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AN OPEN SET SATISFYING A LOCAL INTERMEDIATE COUSIN-I CONDITION IN A COMPLEX SPACE

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Abstract

Let D be an open set in a pure n-dimensional complex space X, q an integer such that $1 \leq q \leq n$ and $S(X)$ the set of singular points in X. In this paper, we prove the following two theorems: If D is exhausted by open sets $\{D_\nu\}$ such that $H^q(D_\nu \cap U, \mathcal{O}) \to H^q(D_\nu \cap U, \mathcal{M})$ is injective for every $\nu \in \mathbb{N}$ and for every relatively compact Stein open set U, then D is locally q-complete with corners at every point $x \in \partial D \setminus S(X)$. If D has a continuous boundary and $H^q(D \cap U, \mathcal{O}) \to H^q(D \cap U, \mathcal{M})$ is injective for every relatively compact Stein open set U, then D is locally q-complete with corners at every point $x \in \partial D \setminus S(X)$.

1. Introduction

In this paper, unless otherwise noted, X stands for a reduced complex space, and $S(X)$ stands for the set of singular points in X. We say that X is *Cousin-I* if any additive Cousin-I problem has a solution. Kajiwara–Kazama [10, Lemma 11] proved that an open set D in a 2-dimensional Stein manifold is Stein if and only if D is Cousin-I. Moreover, Abe–Abe [1, Corollary] generalized this result and showed that the equivalence of Cousin-I property and pseudoconvexity holds for unbranched Riemann domains over 2-dimensional Stein manifolds.

Furthermore, Kajiwara [8, Proposition 1] showed that an open set D in $\mathbb{C}^n$ is Stein if and only if D is exhausted by regular open sets $\{D_\nu\}$, where an open set D is called *regular* if $D \cap \mathsf{P}$ is Cousin-I for any polydisc P.

This paper aims to extend Kajiwara's regular exhaustion sequence theorem. For the sake of this goal, we define intermediate pseudoconvexity and intermediate Cousin-I property in complex spaces as follows: An open set D in X is said to be *locally q-complete with corners* at $x \in \partial D$, $1 \leq q \leq n$, if there exists a neighborhood U of x such that $U \cap D$ is q-complete with corners, that is, there exists a q-convex with corners exhaustion function on $U \cap D$ (see Peternell [13, Definition 3]). An open set D in X is said to be *q-Cousin-I*, $1 \leq q \leq n$, if the canonical map $H^q(D, \mathcal{O}) \to H^q(D, \mathcal{M})$ is injective where $\mathcal{M}$ denotes the sheaf of all germs of meromorphic functions on D.

In this paper, we prove that an open set D in a pure n-dimensional complex space X is locally q-complete with corners at every point $x \in \partial D \setminus S(X)$ if D is exhausted by open sets $\{D_\nu\}$ such that $D_\nu \cap U$ is q-Cousin-I for every $\nu \in \mathbb{N}$ and for every relatively compact Stein open set U. Moreover, we obtain that an open set D in a pure n-dimensional complex space X with a continuous boundary is locally q-complete with corners at every point $x \in \partial D \setminus S(X)$ if $D \cap U$ is q-Cousin-I for every relatively compact Stein open set U.

2. Preliminaries

Throughout this paper, we denote by $|\cdot|$ the maximum norm on $\mathbb{C}^n$. Let $\mathsf{P}_n(c, r) = \{z \in \mathbb{C}^n ; |z_j - c_j| < r_j \ (j = 1, \ldots, n)\}$ for every $c = (c_1, \ldots, c_n) \in \mathbb{C}^n$ and $r = (r_1, \ldots, r_n)$, where $r_j > 0$ $(j = 1, \ldots, n)$. We call the set $\mathsf{P}_n(c, r)$ the *polydisc* of polyradius r with center c in $\mathbb{C}^n$. We write $0_k = (0, \ldots, 0) \in \mathbb{C}^k$. Put

