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Abstract

In persistent homology analysis, interval modules play a central role in describing the birth and death of topological features across a filtration. In this work, we extend this setting, and propose the use of *bipath persistent homology*, which can be used to study the persistence of topological features across a pair of filtrations connected at their ends, to compare the two filtrations. In this setting, interval-decomposability is guaranteed, and we provide an algorithm for computing persistence diagrams for bipath persistent homology and discuss the interpretation of bipath persistence diagrams.

Keywords Persistence modules · Bipath persistence · Interval modules · Computational topology · Algorithms

Mathematics Subject Classification 55N31 · 68W30 · 16G20 · 16Z05

1 Introduction

Persistent homology [20, 25] has been successfully applied in diverse domains, such as material science [30], evolutionary biology [15], and others [1, 40, 41]. Let us recall how persistent homology arises in a typical way. For point cloud data, one can construct a filtration of simplicial complexes

$$S: S_1 \subseteq S_2 \subseteq \cdots \subseteq S_n$$

for example by using the Čech complex, the Vietoris–Rips complex, or the alpha complex construction. More generally, one can consider a filtration of topological

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spaces. Applying the q th degree homology functor $H_q(-; k)$ (with coefficient field k) to the filtration S , we obtain a *persistence module*

$$H_q(S; k) : H_q(S_1; k) \rightarrow H_q(S_2; k) \rightarrow \cdots \rightarrow H_q(S_n; k)$$

where each linear map $H_q(S_a; k) \rightarrow H_q(S_b; k)$ is induced by the inclusion map $S_a \hookrightarrow S_b$. A structure theorem (Gabriel's theorem [27], etc.) guarantees that the persistence module $H_q(S; k)$ decomposes into interval modules. Going back to the filtration, these intervals $[b, d]$ are then interpreted as describing some topological feature's persistence (born at b , dies at d). Thus, the interval modules (and interval-decomposition) play a central role in describing the birth and death of topological features (e.g. connected components, rings, cavities, etc) of data, and is visualized as a *persistence diagram*.

In order to accommodate different diagrams of spaces arising from data, the above standard setting has been naturally generalized in various ways, for example, zigzag persistence [16] or multiparameter persistence [21]. It is a fact that zigzag persistence modules are also guaranteed to admit interval-decomposability, while multiparameter persistence modules are not in general. For their motivation and applications, see [18, 32, 35, 42] and [11, 21], respectively, for example.

In this work, we propose a new method for studying the persistence of topological features across a family of spaces along a bipath, or a pair of filtrations connected at their ends, as below:

$$S: \begin{array}{c} S_0 \nearrow S_1 \hookrightarrow S_2 \hookrightarrow \cdots \hookrightarrow S_n \searrow S_1' \\ S_0 \searrow S_{1'} \hookrightarrow S_{2'} \hookrightarrow \cdots \hookrightarrow S_{m'} \end{array}$$

We call such a diagram of spaces a *bipath filtration*. From its definition, a bipath filtration can be thought of as two filtrations sharing the same starting space S_0 and ending space S_1 but going through possibly different processes in the middle.

This leads us to consider persistence modules over bipath posets, or *bipath persistence modules* for short. We provide three motivations to consider bipath persistence modules.

First, and from a theoretical point of view, the bipath persistence modules are a special class of persistence modules that arises naturally when considering interval-decomposability. This follows from the fact (see for example [4, Theorem 1.3]) that a bipath persistence module is guaranteed to be interval-decomposable, and thus we can easily define its persistence diagram.

In fact, the result of [4, Theorem 1.3], states that for a finite connected poset P , every P -persistence module is interval decomposable is equivalent to P being either an A -type poset or a bipath poset. Noting that persistence modules over an A -type poset are exactly the zigzag persistence modules, the result [4, Theorem 1.3] says that apart from zigzag persistence (including classical persistence), the only other setting for finite posets for which interval-decomposability is guaranteed is our setting of

bipaths. Thus, our second reason for considering bipaths is that it fills a gap in the literature in exploiting interval-decomposability.

Finally, and perhaps most importantly, bipath persistence modules can be used to study the persistence of topological features across a pair of filtrations connected at their ends, to compare the two filtrations. This setting can be compared to the setting of a filtration over a commutative ladder [24], which connects two filtrations at every point. However, persistence modules over commutative ladders can be very complicated for sufficiently long ladders [24, Theorem 4], in contrast to the bipath persistence modules which are always interval-decomposable. Furthermore, bipath persistence modules are naturally obtained by restriction of multiparameter persistence modules (see Example 2.9), and thus can be thought of as a simpler version of multiparameter persistence.

In this paper, we propose methods for decomposing bipath persistence modules. In Sect. 3, we provide a straightforward way for computing a decomposition by changing the bases of vector spaces consisting bipath persistence modules. In Sect. 4, we provide our main algorithm, which uses the idea of so-called matrix problems methods (see for example [28, Chapter 1], [3]) to compute a decomposition. We use column and row operations as in elementary linear algebra, but only certain operations are deemed permissible as determined by the structure of homomorphisms between intervals. Using these permissible operations, we find a normal form corresponding to an interval decomposition of the input bipath persistence module. Assuming that the bipath persistence module is already in a specific form, or can be easily transformed to a such a form, say for example starting from a bipath filtration of simplicial complexes, the algorithm in Sect. 4 is more efficient (see the last part of Sect. 4.4). Nevertheless, we provide the method in Sect. 3 because of its simplicity and the fact that it highlights a connection with zigzag persistence.

In addition, thanks to interval-decomposability we are able to define persistence diagrams for bipath persistence modules, which we call *bipath persistence diagram*. We remark that our bipath persistence diagram naturally includes the standard persistence diagram. We define it in Definition 2.7 and provide examples in Sect. 5.

1.1 Related work

In [8], persistence of circle valued maps, which can be understood as quiver representations over a bipartite quiver of type $\tilde{A}$, is studied. In the paper, algorithms for the decomposition of those representations are proposed. We remark that bipath persistence modules and quiver representations over $\tilde{A}$ have different algebraic structures, and therefore, their algorithm is not applicable as-is to our bipath persistence modules.

On the other hand, all indecomposable modules over bipath posets are classified up to isomorphisms by using string combinatorics and it is shown that they are exactly interval modules by [4] (see [26] for basic knowledge of string algebras). In contrast to our method (and purpose) of this paper, such a classification result does not imply giving algorithms of decomposition into indecomposable modules.

2 Preliminaries

In this section, we recall the notion of persistence modules. In particular, we study persistence modules over A -type posets and bipath posets which naturally arise from the point of view of interval-decomposability (Theorem 2.4), and discuss their persistence diagrams.

2.1 Persistence modules

We recall basic definitions of persistence modules [7, 12, 17]. Let k be a field. Let P be a (not necessarily finite) poset equipped with a partial order $\leq$. We regard a given poset P as a category in such a way that its objects are elements of P and there exists a unique morphism from a to b whenever $a \leq b$. For a poset P , we denote the category of (covariant) functors from P to the category vect_k of finite dimensional k -vector spaces by $\text{rep}_k(P)$, which is abelian. Each object of $\text{rep}_k(P)$ is called a P -persistence module. More explicitly, objects and morphisms of this category are given as follows. A P -persistence module V is a correspondence such that

- it associates each element $a \in P$ to a finite dimensional k -vector space $V(a)$, and
- it associates each relation $a \leq b$ in P to a k -linear map $V(a \leq b): V(a) \rightarrow V(b)$ such that $V(a \leq a) = 1_{V(a)}$ and $V(b \leq c) \circ V(a \leq b) = V(a \leq c)$.

For two P -persistence modules V and W , a morphism $f: V \rightarrow W$ is a family of k -linear maps $f_a: V(a) \rightarrow W(a)$ indexed by $a \in P$ and satisfying $f_b \circ V(a \leq b) = W(a \leq b) \circ f_a$ for every relation $a \leq b$ in P , that is, the following diagram is commutative.

$$\begin{array}{ccc} V(a) & \xrightarrow{V(a \leq b)} & V(b) \\ \downarrow f_a & \circlearrowright & \downarrow f_b \\ W(a) & \xrightarrow{W(a \leq b)} & W(b) \end{array}$$

A morphism f is called an *isomorphism* if all the f_a 's are isomorphisms of k -vector spaces. If there is an isomorphism $f: V \rightarrow W$, we write $V \cong W$.

A *direct sum* $V \oplus W$ of two P -persistence modules V and W is provided by

$$(V \oplus W)(a) := V(a) \oplus W(a) \quad \text{and} \quad (V \oplus W)(a \leq b) := \begin{bmatrix} V(a \leq b) & 0 \\ 0 & W(a \leq b) \end{bmatrix}.$$

We say that V is *decomposable* if it is isomorphic to a direct sum of non-zero objects, and *indecomposable* otherwise. The Krull–Schmidt theorem asserts that every P -persistence module decomposes into indecomposable objects uniquely, up to isomorphism and order of summands.

In the rest of this subsection, let P be a finite poset. We denote by $\text{Hasse}(P)$ the *Hasse quiver* of P , that is, a quiver whose set of vertices is P , and having an arrow $a \rightarrow b$ for $a, b \in P$ if and only if $a < b$ and there is no $z \in P$ such that $a < z < b$. We

remark that any finite poset is determined by its Hasse quiver. We also remark that it suffices to specify the vector spaces $V(a)$ for all $a \in P$ and the linear maps $V(a \leq b)$ for arrows $a \rightarrow b$ in $\text{Hasse}(P)$ to uniquely specify a P -persistence module V .

Now, we define intervals and interval modules over finite posets.

Definition 2.1 Let $I \subseteq P$ be a subset. We say that I is

- *convex* if, for any $a, b \in I$ and $z \in P$, $a < z < b$ implies $z \in I$.
- *connected* if, for any $a, b \in I$, there exists a sequence $a = z_0, z_1, \dots, z_l = b$ of elements of I such that z_{i-1} and z_i are comparable for every $i \in \{1, \dots, l\}$.
- *interval* if it is convex and connected in P .

We denote by $\mathbb{I}(P)$ the set of intervals of P .

Definition 2.2 For an interval $I \in \mathbb{I}(P)$, let k_I be a P -persistence module defined as follows.

$$k_I(a) = \begin{cases} k & \text{if } a \in I, \\ 0 & \text{otherwise,} \end{cases} \quad k_I(a \leq b) = \begin{cases} 1_k & \text{if } a, b \in I, \\ 0 & \text{otherwise.} \end{cases} \quad (2.1)$$

A P -persistence module M is said to be *interval* if $M \cong k_I$ for some interval $I \in \mathbb{I}(P)$, and *interval-decomposable* if it is isomorphic to a direct sum of finite copies of some interval modules. Clearly, all interval modules are indecomposable.

In [4], they classified finite posets whose persistence modules are always interval-decomposable. This is done by using A -type posets and bipath posets defined as follows.

Definition 2.3 (1) For an integer $n > 0$, let $A_n(a)$ be a poset whose Hasse quiver is of the form

$$A_n(a): 1 \longleftrightarrow 2 \longleftrightarrow \dots \longleftrightarrow n, \quad (2.2)$$

where $\leftrightarrow$ is either $\rightarrow$ or $\leftarrow$ as assigned by the orientation a . We call them *A-type posets*. The *equioriented* A -type poset $A_n(e)$ is provided by an orientation

$$A_n(e): 1 \rightarrow 2 \rightarrow \dots \rightarrow n. \quad (2.3)$$

- (2) For two integers $n, m > 0$, let $B_{n,m}$ be a poset given by adding two distinguished points $\hat{0}$ and $\hat{1}$ to a disjoint union of $A_n(e)$ and $A_m(e)$ so that $\hat{0}$ (resp., $\hat{1}$) is a global minimum (resp., maximum). By definition, the Hasse quiver of $B_{n,m}$ is given by

$$B_{n,m}: \begin{array}{c} \hat{0} \begin{array}{l} \nearrow \\ \searrow \end{array} \begin{array}{c} 1 \rightarrow 2 \rightarrow \dots \rightarrow n \\ 1' \rightarrow 2' \rightarrow \dots \rightarrow m' \end{array} \begin{array}{l} \nwarrow \\ \nearrow \end{array} \hat{1} \end{array} \quad (2.4)$$

where we label elements of $A_n(e)$ (resp., $A_m(e)$) by $1, \dots, n$ (resp., $1', \dots, m'$). We call $B_{n,m}$ a *bipath poset*.

