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GROUPS HAVING WIRTINGER PRESENTATIONS AND THE SECOND GROUP HOMOLOGY

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Abstract

Kuz′min (1996) characterized groups having Wirtinger presentations in relation to their second group homology. In this paper, we further refine the relation between these groups and their second group homology.

1. Introduction

A *Wirtinger presentation* is a group presentation $\langle X \mid R\rangle$ where all defining relations in R have the following form:

$$x_\beta^{-1}x_\alpha x_\beta = x_\gamma \quad (x_\alpha, x_\beta, x_\gamma \in X).$$

We call a group having a Wirtinger presentation a *Wirtinger group*. Wirtinger groups are also called (orientable) C-groups [6–8] or labelled oriented graph (LOG) groups [4, 5] in the literature. A Wirtinger group is said to be *irreducible* if all the generators of its Wirtinger presentation are conjugate each other.

Wirtinger groups appear in various contexts of topology and algebra. Knot groups and link groups are well-known examples of Wirtinger groups. More generally, for any smooth n-dimensional closed orientable manifold smoothly embedded in $\mathbb{R}^{n+2}$, the fundamental group of its complement is known to be a Wirtinger group [7, 10]. Further examples of Wirtinger groups include braid groups, pure braid groups, Artin groups, Thompson's group F, and associated groups of quandles [9].

After pioneering studies by Yajima [11] and Simon [10], Kuz′min obtained necessary and sufficient conditions for a group to be a Wirtinger group [8]. Before quoting Kuz′min's results, we should notice that the abelianization G_{ab} of a Wirtinger group G is a free abelian group. In particular, if G is an irreducible Wirtinger group, then G_{ab} is an infinite cyclic group.

THEOREM 1(KUZ′MIN [8, Theorem 1]). *A group G is a Wirtinger group if and only if G_{ab} is free abelian and there exists a subset $B \subset G$ satisfying the following three conditions:*

(1) *G coincides with the normal closure of B.*
(2) *The classes in G_{ab} of elements in B are linearly independent.*
(3) *The homomorphism*

$$\zeta\colon \bigoplus_{a\in B} C_G(a) \to H_2G, \quad (g_a)_{a\in B} \mapsto \sum_{a\in B} [a|g_a] - [g_a|a]$$

is surjective, where $C_G(a)$ denotes the centralizer of a in G, H_2G is the second integral homology group of G, $[a|g_a]-[g_a|a]$ is a normalized 2-cycle of G and $\bigoplus$ is the external direct sum in the category of groups.

In case that G is an irreducible Wirtinger group, a subset B must be a singleton $B = \{a\}$, and Theorem 1 implies the following result:

COROLLARY 2(KUZ′MIN [8, Corollary 4]). *A group G is an irreducible Wirtinger group if and only if $G_{\mathrm{ab}} \cong \mathbb{Z}$ and there is an element $a \in G$ satisfying the following condtions:*

(1) *G coincides with the normal closure of $a \in G$.*
(2) *The homomorphism*

$$\zeta\colon C_G(a) \to H_2G, \quad g \mapsto [a|g] - [g|a]$$

is surjective.

The results of Kuz′min unveil intrinsic relationships between Wirtinger groups and the second group homology. The purpose of this paper is to prove the following theorem which refines such relationships further:

THEOREM 3. *Let G be a Wirtinger group and let B be a subset of G which satisfies the conditions in Theorem 1. For each $a \in B$, define a homomorphism $\varepsilon_a\colon G \to \mathbb{Z}$ by*

$$\varepsilon_a(b) = \begin{cases} 1 & \text{if } b = a, \\ 0 & \text{otherwise} \end{cases}$$

for $b \in B$. Then the homomorphism

$$\zeta\colon \bigoplus_{a\in B} C_G(a) \cap \operatorname{Ker} \varepsilon_a \to H_2G, \quad (g_a)_{a\in B} \mapsto \sum_{a\in B} [a|g_a] - [g_a|a]$$

is surjective.

If G is an irreducible Wirtinger group, then $B = \{a\}$ for some $a \in G$ and the homomorphism $\varepsilon_a\colon G \to \mathbb{Z}$ is nothing but the projection to the abelianization $G_{\mathrm{ab}} \cong \mathbb{Z}$. As an immediate corollary of Theorem 3, we obtain the following result:

Corollary 4. *Let G be an irreducible Wirtinger group. Then there exists an element $a \in G$ satisfying*

(i) *G coincides with the normal closure of a,*
(ii) *the class of a in G_{ab} generates G_{ab}.*

Furthermore, the homomorphism

$$\zeta\colon C_G(a) \cap [G,G] \to H_2G, \quad g \mapsto [a|g] - [g|a]$$

is surjective.