$$\begin{aligned}
\mathsf{H}^n_q(\varepsilon) &= \{(\zeta_1, \zeta_2) \in \mathbb{C}^q \times \mathbb{C}^{n-q} ; 1 - \varepsilon < |\zeta_1| < 1 + \varepsilon, |\zeta_2| < 1 + \varepsilon\} \\
&\quad \cup \{(\zeta_1, \zeta_2) \in \mathbb{C}^q \times \mathbb{C}^{n-q} ; |\zeta_1| < 1 + \varepsilon, |\zeta_2| < 1\} \quad \text{for } 0 < \varepsilon < 1, \\
\mathsf{S}_q(\delta) &= \{0_q\} \times [0, \delta) \times \{0_{n-(q+1)}\} \quad \text{for } 1 \leq \delta < 1 + \varepsilon \quad \text{and} \\
\mathsf{HS}^n_q(\varepsilon, \delta) &= \mathsf{H}^n_q(\varepsilon) \cup \mathsf{S}_q(\delta).
\end{aligned}$$

The set $\mathsf{H}^n_q(\varepsilon)$ (resp. $\mathsf{HS}^n_q(\varepsilon, \delta)$) is called a *$q$-Hartogs figure* (resp. *q-Hartogs figure with a spike*).

By using lemmata Kajiwara–Kazama [10] and Watanabe [16], we have the following lemma.

Lemma 2.1 (cf. Watanabe [16, Lemma 4]). *Let $n \geq 2$ and $a \in \mathsf{P}_n(0, 1+\varepsilon) \setminus \mathsf{H}^n_q(\varepsilon)$. The function $g = 1/(z_1 - a_1) \cdots (z_n - a_n)$ is not trivial in $H^{n-1}(\mathsf{H}^n_q(\varepsilon), \mathcal{O})$.*

Proof. When $n \geq 3$, this is a direct consequence of Watanabe [16, Lemma 4]. The case $n = 2$ and $q = 1$ is clear from lemma of Kajiwara–Kazama [10, Lemma 4] (see also Sugiyama [14, Lemma 4.1]). □

DEFINITION 2.2.

(1) Let D be an open set in $\mathbb{C}^n$. A real-valued function $\varphi\colon D \to \mathbb{R}$ is said to be *q-convex* if φ is smooth and the Levi form $Lev(\varphi : z) = \left(\frac{\partial^2 \varphi}{\partial z_i \partial \bar{z}_j}(z)\right)_{i,j=1,\dots,n}$ has at least $(n-q+1)$-positive eigenvalues for every $z \in D$.
(2) Let D be an open set in X. A real-valued function $\varphi\colon D \to \mathbb{R}$ is said to be *smooth* if for any $x \in D$ there exist an open neighborhood U of x, an open set Ω, an analytic set A in Ω, a holomorphic embedding $i\colon U \to A \subset \Omega \subset \mathbb{C}^n$ and a C^∞-function $\hat{\varphi}\colon \Omega \to \mathbb{R}$ such that $\hat{\varphi} \circ i = \varphi$.
(3) A smooth function φ is said to be *q-convex* at $x \in D$ if φ is smooth and $\hat{\varphi}$ is q-convex at $i(x)$. Moreover, φ is called *q-convex* if φ is q-convex at every point $x \in D$.
(4) A real-valued function $\varphi\colon D \to \mathbb{R}$ is said to be *q-convex with corners* if every point $x \in D$ has an open neighborhood U of x with a finite number of q-convex functions $\varphi_1, \dots, \varphi_l$ on U such that $\varphi|_U = \max\{\varphi_i\,;\, i = 1, \dots, l\}$.
(5) An open set D is said to be *q-complete with corners* if D has a q-convex with corners exhaustion function.
(6) An open set D is said to be *locally q-complete with corners* at $x \in \partial D$ if there exists an open neighborhood U of x such that $U \cap D$ is q-complete with corners.

REMARK 2.3. According to the definition above and theorem of Matsumoto [12, Theorem 7.3], an open set D in Stein manifold is locally q-complete with corners at every point $x \in \partial D$ if and only if D is pseudoconvex of order $n-q$ in the original sense (see Tadokoro [15], Fujita [6] or Matsumoto [11]). Note that if an open set D is locally 1-complete with corners at every point $x \in \partial D \setminus S(X)$ then D is locally Stein at every point $x \in \partial D \setminus S(X)$.