Theorem 2.4 [4, Theorem 1.3] *For a finite connected poset P , the following conditions are equivalent.*

- (a) *Every P -persistence module is interval-decomposable.*
- (b) *P is either an A -type poset or a bipath poset.*

2.2 Persistence diagrams

In this section, we discuss the persistence diagrams which give a practical tool for persistent homology.

2.2.1 Zigzag persistence modules

Firstly, we study an A -type poset $P := A_n(a)$, where a is an arbitrary orientation. P -persistence modules are known as *zigzag persistence modules*. From its definition, intervals of P are exactly the subsets of the form $[b, d] := \{b, b+1, \dots, d\}$ for some $1 \leq b \leq d \leq n$. Namely,

$$\mathbb{I}(P) = \{[b, d] \mid 1 \leq b \leq d \leq n\}. \quad (2.5)$$

For $1 \leq b \leq d \leq n$, the interval $[b, d]$ and the corresponding interval module $k_{[b,d]}$ can be described in the following picture, where $\leftrightarrow$ is either $\rightarrow$ or $\leftarrow$ as assigned by the orientation a .

$$\begin{array}{l} [b, d]: \quad \cdots \quad \text{---} \quad \overset{b}{\bullet} \quad \text{---} \quad \text{---} \quad \text{---} \quad \overset{d}{\bullet} \quad \text{---} \quad \cdots \\ k_{[b,d]}: \quad \cdots \quad \longleftrightarrow 0 \longleftrightarrow k \xleftarrow{1_k} k \xleftarrow{1_k} \cdots \quad \xleftarrow{1_k} k \xleftarrow{1_k} k \longleftrightarrow 0 \longleftrightarrow \cdots \end{array}$$

As we mentioned before, every P -persistence module V is interval-decomposable. Thus, there exists an isomorphism

$$V \cong \bigoplus_{I \in \mathbb{I}(P)} k_I^{m(I)} \quad (2.6)$$

in $\text{rep}_k(P)$, where $m(I)$ denotes the multiplicity of the interval module k_I as its summands. Notice that such a decomposition is unique up to isomorphism and order of summands. In this case, we define the *persistence diagram* of V by a multiset

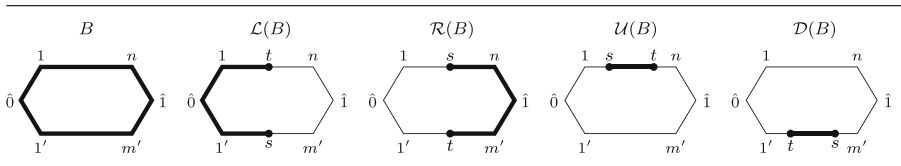
$$\{I \in \mathbb{I}(P) \text{ with multiplicity } m(I)\}.$$

When P is equioriented, we call it the *standard persistence diagram*.

Using the description (2.5), this admits a realization in the plane $\mathbb{R}^2$ by (see [16, 37, 38] for example)

$$\{(b, d) \in \mathbb{R}^2 \text{ for } I = [b, d] \text{ with multiplicity } m(I)\}.$$

Table 1 All intervals of B



The left most one is B itself. For each $\mathcal{X} \in \{\mathcal{L}, \mathcal{R}, \mathcal{U}, \mathcal{D}\}$, we describe intervals $J = \langle s, t \rangle \in \mathcal{X}(B)$ by thick lines. The corresponding interval modules are given by putting the 1-dimensional vector spaces k on J , the identity maps 1_k between them, and 0 elsewhere

2.2.2 Bipath persistence modules

Next, we study persistence modules over bipath posets (or *bipath persistence modules* for short). Let $B := B_{n,m}$ be a bipath poset as in Definition 2.3(2) for fixed $n, m > 0$. In terms of its Hasse quiver, we recall that $V \in \text{rep}_k(B)$ is determined by a diagram of vector spaces and linear maps

$$\begin{array}{ccccccc} & & V(\hat{0} \leq 1) & & & & \\ & \nearrow & & \xrightarrow{V(1 \leq 2)} & \xrightarrow{V(2 \leq 3)} & \cdots & \xrightarrow{V(n-1 \leq n)} & V(n) & \searrow & V(n \leq \hat{1}) \\ V(\hat{0}) & & V(1) & & V(2) & & \cdots & & & V(\hat{1}) \\ & \searrow & & & & & & & \nearrow & \\ & & V(\hat{0} \leq 1') & & & & & & & V(m' \leq \hat{1}) \\ & & & \xrightarrow{V(1' \leq 2')} & \xrightarrow{V(2' \leq 3')} & \cdots & \xrightarrow{V(m-1' \leq m')} & V(m') & & \end{array}$$

where they need to satisfy the commutative relation

$$V(n \leq \hat{1}) \cdots V(1 \leq 2) V(\hat{0} \leq 1) = V(m' \leq \hat{1}) \cdots V(1' \leq 2') V(\hat{0} \leq 1'). \quad (2.7)$$

To define its persistence diagram, we begin with enumerating all intervals over B . Now, let U and D be the upper path and the lower path of B respectively.

$$U: \hat{0} \rightarrow 1 \rightarrow 2 \rightarrow \cdots \rightarrow n \rightarrow \hat{1} \quad \text{and} \quad D: \hat{0} \rightarrow 1' \rightarrow 2' \rightarrow \cdots \rightarrow m' \rightarrow \hat{1}. \quad (2.8)$$

Definition 2.5 With the above notation, we define the following sets of intervals of B .

- The set $\mathcal{L}(B)$ consists of $[\hat{0}, t] \cup [\hat{0}, s]$ for some $t \in U$ and $s \in D$ with $s, t \neq \hat{1}$.
- The set $\mathcal{R}(B)$ consists of $[s, \hat{1}] \cup [t, \hat{1}]$ for some $s \in U$ and $t \in D$ with $s, t \neq \hat{0}$.
- The set $\mathcal{U}(B)$ consists of $[s, t]$ for some $s, t \in U$ with $s, t \notin \{\hat{0}, \hat{1}\}$.
- The set $\mathcal{D}(B)$ consists of $[t, s]$ for some $s, t \in D$ with $s, t \notin \{\hat{0}, \hat{1}\}$.

Here, the symbol $[a, b]$ means the interval considered in the A -type poset U or D . If an interval J of B is given by one of the above form using s and t , then we write $J = \langle s, t \rangle$.

Our notation $\langle s, t \rangle$ for intervals can be understood as the walk from s to t in a clockwise direction in the Hasse quiver of B . See Table 1. For instance, the interval $[t, s]$ of the lower path D corresponds to the interval $\langle s, t \rangle$ of B . We explain our reasoning for this in Remark 2.8, after introducing the persistence diagram.

Under the above notation, we have the following result.

Proposition 2.6 Let $B := B_{n,m}$ be a bipath poset.

- (1) We have $\mathbb{I}(B) = \{B\} \sqcup \mathcal{L}(B) \sqcup \mathcal{R}(B) \sqcup \mathcal{U}(B) \sqcup \mathcal{D}(B)$.
- (2) For any $V \in \text{rep}_k(B)$, we have a decomposition $V = V_B \oplus V_{\mathcal{L}} \oplus V_{\mathcal{R}} \oplus V_{\mathcal{U}} \oplus V_{\mathcal{D}}$, where

$$V_B \cong k_B^{m(B)} \quad \text{and} \quad V_{\mathcal{X}} \cong \bigoplus_{J \in \mathcal{X}(B)} k_J^{m(J)} \quad \text{for } \mathcal{X} \in \{\mathcal{L}, \mathcal{R}, \mathcal{U}, \mathcal{D}\}.$$

Proof It is immediate from [4, Proposition 5.5 and Corollary 5.7]. $\square$

Thanks to interval-decomposability, we can define a persistence diagram for every bipath persistence module as follows.

Definition 2.7 Suppose that a bipath persistence module V is decomposed as in Proposition 2.6(2). The *persistence diagram* of V is defined to be a multiset

$$\begin{aligned} \mathcal{D}(V) &:= \{J \in \mathbb{I}(B) \text{ with multiplicity } m(J)\} \\ &= \mathcal{D}(V_B) \sqcup \mathcal{D}(V_{\mathcal{L}}) \sqcup \mathcal{D}(V_{\mathcal{R}}) \sqcup \mathcal{D}(V_{\mathcal{U}}) \sqcup \mathcal{D}(V_{\mathcal{D}}). \end{aligned}$$

It is natural to ask a visualization of persistence diagrams of bipath persistence modules similar to the case of A -type posets. To do this, we adopt the following convention for it in this paper.

Visualization of persistence diagrams. We describe the persistence diagram $\mathcal{D}(V)$ of a given bipath persistence module V using Fig. 1. Namely, we use the diagram provided by the following procedure:

- Plot points at the upper left corner of Fig. 1 with multiplicity $m(B)$, to correspond to the intervals B .
- Plot points (s, t) with multiplicity $m(\langle s, t \rangle)$ to correspond to the intervals $\langle s, t \rangle$.

We will denote it by $p(\mathcal{D}(V))$. By definition, it can be divided into 5 components as

$$p(\mathcal{D}(V)) = p(\mathcal{D}(V_B)) \sqcup p(\mathcal{D}(V_{\mathcal{L}})) \sqcup p(\mathcal{D}(V_{\mathcal{R}})) \sqcup p(\mathcal{D}(V_{\mathcal{U}})) \sqcup p(\mathcal{D}(V_{\mathcal{D}})), \quad (2.9)$$

where the corresponding regions are explained and visualized in Fig. 1.

Remark 2.8 We provide the following observations on interpreting bipath persistence diagrams, together with some reasoning explaining why we laid out our visualization as we have.

1. In each region, points to the left and/or above correspond to intervals with longer lengths. Assuming we fix the ordering for the points of the upper path as $1, 2, \dots, n$, this forces us to order the points of the lower path as $m', \dots, 2', 1'$ for this to hold. Once this ordering is fixed, this forces us to denote the intervals $[t, s]$ of the lower path by $\langle s, t \rangle$ as we have, to have this hold for region $\mathcal{D}$.

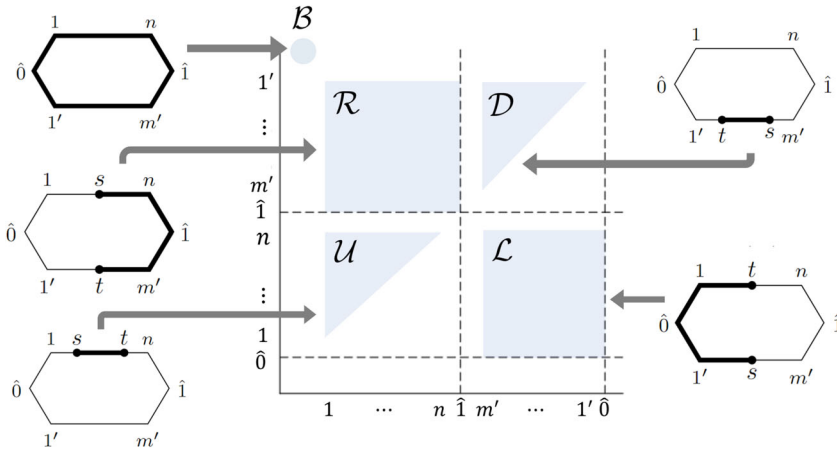


Fig. 1 A diagram for a visualization of persistence diagrams. The interval B corresponds to region $\mathcal{B}$ (i.e., the point) in the upper left corner, and each interval $\langle s, t \rangle \in \mathcal{X}(B)$ corresponds to a point (s, t) in region $\mathcal{X}$ for $\mathcal{X} \in \{\mathcal{L}, \mathcal{R}, \mathcal{U}, \mathcal{D}\}$. The points in the other regions (in white) do not correspond to any intervals. Each point has to be considered with the multiplicity when we visualize a given persistence diagram. We note that region $\mathcal{U}$ can be thought as a standard persistence diagram. On the other hand, the interval denoted $\langle s, t \rangle$ associated to the point (s, t) in region $\mathcal{D}$ corresponds to a birth t and death s in the lower path. This notation for $\mathcal{D}$ is opposite to the one for the standard persistence diagrams, and we explain our reasoning in Remark 2.8(1)

2. Across regions, the following regions can be thought of as “connected” (taken to mean the existence of pairs of intervals with lengths differing by just 1). First, $\mathcal{U}$ and $\mathcal{R}$ (and $\mathcal{R}$ and $\mathcal{D}$) are connected across their shared boundaries in Fig. 1. Furthermore, the left boundary of $\mathcal{U}$ and the right boundary of $\mathcal{L}$ are connected, and lower boundary of $\mathcal{L}$ and the upper boundary of $\mathcal{D}$ are connected. Finally, the interval B is connected to the upper left corner of $\mathcal{R}$, and also to the upper left corner of $\mathcal{L}$.

