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BEHAVIOR OF CURVES THROUGH INTRINSIC CROSS CAPS

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Abstract

We prove the boundedness of geodesic curvature measures of regular curves through intrinsic cross caps, which is a generalization of cross cap singular points introduced by [3]. As an application, we give the Gauss-Bonnet type formula for surfaces with boundary that admit intrinsic cross caps.

1. Introduction

The classical Gauss-Bonnet theorem for a compact oriented Riemannian 2-manifold (M, g) with boundary ∂M is given by

$$\int_M K dA + \int_{\partial M} \kappa_g d\tau = 2\pi\chi(M),$$

where K is the Gaussian curvature of the Riemannian metric g, dA (respectively, $\chi(M)$) is the area form (respectively, the Euler characteristic) of M, κ_g is the geodesic curvature of ∂M, and $d\tau$ is the arc-length measure with respect to g (see, for example, [5, Theorem 14.2]). In [4, (7a)], the Gauss-Bonnet type formula is derived for compact surfaces without boundary that admit cross caps in $\mathbb{R}^3$. Here, a smooth map-germ $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ is a *cross cap* if f is $\mathcal{A}$-equivalent to the smooth map-germ $f_{\mathrm{CC}} : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ defined by

$$f_{\mathrm{CC}}(u, v) := (u, uv, v^2) \tag{1}$$

at the origin 0, where two smooth map-germs $f_i : (\mathbb{R}^m, x_i) \to (\mathbb{R}^n, y_i)$, $i = 1, 2$, are *$\mathcal{A}$-equivalent* if there exist two diffeomorphism-germs $\varphi : (\mathbb{R}^m, x_1) \to (\mathbb{R}^m, x_2)$

and $\Phi : (\mathbb{R}^n, y_1) \to (\mathbb{R}^n, y_2)$ such that $f_2 = \Phi \circ f_1 \circ \varphi^{-1}$. In [3, Definition 4.1], a notion of intrinsic cross cap, which is a generalization of cross cap is introduced, and as a generalization of the formula [4, (7a)], the following Gauss-Bonnet type formula is shown.

THEOREM 1.1(THEOREM 5.1 OF [3]). *Let M be a compact oriented smooth 2-manifold without boundary and ds^2 a positive semi-definite metric that admits intrinsic cross caps as singular points,* called a *Whitney metric* (see [3, Definition 4.1] and Definition 2.1)*, on M. Then*

$$\int_M K dA = 2\pi\chi(M) \tag{2}$$

holds.

Note that the integral on the left-hand side of the formula (2) is well-defined by [3, Corollary 4.18].

In this paper, we prove the following theorem on the boundedness of geodesic curvature measures of regular curves through intrinsic cross caps, which is indispensable in generalizing Theorem 1.1 to the case where a 2-manifold has boundary and intrinsic cross caps on the boundary.

THEOREM 1.2. *Let M be a smooth 2-manifold (possibly with boundary) and ds^2 a positive semi-definite metric on M. Let $\gamma : [0, \varepsilon) \to M$ a smooth regular curve starting from an intrinsic cross cap, where $\varepsilon > 0$ is a number such that $\gamma((0, \varepsilon))$ consists of regular points of ds^2. Then the geodesic curvature measure $\kappa_g d\tau$ defined on $(0, \varepsilon)$ is continuously extended on $[0, \varepsilon)$, where $d\tau$ is the arc-length measure with respect to ds^2, that is, $d\tau = |\dot{\gamma}(t)| dt$, where we set $\dot{\gamma}(t) = d\gamma/dt(t)$ and $|\dot{\gamma}(t)| = \sqrt{(ds^2)_{\gamma(t)}(\dot{\gamma}(t), \dot{\gamma}(t))}$.*

Note that by Theorem 1.2, $\kappa_g d\tau$ can be continuously extended to an intrinsic cross cap, but κ_g itself is not necessarily continuously extendable (see Corollary 2.3). Then, as a consequence of [3, Corollary 4.18] and Theorem 1.2, we prove the following assertion, which is a generalization of Theorem 1.1 to the case of 2-manifolds with boundary.

