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GOERITZ MATRICES OF SPATIAL GRAPHS RELATED TO DEHN COLORINGS WITH ALTERNATING VERTEX CONDITIONS

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Abstract

We construct Goeritz matrices of spatial graph diagrams whose vertices are of even valency, and show that the equivalence classes of them are invariants of spatial graphs. We also discuss a relationship between the Goeritz matrices and Dehn colorings with alternating vertex conditions, where the alternating vertex condition is one of special cases and it corresponds to the classical vertex condition of Fox colorings which is often dealt with.

1. Introduction

Goeritz introduced his matrices associated to knot diagrams and gave Goeritz invariants for knots in [3]. The notion was extended and modified to the case of links in [2, 13], and several studies related to Goeritz invariants were taken so far (see [4, 10, 11, 12] for example). Dehn colorings, which are region colorings on link diagrams and related to the Dehn presentations of knot groups, were studied in [1, 5]. In [14], Traldi diagrammatically gave the relationship between Dehn colorings and Goeritz matrices for links. His diagrammatic method motivated us to consider the relationship between Dehn colorings and Goeritz matrices for spatial graphs, where Goeritz matrices of spatial graphs have not been studied before.

In this paper, we extend the definition of Goeritz matrices of link diagrams for spatial graph diagrams whose vertices are of even valency. Although we can construct the Goeritz matrices from spatial graph diagrams in the same manner as the case of link diagrams, they can not produce invariants of spatial graphs. Hence, we need some additional information to obtain Goeritz matrices which induce an invariant for spatial graphs. In this paper, we focus on the alternating vertex condition related to a kind of Dehn coloring and extract some informations added to the Goeritz matrices. We note that the alternating vertex condition is one of special cases and it corresponds to the classical vertex condition of Fox colorings which is often dealt with. Moreover, we also discuss a relationship between the Dehn colorings and the Goeritz matrices for spatial graphs whose vertices are of even valency.

This paper is organized as follows: In Section 2, we review the definitions of spatial graphs and their diagrams. The Goeritz matrices of spatial graph diagrams whose vertices are of even valency and the Goeritz invariants of spatial graphs are introduced in Section 3, and we show that the equivalence classes of the Goeritz matrices are invariants of spatial graphs in Section 4. In Section 5, we define Dehn colorings of alternating vertex conditions for spatial graph diagrams. Finally, we provide the relationship between the Goeritz matrices and the Dehn colorings of spatial graphs whose vertices are of even valency in Section 6.

2. Spatial graphs

A *spatial graph* is a graph embedded in the 3-dimensional Euclidean space $\mathbb{R}^3$. Two spatial graphs are *equivalent* if one can be deformed by an ambient isotopy into the other. A *spatial graph diagram* D of a spatial graph G is an image of G by a regular projection onto the 2-dimensional Euclidean space $\mathbb{R}^2$ with a crossing information at each double point. It is well-known that two spatial graph diagrams represent an equivalent spatial graph if and only if they are related by a finite sequence of the generalized Reidemeister moves as in FIGURE 1.

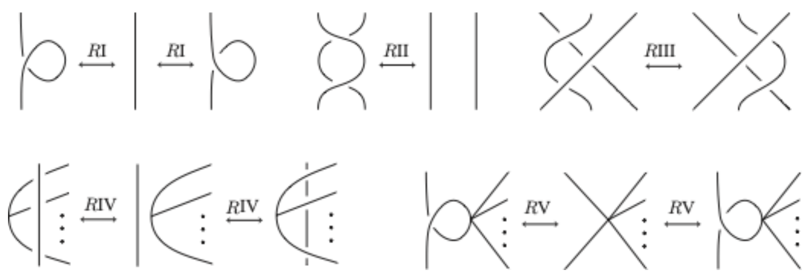


FIGURE 1

The 2-dimensional Euclidean space $\mathbb{R}^2$ is separated into some pieces by a spatial graph diagram. These pieces are said to be *regions*.

3. Goeritz invariants of spatial graphs whose vertices are of even valency

We expand the definition of Goeritz matrices of link diagrams for spatial graph diagrams whose vertices are of even valency. We note that in this paper, Goeritz matrices mean those related to Dehn colorings with alternating vertex conditions.

Let G be a spatial graph whose vertices are of even valency. Let D be a diagram of G. A *checkerboard coloring* of D is an assignment of one of two colors, black and white, to each region of D such that any two adjacent regions have different colors. Any diagram admits two checkerboard colorings. We choose one of them of D, and denote it by σ. We tack each vertex of D as a point, shrink each black region of (D, σ) to a point, and at the same time, the black regions around crossings and vertices shrink into narrow segments (see FIGURE 2). Thus, we have a graph, which we call the *checkerboard graph* (with respect to the black regions) and denote it by $\Gamma_{(D,\sigma)}$. Here we distinguish the vertices of $\Gamma_{(D,\sigma)}$ corresponding to the vertices of (D, σ) from those corresponding to the black regions of (D, σ) by assigning the *white vertices* to the former and the *black vertices* to the latter as shown in the lower-right of FIGURE 2. Let n denote the number of connected components of $\Gamma_{(D,\sigma)}$. Let $x_1, \ldots, x_m$ be the white regions of D, where m is the number of white regions of (D, σ). Let $v_1, \ldots, v_s$ be the vertices of D, where s is the number of vertices of (D, σ). For each vertex v_i of valency $2t_i$, t_i white regions, called the *local white regions* of v_i, appear in a small neighborhood of v_i, where we always suppose that a small neighborhood, say $\mathcal{N}_v$, of a vertex v of (D, σ) is an open disk with the center at v such that $\mathcal{N}_v$ includes no crossings, no other vertices, and no loops. We choose one of the local white regions, and call it the *local specified region* of v_i. As shown in the left of FIGURE 4, we represent the local specified region of v_i by putting the symbol $\star$ on it near v_i.

For each crossing c of (D, σ), we define the Goeritz index $\varepsilon(c) \in \{-1, 1\}$ as follows:

- With a slight rotation, a crossing c is arranged so that the black regions around c are placed vertically. Then $\varepsilon(c) = 1$ if the over-arc of c goes from the upper-right to the lower-left, and $\varepsilon(c) = -1$ otherwise. See FIGURE 3.

For each vertex v of (D, σ), we set the Goeritz index of v by $\varepsilon(v) = 0$.

In this paper, for matrices A and B, $A \oplus B$ means the direct sum of A and B, that is $\begin{pmatrix} A & \boldsymbol{O} \\ \boldsymbol{O} & B \end{pmatrix}$, where $\boldsymbol{O}$ represents some zero matrix.

DEFINITION 3.1. The *Goeritz matrix* $M^{\mathrm{G}}(D, \sigma)$ *of* (D, σ) (with the above situation) is obtained by taking the matrix consisting of the CR-part $M^{\mathrm{CR}}(D, \sigma)$ and the VR-part $M^{\mathrm{VR}}(D, \sigma)$ and taking the direct sum of the zero (n, n)-matrix

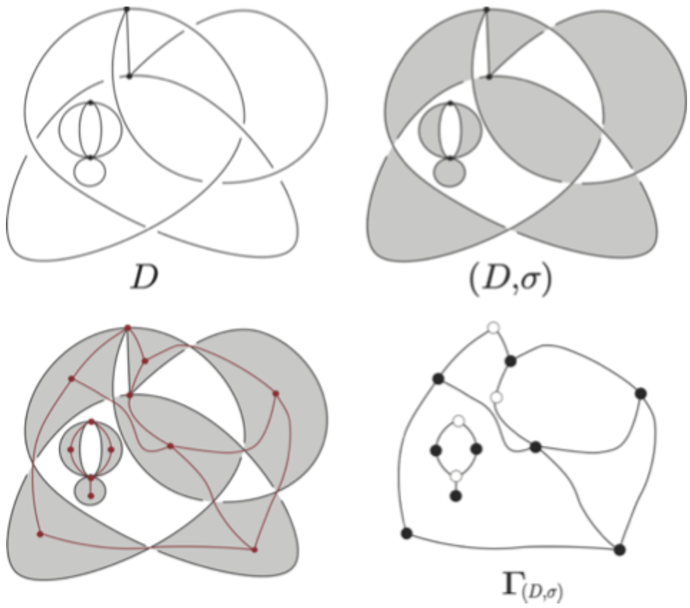


FIGURE 2

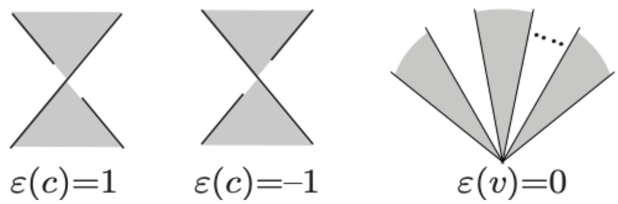


FIGURE 3

$O_{n,n}$ as

$$M^{\mathrm{G}}(D,\sigma) = \begin{pmatrix} M^{\mathrm{CR}}(D,\sigma) \\ M^{\mathrm{VR}}(D,\sigma) \end{pmatrix} \oplus O_{n,n}.$$

To make it easier to see the Goeritz matrix, we also represent it with a dotted line as $\begin{pmatrix} M^{\mathrm{CR}}(D,\sigma) \\ \hdashline M^{\mathrm{VR}}(D,\sigma) \end{pmatrix} \oplus O_{n,n}$.

The *CR-part* $M^{\mathrm{CR}}(D,\sigma)$ of the Goeritz matrix of (D,σ) (with the above situation) is the $m \times m$ matrix $M^{\mathrm{CR}}(D,\sigma) = (a_{ij})$ defined as follows: For $i,j \in \{1,\ldots,m\}$, let C_{ij} denote the set of crossings each of which is incident to the both of x_i and x_j. Then

$$a_{ij} = \begin{cases} -\displaystyle\sum_{c \in C_{ij}} \varepsilon(c) & \text{if } i \neq j, \\ -\displaystyle\sum_{k \neq i} a_{ik} & \text{if } i = j. \end{cases}$$

The *VR-part* $M^{\mathrm{VR}}(D,\sigma)$ of the Goeritz matrix of (D,σ) (with the above situation) is defined as follows: For $i \in \{1,\ldots,s\}$, let $x_{i_1},\ldots,x_{i_t}$ be the local white regions of the vertex v_i, where t is the half of the number of the valency of v_i. We suppose that $x_{i_1},\ldots,x_{i_t}$ appear clockwisely in this order, and x_{i_1} is the local specified region of v_i. Here we note that two regions x_{i_j} and x_{i_k} for $j \neq k$ might be the same. We then set the equations

$$\begin{cases} x_{i_1} - x_{i_2} = 0, \\ x_{i_1} - x_{i_3} = 0, \\ \qquad \vdots \\ x_{i_1} - x_{i_t} = 0, \end{cases} \tag{1}$$

which we call the *vertex relations with respect to* v_i. We write the coefficient matrix of the simultaneous equations (1) with respect to the variables $x_1,\ldots,x_m$ as $M^{\mathrm{VR}}_{v_i}(D,\sigma)$, which we call the *VR-part of the Goeritz matrix with respect to* v_i. Then we define $M^{\mathrm{VR}}(D,\sigma)$ by

$$M^{\mathrm{VR}}(D,\sigma) = \begin{pmatrix} M^{\mathrm{VR}}_{v_1}(D,\sigma) \\ \vdots \\ M^{\mathrm{VR}}_{v_s}(D,\sigma) \end{pmatrix}.$$

Example 3.2. Let us consider the checkerboard colored diagram (D,σ) with the white regions $x_1,\ldots,x_4$, and the vertices $v_1,\ldots,v_4$ as depicted in the left of FIGURE 4. For each vertex, we choose a local specified region as $\star$ in the left of FIGURE 4. We then have

$$M^{\mathrm{CR}}(D,\sigma) = \begin{pmatrix} 3 & -3 & 0 & 0 \\ -3 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ and } M^{\mathrm{VR}}(D,\sigma) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ -1 & 0 & 1 & 0 \\ 1 & 0 & 0 & -1 \\ -1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 1 \end{pmatrix},$$

where the vertex relations with respect to v_1, v_2, v_3 and v_4 are

$$\begin{cases} x_1 - x_1 = 0, \\ x_1 - x_3 = 0, \end{cases} \quad x_3 - x_1 = 0, \quad x_1 - x_4 = 0, \quad \text{and} \quad \begin{cases} x_4 - x_1 = 0, \\ x_4 - x_1 = 0, \end{cases}$$

respectively. Besides, the checkerboard graph of (D,σ) has two connected components (see the right of FIGURE 4). Therefore the Goeritz matrix of (D,σ) is

$$M^{\mathrm{G}}(D,\sigma) = \left(\begin{array}{cccc} 3 & -3 & 0 & 0 \\ -3 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ -1 & 0 & 1 & 0 \\ 1 & 0 & 0 & -1 \\ -1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 1 \end{array}\right) \oplus O_{2,2} = \begin{pmatrix} 3 & -3 & 0 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