That is, removing the interval B , the bipath persistence diagram naturally lives on a torus, with the valid regions forming a cycle $\cdots - \mathcal{U} - \mathcal{R} - \mathcal{D} - \mathcal{L} - \cdots$. For example, starting from an interval in $\mathcal{U}$, we can progressively lengthen it to the right until it reaches $\hat{1}$ and extends to the lower path. We can then start shortening it from the left until it is in $\mathcal{D}$. Then, lengthening it again, we can make it reach $\hat{0}$ and beyond, giving an interval in $\mathcal{L}$. Shortening it again gives an interval in $\mathcal{U}$; and so on.

In Sect. 5, we will compute the persistence diagram together with its realization for a small example.

2.3 Bipath filtrations

For a poset P , we say that a P -filtration is a functor $S: P \rightarrow \mathbf{Top}$ such that $S(a \leq b): S(a) \hookrightarrow S(b)$ is an inclusion map for every $a \leq b$ in P , where $\mathbf{Top}$ is the category of topological spaces. Applying the q th homology functor to S (with coefficient field k), we obtain a P -persistence module $H_q(S; k)$ if it is

pointwise finite (i.e., $H_q(S; k)(a)$ is finite dimensional for all $a \in P$). That is, $H_q(S; k)(a) := H_q(S(a); k)$ is the q th homology group of the space $S(a)$, and each linear map $H_q(S; k)(a \leq b): H_q(S(a); k) \rightarrow H_q(S(b); k)$ is induced by the inclusion map $S(a) \hookrightarrow S(b)$.

Among others, a $B_{n,m}$ -filtration (a bipath filtration) can be illustrated as follows:

$$S: \begin{array}{ccccccc} & & S_1 & \hookrightarrow & S_2 & \hookrightarrow & \dots \hookrightarrow S_n \\ & \nearrow & & & & & \searrow \\ S_0 & & & & & & S_{\hat{1}} \\ & \searrow & & & & & \nearrow \\ & & S_{1'} & \hookrightarrow & S_{2'} & \hookrightarrow & \dots \hookrightarrow S_{m'} \end{array}.$$

If $H_q(S; k)$ is pointwise finite, then we obtain $\mathcal{D}_q(S) := \mathcal{D}(H_q(S; k))$ and call it the q th bipath persistence diagram of S (with coefficient field k). Notice that it admits a realization $p(\mathcal{D}_q(S))$ in the plane as in Sect. 2.2.2.

Now, we explain how bipath persistence modules (and hence bipath persistence diagrams) arise from P -filtrations, including multiparameter filtrations (the case $P = \mathbb{R}^d$).

Example 2.9 Let P be a poset and $B := B_{n,m}$ a bipath poset for some $n, m > 0$. Regarding as a functor, an order embedding $f: B \hookrightarrow P$ naturally induces a functor $f^*: \text{rep}_k(P) \rightarrow \text{rep}_k(B)$ called the *restriction functor*, where $f^*(V) \in \text{rep}_k(B)$ is defined by

$$f^*(V)(a) := V(f(a)) \quad \text{and} \quad f^*(V)(a \leq b) := V(f(a) \leq f(b))$$

for any P -persistence module V .

Let $S: P \rightarrow \text{Top}$ be a P -filtration. On one hand, applying the q th degree homology functor, one obtains a P -persistence module $H_q(S; k)$ whenever it is pointwise finite. Then, it gives a bipath persistence module $f^*(H_q(S; k))$ via the restriction functor. On the other hand, we have a bipath filtration $S \circ f: B \rightarrow \text{Top}$ as a composition. Then, it induces a bipath persistence module $H_q(S \circ f; k)$ by taking homology. Since both constructions are functorial, we have an isomorphism

$$f^*(H_q(S; k)) \cong H_q(S \circ f; k)$$

in $\text{rep}_k(B)$. Finally, we obtain its persistence diagram via the interval-decomposition.

The above construction yields large family of q th bipath persistence diagrams starting from a given P -filtration S by changing order embeddings. For more on invariants defined using order embeddings, see [2].

3 Decomposition of bipath persistence modules

In this section, we study the structure of bipath persistence modules. As a result, we give a conceptually simple method for the decomposition. In particular, we will give a construction of submodules which provide an internal direct sum for it. Note that in

Sect. 4 we give a potentially more efficient algorithm for computing decompositions using ideas from matrix problems.

Throughout this section, let V be a bipath persistence module over $B := B_{n,m}$. We recall from Proposition 2.6(2) that V is decomposed as $V = V_B \oplus V_{\mathcal{L}} \oplus V_{\mathcal{R}} \oplus V_{\mathcal{U}} \oplus V_{\mathcal{D}}$. Thus, we describe each summand in order to obtain the (interval-)decomposition.

Our strategy is the following.

- In Sect. 3.1, we will construct submodules $Y, W \subseteq V$ and prove that they provide an internal direct sum $V = Y \oplus W$ such that $Y = V_B$. Furthermore, we compute the multiplicity $m(B)$ of k_B by $m(B) = \text{rank } V(\hat{0} \leq \hat{1})$.
- In Sect. 3.2, we decompose the complement W obtained in the previous step. In fact, we will construct submodules $W', W'', Z \subseteq W$ and prove that they satisfy $W = W' \oplus W'' \oplus Z$, $W' = W_{\mathcal{L}} = V_{\mathcal{L}}$, and $W'' = W_{\mathcal{R}} = V_{\mathcal{R}}$. The constructions of W' and W'' are the dual to each other.
- The remaining part Z satisfies $Z(\hat{0}) = Z(\hat{1}) = 0$. Then, we find that the restriction Z' and Z'' of Z to the upper path U and the lower path D in (2.8) coincides with $Z_{\mathcal{U}} = V_{\mathcal{U}}$ and $Z_{\mathcal{D}} = V_{\mathcal{D}}$ respectively.

Consequently, we obtain a decomposition

$$V = V_B \oplus V_{\mathcal{L}} \oplus V_{\mathcal{R}} \oplus V_{\mathcal{U}} \oplus V_{\mathcal{D}} = Y \oplus W' \oplus W'' \oplus Z' \oplus Z''. \quad (3.1)$$

It is worth mentioning that each summand W', W'', Z', Z'' can be regarded as zigzag persistence modules. Hence, one can use methods for zigzag persistence modules in order to decompose them.

3.1 A construction of V_B and its complement

For a bipath persistence module V , we will construct submodules Y and W with the desired property that $V = Y \oplus W$ and $Y = V_B$.

If we have $V(\hat{0} \leq \hat{1}) = 0$, then let $Y := 0$ and $W := V$. Otherwise, we define Y and W in the following way. By our assumption, we have a non-zero linear map

$$V(\hat{0} \leq \hat{1}): V(\hat{0}) \rightarrow V(\hat{1}).$$

We regard its image $N := \text{Im } V(\hat{0} \leq \hat{1})$ as a subspace of $V(\hat{0})$ via the retraction $\iota: N \rightarrow V(\hat{0})$ such that $V(\hat{0} \leq \hat{1}) \circ \iota = 1_N$. Then, let Y be a submodule of V provided by $Y(a) := V(\hat{0} \leq a)(N)$ for all $a \in B$. Moreover, we take a complement $\bar{Y}(a)$ of $Y(a)$ in $V(a)$ as k -vector spaces for every $a \in B$. Then, let W be a B -persistence module defined by $W(a) := \bar{Y}(a)$ and $W(a \leq b) := \pi_b \circ V(a \leq b) \circ \iota_a$, where $\iota_a: W(a) \rightarrow V(a)$ and $\pi_a: V(a) \rightarrow W(a)$ are canonical inclusions and projections respectively. By definition, for every arrow $a \rightarrow b$ in $\text{Hasse}(B)$, we have the following

commutative diagram of vector spaces and linear maps

$$\begin{array}{ccc}
 V(a) & \xrightarrow{V(a \leq b)} & V(b) \\
 \parallel & & \parallel \\
 Y(a) \oplus W(a) & \xrightarrow{\begin{bmatrix} Y(a \leq b) & \circ \\ 0 & W(a \leq b) \end{bmatrix} g_{a,b}} & Y(b) \oplus W(b).
 \end{array} \quad (3.2)$$

Thus, we obtain an exact sequence

$$0 \rightarrow Y \rightarrow V \rightarrow W \rightarrow 0 \quad (3.3)$$

in $\text{rep}_k(B)$.

Our claim is the following.

Proposition 3.1 *We have an automorphism $\rho: V \xrightarrow{\sim} V$ such that $\rho(Y) = Y$ and $V = Y \oplus \rho(W)$ in $\text{rep}_k(B)$. In particular, the exact sequence (3.3) splits. Moreover, we have $Y \cong k_B^{\text{rank } V(\hat{0} \leq \hat{1})}$ and $W(\hat{0} \leq \hat{1}) = 0$.*

Therefore, we may take $Y = V_B$ in the decomposition of V . The automorphism ρ in the statement can be obtained by using the following elementary result in linear algebra.

Lemma 3.2 (see [5, Section 3.5] for example) *Suppose that*

$$U_1 \oplus U_2 \xrightarrow{\begin{bmatrix} S & M \\ 0 & T \end{bmatrix}} U'_1 \oplus U'_2 \xrightarrow{\begin{bmatrix} S' & M' \\ 0 & T' \end{bmatrix}} U''_1 \oplus U''_2 \quad (3.4)$$

or

$$U_1 \oplus U_2 \xleftarrow{\begin{bmatrix} S'' & M'' \\ 0 & T'' \end{bmatrix}} U'_1 \oplus U'_2 \xrightarrow{\begin{bmatrix} S' & M' \\ 0 & T' \end{bmatrix}} U''_1 \oplus U''_2. \quad (3.5)$$

is a sequence of linear maps given by block matrices. Now, let $u_1, \dots, u_s$ be a basis of U'_1 . If the number of rows of the matrix S' equals its rank, then the matrix M' can be reduced to the zero matrix by applying the following operation finitely many times:

- If $v_1, \dots, v_t$ is a basis of U'_2 , then for some $1 \leq i \leq s$, $1 \leq j \leq t$ and a scalar $\lambda \in k$, replace an element v_j with $v_j + \lambda u_i$ to obtain a new basis of U'_2 .

For sequence (3.4), the above operations correspond to elementary column (resp., row) operations on the right (resp., left) block matrix, while for sequence (3.5), the above operations correspond to elementary column operations of both the right and left block matrices. Furthermore, they do not change the matrices S , T , S' , and T' in (3.4) and the matrices S'' , T'' , S' , and T' in (3.5).