COROLLARY 1.3(GAUSS-BONNET TYPE FORMULA). *Let M be a compact oriented smooth 2-manifold with boundary ∂M and ds^2 a Whitney metric on M. Suppose that ∂M is transversal to the null space* (see (4)) *at each singular point of ds^2 on ∂M if the set of singular points of ds^2 on ∂M is non-empty. Then*

$$\int_M K dA + \int_{\partial M} \kappa_g d\tau = 2\pi\chi(M) \tag{3}$$

holds.

Note that the second term on the left-hand side of the equation (3) is well-defined by Theorem 1.2.

In Section 2, we briefly review the theory of intrinsic cross caps based on [3], and then prove Theorem 1.2 and Corollary 1.3. In Section 3, we give concrete examples to illustrate our results (see Example 3.1).

All maps and manifolds considered in this paper are differentiable of class C^∞ unless stated otherwise.

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2. Boundedness of the geodesic curvature measure at an intrinsic cross cap and the Gauss-Bonnet type formula

Let M be a smooth 2-manifold (possibly with boundary) and ds^2 a positive semi-definite metric on M. A point $p \in M$ is a *regular* (respectively, *singular*) *point* of ds^2 if the 2-tensor $\left(ds^2\right)_p : T_pM \times T_pM \to \mathbb{R}$ is positive-definite (respectively, degenerate). Recall that if $p \in \partial M$, then the metric ds^2 is, by definition, extended to a C^∞-metric beyond ∂M via a compatible coordinate neighborhood around p, and the above notions are well-defined independently of the choice of a compatible coordinate neighborhood. Also, a regular point and a singular point of a smooth map on M are well-defined on ∂M, because the map is extended to a C^∞-map beyond ∂M in the same way.

Definition 2.1(Definition 4.1 of [3]). A singular point $p \in M$ of ds^2 is an *intrinsic cross cap* if we can take local coordinates (u, v) centered at p such that

$$F(0,0) = G(0,0) = G_u(0,0) = G_v(0,0) = 0 \quad \text{and} \quad E_v(0,0) = 2F_u(0,0)$$

hold and such that the Hessian of a smooth function $EG - F^2 \, (\geq 0)$ does not vanish at the origin, where we set $ds^2 = Edu^2 + 2Fdudv + Gdv^2$ and $E_v = \partial E/\partial v$, $F_u = \partial F/\partial u$, $G_u = \partial G/\partial u$, $G_v = \partial G/\partial v$. If ds^2 admits only intrinsic cross caps as singular points, then ds^2 is called a *Whitney metric.*

By definition, we see that an intrinsic cross cap p is a minimal point of $EG-F^2$, so p is an isolated singular point of ds^2. Also, we can check that p is a rank one singular point of ds^2, that is, the *null space*

$$\{\boldsymbol{v} \in T_pM \mid \langle \boldsymbol{v}, \boldsymbol{w} \rangle = 0 \text{ for all } \boldsymbol{w} \in T_pM\} \tag{4}$$

at p is a 1-dimensional linear subspace of T_pM, where we set $\langle \boldsymbol{v}, \boldsymbol{w} \rangle = \left(ds^2\right)_p (\boldsymbol{v}, \boldsymbol{w})$ for each point $p \in M$ and each vectors $\boldsymbol{v}, \boldsymbol{w} \in T_pM$ (see [3, Proposition 4.3]). Furthermore, we can show that a singular point of a cross cap $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ is an intrinsic cross cap of the first fundamental form of f, that is, an intrinsic cross cap is a generalization of a cross cap (see [3, Corollary 4.5]).

To prove Theorem 1.2, we state the following fact given in [3].