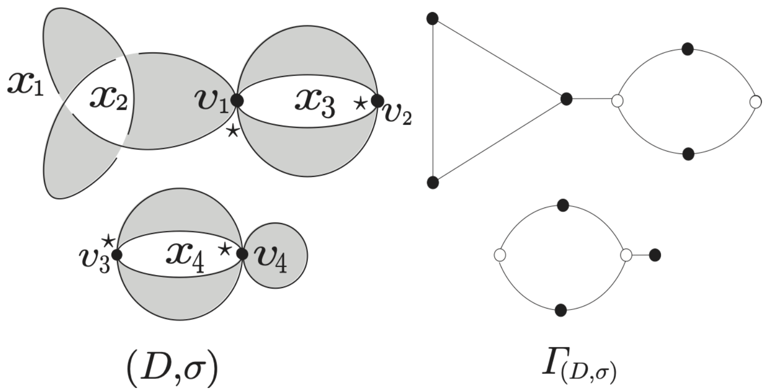


FIGURE 4

DEFINITION 3.3. Two integer matrices M_1 and M_2 are *equivalent* ($M_1 \sim M_2$) if they are transformed into each other by a finite sequence of the following *elementary transformations*:

(T1) $\quad (\boldsymbol{a_1},\dots,\boldsymbol{a_i},\dots,\boldsymbol{a_j},\dots,\boldsymbol{a_n}) \quad \longleftrightarrow \quad (\boldsymbol{a_1},\dots,\boldsymbol{a_j},\dots,\boldsymbol{a_i},\dots,\boldsymbol{a_n}),$

$\quad {}^{\mathrm{t}}(\boldsymbol{a_1},\dots,\boldsymbol{a_i},\dots,\boldsymbol{a_j},\dots,\boldsymbol{a_n}) \quad \longleftrightarrow \quad {}^{\mathrm{t}}(\boldsymbol{a_1},\dots,\boldsymbol{a_j},\dots,\boldsymbol{a_i},\dots,\boldsymbol{a_n}),$

(T2) $\quad (\boldsymbol{a_1},\dots,\boldsymbol{a_i},\dots,\boldsymbol{a_j},\dots,\boldsymbol{a_n}) \quad \longleftrightarrow \quad (\boldsymbol{a_1},\dots,\boldsymbol{a_i}+r\boldsymbol{a_j},\dots,\boldsymbol{a_j},\dots,\boldsymbol{a_n}),$

$\quad {}^{\mathrm{t}}(\boldsymbol{a_1},\dots,\boldsymbol{a_i},\dots,\boldsymbol{a_j},\dots,\boldsymbol{a_n}) \quad \longleftrightarrow \quad {}^{\mathrm{t}}(\boldsymbol{a_1},\dots,\boldsymbol{a_i}+r\boldsymbol{a_j},\dots,\boldsymbol{a_j},\dots,\boldsymbol{a_n}),$

for some $r \in \mathbb{R}$,

(T3) $\quad M \longleftrightarrow \begin{pmatrix} M \\ \mathbf{0} \end{pmatrix},$

(T4) $\quad M \longleftrightarrow \begin{pmatrix} M & \mathbf{0} \\ \mathbf{0} & 1 \end{pmatrix}.$

We have the following lemma.

LEMMA 3.4. *If two integer matrices are transformed into each other by the following transformations, then they are equivalent.*

(T5) $\quad {}^{\mathrm{t}}(\boldsymbol{a_1},\dots,\boldsymbol{a_i},\dots,\boldsymbol{a_n}) \quad \longleftrightarrow \quad {}^{\mathrm{t}}(\boldsymbol{a_1},\dots,-\boldsymbol{a_i},\dots,\boldsymbol{a_n}),$

(T6) $\quad \begin{pmatrix} M_1 \\ M_2 \end{pmatrix} \longleftrightarrow \begin{pmatrix} M_1 \\ \mathbf{0} \\ M_2 \end{pmatrix},$

(T7) $\quad \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix} \longleftrightarrow \begin{pmatrix} M_1 & \mathbf{0} & M_2 \\ \mathbf{0} & \pm 1 & \mathbf{0} \\ M_3 & \mathbf{0} & M_4 \end{pmatrix}.$

PROOF. (T5) follows from

$$\begin{aligned} &{}^{\mathrm{t}}(\boldsymbol{a_1},\dots,\boldsymbol{a_i},\dots,\boldsymbol{a_n}) \overset{T3}{\longleftrightarrow} {}^{\mathrm{t}}(\boldsymbol{a_1},\dots,\boldsymbol{a_i},\dots,\boldsymbol{a_n},\mathbf{0}) \\ &\overset{T2}{\longleftrightarrow} {}^{\mathrm{t}}(\boldsymbol{a_1},\dots,\boldsymbol{a_i},\dots,\boldsymbol{a_n},-\boldsymbol{a_i}) \overset{T1}{\longleftrightarrow} {}^{\mathrm{t}}(\boldsymbol{a_1},\dots,-\boldsymbol{a_i},\dots,\boldsymbol{a_n},\boldsymbol{a_i}) \\ &\overset{T2}{\longleftrightarrow} {}^{\mathrm{t}}(\boldsymbol{a_1},\dots,-\boldsymbol{a_i},\dots,\boldsymbol{a_n},\mathbf{0}) \overset{T3}{\longleftrightarrow} {}^{\mathrm{t}}(\boldsymbol{a_1},\dots,-\boldsymbol{a_i},\dots,\boldsymbol{a_n}). \end{aligned}$$

(T6) follows from

$$\begin{pmatrix} M_1 \\ M_2 \end{pmatrix} \overset{T3}{\longleftrightarrow} \begin{pmatrix} M_1 \\ M_2 \\ \mathbf{0} \end{pmatrix} \overset{T1}{\longleftrightarrow} \begin{pmatrix} M_1 \\ \mathbf{0} \\ M_2 \end{pmatrix}.$$

(T7) follows from

$$\begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix} \xleftrightarrow{T4} \begin{pmatrix} M_1 & M_2 & \mathbf{0} \\ M_3 & M_4 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & 1 \end{pmatrix} \xleftrightarrow{T1} \begin{pmatrix} M_1 & \mathbf{0} & M_2 \\ M_3 & \mathbf{0} & M_4 \\ \mathbf{0} & 1 & \mathbf{0} \end{pmatrix}$$
$$\xleftrightarrow{T1} \begin{pmatrix} M_1 & \mathbf{0} & M_2 \\ \mathbf{0} & 1 & \mathbf{0} \\ M_3 & \mathbf{0} & M_4 \end{pmatrix}, \text{ and}$$

$$\begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix} \xleftrightarrow{T4} \begin{pmatrix} M_1 & M_2 & \mathbf{0} \\ M_3 & M_4 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & 1 \end{pmatrix} \xleftrightarrow{T5} \begin{pmatrix} M_1 & M_2 & \mathbf{0} \\ M_3 & M_4 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -1 \end{pmatrix}$$
$$\xleftrightarrow{T1} \begin{pmatrix} M_1 & \mathbf{0} & M_2 \\ M_3 & \mathbf{0} & M_4 \\ \mathbf{0} & -1 & \mathbf{0} \end{pmatrix} \xleftrightarrow{T1} \begin{pmatrix} M_1 & \mathbf{0} & M_2 \\ \mathbf{0} & -1 & \mathbf{0} \\ M_3 & \mathbf{0} & M_4 \end{pmatrix}.$$

□

THEOREM 3.5. *Let G be a spatial graph whose vertices are of even valency, and (D, σ) a checkerboard colored diagram of G. Let $M^{\mathrm{G}}(D, \sigma)$ be the Goeritz matrix of (D, σ) obtained by setting the orderings of the white regions and the vertices, and by deciding the local specified region of each vertex. Then the equivalence class of $M^{\mathrm{G}}(D, \sigma)$ is an invariant of G.*

For any integer matrix M, there exists a unique diagonal integer matrix

$$M' = (d_1) \oplus (d_2) \oplus \cdots \oplus (d_k) = \begin{pmatrix} d_1 & & \boldsymbol{O} \\ & \ddots & \\ \boldsymbol{O} & & d_k \end{pmatrix}$$

such that $M \sim M'$, where

$$(2) \qquad \begin{cases} d_1, \ldots, d_k \geq 0, d_1, \ldots, d_k \neq 1, d_1 | d_2 | \cdots | d_k & \text{if } k > 1, \text{ and} \\ d_1 \geq 0 & \text{if } k = 1. \end{cases}$$

Hence, for a Goeritz matrix $M^{\mathrm{G}}(D, \sigma)$ of a spatial graph G, when $M^{\mathrm{G}}(D, \sigma) \sim (d_1) \oplus (d_2) \oplus \cdots \oplus (d_k)$ with the condition (2), the sequence $(d_1, d_2, \ldots, d_k)$ is uniquely determined for G, and thus, we have an invariant of G. We denote by $gz(G)$ the invariant $(d_1, d_2, \ldots, d_k)$, and call it the *Goeritz invariant* of G.

EXAMPLE 3.6. For Example 3.2, we have

$$M^{\mathrm{G}}(D,\sigma)=\begin{pmatrix}3&-3&0&0\\-3&3&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\\1&0&-1&0\\-1&0&1&0\\1&0&0&-1\\-1&0&0&1\\-1&0&0&1\end{pmatrix}\oplus O_{2,2}\overset{T2}{\longleftrightarrow}\begin{pmatrix}3&-3&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\\1&0&-1&0\\0&0&0&0\\1&0&0&-1\\0&0&0&0\\0&0&0&0\end{pmatrix}\oplus O_{2,2}$$

$$\overset{T6}{\longleftrightarrow}\begin{pmatrix}3&-3&0&0\\0&0&0&0\\1&0&-1&0\\1&0&0&-1\end{pmatrix}\oplus O_{2,2}\overset{T2}{\longleftrightarrow}\begin{pmatrix}3&0&0&0\\0&0&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}\oplus O_{2,2}$$

$$\overset{T7}{\longleftrightarrow}\begin{pmatrix}3&0\\0&0\end{pmatrix}\oplus O_{2,2}\sim(3)\ \oplus(0)\oplus(0)\oplus(0).$$

Therefore the Goeritz invariant of the spatial graph G which D represents is

$$gz(G)=(3,0,0,0).$$

4. Proof of Theorem 3.5

In this section, the bold $\boldsymbol{A},\boldsymbol{A_1},\boldsymbol{A_2},\boldsymbol{A_3},\boldsymbol{B},\boldsymbol{B_1},\boldsymbol{B_2}$ represent some integer matrices, $\boldsymbol{O}$ represents (a part of) some zero matrix, $\boldsymbol{\alpha},\boldsymbol{\alpha_1},\boldsymbol{\alpha_2},\boldsymbol{\alpha_3},\boldsymbol{\beta},\boldsymbol{\beta_1},\boldsymbol{\beta_2},\boldsymbol{\beta_3}$ represent some column vectors, and the bold $\mathbf{0}$ (resp. $\mathbf{1}$) represents a column vector whose elements are all 0 (resp. all 1).

Let $M^{\mathrm{G}}(D,\sigma)$ be the Goeritz matrix of a checkerboard colored diagram (D,σ) with given orderings of the white regions and the vertices and given local specified regions.

LEMMA 4.1. *The equivalence class of $M^{\mathrm{G}}(D,\sigma)$ is unchanged under the replacement of the ordering of the white regions of (D,σ).*

PROOF. When we swap the ordering of the ith and jth white regions, the corresponding Goeritz matrix is obtained from $M^{\mathrm{G}}(D,\sigma)$ by permuting the ith and jth rows and the ith and jth columns. Hence the equivalence classes of the Goeritz matrices are unchanged. □

LEMMA 4.2. *The equivalence class of $M^{\mathrm{G}}(D,\sigma)$ is unchanged under the replacement of the ordering of vertices of (D,σ).*

PROOF. When we swap the ordering of the ith and jth vertices, the corresponding Goeritz matrix is obtained from $M^{\mathrm{G}}(D,\sigma)$ by permuting the VR-part with respect to v_i and that with respect to v_j, that is, by permuting some row vectors. Hence the equivalence classes of the Goeritz matrices are unchanged. □

LEMMA 4.3. *The equivalence class of $M^{\mathrm{G}}(D,\sigma)$ is unchanged under the RI-moves.*

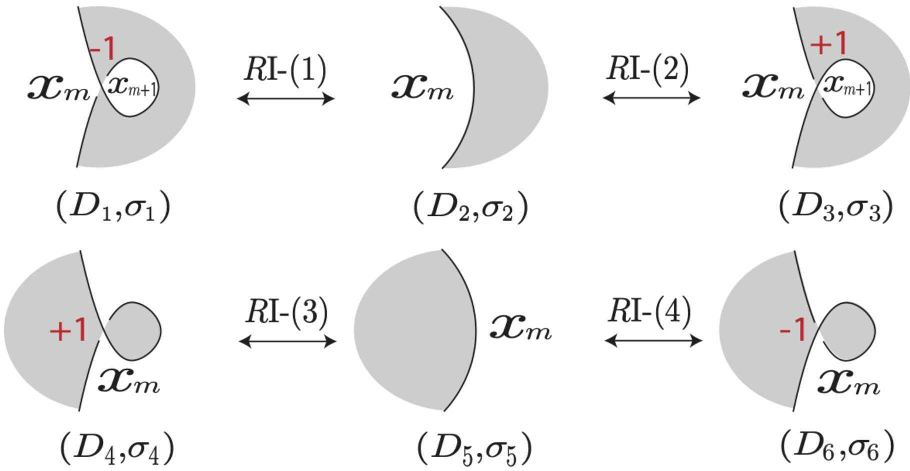


FIGURE 5

PROOF. For the invariance under the RI-moves, we may consider the four types of the RI-moves depicted in FIGURE 5.