Proof of Proposition 3.1 By the definition of Y , the k -linear map $Y(a \leq b)$ is an isomorphism for every $a, b \in B$. Thus, we clearly have $Y \cong k_B^{\text{rank } V(\hat{0} \leq \hat{1})}$, where we use $\text{rank } V(\hat{0} \leq \hat{1}) = \text{rank } Y(\hat{0} \leq \hat{1})$.

In the diagram (3.2), we have a sequence of the representation matrices (with fixed bases) given by

$$Y(c) \oplus W(c) \xrightarrow{\begin{bmatrix} Y(c \leq a) & g_{c,a} \\ 0 & W(c \leq a) \end{bmatrix}} Y(a) \oplus W(a) \xrightarrow{\begin{bmatrix} Y(a \leq b) & g_{a,b} \\ 0 & W(a \leq b) \end{bmatrix}} Y(b) \oplus W(b).$$

Since the number of rows of the matrix $Y(x \leq y)$ equals its rank for any $x \leq y$ in B by our construction, we apply Lemma 3.2 to the above diagram to reduce $g_{a,b}$ to 0 for every $a = n, \dots, 1$ (in this order) and $a = m', \dots, 1'$ (in this order).

With respect to the new basis of $V(1)$ and $V(1')$, we have the following diagram:

$$Y(1') \oplus W(1') \xleftarrow{\begin{bmatrix} Y(\hat{0} \leq 1') & \tilde{g}_{\hat{0},1'} \\ 0 & W(\hat{0} \leq 1') \end{bmatrix}} Y(\hat{0}) \oplus W(\hat{0}) \xrightarrow{\begin{bmatrix} Y(\hat{0} \leq 1) & \tilde{g}_{\hat{0},1} \\ 0 & W(\hat{0} \leq 1) \end{bmatrix}} Y(1) \oplus W(1)$$

for some $\tilde{g}_{\hat{0},1'}$ and $\tilde{g}_{\hat{0},1}$. We can reduce $\tilde{g}_{\hat{0},1}$ to 0 by Lemma 3.2. Then, the representation matrix of $V(\hat{0} \leq 1')$ with respect to new bases of $V(\hat{0})$ and $V(1')$ is of the form

$$\begin{bmatrix} Y(\hat{0} \leq 1') & \tilde{g}_{\hat{0},1'} \\ 0 & W(\hat{0} \leq 1') \end{bmatrix}$$

for some matrix $\tilde{g}_{\hat{0},1'}$. By the commutativity (2.7), we can deduce $\tilde{g}_{\hat{0},1'} = 0$. Consequently, they give rise to an automorphism $\rho: V \rightarrow V$ and a decomposition $V = Y \oplus \rho(W)$, as desired.

The latter assertion is clear from our construction. $\square$

3.2 Constructions of other summands

Let W be a bipath persistence module satisfying $W(\hat{0} \leq \hat{1}) = 0$ (e.g. we may take $\rho(W)$ for V of the previous subsection).

We define a submodule W' of W by letting $W'(a) := W(\hat{0} \leq a)(W(\hat{0}))$ for all $a \in B$. In addition, let X be a B -persistence module such that $X(a)$ is a complement of $W'(a)$ in $W(a)$ and $X(a \leq b) := \pi'_b \circ W(a \leq b) \circ \iota'_a$, where $\iota'_a: X(a) \rightarrow W(a)$ and $\pi'_a: X(a) \rightarrow W(a)$ are the canonical inclusion and projection respectively. From our

construction, for every arrow $a \rightarrow b$ in $\text{Hasse}(B)$, we have a commutative diagram

$$\begin{array}{ccc}
 W(a) & \xrightarrow{V(a \leq b)} & W(b) \\
 \parallel & & \parallel \\
 W'(a) \oplus X(a) & \xrightarrow{\begin{bmatrix} W'(a \leq b) & g'_{a,b} \\ 0 & X(a \leq b) \end{bmatrix}} & W'(b) \oplus X(b)
 \end{array} \quad (3.6)$$

and an exact sequence

$$0 \rightarrow W' \rightarrow W \rightarrow X \rightarrow 0 \quad (3.7)$$

in $\text{rep}_k(B)$.

We have the following result.

Proposition 3.3 *We have an automorphism $\sigma: W \xrightarrow{\sim} W$ such that $\sigma(W') = W'$ and $W = W' \oplus \sigma(X)$ in $\text{rep}_k(B)$. Thus, the exact sequence (3.7) splits. Moreover, they satisfy $W'(\hat{1}) = 0$ and $X(\hat{0}) = 0$.*

Proof By the definition of W , the map $W'(a \leq b): W'(a) \rightarrow W'(b)$ is surjective for every $a \rightarrow b$ in $\text{Hasse}(B)$. Then, we can use Lemma 3.2 and get the former assertion by a similar discussion to a proof of Proposition 3.1. The latter assertion is obvious from the former one. $\square$

In the above situation, we have $W' = W_{\mathcal{L}}$. Moreover, we can apply the dual construction to $\sigma(X)$ to obtain a decomposition $W = W' \oplus \sigma(X) = W' \oplus W'' \oplus Z$ such that $W'' = W_{\mathcal{R}}$ and $Z(\hat{0}) = Z(\hat{1}) = 0$ by the dual statement of Proposition 3.3. For Z , the restrictions Z' and Z'' to the upper path U and the lower path D give a decomposition of Z , that is, $Z = Z' \oplus Z''$ satisfying $Z' = Z_{\mathcal{U}}$ and $Z'' = Z_{\mathcal{D}}$ respectively.

Now, we go back to the decomposition of V . As a summary of this section, we have the following.

Proposition 3.4 *Let Y, W', W'', Z', Z'' be bipath persistence modules defined for V in Sects. 3.1 and 3.2. Then, they give the desired decomposition of V in (3.1).*

Proof All modules in the statement can be taken as submodules of V satisfying the desired properties. $\square$

4 Main algorithm: matrix method

In this section, we use the idea of matrix problems methods (see for example [28, Chapter 1], [3]) to provide an algorithm for computing interval decompositions of bipath persistence modules, independent of the decomposition given in Sect. 3. Note

that matrix problems can be understood within the theoretical framework of the representation theory of bocses [14, 39] or modules over ditalgebras [13], but to keep the necessary theoretical background to a minimum we do not use these formulations.

We first give an overview of the idea for this algorithm. We fix a bipath poset $B := B_{n,m}$ and a bipath persistence module V over B . Recall that V is described by a diagram

$$\begin{array}{ccccccc} & & V(1) & \xrightarrow{V(1 \leq 2)} & V(2) & \xrightarrow{V(2 \leq 3)} & \dots & \xrightarrow{V(n-1 \leq n)} & V(n) & \xrightarrow{V(n \leq \hat{1})} & V(\hat{1}) \\ V(\hat{0}) & \xrightarrow{V(\hat{0} \leq 1)} & & & & & & & & & \\ & & V(1') & \xrightarrow{V(1' \leq 2')} & V(2') & \xrightarrow{V(2' \leq 3')} & \dots & \xrightarrow{V(m-1' \leq m')} & V(m') & \xrightarrow{V(m' \leq \hat{1})} & \\ & & & & & & & & & & \end{array}$$

For the upper path U and the lower path D in (2.8), we restrict V to obtain persistence modules V_U and V_D over U and D respectively.

$$\begin{aligned} V_U: V(\hat{0}) &\xrightarrow{V(\hat{0} \leq 1)} V(1) \xrightarrow{V(1 \leq 2)} V(2) \xrightarrow{V(2 \leq 3)} \dots \xrightarrow{V(n-1 \leq n)} V(n) \xrightarrow{V(n \leq \hat{1})} V(\hat{1}), \\ V_D: V(\hat{0}) &\xrightarrow{V(\hat{0} \leq 1')} V(1') \xrightarrow{V(1' \leq 2')} V(2') \xrightarrow{V(2' \leq 3')} \dots \xrightarrow{V(m-1' \leq m')} V(m') \xrightarrow{V(m' \leq \hat{1})} V(\hat{1}). \end{aligned}$$

Since U and D are the equioriented A -type posets, both are interval-decomposable as in Sect. 2.2. That is, there exist isomorphisms of persistence modules

$$\begin{aligned} V_U &\xrightarrow{\phi} \bigoplus_{I \in \mathbb{I}(U)} k_I^{m_U(I)} =: X \quad (\text{in } \text{rep}_k(U)), \\ V_D &\xrightarrow{\psi} \bigoplus_{I \in \mathbb{I}(D)} k_I^{m_D(I)} =: Y \quad (\text{in } \text{rep}_k(D)), \end{aligned}$$

which give interval decompositions for V_U and V_D independently of each other. However, V_U and V_D share the same vector space at $\hat{0}$ and $\hat{1}$, and we may not be able to combine ϕ and ψ as-is in order to obtain an isomorphism of V with an interval-decomposable representation of $\text{rep}_k(B)$.

Thus, we consider two change-of-basis matrices: one (which we call Λ below) at $\hat{0}$ for the vector space $V_U(\hat{0}) = V_D(\hat{0})$ connecting the basis specified by $\phi_{\hat{0}}$ and the basis specified by $\psi_{\hat{0}}$ and another one (which we call Γ below) at $\hat{1}$ for the vector space $V_U(\hat{1}) = V_D(\hat{1})$ defined similarly. Using elementary matrix operations we transform them in a way (coinciding with changes of bases) to get bases for all the vector spaces of V coinciding with an interval-decomposition. While Λ and Γ are defined as matrices, we declare certain elementary operations as not permissible, and perform only the permissible elementary operations. This is to ensure that the interval decompositions already obtained for the upper and lower paths are not destroyed.

That is, the above defines a matrix problem $M(\phi, \psi)$ which is given as a block matrix together with certain permissible operations. Solving this matrix problem (i.e. transforming it to a standard form) corresponds with computing an interval decomposition of V . This ends the high-level overview of the algorithm.

4.1 Defining the block matrix problem

Keeping the above notation, we consider the isomorphisms of vector spaces $\Lambda := \psi_{\hat{0}}\phi_{\hat{0}}^{-1}$ at $\hat{0}$ and $\Gamma := \psi_{\hat{1}}\phi_{\hat{1}}^{-1}$ at $\hat{1}$, which makes the diagrams

$$\begin{array}{ccc} V_U(\hat{0}) & \xrightarrow{\phi_{\hat{0}}} & X(\hat{0}) \\ \parallel & & \downarrow \Lambda := \psi_{\hat{0}}\phi_{\hat{0}}^{-1} \\ V_D(\hat{0}) & \xrightarrow{\psi_{\hat{0}}} & Y(\hat{0}) \end{array} \quad \text{and} \quad \begin{array}{ccc} V_U(\hat{1}) & \xrightarrow{\phi_{\hat{1}}} & X(\hat{1}) \\ \parallel & & \downarrow \Gamma := \psi_{\hat{1}}\phi_{\hat{1}}^{-1} \\ V_D(\hat{1}) & \xrightarrow{\psi_{\hat{1}}} & Y(\hat{1}) \end{array}$$

commute. These form the front and back faces of the following cube:

$$\begin{array}{ccccc} V_U(\hat{0}) & \xrightarrow{\phi_{\hat{0}}} & X(\hat{0}) & \xrightarrow{X(\hat{0} \leq \hat{1})} & V_U(\hat{1}) & \xrightarrow{\phi_{\hat{1}}} & X(\hat{1}) \\ \parallel & & \downarrow \Lambda & & \parallel & & \downarrow \Gamma \\ V_D(\hat{0}) & \xrightarrow{\psi_{\hat{0}}} & Y(\hat{0}) & \xrightarrow{Y(\hat{0} \leq \hat{1})} & V_D(\hat{1}) & \xrightarrow{\psi_{\hat{1}}} & Y(\hat{1}) \end{array} \quad (4.1)$$

The left face commutes because $V_U(\hat{0} \leq \hat{1}) = V(\hat{0} \leq \hat{1}) = V_D(\hat{0} \leq \hat{1})$. The top and bottom faces commute because ϕ and ψ are morphisms of representations. Thus, the right face

$$\begin{array}{ccc} X(\hat{0}) & \xrightarrow{X(\hat{0} \leq \hat{1})} & X(\hat{1}) \\ \downarrow \Lambda & & \downarrow \Gamma \\ Y(\hat{0}) & \xrightarrow{Y(\hat{0} \leq \hat{1})} & Y(\hat{1}) \end{array} \quad (4.2)$$

also commutes.

