y_2,y_3,y_4 : y_2 = (y_4y_3^{-1}y_2^{-1}y_1^{-1})^{-1}y_1(y_4y_3^{-1}y_2^{-1}y_1^{-1}), \\
&\qquad\qquad y_3 = (y_4y_3^{-1}y_2^{-1}y_1^{-1})^{-2}y_1(y_4y_3^{-1}y_2^{-1}y_1^{-1})^2, \\
&\qquad\qquad y_4 = (y_4y_3^{-1}y_2^{-1}y_1^{-1})^2\, y_1(y_4y_3^{-1}y_2^{-1}y_1^{-1})^{-2} > .
\end{aligned}$$

This implies that $B(G,y_1) \leqq 4$. To calculate $B(G,y_1)$, we consider the mapping h from the generators y_1,y_2,y_3 and y_4 into the symmetric group S_5 of degree 5 such that $h(y_1) = (15)$, $h(y_2) = (45)$, $h(y_3) = (34)$ and $h(y_4) = (23)$. This mapping h induces a homomorphism from G onto S_5. Since we need at least four transpositions to generate S_5, $B(G,y_1) \geqq 4$. Therefore, $B(G,y_1) = 4$. This implies that $B(G,x_1) \neq B(G,y_1)$.

2. Classes of meridian elements

If f is an automorphism of G, it is clear that $B(G,x) = B(G,f(x))$ for an arbitrary meridian element x of G. Using this fact,we can define an equivalence relation for the set of all meridian elements of G as follows:

Definition 4. Let x and x' be two meridian elements of G. x and x' are *equivalent in a weak sense*, denoted by $x \sim x'$, if there exists an automorphism f of G such that $f(x) = x'$.

We will denote by $[x]$ the equivalence class of x in the set of all meridian elements of G modulo " $\sim$ ". Then we can define *the algebraic* *bridge index of* $(G,[x])$, denoted by $B(G,[x])$, as $B(G,x)$.

The example in § 1 shows that there exists G which has some different equivalence classes. The group N = (the binary icosahedral group) $\times$ Z gives us the same result, because N has at least three classes (see [7], p.107). Moreover, we can show that (the generators of the direct factor Z of N)$\cdot$(the center of the binary icosahedral group) can not be meridian elements of N but are mapped to the generators of the infinite cyclic group $H_1(N)$ under the commutator homomorphism.

For a group G with an abelian commutator subgroup G',the following facts are trivial. The set of all meridian elements of G is $xG' \cup x^{-1}G'$. The number of the equivalence classes is 1 if and only if for the automorphism f of G' which is induced by an meridian element of G , there exists an automorphism g of G' such that $g^{-1}\circ f\circ g = f^{-1}$. The other case, the number of the equivalence classes is 2, i.e. xG' and $x^{-1}G'$.

3. Algebraic bridge index of composite knot groups

Definition 5. $(G,[z])$ is *the composition of* $(G_1,[x])$ *and* $(G_2,[y])$, denoted by $(G_1,[x])\#(G_2,[y])$, if G is the amalgamated product group of G_1 and G_2 under the mapping $x \to y$ and z is the image of x under the natural inclusion from G_1 into G.

It is clear that G has a Wirtinger presentation whose generators include the element z. Therefore, this definition is well defined.Moreover, all of $(G,[x])$ obviousely form a commutative semigroup $\mathcal{G}$ with identity $(Z,[1])$. The commutator subgroup G' of G such that $(G,[z]) = (G_1,[x])$ $\#(G_2,[y])$ is the free product of the commutator subgroups of G_1 and G_2

by (3.3) in [7], p.108.

All of knot groups form a commutative subsemigroup of $\mathcal{G}$, because $H_2(G) = H_2(G_1) + H_2(G_2)$ for all $(G,[z]) = (G_1,[x]) \# (G_2,[y])$.

Since the above definition of the composition is similar to the composition of knots, we can raise the following question :

Question. *Does the equation*

$$(3.1) \qquad B((G_1,[x])\#(G_2,[y])) = B(G_1,[x]) + B(G_2,[y]) - 1$$

hold ?

We get infinite examples which induce the negative answer to the question. Let $G_{p,q}$ be the group with the following presentation

$$(3.2) \quad < x,a_1 \ : \ x^{-1}a_1x = a_1^q,\ a^p = 1 >,\ (p,q) = (p,q-1) = 1,\ p > q > 1.$$