Let us consider the move RI-(1) between the checkerboard colored diagrams (D_1,σ_1) and (D_2,σ_2), and the move RI-(2) between the checkerboard colored diagrams (D_2,σ_2) and (D_3,σ_3). Let $x_1,\dots,x_m$(, and x_{m+1}) (resp. $v_1,\dots,v_s$) be the white regions (resp. the vertices) of $(D_1,\sigma_1),(D_2,\sigma_2)$ and (D_3,σ_3). We suppose that $x_1,\dots,x_m$, $v_1,\dots,v_s$ and their local specified regions stay outside the disk in which the moves are applied, and x_m and x_{m+1} are changed inside the disk as in FIGURE 5. For $i\in\{1,2,3\}$, let n_i denote the number of the connected components of the checkerboard graph of (D_i,σ_i), and M_i the Goeritz matrix of

(D_i, σ_i). We then have $n := n_1 = n_2 = n_3$ and

$$M_1 = \left(\begin{array}{ccc} \boldsymbol{A} & \boldsymbol{\alpha} & \boldsymbol{0} \\ {}^{\mathrm{t}}\boldsymbol{\alpha} & a-1 & 1 \\ \boldsymbol{0} & 1 & -1 \\ \hdashline \boldsymbol{B} & \boldsymbol{\beta} & \boldsymbol{0} \end{array}\right) \oplus O_{n,n}, \qquad M_2 = \left(\begin{array}{cc} \boldsymbol{A} & \boldsymbol{\alpha} \\ {}^{\mathrm{t}}\boldsymbol{\alpha} & a \\ \hdashline \boldsymbol{B} & \boldsymbol{\beta} \end{array}\right) \oplus O_{n,n},$$

$$M_3 = \left(\begin{array}{ccc} \boldsymbol{A} & \boldsymbol{\alpha} & \boldsymbol{0} \\ {}^{\mathrm{t}}\boldsymbol{\alpha} & a+1 & -1 \\ \boldsymbol{0} & -1 & 1 \\ \hdashline \boldsymbol{B} & \boldsymbol{\beta} & \boldsymbol{0} \end{array}\right) \oplus O_{n,n}.$$

Then we have $M_1 \sim M_2$ and $M_3 \sim M_2$, which follow from

$$\left(\begin{array}{ccc} \boldsymbol{A} & \boldsymbol{\alpha} & \boldsymbol{0} \\ {}^{\mathrm{t}}\boldsymbol{\alpha} & a \mp 1 & \pm 1 \\ \boldsymbol{0} & \pm 1 & \mp 1 \\ \hdashline \boldsymbol{B} & \boldsymbol{\beta} & \boldsymbol{0} \end{array}\right) \oplus O_{n,n} \xleftrightarrow{T2} \left(\begin{array}{ccc} \boldsymbol{A} & \boldsymbol{\alpha} & \boldsymbol{0} \\ {}^{\mathrm{t}}\boldsymbol{\alpha} & a & \pm 1 \\ {}^{\mathrm{t}}\boldsymbol{0} & 0 & \mp 1 \\ \boldsymbol{B} & \boldsymbol{\beta} & \boldsymbol{0} \end{array}\right) \oplus O_{n,n}$$

$$\xleftrightarrow{T2} \left(\begin{array}{ccc} \boldsymbol{A} & \boldsymbol{\alpha} & \boldsymbol{0} \\ {}^{\mathrm{t}}\boldsymbol{\alpha} & a & 0 \\ {}^{\mathrm{t}}\boldsymbol{0} & 0 & \mp 1 \\ \boldsymbol{B} & \boldsymbol{\beta} & \boldsymbol{0} \end{array}\right) \oplus O_{n,n} \xleftrightarrow{T7} \left(\begin{array}{cc} \boldsymbol{A} & \boldsymbol{\alpha} \\ {}^{\mathrm{t}}\boldsymbol{\alpha} & a \\ \hdashline \boldsymbol{B} & \boldsymbol{\beta} \end{array}\right) \oplus O_{n,n} = M_2,$$

where all the double signs correspond to each other.

For the moves RI-(3) and RI-(4), we set the white regions $x_1, \ldots, x_m$, the vertices $v_1, \ldots, v_s$ and their local specified regions so that they stay outside the disk in which the moves are applied. Then the Goeritz matrices M_4, M_5 and M_6, respectively, of (D_4, σ_4), (D_5, σ_5), and (D_6, σ_6) are the same, since the crossings in the disk in which the moves are applied are incident to the unique white region x_m. This clearly gives $M_4 \sim M_5 \sim M_6$.

□

LEMMA 4.4. *The equivalence class of $M^{\mathrm{G}}(D, \sigma)$ is unchanged under the RII-moves.*

PROOF. For the invariance under the RII-moves, we may consider the two types of the RII-moves depicted in FIGURE 6. More precisely, we need to investigate the four cases: (i) the move RII-(1) such that $x_{m-1} \neq x_m$, (ii) the move RII-(1) such that $x_{m-1} = x_m$, (iii) the move RII-(2) such that $x_m \neq x_{m+2}$, and (iv) the move RII-(2) such that $x_m = x_{m+2}$.

Let us consider the case (i), that is, the move RII-(1) between the checkerboard colored diagrams (D_1, σ_1) and (D_2, σ_2) such that $x_{m-1} \neq x_m$. Let $x_1, \ldots, x_m$

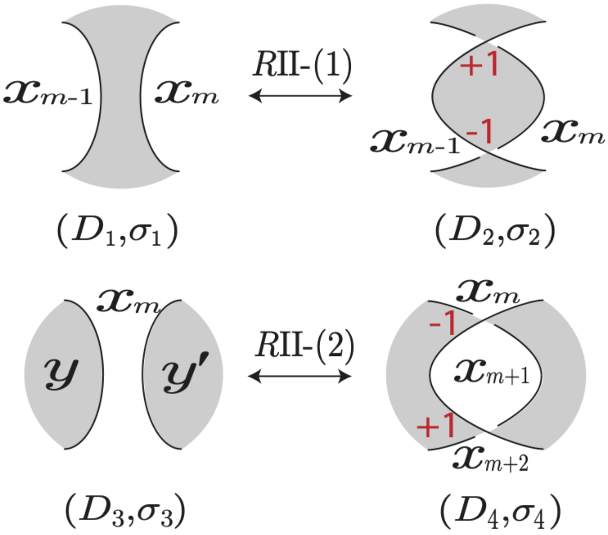


FIGURE 6

(resp. $v_1, \dots, v_s$) be the white regions (resp. the vertices) of (D_1, σ_1) and (D_2, σ_2). We suppose that $x_1, \dots, x_m$, $v_1, \dots, v_s$ and their local specified regions stay outside the disk in which the moves are applied, and x_{m-1} and x_m are changed inside the disk as in FIGURE 6. For $i \in \{1, 2\}$, let n_i denote the number of the connected components of the checkerboard graph of (D_i, σ_i), and M_i the Goeritz matrix of (D_i, σ_i). We then have $n := n_1 = n_2$ and

$$M_1 = \left(\begin{array}{ccc} \boldsymbol{A} & \boldsymbol{\alpha_1} & \boldsymbol{\alpha_2} \\ {}^{\mathrm{t}}\boldsymbol{\alpha_1} & a_1 & a_2 \\ {}^{\mathrm{t}}\boldsymbol{\alpha_2} & a_3 & a_4 \\ \hdashline \boldsymbol{B} & \boldsymbol{\beta_1} & \boldsymbol{\beta_2} \end{array}\right) \oplus O_{n,n} = \left(\begin{array}{ccc} \boldsymbol{A} & \boldsymbol{\alpha_1} & \boldsymbol{\alpha_2} \\ {}^{\mathrm{t}}\boldsymbol{\alpha_1} & a_1 - 1 + 1 & a_2 + 1 - 1 \\ {}^{\mathrm{t}}\boldsymbol{\alpha_2} & a_3 + 1 - 1 & a_4 - 1 + 1 \\ \hdashline \boldsymbol{B} & \boldsymbol{\beta_1} & \boldsymbol{\beta_2} \end{array}\right) \oplus O_{n,n} = M_2,$$

which clearly gives $M_1 \sim M_2$.

For the case (ii), that is, the move RII-(1) between the checkerboard colored diagrams (D_1, σ_1) and (D_2, σ_2) such that $x_{m-1} = x_m$, we set the white regions $x_1, \dots, x_{m-1}$, the vertices $v_1, \dots, v_s$ and their local specified regions so that they stay outside the disk in which the moves are applied. Then the Goeritz matrices M_1 and M_2, respectively, of (D_1, σ_1) and (D_2, σ_2) are the same, since the crossings in the disk in which the moves are applied are incident to the unique white region x_{m-1}. This clearly gives $M_1 \sim M_2$.

Let us consider the case (iii), that is, the move RII-(2) between the checkerboard colored diagrams (D_3, σ_3) and (D_4, σ_4), such that $x_m \neq x_{m+2}$. We note that two black vertices in $\Gamma_{(D_3,\sigma_3)}$ corresponding to the black regions y and y', respectively, are in the same connected component of the checkerboard graph. Let

$x_1, \dots, x_m(, x_{m+1}$ and $x_{m+2})$ (resp. $v_1, \dots, v_s$) be the white regions (resp. the vertices) of (D_3, σ_3) and (D_4, σ_4). We suppose that $x_1, \dots, x_{m-1}$, $v_1, \dots, v_s$ and their local specified regions stay outside the disk in which the moves are applied, and x_m, x_{m+1} and x_{m+2} are changed inside the disk as in FIGURE 6. Note that the white region x_m of (D_3, σ_3) are separated to x_m, x_{m+1} and x_{m+2} of (D_4, σ_4) as in FIGURE 6. For $i \in \{3, 4\}$, let n_i denote the number of the connected components of the checkerboard graph of (D_i, σ_i), and M_i the Goeritz matrix of (D_i, σ_i). We then have $n := n_3 = n_4$ and

$$M_3 = \left(\begin{array}{cc} \boldsymbol{A} & \boldsymbol{\alpha_1} + \boldsymbol{\alpha_2} \\ {}^{\mathrm{t}}\boldsymbol{\alpha_1} + {}^{\mathrm{t}}\boldsymbol{\alpha_2} & a_1 + 2a_2 + a_3 \\ \hdashline \boldsymbol{B} & \boldsymbol{\beta_1} + \boldsymbol{\beta_2} \end{array}\right) \oplus O_{n,n},$$

$$M_4 = \left(\begin{array}{cccc} \boldsymbol{A} & \boldsymbol{\alpha_1} & \boldsymbol{0} & \boldsymbol{\alpha_2} \\ {}^{\mathrm{t}}\boldsymbol{\alpha_1} & a_1 & 1 & a_2 \\ {}^{\mathrm{t}}\boldsymbol{0} & 1 & 0 & -1 \\ {}^{\mathrm{t}}\boldsymbol{\alpha_2} & a_2 & -1 & a_3 \\ \hdashline \boldsymbol{B} & \boldsymbol{\beta_1} & \boldsymbol{0} & \boldsymbol{\beta_2} \end{array}\right) \oplus O_{n,n}.$$

Then we have $M_3 \sim M_4$, which follows from

$$M_4 \stackrel{T2}{\longleftrightarrow} \begin{pmatrix} \boldsymbol{A} & \boldsymbol{\alpha_1} + \boldsymbol{\alpha_2} & \boldsymbol{0} & \boldsymbol{\alpha_2} \\ {}^{\mathrm{t}}\boldsymbol{\alpha_1} & a_1 + a_2 & 1 & a_2 \\ {}^{\mathrm{t}}\boldsymbol{0} & 0 & 0 & -1 \\ {}^{\mathrm{t}}\boldsymbol{\alpha_2} & a_2 + a_3 & -1 & a_3 \\ \boldsymbol{B} & \boldsymbol{\beta_1} + \boldsymbol{\beta_2} & \boldsymbol{0} & \boldsymbol{\beta_2} \end{pmatrix} \oplus O_{n,n}$$

$$\stackrel{T2}{\longleftrightarrow} \begin{pmatrix} \boldsymbol{A} & \boldsymbol{\alpha_1} + \boldsymbol{\alpha_2} & \boldsymbol{0} & \boldsymbol{\alpha_2} \\ {}^{\mathrm{t}}\boldsymbol{\alpha_1} + {}^{\mathrm{t}}\boldsymbol{\alpha_2} & a_1 + 2a_2 + a_3 & 0 & a_2 + a_3 \\ {}^{\mathrm{t}}\boldsymbol{0} & 0 & 0 & -1 \\ {}^{\mathrm{t}}\boldsymbol{\alpha_2} & a_2 + a_3 & -1 & a_3 \\ \boldsymbol{B} & \boldsymbol{\beta_1} + \boldsymbol{\beta_2} & \boldsymbol{0} & \boldsymbol{\beta_2} \end{pmatrix} \oplus O_{n,n}$$

$$\stackrel{T2}{\longleftrightarrow} \begin{pmatrix} \boldsymbol{A} & \boldsymbol{\alpha_1} + \boldsymbol{\alpha_2} & \boldsymbol{0} & \boldsymbol{0} \\ {}^{\mathrm{t}}\boldsymbol{\alpha_1} + {}^{\mathrm{t}}\boldsymbol{\alpha_2} & a_1 + 2a_2 + a_3 & 0 & 0 \\ {}^{\mathrm{t}}\boldsymbol{0} & 0 & 0 & -1 \\ {}^{\mathrm{t}}\boldsymbol{0} & 0 & -1 & 0 \\ \boldsymbol{B} & \boldsymbol{\beta_1} + \boldsymbol{\beta_2} & \boldsymbol{0} & \boldsymbol{0} \end{pmatrix} \oplus O_{n,n} \stackrel{T7}{\longleftrightarrow} M_3.$$