As we will need it later, and to give some theoretical framework for our manipulations, we define the poset

$$B': \begin{array}{ccccccc} \hat{0} & \longrightarrow & 1 & \longrightarrow & 2 & \longrightarrow & \cdots & \longrightarrow & n & \longrightarrow & \hat{1}^* \\ \downarrow & & & & & & & & & & \downarrow \\ \hat{0}^* & \longrightarrow & 1' & \longrightarrow & 2' & \longrightarrow & \cdots & \longrightarrow & m' & \longrightarrow & \hat{1} \end{array}, \quad (4.3)$$

which contains B as a subposet, and consider the inclusion functor

$$\iota: \text{rep}_k(B) \hookrightarrow \text{rep}_k(B')$$

where for $V \in \text{rep}_k(B)$, we set

$$\begin{aligned}\iota(V)(a) &:= V(a) \text{ for } a \in B, \\ \iota(V)(x^*) &:= V(x) \text{ for } x \in \{\hat{0}, \hat{1}\}, \\ \iota(V)(\hat{0} \leq \hat{0}^*) &:= 1_{V(\hat{0})}, \\ \iota(V)(\hat{1}^* \leq \hat{1}) &:= 1_{V(\hat{1})},\end{aligned}$$

and a similar definition for morphisms $f: V \rightarrow W$ in $\text{rep}_k(B)$.

Lemma 4.1 *The functor ι as defined above is a fully faithful k -linear functor.*

Then, the right face of Diagram (4.1) is simply the representation $\iota(V)$ restricted to the corners $\hat{0}, \hat{0}^*, \hat{1}, \hat{1}^*$ of B' , and the above argument shows the following.

Lemma 4.2 *The isomorphisms ϕ and ψ induce an isomorphism $f: \iota(V) \rightarrow Z$ in $\text{rep}_k(B')$, where Z is the B' -persistence module given by*

$$\begin{array}{ccccccc} X(\hat{0}) & \longrightarrow & X(1) & \longrightarrow & \cdots & \longrightarrow & X(n) \longrightarrow X(\hat{1}) \\ \Lambda \downarrow & & & & & & \downarrow \Gamma \\ Y(\hat{0}) & \longrightarrow & Y(1') & \longrightarrow & \cdots & \longrightarrow & Y(m') \longrightarrow Y(\hat{1}). \end{array}$$

Note that the representation Z (restricted to the corners $\hat{0}, \hat{0}^*, \hat{1}, \hat{1}^*$) is the right face of Diagram (4.1), and the proof is essentially the commutativity of Diagram (4.1).

In the rest of this subsection until the start of Sect. 4.3, we will define a block matrix problem and provide an algorithm for computing its normal form. While we will not directly refer to the above formalization, the block matrix problem and computing its normal form corresponds to finding an isomorphism of $\iota(V)$ with a representation Z in $\text{rep}_k(B')$ with Λ and Γ essentially identity maps (in fact, permutation maps). From such a form we can read out an interval decomposition of $\iota(V)$, and thus of V . This connection will be explained in Sect. 4.3.

We partition the direct summands of $X := \bigoplus_{I \in \mathbb{I}(U)} k_I^{m(I)}$ as $X = X_\ell \oplus X_c \oplus X_r \oplus X_o$

where X_c consists of all terms that are nonzero at $\hat{0}$ and $\hat{1}$, X_ℓ consists all terms that are nonzero at $\hat{0}$ and zero at $\hat{1}$, X_r consists all terms that are zero at $\hat{0}$ and nonzero at $\hat{1}$, and X_o are the remaining terms. We also define the decomposition $Y = Y_\ell \oplus Y_c \oplus Y_r \oplus Y_o$ similarly. The intervals in X_o and Y_o do not intersect $\hat{0}$ nor $\hat{1}$, and thus are themselves also (isomorphic to) intervals for the original bipath persistence module V .

Thus, commutative Diagram (4.2) becomes

$$\begin{array}{ccc} X_\ell(\hat{0}) \oplus X_c(\hat{0}) & \xrightarrow{\begin{bmatrix} 0 & E \\ 0 & 0 \end{bmatrix}} & X_c(\hat{1}) \oplus X_r(\hat{1}) \\ \downarrow \Lambda & & \downarrow \Gamma \\ Y_\ell(\hat{0}) \oplus Y_c(\hat{0}) & \xrightarrow{\begin{bmatrix} 0 & E \\ 0 & 0 \end{bmatrix}} & Y_c(\hat{1}) \oplus Y_r(\hat{1}) \end{array} \quad (4.4)$$

where

- we identify the linear maps with their matrix representations relative to the standard basis given by the decomposition of $X(\hat{0})$, $X(\hat{1})$, $Y(\hat{0})$, and $Y(\hat{1})$,
- 0 and E are block matrices of appropriate sizes, and
- $X(\hat{0} \leq \hat{1})$ and $Y(\hat{0} \leq \hat{1})$ take on the given forms due to the definition of X_ℓ, X_c, X_r, X_o and of Y_ℓ, Y_c, Y_r, Y_o .

Then, writing Λ and Γ in block matrix form

$$\Lambda = \begin{bmatrix} \pi_\ell \Lambda_{\iota_\ell} & \pi_\ell \Lambda_{\iota_c} \\ \pi_c \Lambda_{\iota_\ell} & \pi_c \Lambda_{\iota_c} \end{bmatrix} : X_\ell(\hat{0}) \oplus X_c(\hat{0}) \rightarrow Y_\ell(\hat{0}) \oplus Y_c(\hat{0})$$

$$\Gamma = \begin{bmatrix} \pi_c \Gamma_{\iota_c} & \pi_c \Gamma_{\iota_r} \\ \pi_r \Gamma_{\iota_c} & \pi_r \Gamma_{\iota_r} \end{bmatrix} : X_c(\hat{1}) \oplus X_r(\hat{1}) \rightarrow Y_c(\hat{1}) \oplus Y_r(\hat{1})$$

the commutativity of Diagram (4.4) implies that

$$\pi_c \Lambda_{\iota_\ell} = 0, \pi_r \Gamma_{\iota_c} = 0, \text{ and } \pi_c \Lambda_{\iota_c} = \pi_c \Gamma_{\iota_c} \quad (4.5)$$

where π and ι indicate the appropriate projections and inclusions. That is, we have block diagonal forms

$$\Lambda = \begin{bmatrix} \pi_\ell \Lambda_{\iota_\ell} & \pi_\ell \Lambda_{\iota_c} \\ 0 & \pi_c \Lambda_{\iota_c} \end{bmatrix} : X_\ell(\hat{0}) \oplus X_c(\hat{0}) \rightarrow Y_\ell(\hat{0}) \oplus Y_c(\hat{0}) \quad (4.6)$$

$$\Gamma = \begin{bmatrix} \pi_c \Gamma_{\iota_c} & \pi_c \Gamma_{\iota_r} \\ 0 & \pi_r \Gamma_{\iota_r} \end{bmatrix} : X_c(\hat{1}) \oplus X_r(\hat{1}) \rightarrow Y_c(\hat{1}) \oplus Y_r(\hat{1}) \quad (4.7)$$

We will need the following property later.

Lemma 4.3 *The blocks $\pi_\ell \Lambda_{\iota_\ell}$, $\pi_c \Lambda_{\iota_c} = \pi_c \Gamma_{\iota_c}$, and $\pi_r \Gamma_{\iota_r}$ in Eqs. (4.6), (4.7) are invertible.*

Proof We have

$$\begin{aligned} \left(\dim X_c(\hat{0}) = \dim X_c(\hat{1}) = \right) \dim \operatorname{Im} X(\hat{0} \leq \hat{1}) \\ = \dim \operatorname{Im} Y(\hat{0} \leq \hat{1}) \left(= \dim Y_c(\hat{0}) = \dim Y_c(\hat{1}) \right), \end{aligned}$$

and thus the blocks in question are square matrices. Furthermore, the block diagonal form implies that $\det \Lambda = \det(\pi_\ell \Lambda_{\iota_\ell}) \det(\pi_c \Lambda_{\iota_c})$ and $\det \Gamma = \det(\pi_c \Gamma_{\iota_c}) \det(\pi_r \Gamma_{\iota_r})$. The result follows from the invertibility of Λ and Γ . $\square$

We gather the two block matrices of Eqs. (4.6), (4.7) into a single block matrix

$$\begin{bmatrix} \pi_\ell \Lambda_{\iota_\ell} & \pi_\ell \Lambda_{\iota_c} \\ 0 & \pi_c \Lambda_{\iota_c} = \pi_c \Gamma_{\iota_c} & \pi_c \Gamma_{\iota_r} \\ & 0 & \pi_r \Gamma_{\iota_r} \end{bmatrix} \quad (4.8)$$

where the upper-right block and the lower-left block are not defined. This can be interpreted as a notational convenience, or alternatively can be thought of as a block matrix with some blocks left always empty or “strongly zero”. When we perform matrix row or column operations on such block matrices, strongly zero blocks remain zero no matter what operations are done to them, in contrast to zero blocks that may become nonzero. To simplify the notation, and denote the possible block statuses as we perform matrix operations, we adapt the notation of [3].

Notation 4.4 We write

- $*$ for unprocessed blocks,
- $\emptyset$ for strongly zero blocks,
- E for identity blocks (of appropriate sizes), and
- 0 for zero blocks (of appropriate sizes).

The blocks marked as $\emptyset$, E , and 0 are considered processed.

The single block matrix will then be expressed as

$$\begin{bmatrix} * & * & \emptyset \\ 0 & * & * \\ \emptyset & 0 & * \end{bmatrix}. \quad (4.9)$$

Next, let us consider the permissible matrix operations that preserve the interval-decomposed form of the upper and lower parts. We reuse the permissible operations discussed in [3, Subsection 3.3 and Definition 6]. Before going into the details, we note the following differences with [3].

1. As opposed to [3] which considers block matrices consisting of morphisms between representations of an A -type poset, Λ and Γ are simply linear maps between vector spaces. However, since Λ and Γ are attached to representations of A -type poset (U and D , at $\hat{0}$ and $\hat{1}$), we use only the operations permitted by this additional structure.
2. In contrast to [3] which considers all intervals, Λ and Γ are attached at $\hat{0}$ and $\hat{1}$ respectively, and thus we only need to consider the structure of intervals of the form $k_{[\hat{0}, ?]}$ and $k_{[?, \hat{1}]}$. These are the injective and projective indecomposable representations of an A -type poset.

Let Q be either U or D (an A -type poset). To treat both cases in parallel to facilitate the explanation, we write

$$Q : \hat{0} =: 1 \longrightarrow 2 \longrightarrow \cdots \longrightarrow n-1 \longrightarrow n := \hat{1}$$

where n is $n+2$ for U and $m+2$ for D . That is, $1 := \hat{0}$, $n := \hat{1}$, and the intermediate numbers represent the points in-between.

Then, it is well known that $\dim \operatorname{Hom}(k_I, k_J)$ is either 0 or 1 for intervals $I, J \in \mathbb{I}(Q)$ (see for example [3, Lemma 1]). Following [3] we define a relation on $\mathbb{I}(Q)$ by

$$I \supseteq J \text{ (or equivalently } J \trianglelefteq I) \text{ if and only if } \dim \operatorname{Hom}(k_I, k_J) \neq 0.$$

We write $I \triangleright J$ (or equivalently $J \triangleleft I$) whenever $I \supseteq J$ and $I \neq J$. We also fix the following morphism $f_{[a,b]}^{[c,d]}: k_{[a,b]} \rightarrow k_{[c,d]}$ defined by

$$\left(f_{[a,b]}^{[c,d]}\right)_i = \begin{cases} 1_k & \text{for } i \in [a, b] \cap [c, d], \\ 0 & \text{otherwise} \end{cases}$$

for each pair of intervals $[a, b]$, $[c, d]$ with $[a, b] \supseteq [c, d]$. If $[a, b] \supseteq [c, d]$, $[c, d] \supseteq [e, f]$, and $[a, b] \supseteq [e, f]$, then it is immediate that

$$f_{[a,b]}^{[e,f]} = f_{[c,d]}^{[e,f]} f_{[a,b]}^{[c,d]}.$$

Note that $\supseteq$ defines a reflexive and antisymmetric (but in general non-transitive) relation on $\mathbb{I}(Q)$.