We will use the following theorem to prove our main theorem.

THEOREM 2.4 (cf. Sugiyama [14, Theorem 3.2]). *Let D be an open set in $\mathbb{C}^n$. Then the following three conditions are equivalent.*

(1) *D is q-complete with corners.*
(2) *D is locally q-complete with corners at every point $x \in \partial D$.*
(3) *There do not exist a q-Hartogs figure with a spike $\mathsf{HS}^n_q(\varepsilon, \delta)$ and a biholomorphic map $\varphi\colon \mathbb{C}^n \to \mathbb{C}^n$ such that $\varphi(\mathsf{HS}^n_q(\varepsilon, \delta)) \subset D$, $\varphi(\mathsf{H}^n_q(\varepsilon)) \Subset D$ and $\varphi(0_q, \delta, 0_{n-(q+1)}) \in \partial D$.*

PROOF. (1) $\Leftrightarrow$ (2). This equivalence is a direct consequence of Matsumoto's proposition ([12, Proposition 2.2]) and Remark 2.3. (2) $\Leftrightarrow$ (3). This is clear from the proof of the author's theorem ([14, Theorem 3.2]) and Remark 2.3. $\square$

COROLLARY 2.5. *Let D be an open set in $\mathbb{C}^n$. Then the following two conditions are equivalent.*

(1) *D is Stein.*

(2) *There do not exist a 1-Hartogs figure with a spike $\mathsf{HS}_1^n(\varepsilon, \delta)$ and a biholomorphic map $\varphi\colon \mathbb{C}^n \to \mathbb{C}^n$ such that $\varphi(\mathsf{HS}_1^n(\varepsilon, \delta)) \subset D$, $\varphi(\mathsf{H}_1^n(\varepsilon)) \Subset D$ and $\varphi(0, \delta, 0_{n-2}) \in \partial D$.*

PROOF. (1) $\Rightarrow$ (2). Since D is Stein, D is Hartogs pseudoconvex (see Fritzsche–Grauert [5, p. 51]). (2) $\Rightarrow$ (1). By Theorem 2.4 with $q = 1$ and Remark 2.3, D is locally Stein. Thus D is Stein because D is an open set in $\mathbb{C}^n$. □

3. Open sets exhausted by locally q-Cousin-I open sets

DEFINITION 3.1.

(1) An open set D in X is *q-Cousin-I*, $1 \leq q \leq n$, if the canonical map $H^q(D, \mathcal{O}) \to H^q(D, \mathcal{M})$ is injective, where $\mathcal{M}$ denotes the sheaf of all germs of meromorphic functions on D.

(2) An open set D in X is *locally q-Cousin-I* if $D \cap W$ is q-Cousin-I for every relatively compact Stein open set W.

REMARK 3.2.

(1) An open set D is Cousin-I if and only if $H^1(D, \mathcal{O}) \to H^1(D, \mathcal{M})$ is injective (see Grauert–Remmert [7, p. 137]). So a q-Cousin-I open set is a generalization of a Cousin-I open set. The author [14, Theorem 5.1] showed that an $(n-1)$-Cousin-I open set in an n-dimensional Stein manifold is pseudoconvex of order 1.

(2) An open set D in a Stein manifold is called strongly regular in the sense of Kajiwara if $D \cap W$ is Cousin-I for every Stein open set W (see Kajiwara [8, p. 195]). Thus a strongly regular open set is locally 1-Cousin-I.

The following lemma is elementary but useful.

LEMMA 3.3. *Let D be a locally q-Cousin-I open set in X and $W \Subset X$ a relatively compact Stein open set. Then $D \cap W$ is locally q-Cousin-I.*

An open set D is *exhausted* by open sets $\{D_\nu\}$ if $D = \bigcup_\nu D_\nu$ and $D_\nu \Subset D_{\nu+1}$ for every ν. An open set $P \subset X$ is called an *analytic polydisc* if P is biholomorphic to a polydisc in $\mathbb{C}^n$ (cf. Kajiwara [8, 9]).