Let us consider the case (iv), that is, the move RII-(2) between the checkerboard colored diagrams (D_3, σ_3) and (D_4, σ_4) such that $x_m = x_{m+2}$. We note that two black vertices corresponding to the black regions y and y', respectively, are in different connected components of the checkerboard graph. Let $x_1, \dots, x_m(,$ and $x_{m+1})$ (resp. $v_1, \dots, v_s$) be the white regions (resp. the vertices) of (D_3, σ_3) and (D_4, σ_4). We suppose that $x_1, \dots, x_{m-1}$, $v_1, \dots, v_s$ and their local specified regions stay outside the disk in which the moves are applied, and x_m and x_{m+1} are changed inside the disk as in FIGURE 6. Note that the white region x_m of (D_3, σ_3) are separated to x_m and x_{m+1} of (D_4, σ_4) as in FIGURE 6. For $i \in \{3, 4\}$, let n_i denote the number of the connected components of the checkerboard graph of (D_i, σ_i), and M_i the Goeritz matrix of (D_i, σ_i). We then have $n := n_3 = n_4 + 1$ and

$$M_3 = \left(\begin{array}{cc} \boldsymbol{A} & \boldsymbol{\alpha} \\ {}^{\mathrm{t}}\boldsymbol{\alpha} & a \\ \hdashline \boldsymbol{B} & \boldsymbol{\beta_1} \end{array}\right) \oplus O_{n,n},$$

$$M_4 = \left(\begin{array}{ccc} \boldsymbol{A} & \boldsymbol{\alpha} & \boldsymbol{0} \\ {}^{\mathrm{t}}\boldsymbol{\alpha} & a & +1-1 \\ {}^{\mathrm{t}}\boldsymbol{0} & +1-1 & 0 \\ \hdashline \boldsymbol{B} & \boldsymbol{\beta} & \boldsymbol{0} \end{array}\right) \oplus O_{n-1,n-1} = \left(\begin{array}{ccc} \boldsymbol{A} & \boldsymbol{\alpha} & \boldsymbol{0} \\ {}^{\mathrm{t}}\boldsymbol{\alpha} & a & 0 \\ {}^{\mathrm{t}}\boldsymbol{0} & 0 & 0 \\ \hdashline \boldsymbol{B} & \boldsymbol{\beta} & \boldsymbol{0} \end{array}\right) \oplus O_{n-1,n-1}.$$

Then we have $M_3 \sim M_4$, which follows from

$$M_4 = \begin{pmatrix} \boldsymbol{A} & \boldsymbol{\alpha} & \boldsymbol{0} \\ {}^{\mathrm{t}}\boldsymbol{\alpha} & a & 0 \\ {}^{\mathrm{t}}\boldsymbol{0} & 0 & 0 \\ \boldsymbol{B} & \boldsymbol{\beta} & \boldsymbol{0} \end{pmatrix} \oplus O_{n-1,n-1} \overset{T1}{\longleftrightarrow} \begin{pmatrix} \boldsymbol{A} & \boldsymbol{\alpha} & \boldsymbol{0} \\ {}^{\mathrm{t}}\boldsymbol{\alpha} & a & 0 \\ \boldsymbol{B} & \boldsymbol{\beta} & \boldsymbol{0} \\ {}^{\mathrm{t}}\boldsymbol{0} & 0 & 0 \end{pmatrix} \oplus O_{n-1,n-1} = M_3.$$

□

LEMMA 4.5. *The equivalence class of $M^{\mathrm{G}}(D, \sigma)$ is unchanged under the RIII-moves.*

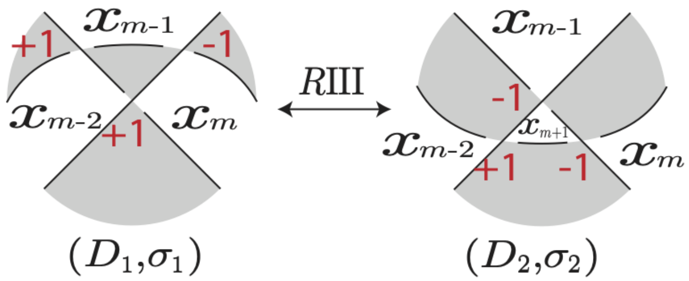


FIGURE 7

PROOF. For the invariance under the RIII-moves, we may consider the RIII-move depicted in FIGURE 7 between the checkerboard colored diagrams (D_1, σ_1) and (D_2, σ_2). We note that the other RIII-move can be obtained by performing some of the RI-moves, the RII-moves and the RIII-move depicted in FIGURE 7.

Let $x_1, \dots, x_m(, \text{and } x_{m+1})$ (resp. $v_1, \dots, v_s$) be the white regions (resp. the vertices) of (D_1, σ_1) and (D_2, σ_2). We suppose that $x_1, \dots, x_m$, $v_1, \dots, v_s$ and their local specified regions stay outside the disk in which the moves are applied, and $x_{m-2}, \cdots, x_{m+1}$ are changed inside the disk as in FIGURE 7. By performing RII-moves if necessary, we may assume that the regions x_{m-2}, x_{m-1} and x_m are mutually distinct. For $i \in \{1, 2\}$, let n_i denote the number of the connected components of the checkerboard graph of (D_i, σ_i), and M_i the Goeritz matrix of (D_i, σ_i). We then have $n := n_1 = n_2$,

$$M_1 = \left(\begin{array}{cccc} \boldsymbol{A} & \boldsymbol{\alpha_1} & \boldsymbol{\alpha_2} & \boldsymbol{\alpha_3} \\ {}^{\mathrm{t}}\boldsymbol{\alpha_1} & a_1+2 & a_2-1 & a_3-1 \\ {}^{\mathrm{t}}\boldsymbol{\alpha_2} & a_2-1 & a_4 & a_5+1 \\ {}^{\mathrm{t}}\boldsymbol{\alpha_3} & a_3-1 & a_5+1 & a_6 \\ \hdashline \boldsymbol{B} & \boldsymbol{\beta_1} & \boldsymbol{\beta_2} & \boldsymbol{\beta_3} \end{array}\right) \oplus O_{n,n}$$

and

$$M_2 = \left(\begin{array}{ccccc} \boldsymbol{A} & \boldsymbol{\alpha_1} & \boldsymbol{\alpha_2} & \boldsymbol{\alpha_3} & \mathbf{0} \\ {}^{\mathrm{t}}\boldsymbol{\alpha_1} & a_1+1 & a_2 & a_3 & -1 \\ {}^{\mathrm{t}}\boldsymbol{\alpha_2} & a_2 & a_4-1 & a_5 & 1 \\ {}^{\mathrm{t}}\boldsymbol{\alpha_3} & a_3 & a_5 & a_6-1 & 1 \\ {}^{\mathrm{t}}\mathbf{0} & -1 & 1 & 1 & -1 \\ \hdashline \boldsymbol{B} & \boldsymbol{\beta_1} & \boldsymbol{\beta_2} & \boldsymbol{\beta_3} & \mathbf{0} \end{array}\right) \oplus O_{n,n}.$$

Then we have $M_1 \sim M_2$, which follows from

$$M_2 \overset{T2}{\longleftrightarrow} \begin{pmatrix} \boldsymbol{A} & \boldsymbol{\alpha_1} & \boldsymbol{\alpha_2} & \boldsymbol{\alpha_3} & \mathbf{0} \\ {}^{\mathrm{t}}\boldsymbol{\alpha_1} & a_1+2 & a_2-1 & a_3-1 & 0 \\ {}^{\mathrm{t}}\boldsymbol{\alpha_2} & a_2-1 & a_4 & a_5+1 & 0 \\ {}^{\mathrm{t}}\boldsymbol{\alpha_3} & a_3-1 & a_5+1 & a_6 & 0 \\ {}^{\mathrm{t}}\mathbf{0} & -1 & 1 & 1 & -1 \\ \boldsymbol{B} & \boldsymbol{\beta_1} & \boldsymbol{\beta_2} & \boldsymbol{\beta_3} & \mathbf{0} \end{pmatrix} \oplus O_{n,n}$$

$$\overset{T2}{\longleftrightarrow} \begin{pmatrix} \boldsymbol{A} & \boldsymbol{\alpha_1} & \boldsymbol{\alpha_2} & \boldsymbol{\alpha_3} & \mathbf{0} \\ {}^{\mathrm{t}}\boldsymbol{\alpha_1} & a_1+2 & a_2-1 & a_3-1 & 0 \\ {}^{\mathrm{t}}\boldsymbol{\alpha_2} & a_2-1 & a_4 & a_5+1 & 0 \\ {}^{\mathrm{t}}\boldsymbol{\alpha_3} & a_3-1 & a_5+1 & a_6 & 0 \\ {}^{\mathrm{t}}\mathbf{0} & 0 & 0 & 0 & -1 \\ \boldsymbol{B} & \boldsymbol{\beta_1} & \boldsymbol{\beta_2} & \boldsymbol{\beta_3} & \mathbf{0} \end{pmatrix} \oplus O_{n,n} \overset{T7}{\longleftrightarrow} M_1.$$

□

LEMMA 4.6. *The equivalence class of $M^{\mathrm{G}}(D, \sigma)$ is unchanged under the replacement of the local specified region of a vertex.*

PROOF. Let v be a vertex of D. By permuting the orderings of the white regions and the vertices of (D, σ) if necessary, we may assume that $v = v_1$ and

v_1 is incident to the white regions $x_1, \dots, x_t$ clockwisely in this order, where $2t$ is the valency of v_1. Additionally, by performing RII-moves if necessary, we may assume that there are no duplicates for the white regions $x_1, \dots, x_t$. It is sufficient to show that the replacement of the local specified region x_1 of v_1 with x_2 does not change the equivalence class of $M^{\mathrm{G}}(D, \sigma)$.

When we set the local specified region of v_1 to be x_1, the VR-part, say $M^{\mathrm{VR}}_{v_1,x_1}$, of the Goeritz matrix with respect to v_1 is the $(t-1) \times m$-matrix

$$M^{\mathrm{VR}}_{v_1,x_1} = \begin{pmatrix} 1 & -1 & & \boldsymbol{O} & 0 & \cdots & 0 \\ \vdots & & \ddots & & \vdots & & \vdots \\ 1 & \boldsymbol{O} & & -1 & 0 & \cdots & 0 \end{pmatrix},$$

where m is the number of the white regions of (D, σ). We denote by M_1 the Goeritz matrix for this case. On the other hand, when we set the local specified region of v_1 to be x_2, the VR-part, say $M^{\mathrm{VR}}_{v_1,x_2}$, of the Goeritz matrix with respect to v_1 is the $(t-1) \times m$-matrix

$$M^{\mathrm{VR}}_{v_1,x_2} = \begin{pmatrix} 0 & 1 & -1 & & \boldsymbol{O} & 0 & \cdots & 0 \\ \vdots & \vdots & & \ddots & & \vdots & & \vdots \\ 0 & 1 & \boldsymbol{O} & & -1 & 0 & \cdots & 0 \\ -1 & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

We denote by M_2 the Goeritz matrix for this case. We note that M_1 and M_2 differ only for the VR-part with respect to v_1. We then have

$$M^{\mathrm{VR}}_{v_1,x_2} \overset{T5}{\longleftrightarrow} \begin{pmatrix} 0 & 1 & -1 & & \boldsymbol{O} & 0 & \cdots & 0 \\ \vdots & \vdots & & \ddots & & \vdots & & \vdots \\ 0 & 1 & \boldsymbol{O} & & -1 & 0 & \cdots & 0 \\ 1 & -1 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix}$$

$$\overset{T2}{\longleftrightarrow} \begin{pmatrix} 1 & 0 & -1 & & \boldsymbol{O} & 0 & \cdots & 0 \\ \vdots & \vdots & & \ddots & & \vdots & & \vdots \\ 1 & 0 & \boldsymbol{O} & & -1 & 0 & \cdots & 0 \\ 1 & -1 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix} \overset{T1}{\longleftrightarrow} M^{\mathrm{VR}}_{v_1,x_1}.$$

Here we note that the above transformations are of row vectors, and they does not change the other parts of the Goeritz matrices. Hence, we have $M_1 \sim M_2$. $\square$

LEMMA 4.7. *The equivalence class of $M^{\mathrm{G}}(D, \sigma)$ is unchanged under the RIV-moves.*

PROOF. For the invariance under the RIV-moves, we may consider the RIV-moves depicted in FIGURE 8. We note that the RIV-moves with the other