THEOREM 2.2(THEOREM 4.11 OF [3]). *There exist local coordinates* (u, v) *centered at an intrinsic cross cap and constants* $\alpha_{02}\,(> 0)$*,* α_{11} *and* α_{20} *satisfying*

$$\begin{aligned}
E(u,v) &= 1 + (\alpha_{20})^2 u^2 + 2\alpha_{11}\alpha_{20}uv + \left(1 + (\alpha_{11})^2\right) v^2 + O(u,v)^3, \\
F(u,v) &= \alpha_{11}\alpha_{20}u^2 + \left(1 + (\alpha_{11})^2 + \alpha_{02}\alpha_{20}\right) uv + \alpha_{02}\alpha_{11}v^2 + O(u,v)^3, \\
G(u,v) &= \left(1 + (\alpha_{11})^2\right) u^2 + 2\alpha_{02}\alpha_{11}uv + (\alpha_{02})^2 v^2 + O(u,v)^3,
\end{aligned}$$

where we set $ds^2 = Edu^2 + 2Fdudv + Gdv^2$ *and* $O(u,v)^3$ *is the term whose order is greater than or equal to* 3 *with respect to* (u, v)*.* (Such local coordinates are called *Bruce-West type coordinates of the second order.*)

One can show that the Bruce-West coordinates are Bruce-West type coordinates of the second order (see [3, Proposition 4.10], [6, Proposition 3.1.1], [1], [2, Proposition 2.1]). Also, it is known that the constants α_{02}, $|\alpha_{11}|$ and α_{20} are independent of the choice of Bruce-West type coordinates of the second order centered at an intrinsic cross cap p, and α_{11} is independent of the choice of Bruce-West type coordinates of the second order compatible with the orientation of a manifold M centered at p (see [3, Proposition 4.6, Theorem 4.17]).

Now, we prove Theorem 1.2.

PROOF OF THEOREM 1.2. Take Bruce-West type coordinates of the second order (u, v) centered at an intrinsic cross cap $p = \gamma(0)$ and write $\gamma(t) = (u(t), v(t))$.

First, we express the geodesic curvature κ_g of $\gamma(t)$, $t \in (0, \varepsilon)$, using the coefficients of $ds^2 = Edu^2 + 2Fdudv + Gdv^2$. The tangential vector field $\dot\gamma$ along γ, the unit normal vector field $\boldsymbol{n}$ along γ such that $\{\dot\gamma(t) / |\dot\gamma(t)|, \boldsymbol{n}(t)\}$ is a positively oriented orthonormal basis for $T_{\gamma(t)}M$, and the covariant derivative $\nabla_{\dot\gamma}\dot\gamma$ along γ

can be written by

$$\begin{aligned}
\dot{\gamma}(t) &= \dot{u}(t)\left(\frac{\partial}{\partial u}\right)_{\gamma(t)} + \dot{v}(t)\left(\frac{\partial}{\partial v}\right)_{\gamma(t)}, && (5)\\
\boldsymbol{n}(t) &= \frac{-\left(\dot{u}(t)F + \dot{v}(t)G\right)}{\sqrt{EG-F^2}\sqrt{\dot{u}(t)^2E + 2\dot{u}(t)\dot{v}(t)F + \dot{v}(t)^2G}}\left(\frac{\partial}{\partial u}\right)_{\gamma(t)} \\
&\quad + \frac{\dot{u}(t)E + \dot{v}(t)F}{\sqrt{EG-F^2}\sqrt{\dot{u}(t)^2E + 2\dot{u}(t)\dot{v}(t)F + \dot{v}(t)^2G}}\left(\frac{\partial}{\partial v}\right)_{\gamma(t)}, && (6)\\
\nabla_{\dot{\gamma}}\dot{\gamma}(t) &= \left\{\ddot{u}(t) + \frac{\dot{u}(t)^2\left(GE_u - 2FF_u + FE_v\right)}{2\left(EG-F^2\right)} + \frac{\dot{u}(t)\dot{v}(t)\left(GE_v - FG_u\right)}{EG-F^2}\right.\\
&\quad \left. + \frac{\dot{v}(t)^2\left(2GF_v - GG_u - FG_v\right)}{2\left(EG-F^2\right)}\right\}\left(\frac{\partial}{\partial u}\right)_{\gamma(t)}\\
&\quad + \left\{\ddot{v}(t) + \frac{\dot{u}(t)^2\left(2EF_u - EE_v - FE_u\right)}{2\left(EG-F^2\right)} + \frac{\dot{u}(t)\dot{v}(t)\left(EG_u - FE_v\right)}{EG-F^2}\right.\\
&\quad \left. + \frac{\dot{v}(t)^2\left(EG_v - 2FF_v + FG_u\right)}{2\left(EG-F^2\right)}\right\}\left(\frac{\partial}{\partial v}\right)_{\gamma(t)}, && (7)
\end{aligned}$$