Restricted to the intervals $[?, n]$ corresponding to the projective representations and the intervals $[1, ?]$ corresponding to the injective representations, we have

$$[1, 1] \triangleleft [1, 2] \triangleleft \cdots \triangleleft [1, n-1] \triangleleft [1, n] \triangleleft [2, n] \triangleleft \cdots \triangleleft [n-1, n] \triangleleft [n, n]. \quad (4.10)$$

Note that $\triangleleft$ is not a total order. However, we can extend it to a total order so that the sequence (4.10) gives the total ordering. Then, to be able to easily describe the permissible operations for our matrix problem, we now impose an order on the bases for the domains (columns) and codomains (rows) of Λ and Γ by following the above total order.

In more detail (in the new notation for points of the poset $Q = U$ or D), recall that $\Lambda: X(\mathfrak{t}) \rightarrow Y(\mathfrak{t})$ where $X(\mathfrak{t}) = X_\ell(\mathfrak{t}) \oplus X_c(\mathfrak{t})$ with X_ℓ a direct sum of interval modules $k_{[1,?]}$ but not equal to $k_{[1,n]}$, and X_c a direct sum of interval modules $k_{[1,n]}$. Now, $X_\ell(\mathfrak{t})$ is just a direct sum of copies of the vector space k , each coming from the vector space k at point $\mathfrak{t}$ of some representation $k_{[1,?]}$. We order the standard basis by the order (4.10) on the corresponding intervals $[1, ?]$ from whence they came. Note that each interval $[1, ?]$ can appear with multiplicity in X_ℓ . We do not impose any particular order within basis elements corresponding to the same interval. The order for the basis of the other vector spaces coming from X (and Y) are defined similarly (i.e. with reference to the order (4.10)).

We can decorate the rows and columns of the block matrix (4.8) (its abbreviated form (4.9)) to indicate the original intervals which they correspond to. Note once again that each interval can appear multiple times, and thus each interval denotes multiple

rows (or columns).

$$\begin{array}{c}
 [1, 1] \quad \dots [1, n-1] [1, n] [2, n] \quad \dots [n, n] \\
 \begin{array}{c} [1, 1] \\ \vdots \\ [1, n-1] \\ [1, n] \\ [2, n] \\ \vdots \\ [n, n] \end{array} \left[\begin{array}{ccccccc}
 * & \cdots & * & * & & & \\
 \vdots & \ddots & \vdots & \vdots & & & \\
 * & \cdots & * & * & & & \\
 0 & \cdots & 0 & * & * & \cdots & * \\
 & & & 0 & * & \cdots & * \\
 & & & \vdots & \vdots & \ddots & \vdots \\
 & & & 0 & * & \cdots & *
 \end{array} \right]. \quad (4.11)
 \end{array}$$

Definition 4.5 Following [3, Definition 6], we declare as *permissible* the following elementary row or column operations on the block matrix (4.9).

1. All elementary operations within rows (or columns) with the same label are permissible.
2. If $[a, b] \triangleleft [c, d]$ then additions of multiples of columns labelled $[a, b]$ to columns labelled $[c, d]$ are permissible.
3. (Dually,) if $[a, b] \triangleleft [c, d]$ then additions of multiples from rows labelled $[c, d]$ to rows labelled $[a, b]$ are permissible.

Let us recap the above considerations as one Definition.

Definition 4.6 Let $V \in \text{rep}_k(B)$ and $\phi: V_U \rightarrow \bigoplus_{I \in \mathbb{I}(U)} k_I^{m_U(I)}$, $\psi: V_D \rightarrow \bigoplus_{I \in \mathbb{I}(D)} k_I^{m_D(I)}$ be isomorphisms to interval decompositions of the upper and lower parts of V . We call the block matrix (4.11) (the block matrix (4.8) with the order of its rows and columns imposed by $\triangleleft$), together with the permissible operations in Definition 4.5, a *block matrix problem associated to the pair (ϕ, ψ)* .

Note that in the above definition, there can be multiple different block matrix problems associated to the same (ϕ, ψ) . These arise because we have not fixed any particular order among the rows (and columns) with the same interval labels. Up to permutations of rows and columns with the same label, there is essentially only one block matrix problem associated to (ϕ, ψ) . We denote this by $M(\phi, \psi)$, allowing for some ambiguity on the order of the rows and columns with the same labels.

4.2 Obtaining a normal form

Next, we perform permissible row and column operations on the block matrix of $M(\phi, \psi)$ to transform it to a normal form. We show that this normal form is a block matrix problem of a pair (ϕ', ψ') , and that we can then combine the isomorphisms ϕ' and ψ' to get an interval decomposition of V .

The block with rows and columns labelled $[1, n]$ (the central block) is nothing but $\pi_c \Lambda \iota_c = \pi_c \Gamma \iota_c$, which was shown to be invertible in Lemma 4.3. Using elementary row and column operations within the rows labelled $[1, n]$ and the columns labelled

$[1, n]$, we can transform it to an identity matrix E , giving the following form for the block matrix.

$$\begin{array}{c} [1, 1] \quad \dots [1, n-1] [1, n] [2, n] \quad \dots [n, n] \\ \begin{array}{c} [1, 1] \\ \vdots \\ [1, n-1] \\ [1, n] \\ [2, n] \\ \vdots \\ [n, n] \end{array} \left[\begin{array}{ccccccc} * & \cdots & * & * & & & \\ \vdots & \ddots & \vdots & \vdots & & & \\ * & \cdots & * & * & & & \\ 0 & \cdots & 0 & E & * & \cdots & * \\ & & & 0 & * & \cdots & * \\ & & & \vdots & \vdots & \ddots & \vdots \\ & & & 0 & * & \cdots & * \end{array} \right]. \end{array}$$

Then, using elementary column operations adding appropriate multiples of the columns labelled $[1, n]$ to the columns labelled $[2, n], \dots, [n, n]$, we can zero out all the blocks to the right of the central E . Similarly, using elementary row operations adding appropriate multiples of the rows labelled $[1, n]$ to the rows labelled $[1, n-1], \dots, [1, 1]$, we can zero out all the blocks above the central E . Note that these are all permissible operations. The block matrix becomes

$$\begin{array}{c} [1, 1] \quad \dots [1, n-1] [1, n] [2, n] \quad \dots [n, n] \\ \begin{array}{c} [1, 1] \\ \vdots \\ [1, n-1] \\ [1, n] \\ [2, n] \\ \vdots \\ [n, n] \end{array} \left[\begin{array}{ccccccc} * & \cdots & * & 0 & & & \\ \vdots & \ddots & \vdots & \vdots & & & \\ * & \cdots & * & 0 & & & \\ 0 & \cdots & 0 & E & 0 & \cdots & 0 \\ & & & 0 & * & \cdots & * \\ & & & \vdots & \vdots & \ddots & \vdots \\ & & & 0 & * & \cdots & * \end{array} \right]. \end{array}$$

At this point, we note that we can perform operations on (the rows and columns of) the group of blocks above-left of E and on (the rows and columns of) the group of blocks below-right of E without affecting the other group (i.e. independently of each other). Thus, we consider the subproblem

$$\begin{array}{c} [1, 1] \quad \dots [1, n-1] \\ \begin{array}{c} [1, 1] \\ \vdots \\ [1, n-1] \end{array} \left[\begin{array}{ccc} * & \cdots & * \\ \vdots & \ddots & \vdots \\ * & \cdots & * \end{array} \right] \end{array}$$

where left-to-right block operations and below-to-above block operations are permissible (Definition 4.5), and where by Lemma 4.3 the entire matrix is invertible. In Definition 4.5, elementary row (resp. column) operations among rows (resp. columns) with the same label are all permissible, but we do not need all of them (see the next Lemma).

The following is from basic linear algebra.

Lemma 4.7 *Let A be an invertible matrix. Then, using only the elementary operations of*

1. *multiplying a row (or column) by a nonzero scalar;*
2. *adding a multiple of a row to a row above it, and*
3. *adding a multiple of a column to a column to its right,*

A can be transformed to a permutation matrix.

For completeness, we provide an algorithm in Algorithm 1, which we call ONEWAYREDUCE because only operations in one direction (one-way) are allowed.

Algorithm 1 Reduction of an invertible $d \times d$ matrix A

```

1: procedure ONEWAYREDUCE( $A$ )
2:   for  $\text{col} \leftarrow 1, 2, \dots, d$  do
3:      $\text{row} \leftarrow \max\{i \mid A_{i,\text{col}} \neq 0\}$ 
4:     Multiply the  $\text{row}$ -th row by  $(A_{\text{row},\text{col}})^{-1}$ 
5:     Use entry at  $(\text{row}, \text{col})$  to zero out all entries to its right using column operations.
6:     Use entry at  $(\text{row}, \text{col})$  to zero out all entries above it using row operations.
7:   end for
8:   return  $A$ 
9: end procedure
    
```

In Algorithm 1, the fact that A is invertible guarantees that the set $\{i \mid A_{i,\text{col}} \neq 0\}$ in line 3 is always nonempty, and thus row is well-defined (a valid row number). Clearly, Algorithm 1 terminates, and returns a matrix that satisfies the following property: each column has exactly one entry “1”, with all other entries “0”, and the “1” entries of different columns are never in the same row. That is, a permutation matrix is returned.

The block matrix becomes

$$M' = \begin{bmatrix} [1, 1] & \dots & [1, n-1] & [1, n] & [2, n] & \dots & [n, n] \\ \vdots & & & & & & \\ [1, n-1] & & & & & & \\ [1, n] & \boxed{P_\ell} & 0 & & & & \\ \vdots & 0 & \dots & 0 & E & 0 & \dots & 0 \\ [2, n] & & & & 0 & & \boxed{P_r} & \\ \vdots & & & & \vdots & & & \\ [n, n] & & & & 0 & & & \end{bmatrix}$$

where P_ℓ and P_r are permutation matrices. Furthermore, by recording the operations performed on the original block matrix $M(\phi, \psi)$, this can be expressed as

$$M' = T' M(\phi, \psi) T \quad (4.12)$$

for some invertible block upper-triangular matrices T and T' of the forms

$$\begin{aligned}
 & \begin{matrix} & [1, 1] & \dots & [1, n-1] & [1, n] & [2, n] & \dots & [n, n] \\ \begin{matrix} [1, 1] \\ \vdots \\ [1, n-1] \\ [1, n] \\ [2, n] \\ \vdots \\ [n, n] \end{matrix} & \begin{bmatrix} * & \dots & * & 0 & & & \\ & \ddots & \vdots & \vdots & & & \\ 0 & & * & 0 & * & \dots & * \\ 0 & \dots & 0 & 0 & * & \dots & * \\ & & 0 & * & \dots & * & \\ & & \vdots & \vdots & \ddots & \vdots & \\ 0 & & 0 & 0 & & & * \end{bmatrix} \end{matrix} \quad \text{and} \quad \begin{matrix} & [1, 1] & \dots & [1, n-1] & [1, n] & [2, n] & \dots & [n, n] \\ \begin{matrix} [1, 1] \\ \vdots \\ [1, n-1] \\ [1, n] \\ [2, n] \\ \vdots \\ [n, n] \end{matrix} & \begin{bmatrix} * & \dots & * & * & & & \\ & \ddots & \vdots & \vdots & & & \\ 0 & & * & * & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & 0 & * & \dots & * \\ & & 0 & * & \dots & * & \vdots & \\ & & \vdots & \vdots & \ddots & \vdots & \vdots & \\ 0 & & 0 & 0 & & & * \end{bmatrix} \end{matrix} \quad (4.13)
 \end{aligned}$$

respectively.