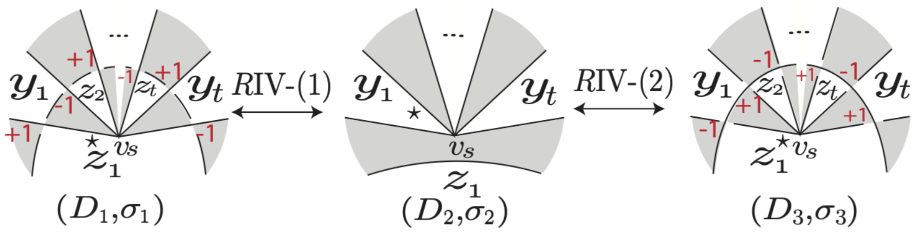


FIGURE 8

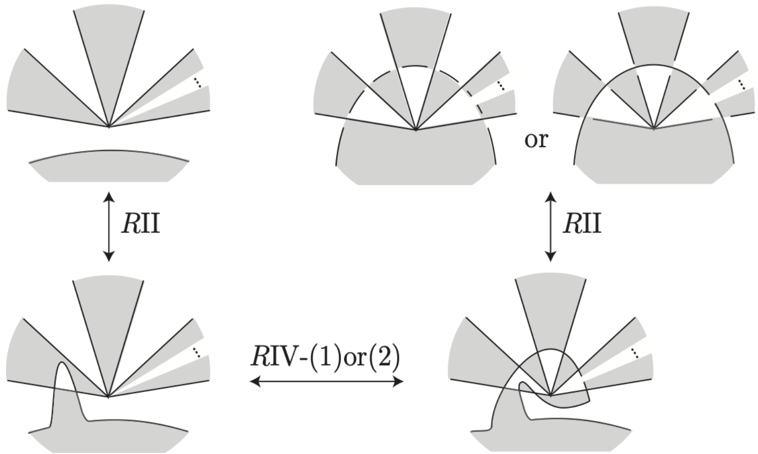


FIGURE 9

checkerboard pattern can be obtained by performing some of the RII-moves and the RIV-move in FIGURE 8 as shown in FIGURE 9.

Let us consider the move RIV-(1) between the checkerboard colored diagrams (D_1, σ_1) and (D_2, σ_2), and the move RIV-(2) between the checkerboard colored diagrams (D_2, σ_2) and (D_3, σ_3). Let $x_1, \ldots, x_m, y_1, \ldots, y_t, z_1$ be the white regions of (D_1, σ_1), (D_2, σ_2) and (D_3, σ_3) which appear outside the disk E in which the moves are applied. Let $z_2, \ldots, z_t$ be the other white regions of (D_1, σ_1) and (D_3, σ_3). Let $v_1, \ldots, v_s$ be the vertices of (D_1, σ_1), (D_2, σ_2) and (D_3, σ_3). We suppose that $x_1, \ldots, x_m, y_1, \ldots, y_t, z_1, v_1, \ldots, v_{s-1}$ and their local specified regions stay outside the disk E, and $y_1, \ldots, y_t, z_1, \ldots, z_t$, v_s and the local specified region of v_s are changed inside the disk E as in FIGURE 8.

For $i \in \{1, 2, 3\}$, let n_i denote the number of the connected components of the checkerboard graph of (D_i, σ_i), and M_i the Goeritz matrix of (D_i, σ_i), where we set the ordering of the white regions to be $x_1, \ldots, x_m, y_1, \ldots, y_t, z_1\,(, z_2, \ldots, z_t)$.

We then have $n := n_1 = n_2 = n_3$ and

$$M_1 = \left(\begin{array}{ccc} \boldsymbol{A_1} & \boldsymbol{A_2} & \boldsymbol{\alpha_1} \bowtie \boldsymbol{O} \\ {}^{\mathrm{t}}\boldsymbol{A_2} & \boldsymbol{A_3} & \boldsymbol{\alpha_2} \bowtie \boldsymbol{O} + \boldsymbol{C} \\ {}^{\mathrm{t}}(\boldsymbol{\alpha_1} \bowtie \boldsymbol{O}) & {}^{\mathrm{t}}(\boldsymbol{\alpha_2} \bowtie \boldsymbol{O} + \boldsymbol{C}) & (a) \oplus \boldsymbol{O} \\ \hdashline \boldsymbol{B_1} & \boldsymbol{B_2} & \boldsymbol{\beta} \bowtie \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{1} \bowtie (-\boldsymbol{E_{t-1}}) \end{array}\right) \oplus O_{n,n},$$

$$M_2 = \left(\begin{array}{ccc} \boldsymbol{A_1} & \boldsymbol{A_2} & \boldsymbol{\alpha_1} \\ {}^{\mathrm{t}}\boldsymbol{A_2} & \boldsymbol{A_3} & \boldsymbol{\alpha_2} \\ {}^{\mathrm{t}}\boldsymbol{\alpha_1} & {}^{\mathrm{t}}\boldsymbol{\alpha_2} & a \\ \hdashline \boldsymbol{B_1} & \boldsymbol{B_2} & \boldsymbol{\beta} \\ \boldsymbol{O} & \boldsymbol{1} \bowtie (-\boldsymbol{E_{t-1}}) & \boldsymbol{0} \end{array}\right) \oplus O_{n,n},$$

$$M_3 = \left(\begin{array}{ccc} \boldsymbol{A_1} & \boldsymbol{A_2} & \boldsymbol{\alpha_1} \bowtie \boldsymbol{O} \\ {}^{\mathrm{t}}\boldsymbol{A_2} & \boldsymbol{A_3} & \boldsymbol{\alpha_2} \bowtie \boldsymbol{O} - \boldsymbol{C} \\ {}^{\mathrm{t}}(\boldsymbol{\alpha_1} \bowtie \boldsymbol{O}) & {}^{\mathrm{t}}(\boldsymbol{\alpha_2} \bowtie \boldsymbol{O} - \boldsymbol{C}) & (a) \oplus \boldsymbol{O} \\ \hdashline \boldsymbol{B_1} & \boldsymbol{B_2} & \boldsymbol{\beta} \bowtie \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{1} \bowtie (-\boldsymbol{E_{t-1}}) \end{array}\right) \oplus O_{n,n},$$

where for two matrices A and B such that the number of row vectors of A is equal to that of B, $A \bowtie B$ means the join of A and B, that is $A \bowtie B = \begin{pmatrix} A & B \end{pmatrix}$,

$$\boldsymbol{C} = \begin{pmatrix} -1 & 1 & & & \boldsymbol{O} \\ & -1 & 1 & & \\ & & \ddots & \ddots & \\ & \boldsymbol{O} & & -1 & 1 \\ 1 & & & & -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \bowtie \left(\begin{pmatrix} \boldsymbol{E_{t-1}} \\ {}^{\mathrm{t}}\boldsymbol{0} \end{pmatrix} + \begin{pmatrix} {}^{\mathrm{t}}\boldsymbol{0} \\ -\boldsymbol{E_{t-1}} \end{pmatrix} \right)$$

and $\boldsymbol{E_{t-1}}$ means the identity matrix of size $t-1$. Then we have $M_1 \sim M_2$ and $M_3 \sim M_2$, which follow from

$$\left(\begin{array}{ccc}
\boldsymbol{A_1} & \boldsymbol{A_2} & \boldsymbol{\alpha_1} \bowtie \boldsymbol{O} \\
{}^{\mathrm{t}}\boldsymbol{A_2} & \boldsymbol{A_3} & \boldsymbol{\alpha_2} \bowtie \boldsymbol{O} \pm \boldsymbol{C} \\
{}^{\mathrm{t}}(\boldsymbol{\alpha_1} \bowtie \boldsymbol{O}) & {}^{\mathrm{t}}(\boldsymbol{\alpha_2} \bowtie \boldsymbol{O} \pm \boldsymbol{C}) & (a) \oplus \boldsymbol{O} \\
\hdashline
\boldsymbol{B_1} & \boldsymbol{B_2} & \boldsymbol{\beta} \bowtie \boldsymbol{O} \\
\boldsymbol{O} & \boldsymbol{O} & \boldsymbol{1} \bowtie (-\boldsymbol{E_{t-1}})
\end{array}\right)$$

$$\xleftrightarrow{T2} \begin{pmatrix}
\boldsymbol{A_1} & \boldsymbol{A_2} & \boldsymbol{\alpha_1} \bowtie \boldsymbol{O} \\
{}^{\mathrm{t}}\boldsymbol{A_2} & \boldsymbol{A_3} & \boldsymbol{\alpha_2} \bowtie \left(\begin{pmatrix} \pm\boldsymbol{E_{t-1}} \\ {}^{\mathrm{t}}\boldsymbol{0} \end{pmatrix} + \begin{pmatrix} {}^{\mathrm{t}}\boldsymbol{0} \\ \mp\boldsymbol{E_{t-1}} \end{pmatrix} \right) \\
{}^{\mathrm{t}}(\boldsymbol{\alpha_1} \bowtie \boldsymbol{O}) & {}^{\mathrm{t}}(\boldsymbol{\alpha_2} \bowtie \boldsymbol{O} \pm \boldsymbol{C}) & (a) \oplus \boldsymbol{O} \\
\boldsymbol{B_1} & \boldsymbol{B_2} & \boldsymbol{\beta} \bowtie \boldsymbol{O} \\
\boldsymbol{O} & \boldsymbol{O} & \boldsymbol{0} \bowtie (-\boldsymbol{E_{t-1}})
\end{pmatrix}$$

$$\xleftrightarrow{T2} \begin{pmatrix}
\boldsymbol{A_1} & \boldsymbol{A_2} & \boldsymbol{\alpha_1} \bowtie \boldsymbol{O} \\
{}^{\mathrm{t}}\boldsymbol{A_2} & \boldsymbol{A_3} & \boldsymbol{\alpha_2} \bowtie \boldsymbol{O} \\
{}^{\mathrm{t}}(\boldsymbol{\alpha_1} \bowtie \boldsymbol{O}) & {}^{\mathrm{t}}(\boldsymbol{\alpha_2} \bowtie \boldsymbol{O} \pm \boldsymbol{C}) & (a) \oplus \boldsymbol{O} \\
\boldsymbol{B_1} & \boldsymbol{B_2} & \boldsymbol{\beta} \bowtie \boldsymbol{O} \\
\boldsymbol{O} & \boldsymbol{O} & \boldsymbol{0} \bowtie (-\boldsymbol{E_{t-1}})
\end{pmatrix}$$

$$\xleftrightarrow{T2} \begin{pmatrix}
\boldsymbol{A_1} & \boldsymbol{A_2} & \boldsymbol{\alpha_1} \bowtie \boldsymbol{O} \\
{}^{\mathrm{t}}\boldsymbol{A_2} & \boldsymbol{A_3} & \boldsymbol{\alpha_2} \bowtie \boldsymbol{O} \\
{}^{\mathrm{t}}(\boldsymbol{\alpha_1} \bowtie \boldsymbol{O}) & {}^{\mathrm{t}}\left(\boldsymbol{\alpha_2} \bowtie \left(\begin{pmatrix} \pm\boldsymbol{E_{t-1}} \\ {}^{\mathrm{t}}\boldsymbol{0} \end{pmatrix} + \begin{pmatrix} {}^{\mathrm{t}}\boldsymbol{0} \\ \mp\boldsymbol{E_{t-1}} \end{pmatrix} \right)\right) & (a) \oplus \boldsymbol{O} \\
\boldsymbol{B_1} & \boldsymbol{B_2} & \boldsymbol{\beta} \bowtie \boldsymbol{O} \\
\boldsymbol{O} & \boldsymbol{O} & \boldsymbol{0} \bowtie (-\boldsymbol{E_{t-1}})
\end{pmatrix}$$

$$\xleftrightarrow{T2} \begin{pmatrix}
\boldsymbol{A_1} & \boldsymbol{A_2} & \boldsymbol{\alpha_1} \bowtie \boldsymbol{O} \\
{}^{\mathrm{t}}\boldsymbol{A_2} & \boldsymbol{A_3} & \boldsymbol{\alpha_2} \bowtie \boldsymbol{O} \\
{}^{\mathrm{t}}(\boldsymbol{\alpha_1} \bowtie \boldsymbol{O}) & {}^{\mathrm{t}}\left(\boldsymbol{\alpha_2} \bowtie \left(\begin{pmatrix} \pm\boldsymbol{E_{t-1}} \\ {}^{\mathrm{t}}\boldsymbol{0} \end{pmatrix} + \begin{pmatrix} {}^{\mathrm{t}}\boldsymbol{0} \\ \mp\boldsymbol{E_{t-1}} \end{pmatrix} \right)\right) & (a) \oplus (\pm\boldsymbol{E_{t-1}}) \\
\boldsymbol{B_1} & \boldsymbol{B_2} & \boldsymbol{\beta} \bowtie \boldsymbol{O} \\
\boldsymbol{O} & \boldsymbol{O} & \boldsymbol{0} \bowtie (-\boldsymbol{E_{t-1}})
\end{pmatrix}$$