respectively, where we set $E = E(\gamma(t))$, $F = F(\gamma(t))$, $G = G(\gamma(t))$, $E_u = E_u(\gamma(t))$, $E_v = E_v(\gamma(t))$, $F_u = F_u(\gamma(t))$, $F_v = F_v(\gamma(t))$, $G_u = G_u(\gamma(t))$, $G_v = G_v(\gamma(t))$, and ∇ is the Levi-Civita connection on the set of regular points of ds^2. Note that we used the expressions of Christoffel's symbols Γ_{ij}^k, $i, j, k = 1, 2$, to obtain the expression (7) (see, for example, [5, (10.6)]). By the equations (5),

(6) and (7), the geodesic curvature $\kappa_g(t)$ of $\gamma(t)$, $t \in (0, \varepsilon)$, is given by

$$\begin{aligned}
\kappa_g(t) &:= \frac{\langle \nabla_{\dot\gamma}\dot\gamma(t), \boldsymbol{n}(t)\rangle}{|\dot\gamma(t)|^2} \\
&= \frac{(\dot u(t)\ddot v(t) - \dot v(t)\ddot u(t))\sqrt{EG-F^2}}{\left(\dot u(t)^2 E + 2\dot u(t)\dot v(t) F + \dot v(t)^2 G\right)^{3/2}} \\
&\quad + \frac{\dot u(t)^3 (2EF_u - EE_v - FE_u)}{2\sqrt{EG-F^2}\left(\dot u(t)^2 E + 2\dot u(t)\dot v(t) F + \dot v(t)^2 G\right)^{3/2}} \\
&\quad - \frac{\dot v(t)^3 (2GF_v - GG_u - FG_v)}{2\sqrt{EG-F^2}\left(\dot u(t)^2 E + 2\dot u(t)\dot v(t) F + \dot v(t)^2 G\right)^{3/2}} \\
&\quad + \frac{\dot u(t)^2 \dot v(t) (2EG_u - 3FE_v - GE_u + 2FF_u)}{2\sqrt{EG-F^2}\left(\dot u(t)^2 E + 2\dot u(t)\dot v(t) F + \dot v(t)^2 G\right)^{3/2}} \\
&\quad - \frac{\dot u(t) \dot v(t)^2 (2GE_v - 3FG_u - EG_v + 2FF_v)}{2\sqrt{EG-F^2}\left(\dot u(t)^2 E + 2\dot u(t)\dot v(t) F + \dot v(t)^2 G\right)^{3/2}}.
\end{aligned} \tag{8}$$

Now, we prove that the geodesic curvature measure is continuously extended to an intrinsic cross cap by using the equation (8).

First, consider the case where the curve γ is transversal to the null space at p, that is, the v-axis. Then we can parameterize γ as

$$\gamma(t) = (t, v(t)), \quad v(0) = 0. \tag{9}$$

Computing E, F, G and their partial derivatives with respect to (9) and substituting these expressions into (8), we obtain

$$\kappa_g(t) = \frac{(\alpha_{11} + \alpha_{02}\dot v(0))(\alpha_{20} + \dot v(0)(2\alpha_{11} + \alpha_{02}\dot v(0))) + 2\dot v(0)}{\sqrt{(\alpha_{11} + \alpha_{02}\dot v(0))^2 + 1}} + O(t)$$

for $t > 0$. Thus κ_g is a continuous function on $[0, \varepsilon)$. In particular, $\kappa_g d\tau$ is a continuous 1-form on $[0, \varepsilon)$ because we obtain

$$d\tau = \sqrt{\dot u(t)^2 E + 2\dot u(t)\dot v(t) F + \dot v(t)^2 G}\,dt = (1 + O(t))\,dt.$$