In [2], the author proved that the group $G_{p,q}$ is a knot group whose commutator subgroup is $< a_1 : a_1^p = 1 >$, and that $G_{p,q}$ has the following presentations

$$(3.3) \qquad < x,a_1 \ : \ x^{-1}a_1x = a_1^q,\ x = a_1^{-p}xa_1^p > \ ,$$
$$< x_1,x_2 \ : \ x_2 = (x_2^{-1}x_1)^{-r}x_1(x_2^{-1}x_1)^r,\ x_1 = (x_2^{-1}x_1)^{-p}x_1(x_2^{-1}x_1)^p >$$

where $r(q-1) + sp = 1$ for some integer s , and $x_1 = x$, $x_2 = xa_1^{-1}$. Since a_1 belongs to the commutator subgroup of $G_{p,q}$,it follows from (3.2) that x and a_1 from a canonical system of $G_{p,q}$. By (3.4), $B(G_{p,q},[x]) = 2$. Let $(G,[z]) = (G_{p,q},[x]) \# (G_{m,n},[y])$ which satisfies the following condition : there exists an integer k such that (i) $q^k +1 \equiv 0 \pmod{p}$, (ii) $n^{k-1}-1 \equiv 0 \pmod{m}$ and (iii) $(n+1,m) = 1$. The existence of the above required integers p,q,m,n and k is easily shown by putting p be an odd number greater than 1 , $q = p-1$, m be a prime number greater than 3 , $2 \leqq n \leqq m-2$ and $k = 1$. Therefore, We will show that $B(G,[z]) = 2$.

$$\begin{aligned} G &= \langle z,a_1,b_1 : z^{-1}a_1z = a_1^q,\ z = a_1^{-p}za_1^p,\ z^{-1}b_1z = b_1^n,\ z = b_1^{-m}zb_1^m \rangle \\ &= \langle z,a_1,b_1,c_1 : z^{-1}a_1z = a_1^q,\ z = a_1^{-p}za_1^p,\ z^{-1}b_1z = b_1^n,\ z = b_1^{-m}zb_1^m, \\ &\qquad c_1 = z^{-1}b_1a_1za_1^{-1}b_1^{-1} \rangle. \end{aligned}$$

Then, $c_1 = b_1^n a_1^{q-1} b_1^{-1}$. Therefore, $z^{-k}c_1z^kc_1 = (b_1^{n^{k+1}} a_1^{q^k(q-1)} b_1^{-n^k})(b_1^n \cdot a_1^{q-1} b_1^{-1})$. Since $n^{k-1}-1 \equiv 0 \pmod m$ and $q^k+1 \equiv 0 \pmod p$, $z^{-k}c_1z^kc_1 = b^{n^{k+1}-1}$. Moreover, $(n^{k+1}-1,m) = 1$ from $(n-1,m) = 1$, $(n+1,m) = 1$ and $n^{k-1}-1 \equiv 0 \pmod m$. Therefore, there exists an integer t such that $t \cdot (n^{k+1}-1) \equiv 1 \pmod m$. Hence, $(z^{-k}c_1z^kc_1)^t = b_1$ and then, $a_1 = ((z^{-k}c_1z^k \cdot c_1)^{-nt}c_1(z^{-k}c_1z^kc_1)^t)^u$, where $u(q-1) \equiv 1 \pmod p$. This implies that G has the following presentation :

$$(3.5) \quad \begin{aligned} \langle z,c_1 : &\ z^{-1}A_1z = A_1^q,\ z = A_1^{-p}zA_1^p,\ z^{-1}B_1z = B_1^n,\ z = B_1^{-m}zB_1^m, \\ &\ c_1 = z^{-1}B_1A_1zA_1^{-1}B_1^{-1} \rangle \end{aligned}$$

where B_1 is the word $(z^{-k}c_1z^kc_1)^t$ and A_1 is the word $(B_1^{-n}c_1B_1)^u$. The presentation (3.5) shows that $\{z,c_1\}$ generates G and that c_1 is an element of the commutator subgroup of G. Therefore, $\{z,c_1\}$ is a canonical system of G. Hence, the Sosinskii's statement is false. To prove $B(G,[z]) = 2$, we change the presentation (3.5) by adjoining $z_1 = z$ and $z_2 = zc_1$, and then, by deleting z and c_1.

$$(3.6) \quad \begin{aligned} \langle z_1,z_2 : &\ R_1 = z_2B_1^*A_1^*z_1^{-1}A_1^{*-1}B_1^{*-1}, \\ &\ W_1' = z_1^{-1}A_1^*z_1A_1^{*-q},\quad W_2' = z_1A_1^{*-p}z_1^{-1}A_1^{*p}, \\ &\ W_3' = z_1^{-1}B_1^*z_1B_1^{*-n},\quad W_4' = z_1B_1^{*-m}z_1^{-1}B_1^{*m} \rangle, \end{aligned}$$

where A_1^* and B_1^* are the words in z_1 and z_2 induced from A_1 and B_1 in the above manner. Moreover, we change the relators W_1', W_2', W_3' and W_4' to W_1, W_2, W_3 and W_4 by replacing all z_2's in W_1', W_2', W_3' and W_4' to $R_1^{-1}z_2$. The presentation obtained in such a manner satisfies the conditions in Lemma 2, [3]. Therefore, G has a Wirtinger presentation

$$(3.7) \qquad < z_1, z_2 \ : \ R_1, \ \ z_i = W_j z_j W_j^{-1} \ , \ i = 1,2 \ \text{ and } \ j = 1,2,3,4 >.$$

Hence, $B(G,[z]) = B(G,[z]) = 2$. This implies that the equation (3.1) does not hold.

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