Lemma 3.4. *Let D be an open set in $\mathbb{C}^n$. If D is exhausted by open sets $\{D_\nu\}$ which satisfy the following condition $(\star)$, then D is locally q-complete with corners at every point $x \in \partial D$.*

$(\star)$ *The open set $D_\nu \cap P$ is q-Cousin-I for every ν and for every analytic polydisc P.*

Proof. We use the argument in Kajiwara [9]. For any positive numbers with $0 \leq \varepsilon_2 < \varepsilon_1 < 1$, we put

$$\begin{aligned}&\mathsf{H}_q^n(\varepsilon_1, \varepsilon_2)\\ &= \{(\zeta_1, \zeta_2) \in \mathbb{C}^q \times \mathbb{C}^{n-q};\, 1 - \varepsilon_1 + \varepsilon_2 < |\zeta_1| < 1 + \varepsilon_1 - \varepsilon_2,\, |\zeta_2| < 1 + \varepsilon_1 - \varepsilon_2\}\\ &\quad \cup \{(\zeta_1, \zeta_2) \in \mathbb{C}^q \times \mathbb{C}^{n-q};\, |\zeta_1| < 1 + \varepsilon_1 - \varepsilon_2,\, |\zeta_2| < 1 - \varepsilon_2\}.\end{aligned}$$

Then we have $\mathsf{H}_q^n(\varepsilon_1, 0) = \mathsf{H}_q^n(\varepsilon_1)$.

Seeking a contradiction, assume that there exists a point $x \in \partial D$ such that D is not locally q-complete with corners at x. By the condition (3) in Theorem 2.4, there exist a biholomorphic map $\varphi\colon \mathbb{C}^n \to \mathbb{C}^n$, $\varepsilon_1 > 0$, $\delta > 0$ and $a \in \mathbb{C}^n$ such that $\varphi(\mathsf{H}_q^n(\varepsilon_1)) \Subset D$, $a = (0_q, \delta, 0_{n-(q+1)})$ and $\varphi(a) \in \partial D$. By changing holomorphic coordinates, we can assume that $\mathsf{H}_q^n(\varepsilon_1) \Subset D$ and $a \in \partial D$. Since D is exhausted by open sets $\{D_\nu\}$, so we can take decreasing sequences $\{\varepsilon_\nu\}$ and $\{\delta_\nu\}$ which satisfy the following three conditions:

- $\lim \varepsilon_\nu = 0$ and $\lim \delta_\nu = 0$.
- $\mathsf{H}_q^n(\varepsilon_1, \varepsilon_\nu) \subset D_\nu$ for every ν.
- The set $G_\nu = \{z \in D_\nu;\, |z_i| < 1 + 2\varepsilon_1\ (i = 1, \dots, q+1),\, |z_j| < \delta_\nu\ (j = q+2, \dots, n)\}$ satisfies $\{(0_q, \delta)\} \times \mathbb{C}^{n-(q+1)} \cap G_\nu = \emptyset$.

We put $U_i = \{z \in \mathbb{C}^n;\, z_i \neq a_i\}$ $(i = 1, \dots, q+1)$ and $\mathcal{U} = \{U_i\}_{i=1,\dots,q+1}$. Then $\mathcal{U}$ is an open covering of G_ν because $\{(0_q, \delta)\} \times \mathbb{C}^{n-(q+1)} \cap G_\nu = \emptyset$. Next we consider the function

$$f = \frac{1}{z_1 \cdots z_q (z_{q+1} - \delta)}.$$

Then we define a $\{f\} \in Z^q(\mathcal{U}, \mathcal{O})$. Since f is meromorphic on $\mathbb{C}^n$, we have $\{f\} = 0 \in H^q(\mathcal{U}, \mathcal{M})$. The canonical map $H^q(G_\nu, \mathcal{O}) \to H^q(G_\nu, \mathcal{M})$ is injective by the assumption of D_ν. Thus we have $\{f\} = 0 \in H^q(G_\nu, \mathcal{O})$. Since $\mathsf{H}_q^{q+1}(\varepsilon_1, \varepsilon_\nu) \times \{0_{n-(q+1)}\} \subset G_\nu$, we have

$$\{f\} = 0 \in H^q(\mathsf{H}_q^{q+1}(\varepsilon_1, \varepsilon_\nu) \times \{0_{n-(q+1)}\}, \mathcal{O}).$$