Next, consider the case where γ is tangential to the null space at p. Then we can parameterize γ as

$$\gamma(t) = (u(t), t), \quad u(0) = \dot u(0) = 0. \tag{10}$$

Computing E, F, G and their partial derivatives with respect to (10) and substituting these expressions into (8), we obtain

$$\begin{aligned}\kappa_g(t) \;=\;& \frac{3\alpha_{11}\ddot{u}(0)^2 - \alpha_{02}u^{(3)}(0)}{2t\left((\alpha_{02})^2 + \ddot{u}(0)^2\right)^{3/2}} - \frac{\alpha_{02}u^{(4)}(0)}{3\left((\alpha_{02})^2 + \ddot{u}(0)^2\right)^{3/2}} \\ & + \frac{\ddot{u}(0)\begin{pmatrix} 9(\alpha_{02})^2 u^{(3)}(0)^2 + \alpha_{02}\alpha_{11}u^{(3)}(0)\left(43(\alpha_{02})^2 - 11\ddot{u}(0)^2\right) \\ +3\begin{pmatrix} -6(\alpha_{02})^4 \\ +\left(-27(\alpha_{11})^2 + 4\alpha_{02}\alpha_{20} - 3\right)(\alpha_{02})^2\ddot{u}(0)^2 \\ +(4\alpha_{02}\alpha_{20} + 3)\ddot{u}(0)^4 \end{pmatrix}\end{pmatrix}}{12\alpha_{02}\left((\alpha_{02})^2 + \ddot{u}(0)^2\right)^{5/2}} \\ & + O(t) \end{aligned} \tag{11}$$

for $t > 0$. Since $d\tau = \left(t\sqrt{(\alpha_{02})^2 + \ddot{u}(0)^2} + O(t)^2\right)dt$, the geodesic curvature measure $\kappa_g d\tau$ is given by

$$\kappa_g(t)\,d\tau = \left(\frac{3\alpha_{11}\ddot{u}(0)^2 - \alpha_{02}u^{(3)}(0)}{2\left((\alpha_{02})^2 + \ddot{u}(0)^2\right)} + O(t)\right)dt$$

for $t > 0$. Recall that $\alpha_{02} > 0$ because (u, v) are Bruce-West type coordinates of the second order (see Theorem 2.2). Hence $\kappa_g d\tau$ is a continuous 1-form on $[0, \varepsilon)$. □

By the equation (11), the following assertion holds.

Corollary 2.3. *If a smooth regular curve $\gamma : [0, \varepsilon) \to M$ starting from an intrinsic cross cap $p \in M$ is tangential to the null space at p, then the geodesic curvature of γ defined on $(0, \varepsilon)$ is continuously extended on $[0, \varepsilon)$ if and only if γ is parameterized by*

$$\gamma(t) = (u(t), t), \quad u(0) = \dot{u}(0) = 0, \quad 3\alpha_{11}\ddot{u}(0)^2 - \alpha_{02}u^{(3)}(0) = 0 \tag{12}$$

with respect to Bruce-West type coordinates of the second order (u, v) centered at p.

Remark 2.4. The formula for the geodesic curvature measure with respect to polar coordinates centered at an intrinsic cross cap for setting $\gamma(t) = (t, 0)$ in (9) is derived in the proof of [3, Theorem 5.1].

Now, we prove Corollary 1.3.

PROOF OF COROLLARY 1.3. First, consider the case where the set of singular points of the Whitney metric ds^2 on the boundary ∂M is empty. Triangulating M so that each singular point of ds^2 is in the interior of triangles, by [3, Corollary 4.18], the local Gauss-Bonnet type formula

$$\int_{\triangle \mathrm{ABC}} K dA + \int_{\partial \triangle \mathrm{ABC}} \kappa_g d\tau = \angle \mathrm{A} + \angle \mathrm{B} + \angle \mathrm{C} - \pi \tag{13}$$

holds for each triangle $\triangle$ABC. Therefore, by taking the sum of the equations (13) for each triangle, we obtain the equation (3).