$$\xleftrightarrow{T2} \begin{pmatrix}
\boldsymbol{A_1} & \boldsymbol{A_2} & \boldsymbol{\alpha_1} \bowtie \boldsymbol{O} \\
{}^{\mathrm{t}}\boldsymbol{A_2} & \boldsymbol{A_3} & \boldsymbol{\alpha_2} \bowtie \boldsymbol{O} \\
{}^{\mathrm{t}}(\boldsymbol{\alpha_1} \bowtie \boldsymbol{O}) & {}^{\mathrm{t}}(\boldsymbol{\alpha_2} \bowtie \boldsymbol{O}) & (a) \oplus (\pm\boldsymbol{E_{t-1}}) \\
\boldsymbol{B_1} & \boldsymbol{B_2} & \boldsymbol{\beta} \bowtie \boldsymbol{O} \\
\boldsymbol{O} & {}^{\mathrm{t}}\left(\begin{pmatrix} \boldsymbol{E_{t-1}} \\ {}^{\mathrm{t}}\boldsymbol{0} \end{pmatrix} + \begin{pmatrix} {}^{\mathrm{t}}\boldsymbol{0} \\ -\boldsymbol{E_{t-1}} \end{pmatrix} \right) & \boldsymbol{0} \bowtie (-\boldsymbol{E_{t-1}})
\end{pmatrix}$$

$$\xleftrightarrow{T2} \begin{pmatrix}
\boldsymbol{A_1} & \boldsymbol{A_2} & \boldsymbol{\alpha_1} \bowtie \boldsymbol{O} \\
{}^{\mathrm{t}}\boldsymbol{A_2} & \boldsymbol{A_3} & \boldsymbol{\alpha_2} \bowtie \boldsymbol{O} \\
{}^{\mathrm{t}}(\boldsymbol{\alpha_1} \bowtie \boldsymbol{O}) & {}^{\mathrm{t}}(\boldsymbol{\alpha_2} \bowtie \boldsymbol{O}) & (a) \oplus (\pm\boldsymbol{E_{t-1}}) \\
\boldsymbol{B_1} & \boldsymbol{B_2} & \boldsymbol{\beta} \bowtie \boldsymbol{O} \\
\boldsymbol{O} & {}^{\mathrm{t}}\left(\begin{pmatrix} \boldsymbol{E_{t-1}} \\ {}^{\mathrm{t}}\boldsymbol{0} \end{pmatrix} + \begin{pmatrix} {}^{\mathrm{t}}\boldsymbol{0} \\ -\boldsymbol{E_{t-1}} \end{pmatrix} \right) & \boldsymbol{0} \bowtie \boldsymbol{O}
\end{pmatrix}$$

$$\xleftrightarrow{T2} \begin{pmatrix}
\boldsymbol{A_1} & \boldsymbol{A_2} & \boldsymbol{\alpha_1} \bowtie \boldsymbol{O} \\
{}^{\mathrm{t}}\boldsymbol{A_2} & \boldsymbol{A_3} & \boldsymbol{\alpha_2} \bowtie \boldsymbol{O} \\
{}^{\mathrm{t}}(\boldsymbol{\alpha_1} \bowtie \boldsymbol{O}) & {}^{\mathrm{t}}(\boldsymbol{\alpha_2} \bowtie \boldsymbol{O}) & (a) \oplus (\pm\boldsymbol{E_{t-1}}) \\
\boldsymbol{B_1} & \boldsymbol{B_2} & \boldsymbol{\beta} \bowtie \boldsymbol{O} \\
\boldsymbol{O} & \boldsymbol{1} \bowtie (-\boldsymbol{E_{t-1}}) & \boldsymbol{0} \bowtie \boldsymbol{O}
\end{pmatrix} \xleftrightarrow{T7} M_2,$$

where all the double signs correspond to each other. □

Lemma 4.8. *The equivalence class of $M^{\mathrm{G}}(D,\sigma)$ is unchanged under the RV-moves.*

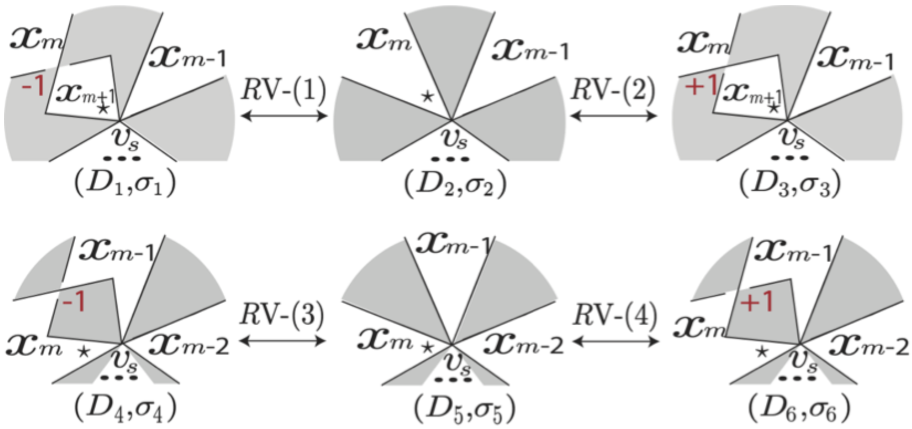


Figure 10

Proof. For the invariance under the RV-moves, we may consider the four types of the RV-moves depicted in FIGURE 10. Let us consider the move RV-(1) between the checkerboard colored diagrams (D_1,σ_1) and (D_2,σ_2), and the move RV-(2) between the checkerboard colored diagrams (D_2,σ_2) and (D_3,σ_3). Let $x_1,\ldots,x_m($, and $x_{m+1})$ (resp. $v_1,\ldots,v_s$) be the white regions (resp. the vertices) of $(D_1,\sigma_1),(D_2,\sigma_2)$ and (D_3,σ_3). We suppose that $x_1,\ldots,x_m$, $v_1,\ldots,v_{s-1}$ and their local specified regions stay outside the disk in which the moves are applied, and x_m, x_{m+1}, v_s and the local specified region of v_s are changed inside the disk as in FIGURE 10. By performing RII-moves if necessary, we may assume that there are no duplicate white regions incident to v_s. For $i\in\{1,2,3\}$, let n_i denote the number of the connected components of the checkerboard graph of (D_i,σ_i), and M_i the Goeritz matrix of (D_i, σ_i). We then have $n := n_1 = n_2 = n_3$ and

$$M_1 = \left(\begin{array}{ccc} \boldsymbol{A} & \boldsymbol{\alpha} & \boldsymbol{0} \\ {}^{\mathrm{t}}\boldsymbol{\alpha} & a-1 & 1 \\ {}^{\mathrm{t}}\boldsymbol{0} & 1 & -1 \\ \hdashline \boldsymbol{B_1} & \boldsymbol{\beta} & \boldsymbol{0} \\ \boldsymbol{B_2} & \boldsymbol{0} & \boldsymbol{1} \end{array}\right) \oplus O_{n,n}, \qquad M_2 = \left(\begin{array}{cc} \boldsymbol{A} & \boldsymbol{\alpha} \\ {}^{\mathrm{t}}\boldsymbol{\alpha} & a \\ \hdashline \boldsymbol{B_1} & \boldsymbol{\beta} \\ \boldsymbol{B_2} & \boldsymbol{1} \end{array}\right) \oplus O_{n,n},$$

$$M_3 = \left(\begin{array}{ccc} \boldsymbol{A} & \boldsymbol{\alpha} & \boldsymbol{0} \\ {}^{\mathrm{t}}\boldsymbol{\alpha} & a+1 & -1 \\ {}^{\mathrm{t}}\boldsymbol{0} & -1 & 1 \\ \hdashline \boldsymbol{B_1} & \boldsymbol{\beta} & \boldsymbol{0} \\ \boldsymbol{B_2} & \boldsymbol{0} & \boldsymbol{1} \end{array}\right) \oplus O_{n,n}.$$

Then we have $M_1 \sim M_2$ and $M_3 \sim M_2$, which follow from

$$\left(\begin{array}{ccc} \boldsymbol{A} & \boldsymbol{\alpha} & \boldsymbol{0} \\ {}^{\mathrm{t}}\boldsymbol{\alpha} & a\mp 1 & \pm 1 \\ {}^{\mathrm{t}}\boldsymbol{0} & \pm 1 & \mp 1 \\ \hdashline \boldsymbol{B_1} & \boldsymbol{\beta} & \boldsymbol{0} \\ \boldsymbol{B_2} & \boldsymbol{0} & \boldsymbol{1} \end{array}\right) \oplus O_{n,n} \overset{T2}{\longleftrightarrow} \left(\begin{array}{ccc} \boldsymbol{A} & \boldsymbol{\alpha} & \boldsymbol{0} \\ {}^{\mathrm{t}}\boldsymbol{\alpha} & a & \pm 1 \\ {}^{\mathrm{t}}\boldsymbol{0} & 0 & \mp 1 \\ \boldsymbol{B_1} & \boldsymbol{\beta} & \boldsymbol{0} \\ \boldsymbol{B_2} & \boldsymbol{1} & \boldsymbol{1} \end{array}\right) \oplus O_{n,n}$$

$$\overset{T2}{\longleftrightarrow} \left(\begin{array}{ccc} \boldsymbol{A} & \boldsymbol{\alpha} & \boldsymbol{0} \\ {}^{\mathrm{t}}\boldsymbol{\alpha} & a & 0 \\ {}^{\mathrm{t}}\boldsymbol{0} & 0 & \mp 1 \\ \boldsymbol{B_1} & \boldsymbol{\beta} & \boldsymbol{0} \\ \boldsymbol{B_2} & \boldsymbol{1} & \boldsymbol{0} \end{array}\right) \oplus O_{n,n} \overset{T7}{\longleftrightarrow} \left(\begin{array}{cc} \boldsymbol{A} & \boldsymbol{\alpha} \\ {}^{\mathrm{t}}\boldsymbol{\alpha} & a \\ \hdashline \boldsymbol{B_1} & \boldsymbol{\beta} \\ \boldsymbol{B_2} & \boldsymbol{1} \end{array}\right) \oplus O_{n,n} = M_2,$$

where all the double signs correspond to each other.

Let us consider the move RV-(3) between the checkerboard colored diagrams (D_4, σ_4) and (D_5, σ_5), and the move RV-(4) between the checkerboard colored diagrams (D_5, σ_5) and (D_6, σ_6). Let $x_1, \ldots, x_m$ (resp. $v_1, \ldots, v_s$) be the white regions (resp. the vertices) of $(D_4, \sigma_4), (D_5, \sigma_5)$ and (D_6, σ_6). We suppose that $x_1, \ldots, x_m$, $v_1, \ldots, v_{s-1}$ and their local specified regions stay outside the disk in which the moves are applied, and x_{m-1}, x_m, v_s and the local specified region of v_s are changed inside the disk as in FIGURE 10. By performing RII-moves if necessary, we may assume that there are no duplicate white regions incident to v_s.

For $i \in \{4, 5, 6\}$, let n_i denote the number of the connected components of the checkerboard graph of (D_i, σ_i), and M_i the Goeritz matrix of (D_i, σ_i). We then have $n := n_4 = n_5 = n_6$ and

$$M_4 = \left(\begin{array}{ccc} \boldsymbol{A} & \boldsymbol{\alpha_1} & \boldsymbol{\alpha_2} \\ {}^{\mathrm{t}}\boldsymbol{\alpha_1} & a_1 - 1 & a_2 + 1 \\ {}^{\mathrm{t}}\boldsymbol{\alpha_2} & a_2 + 1 & a_3 - 1 \\ \hdashline \boldsymbol{B_1} & \boldsymbol{\beta_1} & \boldsymbol{\beta_2} \\ \mathbf{0} & -1 & 1 \\ \boldsymbol{B_2} & \mathbf{0} & \mathbf{1} \end{array}\right) \oplus O_{n,n}, \quad M_5 = \left(\begin{array}{ccc} \boldsymbol{A} & \boldsymbol{\alpha_1} & \boldsymbol{\alpha_2} \\ {}^{\mathrm{t}}\boldsymbol{\alpha_1} & a_1 & a_2 \\ {}^{\mathrm{t}}\boldsymbol{\alpha_2} & a_2 & a_3 \\ \hdashline \boldsymbol{B_1} & \boldsymbol{\beta_1} & \boldsymbol{\beta_2} \\ \mathbf{0} & -1 & 1 \\ \boldsymbol{B_2} & \mathbf{0} & \mathbf{1} \end{array}\right) \oplus O_{n,n},$$

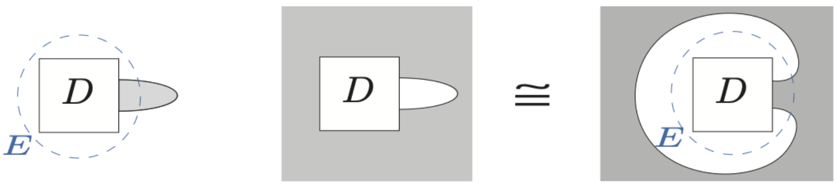
$$M_6 = \left(\begin{array}{ccc} \boldsymbol{A} & \boldsymbol{\alpha_1} & \boldsymbol{\alpha_2} \\ {}^{\mathrm{t}}\boldsymbol{\alpha_1} & a_1 + 1 & a_2 - 1 \\ {}^{\mathrm{t}}\boldsymbol{\alpha_2} & a_2 - 1 & a_3 + 1 \\ \hdashline \boldsymbol{B_1} & \boldsymbol{\beta_1} & \boldsymbol{\beta_2} \\ \mathbf{0} & -1 & 1 \\ \boldsymbol{B_2} & \mathbf{0} & \mathbf{1} \end{array}\right) \oplus O_{n,n}.$$