This contradicts Lemma 2.1. Therefore D is locally q-complete with corners at every point $x \in \partial D$. □

Theorem 3.5. *Let X be a pure n-dimensional complex space and D an open set in X. If D is exhausted by locally q-Cousin-I open sets $\{D_\nu\}$, then D is locally q-complete with corners at every point $x \in \partial D \setminus S(X)$.*

Proof. The point x is regular, so we can take an open neighborhood W of x and $f_1, \ldots, f_n \in \mathcal{O}(W)$ such that $F = (f_1, \ldots, f_n)\colon W \to F(W) \subset \mathbb{C}^n$ is biholomorphic. Without loss of generality, we assume that W is Stein. Since $F\colon W \to F(W)$ is biholomorphic, we can regard $D \cap W$ as an open set in $\mathbb{C}^n$. The open set W is exhausted by Stein open sets $\{W_\nu\}$. By the assumption of D, D is exhausted by locally q-Cousin-I open sets $\{D_\nu\}$. Then $D \cap W$ is exhausted by locally q-Cousin-I open sets $\{D_\nu \cap W_\nu\}$ by Lemma 3.3. It follows from Lemma 3.4 that $D \cap W$ is locally q-complete with corners at every point $x \in \partial(D \cap W)$. By Theorem 2.4, $D \cap W$ is q-complete with corners. Therefore D is locally q-complete with corners at $x \in \partial D \setminus S(X)$. □

In the case where $q = 1$, we have the following corollary.

Corollary 3.6. *Let X be a pure n-dimensional complex space and D an open set in X. If D is exhausted by locally 1-Cousin-I open sets $\{D_\nu\}$, then D is locally Stein at every point $x \in \partial D \setminus S(X)$.*

We give another proof of the following theorem.

Corollary 3.7 (cf. Kajiwara [8, Proposition 3]). *Let X be an n-dimensional Stein manifold and D an open set in X. Then the following two conditions are equivalent.*

(1) *D is Stein.*
(2) *D is exhausted by locally 1-Cousin-I open sets $\{D_\nu\}$.*

Proof. (1) ⇒ (2). Since D is Stein, D is exhausted by Stein open sets $\{D_\nu\}$. Then D_ν is locally 1-Cousin-I. So D is exhausted by locally 1-Cousin-I open sets. (2) ⇒ (1). By Theorem 3.5, D is locally 1-complete with corners at every point $x \in \partial D$. Therefore D is locally Stein. Thus D is Stein by Docquier–Grauert [4]. □

Corollary 3.8. *Let X be a Stein manifold and D an open set in X. If D is exhausted by locally q-Cousin-I open sets $\{D_\nu\}$, then D is q-complete with corners. Therefore D is $\tilde{q}$-complete and $H^k(D, \mathcal{F}) = 0$ for every coherent analytic sheaf $\mathcal{F}$ on D and for every $k \geq \tilde{q}$. Where $\tilde{q} = n - [n/q] + 1$ and $[n/q]$ denotes as the largest integer $\leq n/q$.*

PROOF. It follows from Theorem 3.5 D is locally q-complete with corners at every point $x \in \partial D$. By Matsumoto [12, Theorem 7.3], D is q-complete with corners. Moreover, D is $\tilde{q}$-complete by Diederich–Fornæss [3, Corollary 1]. Hence we have $H^k(D, \mathcal{F}) = 0$ for every coherent analytic sheaf $\mathcal{F}$ on D and for every $k \geq \tilde{q}$ by Andreotti–Grauert's theorem [2]. □

REMARK 3.9. The open set $D = \mathbb{C}^4 \setminus (\{z_1 = z_2 = 0\} \cup \{z_3 = z_4 = 0\})$ is 2-complete with corners but not 2-complete (see Watanabe [16]). Thus the same equivalence result as Corollary 3.7 cannot be obtained in the case of intermediate pseudoconvexity.

4. Open sets with a continuous boundary

DEFINITION 4.1.