2. The proof of Theorem 3

Proposition 5(Brown [1, §II.5 Exercise 4]). *Let $G = F/R$ be a group where F is a free group and R is a normal subgroup of F. Choose a set-theoretical section $s\colon G \to F$, and define a homomorphism $\varphi\colon H_2G \to R \cap [F,F]/[R,F]$ to be the one induced by a homomorphism $C_2G \to R/[R,R]$, $[g|h] \mapsto s(g)s(h)s(gh)^{-1}[R,R]$, where C_2G is the group of normalized 2-chains of G. In addition, define a homomorphism in the opposite direction $\psi\colon R \cap [F,F]/[R,F] \to H_2G$ by*

$$\prod_{i=1}^{n}[a_i, b_i][R,F] \mapsto \sum_{i=1}^{n}\{[I_{i-1}|\bar{a_i}] + [I_{i-1}\bar{a_i}|\bar{b_i}] - [I_{i-1}\bar{a_i}\bar{b_i}\bar{a_i}^{-1}|\bar{a_i}] - [I_i|\bar{b_i}]\},$$

where $\bar{a_i}$ is the class of $a_i \in F$ in $G = F/R$ and $I_i := [\bar{a_1}, \bar{b_1}] \cdots [\bar{a_i}, \bar{b_i}]$. Then φ is an isomorphism whose inverse is ψ.

Proposition 6. *Under the same notation as Proposition 5, let*

$$\zeta'\colon \bigoplus_{a \in B} C_G(a) \to R \cap [F,F]/[R,F]$$

be the homomorphism defined by setting

$$\zeta'((g_a)_{a\in B}) = \prod_{a \in B}[s(a), s(g_a)][R,F].$$

Then the composition of ζ' with the isomorphism ψ in Proposition 5 coincides with the homomorphism

$$\zeta\colon \bigoplus_{a\in B} C_G(a) \to H_2G, \quad (g_a)_{a\in B} \mapsto \sum_{a\in B}[a|g_a] - [g_a|a].$$

In particular, the homomorphism ζ' does not depend on the choice of section s.

Proof. It is enough to show that for each a and $g_a \in C_G(a)$, the equation $\psi \circ \zeta'(g_a) = \zeta(g_a)$ holds. It follows from Proposition 5 that

$$\begin{aligned} \psi \circ \zeta'(g_a) &= \psi([s(a), s(g_a)][R, F]) \\ &= [1|a] + [a|g_a] - [ag_a a^{-1}|a] - [[a, g_a]|g_a] \\ &= [a|g_a] - [g_a|a] \\ &= \zeta(g_a). \end{aligned}$$

□

Proof of Theorem 3. Let a be an arbitrary element of B. Since the right coset $\operatorname{Ker}\varepsilon_a \cdot a$ is a generator of an infinite cyclic group $\operatorname{Ker}\varepsilon_a \backslash G$, for any element g_a of G there is a unique pair $(n, g'_a) \in \mathbb{Z} \times \operatorname{Ker}\varepsilon_a$ such that $g_a = g'_a a^n$. We take a set-theoretical section $s_a \colon G \to F$ for each $a \in B$ such that

$$s_a(g_a) = s_a(g'_a)s_a(a)^n.$$

A simple calculation yields $[s_a(a), s_a(g_a)] = [s_a(a), s_a(g'_a)]$ and hence $[a, g_a] = [a, g'_a]$, from which it follows that $g_a \in C_G(a)$ if and only if $g'_a \in C_G(a) \cap \operatorname{Ker}\varepsilon_a$. Therefore, for any $(g_a)_{a\in B} \in \bigoplus_{a\in B} C_G(a)$, there is $(g'_a)_{a\in B} \in \bigoplus_{a\in B} C_G(a) \cap \operatorname{Ker}\varepsilon_a$ which satisfies

$$\zeta'((g_a)_{a\in B}) = \zeta'((g'_a)_{a\in B}).$$

Finally, the composition with ψ gives rise to $\zeta((g_a)_{a\in B}) = \zeta((g'_a)_{a\in B})$. This completes the proof of Theorem 3. □

3. Examples

Let G be an irreducible Wirtinger group and $a \in G$ is an element as in Corollary 2 and 4. In general, $C_G(a) \cap [G, G]$ is a proper subgroup of $C_G(a)$, and therefore, Theorem 3 (*resp.* Corollary 4) actually improves upon Theorem 1 (*resp.* Corollary 2). In this section, we give examples of G and $a \in G$ satisfying $C_G(a) \cap [G, G]$ is a proper subgroup of $C_G(a)$.