Next, consider the case where the set of singular points on ∂M is non-empty. Take a positively oriented Bruce-West type coordinate chart of the second order $(U; u, v)$ centered at a singular point p on ∂M. Let $D(r)$ be a closed disc contained in U with center p and radius $r > 0$, and $\{\mathrm{P}_1, \mathrm{P}_2\}$ the intersection of the boundary $\partial D(r)$ with ∂M. Triangulating $D(r)$ so that p is a vertex and edges starting from p are transversal to the null space at p, by [3, Corollary 4.18] and Theorem 1.2, the local Gauss-Bonnet type formula (13) holds for each triangle $\triangle$ABC. Note that even if the vertices A, B and C are singular points, the angles $\angle$A, $\angle$B and $\angle$C on the right-hand side of the equation (13) are well-defined because, by the assumption, there exists an initial vector of edges starting from A, B and C, respectively. Therefore, by taking the sum of the equations (13) for each triangle, we obtain

$$\int_{D(r)} K dA + \int_{\partial D(r)} \kappa_g d\tau = 2\pi - \sum_{i=1,2} (\pi - \angle \mathrm{P}_i).$$

This completes the proof of Corollary 1.3. □

3. Examples

we give examples to illustrate Theorem 1.2 and Corollary 2.3.

EXAMPLE 3.1. Let $f_{\mathrm{CC}} : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ be a cross cap defined by (1). Then the local coordinates of the domain of f_{CC} are Bruce-West type coordinates of the second order (see Theorem 2.2) and the null space at the origin 0 is the v-axis. Consider the following curves $\gamma_i,\ i = 1, 2, 3, 4$, in the domain of f_{CC}:

$$\gamma_1(t) = (t, t),\ \gamma_2(t) = \left(\frac{t^2}{2}, t\right),\ \gamma_3(t) = \left(\frac{t^3}{6}, t\right),\ \gamma_4(t) = \left(\frac{t^4}{24}, t\right).$$

The curve γ_1 is transversal to the v-axis and $\gamma_i,\ i = 2, 3, 4$, are tangential to the

v-axis. The images of these curves $\widehat{\gamma}_i = f_{\mathrm{CC}} \circ \gamma_i$ are expressed as

$$\widehat{\gamma}_1(t) = \left(t, t^2, t^2\right), \quad \widehat{\gamma}_2(t) = \left(\frac{t^2}{2}, \frac{t^3}{2}, t^2\right),$$
$$\widehat{\gamma}_3(t) = \left(\frac{t^3}{6}, \frac{t^4}{6}, t^2\right), \quad \widehat{\gamma}_4(t) = \left(\frac{t^4}{24}, \frac{t^5}{24}, t^2\right),$$

respectively (see Figure. 1). The geodesic curvatures κ_g^i of $\widehat{\gamma}_i$ are calculated as

$$\begin{aligned}
\kappa_g^1(t) &= \frac{6}{\sqrt{4t^2+5}\left(8t^2+1\right)^{3/2}} \to \frac{6}{\sqrt{5}} \quad (t \to 0),\\
\kappa_g^2(t) &= \frac{-\operatorname{sgn}(t)\, 84}{\left(9t^2+20\right)^{3/2}\sqrt{17t^2+16}} \to \mp\frac{21}{40\sqrt{5}} \quad (t \to 0 \pm 0),\\
\kappa_g^3(t) &= \frac{72\left(t^4-96t^2-36\right)}{t\sqrt{t^4+144t^2+144}\left(16t^4+9t^2+144\right)^{3/2}} \to -\infty \quad (t \to 0),\\
\kappa_g^4(t) &= \frac{\operatorname{sgn}(t)\, 96\left(5t^6-8640t^2-4608\right)}{\sqrt{t^6+2304t^2+2304}\left(25t^6+16t^4+2304\right)^{3/2}} \to \mp\frac{1}{12} \quad (t \to 0 \pm 0),
\end{aligned}$$

respectively. Note that $\kappa_g^3(t)$ diverges to $-\infty$ as $t \to 0$. In fact, γ_3 does not satisfy the condition (12). Also, the geodesic curvature measures $\kappa_g^i d\tau_i$ of $\widehat{\gamma}_i$ are computed as