(1) A boundary point ξ of an open set D in X is said to be a *continuous boundary point* of D, if there exist an open neighborhood V of ξ, a biholomorphic map $\varphi\colon V \to \Gamma^n$, where $\Gamma = \{x + iy \in \mathbb{C}\,;\, x, y \in (-1, 1)\}$, and a continuous function $g\colon \Gamma^{n-1} \times (-1, 1) \to (-1, 1)$ such that $\varphi(\partial D \cap V) = \{(z_1, \ldots, z_{n-1}, x_n + iy_n) \in \Gamma^n\,;\, y_n = g(z_1, \ldots, z_{n-1}, x_n)\}$.

(2) We say that an open set D in X has a *continuous boundary* if each point $\xi \in \partial D$ is a continuous boundary point of D (see Kajiwara [8, 9]).

THEOREM 4.2. *Let X be a pure n-dimensional complex space, D an open set in X which has a continuous boundary. If D is locally q-Cousin-I, then D is locally q-complete with corners at every point $x \in \partial D \setminus S(X)$.*

PROOF. The following argument is almost the same as the proof of Proposition 2 of Kajiwara [8]. Since the boundary of D is continuous and x is regular, there exist an open neighborhood V of x, a biholomorphic map $\varphi\colon V \to I$ and a continuous function $g\colon \Gamma^{n-1} \times (-1, 1) \to (-1, 1)$ such that $\varphi(\partial D \cap V) = \{(z_1, \ldots, z_{n-1}, x_n + iy_n) \in I\,;\, y_n = g(z_1, \ldots, z_{n-1}, x_n)\}$ where $\Gamma = \{x + iy \in \mathbb{C}\,;\, x, y \in (-1, 1)\}$ and $I = \Gamma^n$. If V is sufficiently small, we can assume that V is relatively compact. Since $\varphi\colon V \to I$ is biholomorphic, we can regard $D \cap V$ as $D \cap I$ and $\partial D \cap V$ as $\{(z_1, \ldots, z_{n-1}, x_n + iy_n) \in I\,;\, y_n = g(z_1, \ldots, z_{n-1}, x_n)\}$. Two cases are possible.

Case 1.

$$D \cap I = \{z \in I\,;\, y_n < g(z_1, \ldots, z_{n-1}, x_n)\} = V_\downarrow \quad \text{or}$$
$$D \cap I = \{z \in I\,;\, y_n > g(z_1, \ldots, z_{n-1}, x_n)\} = V_\uparrow.$$

Considering the case $V_\downarrow$ is sufficient. Notice that the set I can be exhausted by $\{I_t\}_{t\in[0,1]} = \big\{(x_1 + iy_1, \ldots, x_n + iy_n) \in \mathbb{C}^n\,;\, x_i, y_i \in \big(\frac{-1}{1+t}, \frac{1}{1+t}\big),\, i = 1, \ldots, n\big\}$.

We show that $E_t = \{z \in I_t\,;\ y_n < g(z_1, \ldots, z_{n-1}, x_n) - t\}$ is locally q-Cousin-I. Let

$$h_t\colon \mathbb{C}^n \to \mathbb{C}^n, \quad (z_1, \ldots, z_{n-1}, z_n) \mapsto (w_1, \ldots, w_{n-1}, w_n) = (z_1, \ldots, z_{n-1}, z_n + it)$$

be a biholomorphic map and U be an arbitrary relatively compact Stein open set. We have

$$h_t(E_t \cap U) = \{w \in h_t(I_t \cap U)\,;\, v_n < g(w_1, \ldots, w_{n-1}, u_n)\} = D \cap I \cap h_t(I_t \cap U),$$

where $w_n = u_n + iv_n$. The set $V_t = I \cap h_t(I_t \cap U)$ is a relatively compact Stein open set. Hence $D \cap V_t$ is q-Cousin-I by the assumption of D. Thus $E_t \cap U$ is q-Cousin-I. So E_t is locally q-Cousin-I. Moreover the open set $D \cap I$ is exhausted by locally q-Cousin-I open sets $\{E_t\}_{t\in[0,1]}$. It follows from Theorem 3.5 that $D \cap I$ is locally q-complete with corners at every point $x \in \partial(D \cap I)$. Thus $D \cap I$ is q-complete with corners by Theorem 2.4.