4.3 From the normal form to the interval decomposition

The form of M' gives matchings:

1. between intervals $[1, ?]$ of the upper and lower path,
2. between intervals $[?, n]$ of the upper and lower path,
3. between intervals $[1, n]$ of the upper and lower path,

given by the permutation matrices P_ℓ , P_r , and the identity matrix E , respectively. We show that by “gluing” these matched intervals, and together with the intervals not intersecting 1 nor n (the terms X_o and Y_o in the partition of the intervals of X and Y), we obtain a bipath persistence module W that is explicitly a direct sum of interval bipath persistence modules and is isomorphic to the original bipath persistence module V . For computations, we only need to do so (i.e. construct/write out the intervals of W). Below, for convenience, we summarize this and explicitly write out what we mean by “gluing”. Note that we return to the original indexing vertices $\{\hat{0}, 1, \dots, i, \dots, n, \hat{1}\}$ for the upper path U and $\{\hat{0}, 1', \dots, j', \dots, m', \hat{1}\}$ for the lower path D . We also recall Definition 2.5 for intervals in the bipath poset.

Construction 4.8 Given the considerations of the above discussion, construct the following intervals of the bipath poset B (allowing for multiplicities).

1. For each matched pair $[\hat{0}, \hat{1}]$ in U and $[\hat{0}, \hat{1}]$ in D , construct the interval B (i.e. the bipath poset itself). These correspond to region $\mathcal{B}$ in Fig. 1.
2. For each matched pair $[\hat{0}, i]$ in U and $[\hat{0}, j']$ in D , construct the interval $\langle j', i \rangle \in \mathcal{L}(B)$. These correspond to intervals in region $\mathcal{L}$ of Fig. 1.
3. For each matched pair $[i, \hat{1}]$ in U and $[j', \hat{1}]$ in D , construct the interval $\langle i, j' \rangle \in \mathcal{R}(B)$. These correspond to intervals in region $\mathcal{R}$ of Fig. 1.
4. For each interval $[i_1, i_2]$ in X_o (intervals of U not intersecting $\hat{0}$ nor $\hat{1}$), construct the interval $\langle i_1, i_2 \rangle \in \mathcal{U}(B)$. These correspond to intervals in region $\mathcal{U}$ of Fig. 1.
5. For each interval $[j'_1, j'_2]$ in Y_o (intervals of D not intersecting $\hat{0}$ nor $\hat{1}$), construct the interval $\langle j'_2, j'_1 \rangle \in \mathcal{D}(B)$. These correspond to intervals in region $\mathcal{D}$ of Fig. 1.

In the rest of this subsection, we provide the proof that this is algebraically valid.

The form of the invertible block upper triangular matrix T as given above in Eq. (4.13) has only nonzero entries in the block located in row $[a, b]$ and column $[c, d]$

only for pairs $[a, b] \leq [c, d]$. By attaching the fixed morphisms $f_{[a,b]}^{[c,d]}: k_{[a,b]} \rightarrow k_{[c,d]}$ to the matrix T (i.e. use the entries of T as coefficients), we obtain an endomorphism of $X_\ell \oplus X_c \oplus X_r$. This is in fact an automorphism, because T is invertible block upper-triangular. Furthermore, taking a direct sum with identity maps for the terms X_ρ , we obtain an automorphism for $X = X_\ell \oplus X_c \oplus X_r \oplus X_\rho$, which we denote by

$$\hat{T}: X \rightarrow X.$$

Similarly, from T' we obtain an automorphism

$$\hat{T}': Y \rightarrow Y.$$

Then, by construction, $M' = T'M(\phi, \psi)T$ is a block matrix problem (Definition 4.6) associated with the pair of isomorphisms $(\phi', \psi') := (\hat{T}^{-1}\phi, \hat{T}'\psi)$:

$$\begin{array}{ccc} V_U & \xrightarrow{\phi} & X \xrightarrow{\hat{T}^{-1}} X \quad (\text{in } \text{rep}_k(U)), \\ V_D & \xrightarrow{\psi} & Y \xrightarrow{\hat{T}'} Y \quad (\text{in } \text{rep}_k(D)). \end{array}$$

Recalling the poset B' in Diagram (4.3), the inclusion functor $\iota: \text{rep}_k(B) \rightarrow \text{rep}_k(B')$ and Lemma 4.2, ϕ' and ψ' induces an isomorphism $\iota(V) \rightarrow Z'$ where Z' is the representation

$$\begin{array}{ccccccc} X(\hat{0}) & \longrightarrow & X(1) & \longrightarrow & \cdots & \longrightarrow & X(n) \longrightarrow X(\hat{1}) \\ \Lambda' := \begin{bmatrix} P_\ell & 0 \\ 0 & E \end{bmatrix} \downarrow & & & & & & \downarrow \Gamma' := \begin{bmatrix} E & 0 \\ 0 & P_r \end{bmatrix} \\ Y(\hat{0}) & \longrightarrow & Y(1') & \longrightarrow & \cdots & \longrightarrow & Y(m') \longrightarrow Y(\hat{1}) \end{array}$$

by construction (i.e. from the normal form of the block matrix problem).

By permuting the summands, say of X (write the permuted representation as X'), we obtain an isomorphism $Z' \cong Z''$ where the “vertical” maps of Z'' are identity; that is

$$\begin{array}{ccccccc} X'(\hat{0}) & \longrightarrow & X'(1) & \longrightarrow & \cdots & \longrightarrow & X'(n) \longrightarrow X'(\hat{1}) \\ Z'' : \Lambda'' := 1 \downarrow & & & & & & \downarrow \Gamma' := 1 \\ Y(\hat{0}) & \longrightarrow & Y(1') & \longrightarrow & \cdots & \longrightarrow & Y(m') \longrightarrow Y(\hat{1}) \end{array}$$

By the definition of the inclusion functor ι , such a representation Z'' can be written as $Z'' = \iota(W)$ where $W \in \text{rep}_k(B)$ is:

$$\begin{array}{ccccc} & & X'(1) \longrightarrow \cdots \longrightarrow X'(n) & & \\ & \nearrow & & \searrow & \\ (X'(\hat{0}) = Y(\hat{0})) & & & & (X'(\hat{1}) = Y(\hat{1})) \\ & \searrow & & \nearrow & \\ & & Y(1') \longrightarrow \cdots \longrightarrow Y(m') & & \end{array}$$

i.e. defined by the restriction of Z'' to B . Combining, we obtain

$$\iota(V) \cong Z' \cong Z'' = \iota(W)$$

in $\text{rep}_k(B')$. By Lemma 4.1, we conclude that

$$V \cong W$$

in $\text{rep}_k(B)$, and we note that W is (explicitly) a direct sum of intervals of $\text{rep}_k(B)$, since both X' and Y are direct sums of intervals (in $\text{rep}_k(U)$ and $\text{rep}_k(D)$ respectively). That is, we have computed an interval decomposition of V , corresponding to Construction 4.8.

4.4 Summary of the algorithm and an algorithm for bipath filtrations

We summarize the entire procedure below as Algorithm 2.

Algorithm 2 Main Algorithm

Require: bipath persistence module V

- 1: **procedure** INTERVALDECOMPOSE(V)
 - 2: Interval-decompose the upper row V_U and lower row V_D .
 - 3: Construct the associated block matrix problem (Definition 4.6).
 - 4: Reduce the block matrix problem to normal form (Sect. 4.2).
 - 5: **return** The intervals of Construction 4.8
 - 6: **end procedure**
-

The above starts with a bipath persistence module V as its input. What if we are starting with a bipath filtration? Consider the bipath filtration

$$S: \begin{array}{ccccccc} & & S_1 \hookrightarrow S_2 \hookrightarrow \cdots \hookrightarrow S_n & & \\ & \nearrow & & \searrow & \\ S_0 & & & & S_1 \\ & \searrow & & \nearrow & \\ & & S_{1'} \hookrightarrow S_{2'} \hookrightarrow \cdots \hookrightarrow S_{m'} & & \end{array}$$

Applying the homology functor $H_q(-; k)$ we obtain the bipath persistence module $V = H_q(S; k)$:

$$\begin{array}{ccccccc} & & H_q(S_1; k) & \rightarrow & H_q(S_2; k) & \rightarrow & \cdots \rightarrow H_q(S_n; k) & & \\ & \nearrow & & & & & & \searrow & \\ H_q(S_0; k) & & & & & & & & H_q(S_{\hat{1}}; k) \\ & \searrow & & & & & & \nearrow & \\ & & H_q(S_{1'}; k) & \rightarrow & H_q(S_{2'}; k) & \rightarrow & \cdots \rightarrow H_q(S_{m'}; k) & & \end{array}$$

whose interval decomposition we want to compute. Instead of explicitly computing V , say by pointwise computing homology vector spaces and the induced linear maps between them, we can instead apply the standard persistent homology algorithm to the two filtrations

$$S_U: S_0 \subseteq S_1 \subseteq S_2 \subseteq \cdots \subseteq S_n \subseteq S_{\hat{1}}$$

and

$$S_D: S_0 \subseteq S_{1'} \subseteq S_{2'} \subseteq \cdots \subseteq S_{m'} \subseteq S_{\hat{1}}$$

separately.

Remark 4.9 In addition to the intervals of the upper and lower rows, the matrices Λ and Γ are needed to construct the associated block matrix problem. For this, we do not need to explicitly compute the isomorphisms ϕ and ψ . Instead, it suffices to record a set of representative cycles associated to the intervals $[b, d]$ of the persistence diagrams of the upper and lower rows (in fact, only the ones associated to the intervals $[\hat{0}, ?]$ and $[?, \hat{1}]$ are needed) together with bases for the image of the boundary at $\hat{0}$ and $\hat{1}$. Note that the standard reduction algorithm for persistent homology provides these information. From these, one can compute the matrices Λ (resp., Γ) as change of basis matrices for the basis at $\hat{0}$ (resp., $\hat{1}$) determined by (homology classes of) the representative cycles of the upper row versus the one determined by the representative cycles of the lower row.

We summarize this in Algorithm 3.

Algorithm 3 Main Algorithm – for bipath filtrations

Require: Bipath filtration S

- 1: **procedure** INTERVALDECOMPOSE(S, q)
 - 2: Compute the standard q th persistence diagram of S_U and S_D
 (with additional information noted in Remark 4.9).
 - 3: Construct the associated block matrix problem with Λ and Γ as in Remark 4.9.
 - 4: Reduce the block matrix problem to normal form (Sect. 4.2).
 - 5: **return** The intervals of Construction 4.8
 - 6: **end procedure**
-

Our main theorem is as follows (which is proven through this section).

Theorem 4.10 *Given a bipath persistence module V (resp., bipath filtration S), Algorithm 2 (resp., Algorithm 3) correctly computes its interval decomposition (resp., q th bipath persistence diagram).*

Furthermore, the computational complexity of the above algorithm is as follows.

Theorem 4.11 *Assuming that the computational complexity of an algorithm for standard persistent homology that gives a set of representative cycles and bases for the images of the boundary as in Remark 4.9 (for example the standard algorithm in [25]) is $O(F)$, the computational complexity of Algorithm 3 is $O(F + m^3)$, where m is the size of the block matrix problem.*

Proof By assumption, computing the standard persistence diagram of S_U and S_D takes $O(F)$ time. Since we can compute Λ and Γ in $O(m^3)$ time, we can obtain the associated block matrix problem with the two matrices in $O(m^3)$ time. The reduction of the block matrix problem to normal form also takes $O(m^3)$. Hence, Algorithm 3 takes $O(F + m^3)$ time. $\square$

Remark 4.12 We mention some of the known results on the time complexity for computing the persistent homology of a filtration.