$$\begin{aligned}
\kappa_g^1(t)\, d\tau_1 &= \frac{6dt}{\sqrt{4t^2+5}\left(8t^2+1\right)} \to \frac{6dt}{\sqrt{5}} \quad (t \to 0),\\
\kappa_g^2(t)\, d\tau_2 &= \frac{-\operatorname{sgn}(t)\, 42t dt}{\left(9t^2+20\right)\sqrt{17t^2+16}} \to 0dt \quad (t \to 0),\\
\kappa_g^3(t)\, d\tau_3 &= \frac{12\left(t^4-96t^2-36\right)dt}{\sqrt{t^4+144t^2+144}\left(16t^4+9t^2+144\right)} \to -\frac{dt}{4} \quad (t \to 0),\\
\kappa_g^4(t)\, d\tau_4 &= \frac{\operatorname{sgn}(t)\, 4t\left(5t^6-8640t^2-4608\right)dt}{\sqrt{t^6+2304t^2+2304}\left(25t^6+16t^4+2304\right)} \to 0dt \quad (t \to 0),
\end{aligned}$$

respectively. Hence the 1-forms $\kappa_g^i d\tau_i$ are continuous at the origin.

Remark 3.2. Consider a sufficiently small perturbation $\Gamma_1 : [0,\varepsilon) \times (-\delta,\delta) \to M$ of the curve $\gamma_1 : [0,\varepsilon) \to M$ in Example 3.1 given by

$$\Gamma_1(t,s) = (t, t+s).$$

Since the geodesic curvature measure $\kappa_g d\tau$ of Γ_1 is calculated as

$$\kappa_g(t,s)\, d\tau = \frac{\left(6t - 4s(s+t-1)(s+t+1)\right)dt}{\left(5s^2+12st+8t^2+1\right)\sqrt{4(s+t)^4+4(s+t)^2+t^2}}$$

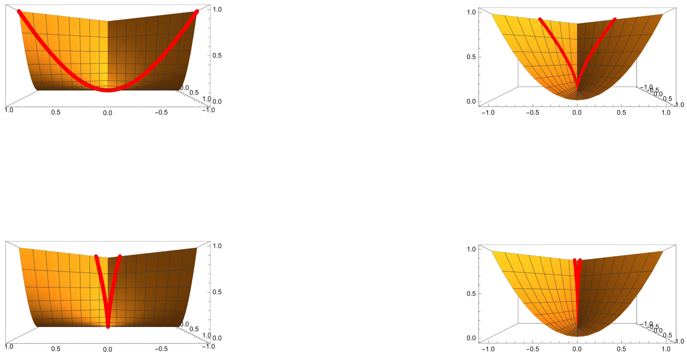

Figure 1: The thick lines from left to right in the top (respectively, bottom) row represent $\widehat{\gamma}_1, \widehat{\gamma}_2$ (respectively, $\widehat{\gamma}_3, \widehat{\gamma}_4$).

for $t > 0$, we have

$$\lim_{t\to 0+0}\left(\lim_{s\to 0}\kappa_g\left(t,s\right)d\tau\right) = \lim_{t\to 0+0}\left(\frac{6dt}{\sqrt{4t^2+5}\left(8t^2+1\right)}\right) = \frac{6dt}{\sqrt{5}}.$$

On the other hand, if $s > 0$ (respectively, $s < 0$), then we have

$$\lim_{s\to 0+0}\left(\lim_{t\to 0+0}\kappa_g d\tau\right) = \lim_{s\to 0+0}\left(-\frac{2\left(s^2-1\right)dt}{\sqrt{s^2+1}\left(5s^2+1\right)}\right) = -2dt$$
$$\left(\text{respectively, } \lim_{s\to 0-0}\left(\lim_{t\to 0+0}\kappa_g d\tau\right) = \lim_{s\to 0-0}\left(\frac{2\left(s^2-1\right)dt}{\sqrt{s^2+1}\left(5s^2+1\right)}\right) = 2dt\right).$$

Hence, when perturbing a curve through an intrinsic cross cap, the geodesic curvature measure defined on $(0,\varepsilon)\times(-\delta,\delta)$ is not necessarily continuously extended to $[0,\varepsilon)\times(-\delta,\delta)$.

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