Case 2. $D \cap I = \{z \in I\,;\, y_n \neq g(z_1, \ldots, z_{n-1}, x_n)\} = V_\uparrow \cup V_\downarrow$. It follows from Case 1 that $V_\uparrow$ and $V_\downarrow$ are exhausted by locally q-Cousin-I open sets. So $V_\uparrow$ (resp. $V_\downarrow$) is locally q-complete with corners at every point $x \in \partial V_\uparrow$ (resp. $x \in \partial V_\downarrow$). By Theorem 2.4, each connected component $V_\uparrow$ and $V_\downarrow$ of $D \cap I$ is q-complete with corners. Thus $D \cap I$ is q-complete with corners by Theorem 2.4. Therefore D is locally q-complete with corners at every point $x \in \partial D \setminus S(X)$. □

If $q = 1$ and X is a Stein manifold, we have the following corollary.

Corollary 4.3 (cf. Kajiwara [8, Proposition 5]). *Let X be an n-dimensional Stein manifold and D an open set in X which has a continuous boundary. Then the following two conditions are equivalent.*

(1) *D is Stein.*
(2) *D is locally 1-Cousin-I.*

Proof. (1) ⇒ (2). Since D is Stein, $D \cap V$ is also Stein for any relatively compact Stein open set $V \Subset X$. Thus D is locally 1-Cousin-I. (2) ⇒ (1). D is locally 1-complete with corners by Theorem 4.2. So D is locally Stein. A locally Stein open set is Stein by Docquier–Grauert [4]. □

Corollary 4.4. *Let X be an n-dimensional Stein manifold and D an open set in X with continuous boundary. If D is locally q-Cousin-I, then D is q-complete with corners. Therefore D is $\tilde{q}$-complete and $H^k(D, \mathcal{F}) = 0$ for every coherent analytic sheaf $\mathcal{F}$ on D and for every $k \geq \tilde{q}$. Where $\tilde{q} = n - [n/q] + 1$ and $[n/q]$ denotes as the largest integer $\leq n/q$.*

Proof. By Theorem 4.2, D is locally q-complete with corners. The rest of the argument is similar to the proof of Corollary 3.8. □

Remark 4.5. In the proof of Theorem 4.2, "relatively compact Stein open set" in the definition of locally q-Cousin-I cannot be replaced by "analytic polydisc" as in Lemma 3.4. This is because, as the example below shows, the intersection of two relatively compact Stein open sets is again a relatively compact Stein open set, but the intersection of two analytic polydiscs is not necessarily an analytic polydisc again.

Example 4.6. Let $\Gamma = \{x + iy \in \mathbb{C};\, x, y \in (-1, 1)\}$. The open set G in $\mathbb{C}^2$ define as follows. Put

$$\begin{aligned}
&V_1 = \{x_1 + iy_1 \in \mathbb{C};\, x_1 \in (-1/4, 1/4),\, y_1 \in (1/4, 2)\},\\
&V_2 = \{x_1 + iy_1 \in \mathbb{C};\, x_1 \in (-1/4, 2),\, y_1 \in (3/2, 2)\},\\
&V_3 = \{x_1 + iy_1 \in \mathbb{C};\, x_1 \in (3/2, 2),\, y_1 \in (-1/4, 2)\},\\
&V_4 = \{x_1 + iy_1 \in \mathbb{C};\, x_1 \in (1/4, 2),\, y_1 \in (-1/4, 1/4)\} \quad \text{and}\\
&G = (V_1 \cup \cdots \cup V_4) \times \Gamma.
\end{aligned}$$

Then G and $\Gamma^2 \subset \mathbb{C}^2$ are biholomorphic to $\mathsf{P}_2(0, 1)$. So G and $\Gamma^2 \subset \mathbb{C}^2$ are analytic polydiscs. On the other hand, $G \cap \Gamma^2$ is not an analytic polydisc because $G \cap \Gamma^2$ is not connected.

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