That is, we assume that the filtration is already given. However, we note that a naive construction (such as the full Vietoris-complex) may give a filtration with number of simplices exponential in the number of input points, and thus additional care is needed.

Let n be the number of simplices in the filtration. It is well-known that the standard algorithm [25] is $O(n^3)$, which is sharp by [36].

The work [33] provides an algorithm with complexity $O(n^\omega)$ (i.e. matrix multiplication time).

Furthermore, there are algorithms proposed that give heuristic speed-ups and reductions in filtration sizes, for example using the “clearing” (“twist”) optimization [19], taking advantage of dualities [22, 23], computing in chunks [9], parallel algorithm [10], reductions using discrete Morse theory [34], apparent and emergent pairs [6], using matroids [29], and so on.

As long as the needed information (Remark 4.9) is computed or can be reconstructed, these speed-ups can be used for our Algorithm.

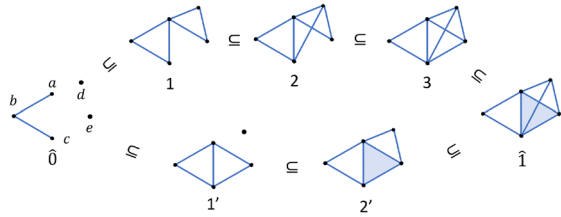
See the survey papers [31, 37] for further details on the computation of persistent homology.

Remark 4.13 We provide an implementation for Algorithm 3 in <https://github.com/ShunsukeTada1357/Bipathposets>.

5 Interpretation of bipath persistence diagrams

In this section, we illustrate the decomposition of a bipath persistence module obtained from a small example of a bipath filtration of simplicial complexes. In addition, we visualize the bipath persistence module (see Sect. 2.2.2) and its persistence diagram and give an interpretation for the visualization.

Fig. 2 A geometric realization of a bipath filtration S



5.1 Simple example

Consider an abstract simplicial complex Δ whose j -faces are given by

$$\Delta^0 := \{a, b, c, d, e\},$$

$$\Delta^1 := \{\{a, b\}, \{a, c\}, \{a, d\}, \{a, e\}, \{b, c\}, \{c, d\}, \{c, e\}, \{d, e\}\},$$

$$\Delta^2 := \{\{a, c, e\}\},$$

and a $B_{3,2}$ -filtration

$$S: \begin{array}{ccccccc} & & S_1 & \hookrightarrow & S_2 & \hookrightarrow & S_3 \\ & \nearrow & & & & & \searrow \\ S_0 & & & & & & S_{\hat{1}} \\ & \searrow & S_{1'} & \longrightarrow & S_{2'} & & \nearrow \end{array}$$

for Δ defined as follows (This is illustrated in Fig. 2): Let $S_0 := \Delta^0 \cup \{\{a, b\}, \{b, c\}\}$ and $S_{\hat{1}} := \Delta$. In addition, let

$$S_1 := S_0 \cup \{\{a, c\}, \{a, d\}, \{a, e\}, \{d, e\}\},$$

$$S_2 := S_1 \cup \{\{c, d\}\},$$

$$S_3 := S_2 \cup \{\{c, e\}\},$$

$$S_{1'} := S_0 \cup \{\{a, c\}, \{a, e\}, \{c, e\}\},$$

$$S_{2'} := S_{1'} \cup \{\{a, d\}, \{d, e\}, \{a, c, e\}\}.$$

We apply the q th homology functor with coefficient field $k = \mathbb{F}_2$ (a field with two elements) to S and obtain a bipath persistence module $V_q := H_q(S; \mathbb{F}_2)$ for each q .

$$\begin{array}{ccccccc} & & H_q(S_1; \mathbb{F}_2) & \longrightarrow & H_q(S_2; \mathbb{F}_2) & \longrightarrow & H_q(S_3; \mathbb{F}_2) \\ & \nearrow & & & & & \searrow \\ H_q(S_0; \mathbb{F}_2) & & & & & & H_q(S_{\hat{1}}; \mathbb{F}_2). \\ & \searrow & H_q(S_{1'}; \mathbb{F}_2) & \longrightarrow & H_q(S_{2'}; \mathbb{F}_2) & & \nearrow \end{array}$$

For computing its persistence diagrams, let us decompose it into interval modules by using Algorithm 2.

We clearly have $V_q = 0$ for all $q \geq 2$. For $q \in \{0, 1\}$, let $(V_q)_U$ and $(V_q)_D$ be restrictions of V_q to the upper path U and lower path D respectively. Then, we have persistence modules

$$X_q = \begin{cases} k_{[\hat{0}, \hat{0}]} \oplus k_{[\hat{0}, \hat{0}]} \oplus k_{[\hat{0}, \hat{1}]} & (q = 0), \\ k_{[1, \hat{1}]} \oplus k_{[1, \hat{1}]} \oplus k_{[2, \hat{1}]} \oplus k_{[3, 3]} & (q = 1), \end{cases} \text{ and}$$

$$Y_q = \begin{cases} k_{[\hat{0}, \hat{0}]} \oplus k_{[\hat{0}, 1']} \oplus k_{[\hat{0}, \hat{1}]} & (q = 0), \\ k_{[1', 1']} \oplus k_{[1', \hat{1}]} \oplus k_{[2', \hat{1}]} \oplus k_{[\hat{1}, \hat{1}]} & (q = 1), \end{cases}$$

over U and D respectively, and a pair of isomorphisms $\phi_q: (V_q)_U \rightarrow X_q$ and $\psi_q: (V_q)_D \rightarrow Y_q$ such that they define the block matrix problem (Definition 4.6) associated with (ϕ_q, ψ_q) of the following form for $q = 0$ and $q = 1$ respectively.

$$\begin{array}{c} [\hat{0}, \hat{0}] [\hat{0}, \hat{0}] [\hat{0}, \hat{1}] \\ \begin{bmatrix} [\hat{0}, \hat{0}] \\ [\hat{0}, 1'] \\ [\hat{0}, \hat{1}] \end{bmatrix} \end{array} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } \begin{array}{c} [1, \hat{1}] [1, \hat{1}] [2, \hat{1}] \\ \begin{bmatrix} [1', \hat{1}] \\ [2', \hat{1}] \\ [\hat{1}, \hat{1}] \end{bmatrix} \end{array} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Solving this (i.e., computing its normal form) for each q , we obtain a collection of intervals of $B_{3,2}$ by Construction 4.8, which corresponds to the interval-decomposition of V_q . We conclude that

$$V_q \cong \begin{cases} k_{B_{3,2}} \oplus k_{\langle \hat{0}, \hat{0} \rangle} \oplus k_{\langle 1', \hat{0} \rangle} & (q = 0), \\ k_{\langle 1, 1' \rangle} \oplus k_{\langle 1, 2' \rangle} \oplus k_{\langle 2, \hat{1} \rangle} \oplus k_{\langle 3, 3 \rangle} \oplus k_{\langle 1', 1' \rangle} & (q = 1), \\ 0 & (q \geq 2), \end{cases}$$

and

$$\mathcal{D}_q(S) = \begin{cases} \{B_{3,2}, \langle \hat{0}, \hat{0} \rangle, \langle 1', \hat{0} \rangle\} & (q = 0), \\ \{\langle 1, 1' \rangle, \langle 1, 2' \rangle, \langle 2, \hat{1} \rangle, \langle 3, 3 \rangle, \langle 1', 1' \rangle\} & (q = 1), \\ \emptyset & (q \geq 2). \end{cases} \quad (5.1)$$

We describe the above bipath persistence diagrams in Fig. 3.

We remark that we may use Algorithm 3 for a bipath filtration S to obtain the persistence diagrams without explicitly computing $H_q(S; \mathbb{F}_2)$. Of course, it returns the same result as (5.1).

Below, we give an interpretation for the persistence diagram, using some geometric intuition. However, we warn that this is dependent on a choice of representative cycles for the homology classes, and thus the interpretation (via representative “holes”) is not unique, in principle.

- For $q = 0$, every connected component is born at $\hat{0}$. The interval $B_{3,2}$ corresponds to a connected component containing a which is alive until the end. Similarly, the

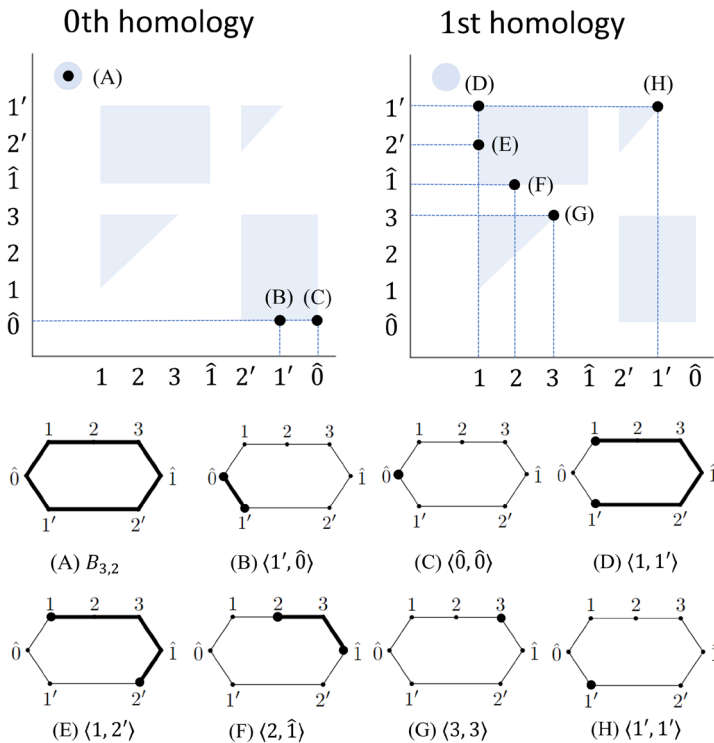


Fig. 3 A visualization of $\mathcal{D}_q(S)$ for $q \in \{0, 1\}$. Each point has the multiplicity one in this case

interval $\langle 1', \hat{0} \rangle$ (resp., $\langle \hat{0}, \hat{0} \rangle$) corresponds to the connected component of d (resp., e) which dies (gets connected to another connected component) at $1'$ (resp., $\hat{0}$). These are illustrated by the points (A), (B), and (C) in Fig. 3.

- For $q = 1$, the interval $\langle 1, 1' \rangle$ corresponds to the hole formed by a, b, c , which is born at both 1 (upper path) and $1'$ (lower path) and is alive until the end in both paths, illustrated by point (D) in Fig. 3. Similarly, the interval $\langle 1, 2' \rangle$ corresponds to a hole formed by a, d, e , born at 1 and $2'$ and is alive until the end in both paths; this is point (E) in Fig. 3.

The interval $\langle 2, \hat{1} \rangle$ corresponds to a hole formed by a, c, d , which is born at 2 (upper path) and $\hat{1}$ (lower path) and is alive until the end $\hat{1}$. Note that it is only present in the lower path at $\hat{1}$. This corresponds to point (F) in Fig. 3.

Both the interval $\langle 3, 3 \rangle$ (point (G) in Fig. 3) and the interval $\langle 1', 1' \rangle$ (point (H) in Fig. 3) can be interpreted as corresponding to the hole formed by a, c, e , which is born and dies at 3 and at $1'$ in the upper and lower path. Since they are not connected at the endpoints $\hat{0}$ and $\hat{1}$, these are two different intervals.

One motivation for our introduction of bipath persistence is that it allows us to study the persistence of topological features across a pair of filtrations to compare the two filtrations. In the above example, the intervals (A) and (D) point to the existence of topological features persisting for long across both upper and lower filtrations, while

intervals (B), (C), (G), and (H) may be seen as noise because of their short persistence. The intervals (E) and (F) correspond to topological features persisting for longer across the upper filtration than the lower filtration. This way, we can see the differences and compare topological features across two filtrations.

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Data availability We provide a code for Algorithm 3 in <https://github.com/ShunsukeTada1357/Bipathposets>.

Declarations

Conflict of interest The authors have no conflict of interest to declare that are relevant to the content of this article.

Ethical approval Not applicable.

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