Then we have $M_4 \sim M_5$ and $M_6 \sim M_5$, which follow from

$$\left(\begin{array}{ccc} \boldsymbol{A} & \boldsymbol{\alpha_1} & \boldsymbol{\alpha_2} \\ {}^{\mathrm{t}}\boldsymbol{\alpha_1} & a_1 \mp 1 & a_2 \pm 1 \\ {}^{\mathrm{t}}\boldsymbol{\alpha_2} & a_2 \pm 1 & a_3 \mp 1 \\ \hdashline \boldsymbol{B_1} & \boldsymbol{\beta_1} & \boldsymbol{\beta_2} \\ \mathbf{0} & -1 & 1 \\ \boldsymbol{B_2} & \mathbf{0} & \mathbf{1} \end{array}\right) \oplus O_{n,n} \overset{T2}{\longleftrightarrow} \left(\begin{array}{ccc} \boldsymbol{A} & \boldsymbol{\alpha_1} & \boldsymbol{\alpha_2} \\ {}^{\mathrm{t}}\boldsymbol{\alpha_1} & a_1 & a_2 \\ {}^{\mathrm{t}}\boldsymbol{\alpha_2} & a_2 & a_3 \\ \hdashline \boldsymbol{B_1} & \boldsymbol{\beta_1} & \boldsymbol{\beta_2} \\ \mathbf{0} & -1 & 1 \\ \boldsymbol{B_2} & \mathbf{0} & \mathbf{1} \end{array}\right) \oplus O_{n,n} = M_5,$$

where all the double signs correspond to each other. □

LEMMA 4.9. *The equivalence class of* $M^{\mathrm{G}}(D, \sigma)$ *is unchanged under the replacement of the checkerboard coloring* σ *of* D.



(D,σ_1) (D,σ_2) (D',σ_2')

FIGURE 11

PROOF. For a checkerboard coloring σ_1 of D, let σ_2 be the other checkerboard coloring of D. As shown in FIGURE 11, by generalized Reidemeister moves,

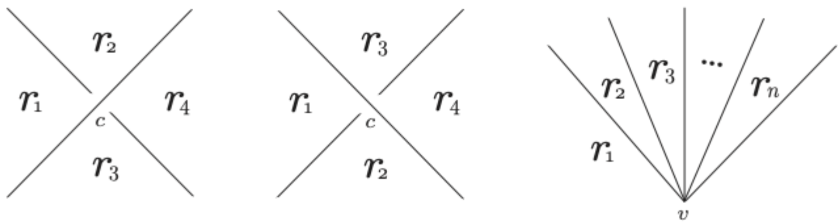


FIGURE 12. The coloring conditions

(D, σ_2) can be transformed onto another checkerboard colored diagram (D', σ_2') as shown in the right of FIGURE 11. Besides, the Goeritz matrices $M^{\mathrm{G}}(D, \sigma_1)$ and $M^{\mathrm{G}}(D', \sigma_2')$ are the same because the checkerboard diagrams are the same inside the open disk E depicted in FIGURE 11. This implies that the equivalence classes of $M^{\mathrm{G}}(D, \sigma_1)$ and $M^{\mathrm{G}}(D, \sigma_2)$ are the same. □

PROOF OF THEOREM 3.5. This follows from Lemmas 4.1–4.9. □

5. Dehn colorings of spatial graph diagrams

We note that in this paper, Dehn colorings mean those with alternating vertex conditions.

DEFINITION 5.1. Let D be a diagram of a spatial graph whose vertices are of even valency, $\mathcal{R}(D)$ the set of regions of D, and A an abelian group. A *Dehn coloring* of D by A is a map $C : \mathcal{R}(D) \to A$ satisfying the following conditions:

- For a crossing c with regions r_1, r_2, r_3 and r_4 such that r_2 is adjacent to an arbitrary chosen r_1 by an under-arc and r_3 is adjacent to r_1 by the over-arc as depicted in FIGURE 12,

$$C(r_1) - C(r_2) + C(r_3) - C(r_4) = 0$$

  holds, which we call the *crossing condition.*
- For a vertex v with regions $r_1, \dots, r_n$ that appear clockwisely as depicted in FIGURE 12,

$$C(r_1) = C(r_3) = \cdots = C(r_{n-1}) \text{ and } C(r_2) = C(r_4) = \cdots = C(r_n)$$

  holds, which we call the *vertex condition.*

We call $C(r)$ the *color* of a region r. We denote by $\mathrm{Col}_A^{\mathrm{Dehn}}(D)$ the set of Dehn colorings of D by A. We denote by (D, C) a diagram D equipped with a Dehn coloring C.

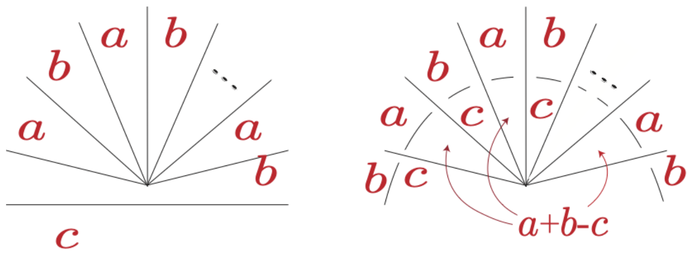


FIGURE 13. Reidemeister moves of type IV

REMARK 5.2. In Definition 5.1, we set the vertex condition so that the regions around each vertex are colored alternatingly. However, there might be other vertex conditions. In [9], we classified the vertex conditions, using *palettes*, for Dehn colorings with $A = \mathbb{Z}_p$. Note that we adopt the alternating palette for Definition 5.1 in this paper.

LEMMA 5.3. *Let D and D' be diagrams of spatial graphs whose vertices are of even valency. If D and D' represent the same spatial graph, then there exists a bijection between* $\mathrm{Col}_A^{\mathrm{Dehn}}(D)$ *and* $\mathrm{Col}_A^{\mathrm{Dehn}}(D')$.

PROOF. Let D and D' be diagrams such that D' is obtained from D by a single Reidemeister move. Let E be a 2-disk in which the move is applied. Let C be a Dehn coloring of D. We define a Dehn coloring C' of D', corresponding to C, by $C'(r) = C(r)$ for each region r appearing in the outside of E. Then the colors of the regions appearing in E, by C', are uniquely determined (see FIGURE 13 for Reidemeister moves of type IV). $\square$

Lemma 5.3 implies that the cardinality of Dehn colorings of D by A, i.e. $\#\mathrm{Col}_A^{\mathrm{Dehn}}(D)$, is an invariant of spatial graphs whose vertices are of even valency.

We can regard $\mathrm{Col}_A^{\mathrm{Dehn}}(D)$ as an abelian group with the addition defined by $(C + C') : \mathcal{R}(D) \to A; (C + C')(x) = C(x) + C'(x)$, for each C, $C' \in \mathrm{Col}_A^{\mathrm{Dehn}}(D)$.

6. A relationship between Goeritz matrices and Dehn colorings

In this section, we discuss a relationship between Goeritz matrices and Dehn colorings of spatial graphs whose vertices are of even valency. Let (D, σ) be a checkerboard colored diagram of a spatial graph whose vertices are of even valency. Let $x_1, \dots, x_m$ be the white regions of D, and $v_1, \dots, v_s$ the vertices of D. Let n denote the number of connected components of the checkerboard graph $\Gamma_{(D,\sigma)}$. We choose a black vertex from each connected component of $\Gamma_{(D,\sigma)}$ and denote by $x_{m+1}, \dots, x_{m+n}$ the black regions of D corresponding to the chosen black vertices. Let $M^{\mathrm{G}}(D,\sigma)$ be the Goeritz matrix of (D,σ) with respect to $x_1,\dots,x_m$, $v_1,\dots,v_s$ and given local specified regions.

From now on, we regard $M^{\mathrm{G}}(D,\sigma)$ as not only a matrix but also the homomorphism defined by

$$\begin{array}{cccc} M^{\mathrm{G}}(D,\sigma): & \displaystyle\bigoplus_{m+n} A & \longrightarrow & \displaystyle\bigoplus_{m+n} A \\ & \cup & & \cup \\ & \begin{pmatrix} a_1 \\ \vdots \\ a_{m+n} \end{pmatrix} & \longmapsto & M^{\mathrm{G}}(D,\sigma)\begin{pmatrix} a_1 \\ \vdots \\ a_{m+n} \end{pmatrix}. \end{array}$$

We denote by $\ker_A M^G(D,\sigma)$ the kernel of the homomorphism

$$M^{\mathrm{G}}(D,\sigma): \bigoplus_{m+n} A \to \bigoplus_{m+n} A.$$

We note that in this section, we fix the above setting. We then have the following lemmas.

Lemma 6.1. *For a Dehn coloring $C \in \mathrm{Col}_A^{\mathrm{Dehn}}(D)$, we have* $\begin{pmatrix} C(x_1) \\ \vdots \\ C(x_{m+n}) \end{pmatrix} \in \ker_A M^G(D,\sigma)$.

Proof. To prove $M^G(D,\sigma)\begin{pmatrix} C(x_1) \\ \vdots \\ C(x_{m+n}) \end{pmatrix} = \mathbf{0}$, it suffices to show that

$$M^{\mathrm{CR}}(D,\sigma)\begin{pmatrix} C(x_1) \\ \vdots \\ C(x_m) \end{pmatrix} = \mathbf{0} \tag{3}$$

and

$$M^{\mathrm{VR}}(D,\sigma)\begin{pmatrix} C(x_1) \\ \vdots \\ C(x_m) \end{pmatrix} = \mathbf{0}. \tag{4}$$

For (3), we may focus on the ith row vector of $M^{\mathrm{CR}}(D,\sigma)$ and denote it by $M^{\mathrm{CR}}(D,\sigma)_i$. Note that $M^{\mathrm{CR}}(D,\sigma)_i$ is associated with the ith white region x_i. When we walk along the boundary of x_i in the clockwise direction, we have a sequence $y_1,\chi_1,y_2,\chi_2,\dots,y_u,\chi_u,y_1$, where y_k is a black region and χ_k is a crossing or a vertex for any $k\in\{1,\dots,u\}$, see FIGURE 14. For $k\in\{1,\dots,u\}$, when χ_k is a crossing, let x_{i_k} be the white region which is incident to χ_k and is located on the opposite side of x_i. On the other hand, when χ_k is a vertex, let x_{i_k} be the white region which is also the local specified region of χ_k. We note that $i_k \in \{1, \dots, m\}$. Then, $M^{\mathrm{CR}}(D,\sigma)_i$ has the form

$$M^{\mathrm{CR}}(D,\sigma)_i = \begin{cases} (\dots, \overbrace{\varepsilon(\chi_k) - \alpha + \beta}^{i\text{th}}, \dots, \overbrace{-\varepsilon(\chi_k) + \alpha}^{i_k\text{th}}, \dots) & \text{if } i \neq i_k, \\ (\dots, \overbrace{\varepsilon(\chi_k) - \varepsilon(\chi_k) + \beta}^{i\text{th}}, \dots) & \text{if } i = i_k, \end{cases}$$

for some $\alpha, \beta \in \mathbb{Z}$. Since by the definition of Dehn colorings, we have

$$\begin{cases} C(x_i) + C(y_{k+1}) = C(x_{i_k}) + C(y_k) & \text{when } \varepsilon(\chi_k) = +1, \\ C(x_i) + C(y_k) = C(x_{i_k}) + C(y_{k+1}) & \text{when } \varepsilon(\chi_k) = -1, \\ C(x_i) = C(x_{i_k}) \text{ and } C(y_k) = C(y_{k+1}) & \text{when } \varepsilon(\chi_k) = 0, \end{cases}$$

it holds that

$$\varepsilon(\chi_k)\Big(C(x_i) - C(x_{i_k})\Big) = C(y_k) - C(y_{k+1}).$$

Thus, we have

$$\begin{aligned}
&M^{\mathrm{CR}}(D,\sigma)_i \begin{pmatrix} C(x_1) \\ \vdots \\ C(x_m) \end{pmatrix} \\
&= \begin{cases} (\dots, \overbrace{\varepsilon(\chi_k) - \alpha + \beta}^{i\text{th}}, \dots, \overbrace{-\varepsilon(\chi_k) + \alpha}^{i_k\text{th}}, \dots) \begin{pmatrix} C(x_1) \\ \vdots \\ C(x_m) \end{pmatrix} & \text{if } i \neq i_k, \\ = (\dots, \overbrace{\varepsilon(\chi_k) - \varepsilon(\chi_k) + \beta}^{i\text{th}}, \dots\dots) \begin{pmatrix} C(x_1) \\ \vdots \\ C(x_m) \end{pmatrix} & \text{if } i = i_k, \end{cases} \\
&= \begin{cases} \dots + (\varepsilon(\chi_k) - \alpha + \beta)C(x_i) + \dots + (-\varepsilon(\chi_k) + \alpha)C(x_{i_k}) + \cdots & \text{if } i \neq i_k, \\ \dots + (\varepsilon(\chi_k) - \varepsilon(\chi_k) + \beta)C(x_i) + \cdots & \text{if } i = i_k, \end{cases} \\
&= \dots + \underbrace{\varepsilon(\chi_k)(C(x_i) - C(x_{i_k}))}_{=C(y_k) - C(y_{k+1})} + \cdots \\
&= (C(y_1) - C(y_2)) + (C(y_2) - C(y_3)) + \dots + (C(y_u) - C(y_1)) \\
&= 0.
\end{aligned}$$