Example 7. Let $K \subset S^3$ be a non-trivial tame knot and $G_K := \pi_1(S^3 \setminus K)$ be the knot group of K. It is well-known that G_K is an irreducible Wirtinger group and that $H_2G_K = 0$. Let (μ, λ) be a pair of a meridian μ and a preferred longitude λ in G_K. Then the meridian $\mu \in G_K$ satisfies the conditions (i)(ii) Corollary 4. Namely,

(1) G_K coincides with the normal closure of μ,
(2) the class of μ in $(G_K)_{\mathrm{ab}} \cong \mathbb{Z}$ generates $(G_K)_{\mathrm{ab}}$.

Now suppose that K is a prime knot. Then the annulus theorem (Cannon-Feustel [2]) implies the isomorphisms

$$C_{G_K}(\mu) = \langle \mu, \lambda \rangle \cong \mathbb{Z} \oplus \mathbb{Z}, \quad C_{G_K}(\mu) \cap [G_K, G_K] = \langle \lambda \rangle \cong \mathbb{Z},$$

as explained in Eisermann [3, §4.3].

Example 8. Let B_n $(n \geq 4)$ be the braid group on n strands, and let σ_i $(1 \leq i \leq n-1)$ be the standard generaters of B_n. Namely,

$$B_n = \langle \sigma_1, \sigma_2, \ldots, \sigma_{n-1} \mid \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j \ (|i-j| = 1),\ \sigma_i\sigma_j = \sigma_j\sigma_i \ (|i-j| \geq 2) \rangle.$$

It is known that B_n is an irreducible Wirtinger group (see Gilbert [4, §3]), and that $H_2B_n \cong \mathbb{Z}/2\mathbb{Z}$. The generator $\sigma_1 \in B_n$ satisfies the conditions (i)(ii) Corollary 4. It follows easily from the presentation of B_n that

$$C_{B_n}(\sigma_1) \supset \langle \sigma_1 \rangle \times \langle \sigma_3, \sigma_4, \ldots, \sigma_{n-1} \rangle \cong \mathbb{Z} \times B_{n-2},$$

and that

$$\sigma_i \notin [B_n, B_n] \quad (1 \leq i \leq n-1),$$

where the latter follows from the fact that each σ_i generates the abelianization $(B_n)_{\mathrm{ab}} \cong \mathbb{Z}$ of B_n. We have verified that $C_{B_n}(\sigma_1) \cap [B_n, B_n]$ is indeed a proper subgroup of $C_{B_n}(\sigma_1)$. Since the homomorphism $\zeta\colon C_{B_n}(\sigma_1) \cap [B_n, B_n] \to H_2B_n$ is surjective and $H_2B_n \cong \mathbb{Z}/2\mathbb{Z}$, the subgroup $C_{B_n}(\sigma_1) \cap [B_n, B_n]$ is non-trivial.

Example 9. Let G be an irreducible Wirtinger group defined by generators g, x, y, z and relations

$$g^{-1}xg = y,\ g^{-1}yg = z,\ g^{-1}zg = x,$$
$$y^{-1}gy = x,\ y^{-1}xy = z,\ y^{-1}zy = g,$$

which was introduced in Kuz′min [8, Example 2]. The group G is a semi-direct product of the quaternion group Q and the infinite cyclic group generated by g. Kuz′min showed that $[G, G] = Q$ and $H_2G = 0$ [8, Example 2]. The generator $g \in G$ satisfies the conditions (i)(ii) Corollary 4. Observe that the order of $C_G(g)$ is infinite because $C_G(g)$ contains the infinite cyclic group generated by g, while the order of $C_G(g) \cap [G, G]$ is finite because $[G, G] = Q$ is a finite group. We conclude that $C_G(g) \cap [G, G]$ is a finite proper subgroup of $C_G(g)$.

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