For (4), we may focus on the ith row vector of $M^{\mathrm{VR}}(D,\sigma)$ and denote it by $M^{\mathrm{VR}}(D,\sigma)_i$. Since we do not need to discuss the case that $M^{\mathrm{VR}}(D,\sigma)_i = \mathbf{0}$, we

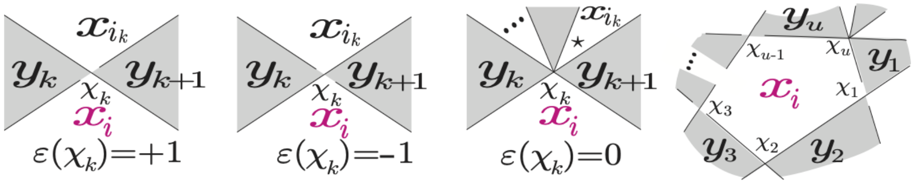


FIGURE 14

suppose that $M^{\mathrm{VR}}(D,\sigma)_i \neq \mathbf{0}$, that is, $M^{\mathrm{VR}}(D,\sigma)_i$ has the form

$$M^{\mathrm{VR}}(D,\sigma)_i = (0,\dots,0,\overbrace{\varepsilon}^{j\text{th}},0\dots,0,\overbrace{-\varepsilon}^{k\text{th}},0,\dots,0)$$

for some $j,k \in \{1,\dots,m\}$ with $j<k$, where $\varepsilon = -1$ or 1. This implies that the white regions x_j and x_k are incident to the same vertex and it follows that $C(x_j) = C(x_k)$. Hence we have

$$\begin{aligned} M^{\mathrm{VR}}(D,\sigma)_i &\begin{pmatrix} C(x_1) \\ \vdots \\ C(x_m) \end{pmatrix} \\ &= (0,\dots,0,\overbrace{\varepsilon}^{j\text{th}},0\dots,0,\overbrace{-\varepsilon}^{k\text{th}},0,\dots,0) \begin{pmatrix} C(x_1) \\ \vdots \\ C(x_m) \end{pmatrix} \\ &= \varepsilon(C(x_j) - C(x_k)) \\ &= 0. \end{aligned}$$

This completes the proof. □

LEMMA 6.2. *For an $(m+n)$ vector $\boldsymbol{a} = \begin{pmatrix} a_1 \\ \vdots \\ a_{m+n} \end{pmatrix} \in \ker_A M^G(D,\sigma)$, we have a unique $C \in \mathrm{Col}_A^{\mathrm{Dehn}}(D)$ which satisfies* $\begin{pmatrix} C(x_1) \\ \vdots \\ C(x_{m+n}) \end{pmatrix} = \begin{pmatrix} a_1 \\ \vdots \\ a_{m+n} \end{pmatrix}$.

PROOF. We will first show the existence of a Dehn coloring C.

For each white region x_j ($j \in \{1,\dots,m\}$), set $C(x_j) = a_j$, which is the jth coordinate of the vector $\boldsymbol{a}$. For each black region x_{m+k} ($k \in \{1,\dots,n\}$), we also set $C(x_{m+k}) = a_{m+k}$, which is the $(m+k)$th coordinate of the vector $\boldsymbol{a}$. Here we note that the colors of all white regions and the colors of the black regions

$x_{m+1}, \dots, x_{m+n}$ are already defined. We shall consider the black regions whose colors have not been defined yet. For each black region x_{m+k} ($k \in \{1, \dots, n\}$), we denote by $\Gamma^k_{(D,\sigma)}$ the connected component of $\Gamma_{(D,\sigma)}$ which contains the black vertex corresponding to the black region x_{m+k}. Let y be the black region corresponding to a black vertex of $\Gamma^k_{(D,\sigma)}$ such that the color of y has not been defined yet. We walk from x_{m+k} to y on a path which is mainly on black regions and passes through a finite number of crossings or vertices. We then have a sequence $x_{m+k} = y_1, \chi_1, y_2, \chi_2, \dots, \chi_{u-1}, y_u = y$, where y_i is a black region and χ_i is a crossing or a vertex for any $i \in \{1, \dots, u\}$. The color of y_2 is determined by the crossing condition or the vertex condition of Dehn colorings at χ_1 since the colors of $x_{m+k} = y_1$ and two white regions incident to χ_1 are already defined. Similarly, we can determine the color of y_3 by the crossing condition or the vertex condition of Dehn colorings at χ_2 since the colors of y_2 and two white regions incident to χ_2 are already defined. With repeating this step, we can determine the color $C(y)$ of $y_u = y$. Thus, we can determine the colors of all black regions.

Here we need to show that the map $C : \mathcal{R}(D) \to A$ is well-definedly determined with the above construction. To verify this, we will prove that any replacement of paths from x_{m+k} to y which we walk on does not change the color $C(y)$. Since the replacement of paths can be given by a finite number of replacements each of which replaces a point on a given path with a lap of the boundary of a white region, it suffices to show that for a white region x_i, the color of a black region adjacent to x_i is not changed after we walk on a path along the boundary of x_i and return to the starting point on the black.

Let us consider a white region x_i. Suppose that when we walk on a path ℓ along the boundary of x_i in the clockwise direction, we have a sequence $y_1, \chi_1, y_2, \chi_2, \dots, \chi_{u-1}, y_u, \chi_u, y_1$ as depicted in the rightmost figure in FIGURE 14, where y_i is a black region and χ_i is a crossing or a vertex for any $i \in \{1, \dots, u\}$. For $k \in \{1, \dots, u\}$, when χ_k is a crossing, let x_{i_k} be the white region which is incident to χ_k and is located on the opposite side of x_i. On the other hand, when χ_k is a vertex, let x_{i_k} be the white region which is also the local specified region of χ_k. We note that $i_k \in \{1, \dots, m\}$. Since we have

$$\begin{cases} C(x_i) + C(y_{k+1}) = C(x_{i_k}) + C(y_k) & \text{when } \varepsilon(\chi_k) = +1, \\ C(x_i) + C(y_k) = C(x_{i_k}) + C(y_{k+1}) & \text{when } \varepsilon(\chi_k) = -1, \\ C(x_i) = C(x_{i_k}) \text{ and } C(y_k) = C(y_{k+1}) & \text{when } \varepsilon(\chi_k) = 0 \end{cases}$$

by the definition of Dehn colorings, it holds that

$$C(y_{k+1}) = C(y_k) - \varepsilon(\chi_k)\Big(C(x_i) - C(x_{i_k})\Big).$$

Now we suppose that we have a color $a := C(y_1)$. Let $\tilde{a}$ be the "new" color of y_1 which is obtained after we walk along the path ℓ from y_1 and return to y_1 again, that is

$$\begin{aligned}
\tilde{a} &= C(y_u) - \varepsilon(\chi_u)\Big(C(x_i) - C(x_{i_u})\Big),\\
C(y_u) &= C(y_{u-1}) - \varepsilon(\chi_{u-1})\Big(C(x_i) - C(x_{i_{u-1}})\Big),\\
&\ \ \vdots\\
C(y_3) &= C(y_2) - \varepsilon(\chi_2)\Big(C(x_i) - C(x_{i_2})\Big), \text{ and}\\
C(y_2) &= a - \varepsilon(\chi_1)\Big(C(x_i) - C(x_{i_1})\Big).
\end{aligned}$$

We then have

$$\begin{aligned}
\tilde{a} &= a - \sum_{k=1}^{u} \varepsilon(\chi_k)\Big(C(x_i) - C(x_{i_k})\Big)\\
&= a - M^{\mathrm{CR}}(D,\sigma)_i \begin{pmatrix} C(x_1) \\ \vdots \\ C(x_m) \end{pmatrix}\\
&= a,
\end{aligned}$$

where the second and third equalities follow from the proof and result of Lemma 6.1, respectively. Thus, the map C is well-definedly determined.

Besides, we can see that the map C is a Dehn coloring of D as follows. For each crossing or vertex χ of D, the Dehn coloring condition for χ by C is satisfied, because for the above construction of C, we can choose a path ℓ so that ℓ passes through χ, and the colors of the black regions which ℓ passes through are determined by the Dehn coloring conditions for all the crossings and vertices on ℓ. Therefore the existence of C was proven.

Finally, any Dehn coloring C of D with $\begin{pmatrix} C(x_1) \\ \vdots \\ C(x_{m+n}) \end{pmatrix} = \begin{pmatrix} a_1 \\ \vdots \\ a_{m+n} \end{pmatrix}$ can be regarded as a Dehn coloring obtained by the above construction using paths. This implies that such a Dehn coloring C is unique.

This completes the proof.

$\square$

By Lemmas 6.1 and 6.2, we have the maps $\phi : \mathrm{Col}_A^{\mathrm{Dehn}}(D) \to \ker_A M^{\mathrm{G}}(D,\sigma)$ and $\psi : \ker_A M^{\mathrm{G}}(D,\sigma) \to \mathrm{Col}_A^{\mathrm{Dehn}}(D)$ defined by

$$\phi(C) = \begin{pmatrix} C(x_1) \\ \vdots \\ C(x_{m+m}) \end{pmatrix}$$

and

$$\psi\left(\begin{pmatrix} a_1 \\ \vdots \\ a_{m+n} \end{pmatrix}\right) = C \text{ such that } \begin{pmatrix} C(x_1) \\ \vdots \\ C(x_{m+n}) \end{pmatrix} = \begin{pmatrix} a_1 \\ \vdots \\ a_{m+n} \end{pmatrix},$$

respectively. We then have the following relationshiop between Goeritz matrices and Dehn colorings for spatial graph diagrams whose vertices are of even valency.

THEOREM 6.3. *$\mathrm{Col}_A^{\mathrm{Dehn}}(D) \cong \ker_A M^{\mathrm{G}}(D,\sigma)$.*

PROOF. It suffices to show that ϕ is an isomorphism. First, we can see that ϕ is a homomorphism, which follows from

$$\begin{aligned} \phi(C+C') &= \begin{pmatrix} (C+C')(x_1) \\ \vdots \\ (C+C')(x_{m+n}) \end{pmatrix} \\ &= \begin{pmatrix} C(x_1)+C'(x_1) \\ \vdots \\ C(x_{m+n})+C'(x_{m+n}) \end{pmatrix} \\ &= \begin{pmatrix} C(x_1) \\ \vdots \\ C(x_{m+n}) \end{pmatrix} + \begin{pmatrix} C'(x_1) \\ \vdots \\ C'(x_{m+n}) \end{pmatrix} \\ &= \phi(C)+\phi(C') \end{aligned}$$

for $C, C' \in \mathrm{Col}_A^{\mathrm{Dehn}}(D)$.

We proceed to show that the maps ϕ and ψ are the inverses of each other. For a Dehn coloring C, we have $\psi \circ \phi(C) = \psi \begin{pmatrix} C(x_1) \\ \vdots \\ C(x_{m+n}) \end{pmatrix} = C$. We also have $\phi \circ \psi\left(\begin{pmatrix} a_1 \\ \vdots \\ a_{m+n} \end{pmatrix}\right) = \phi(C) = \begin{pmatrix} C(x_1) \\ \vdots \\ C(x_{m+n}) \end{pmatrix} = \begin{pmatrix} a_1 \\ \vdots \\ a_{m+n} \end{pmatrix}$, where C is a Dehn coloring which satisfies $C(x_1) = a_1, \ldots, C(x_{m+n}) = a_{m+n}$. Thus, the maps ϕ and ψ are the inverses of each other, and the proof is complete. $\square$

EXAMPLE 6.4. Let p be an integer with $p \geq 2$. For Examples 3.2 and 3.6, we have

$$M^{\mathrm{G}}(D,\sigma) \sim (3) \oplus (0) \oplus (0) \oplus (0).$$

This implies that

$$\ker_{\mathbb{Z}_p} M^{\mathrm{G}}(D,\sigma) \cong \begin{cases} \mathbb{Z}_3 \oplus \mathbb{Z}_p \oplus \mathbb{Z}_p \oplus \mathbb{Z}_p & \text{if } 3|p, \\ \mathbb{Z}_p \oplus \mathbb{Z}_p \oplus \mathbb{Z}_p & \text{if } 3 \not| p. \end{cases}$$

Therefore, by Theorem 6.3, we have

$$\mathrm{Col}^{\mathrm{Dehn}}_{\mathbb{Z}_p}(D) \cong \begin{cases} \mathbb{Z}_3 \oplus \mathbb{Z}_p \oplus \mathbb{Z}_p \oplus \mathbb{Z}_p & \text{if } 3|p, \\ \mathbb{Z}_p \oplus \mathbb{Z}_p \oplus \mathbb{Z}_p & \text{if } 3 \not| p. \end{cases}$$

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