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On Rothschild-Stiglitz's Characterization of Increasing Risk*

Koji Abe[a]

We give a proof for some part of a celebrated Rothschild and Stiglitz's Theorem on increasing in risk, which is more direct in a sense than the existing research has given. Specifically, we give a constructive proof for the fact that if two simple monetary lotteries have equal means and one dominates another in the second-order stochastic dominance, the dominated lottery is obtained from the dominant lottery by a sequence of mean-preserving spreads. Our construction of a sequence of mean-preserving spreads is indeed extremely simple and intuitive.

Keywords　Expected Utility, Risk, Mean Preserving Spread, Probability Shift

1 Introduction

In a celebrated paper Rothschild and Stiglitz (1970), the authors present one of the most important results in the expected utility theory of individual decision-making under risk: the characterization of increasing risk. They consider several formalizations regarding a monetary lottery being riskier than another lottery and prove their equivalence. Although they acknowledge in Rothschild and Stiglitz (1972) that they later realized that some of the equivalences had already been available in existing studies done outside of economics, but even so, it is worth praising that they bring the important result to the field of economics and give a considerably accessible and self-contained proof of the equivalence result of multiple formalizations.

In this note, we give a proof for some part of their theorem, which is more direct in a sense than the existing research has given. Specifically, we give a constructive proof for the fact that if two simple monetary lotteries have equal means and one dominates another in the second-order stochastic dominance, the dominated lottery is obtained from the dominant lottery by a sequence of mean-preserving spreads.[1)]

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Originally, the proof for the corresponding part of Rothschild and Stiglitz's theorem (Rothschild and Stiglitz, 1970, Lemma 1) has an error as pointed out in Rasmusen and Petrakis (1992) and Leshno, Levy and Spector (1997). These authors argue that the original theorem could still be proved correct with appropriate modifications to the proof. But, Deb and Seo (2011) disprove this. They instead argue that the part of the theorem is valid if the definition of mean-preserving spread is replaced with another definition and that one can prove it by appropriately and slightly modifying the technique that Leshno, Levy and Spector (1997) develop to try to prove the part of the theorem. Perhaps because of this tortuous background, the proposed proof requires combining the results of several papers and is somewhat less direct.

We instead provide a more direct constructive proof for the part of Rothschild and Stiglitz's theorem. The proposed way to construct a sequence of mean-preserving spreads from a lottery is extremely simple and intuitive.

Finally, we briefly mention about Machina and Pratt (1997) and Müller and Stoyan (2002). Machina and Pratt (1997) formalize a concept of mean-preserving spread, which is more general than Rothschild and Stiglitz (1970), in terms of cumulative distribution functions, and then prove the corresponding part of Rothschild and Stiglitz's theorem based on their general definition by the successful use of convex analysis techniques. Müller and Stoyan (2002) formalize a concept of local mean-preserving spread and prove that the corresponding part of Rothschild and Stiglitz's theorem holds even if we adopt this very particular form of mean-preserving spread. In Section 4, we briefly compare, by an example, how the sequences of mean-preserving spreads constructed in the corresponding part of Rothschild and Stiglitz's theorem differ between these papers and this paper.

2 A Critical Part of Rothschild and Stiglitz's Theorem

2.1 Setup and Notation

In this note, we focus on simple monetary lotteries as choice objects because they facilitate understanding of the essence of Rothschild and Stiglitz's theorem without advanced mathematical knowledge. A simple monetary lottery corresponds to a finite probability distribution over reals. Let p be a simple monetary lottery. We denote by $p(z)$ the probability that the prize z is realized according to p and by $F_p(z)$ the cumulative probability for the prize z. Let $\mathrm{supp}(p)$ be the support of p, that is, $\mathrm{supp}(p)=\{z\in\mathbb{R} \mid p(z)>0\}$. For two lotteries p and q, we will later abuse the notation $\mathrm{supp}(p, q)$ to mean $\mathrm{supp}(p)\cup\mathrm{supp}(q)$. With these notations, we have $F_p(z)=\sum_{z'\in\mathrm{supp}(p)\cap(-\infty,z]}p(z')$. Let $I_p(z)$ be the corresponding integrated cumulative probability (or the second-order cumulative probability) for the prize z:

$$I_p(z)=\int_{-\infty}^{z} F_p(x)dx\left(=\int_{\min \mathrm{supp}(p)}^{z} F_p(x)dx\right).$$

Note that if $\mathrm{supp}(p)=\{z_1, z_2, \ldots, z_N\}$ for some $N\geq 2$ and for all $n=1,\ldots, N-1$, $z_n<z_{n+1}$, then we have $I_p(z_n)=\sum_{i=1}^{n-1}F_p(z_i)(z_{i+1}-z_i)$ for all $n=1, 2, \ldots, N$, where the empty sum is 0 by convention. Note then from a well-known summation-by-parts method that $I_p(z_n)=z_n-\mathbb{E}_p[\mathrm{wins}(\tilde{z}\mid\tilde{z}\leq z_n)]$, where $\mathbb{E}_p$ means the expectation operation with respect to random variable $\tilde{z}$ according to p and $\mathrm{wins}(\tilde{z}\mid\tilde{z}\leq z_n)$ is the random variable that is obtained from $\tilde{z}$ by winsorizing on upper side of prize z_n, that is, $\mathbb{E}_p[\mathrm{wins}(\tilde{z}\mid\tilde{z}\leq z_n)]=p(z_1)z_1+\cdots+p(z_{n-1})z_{n-1}+\{p(z_n)+\cdots+p(z_N)\}z_n$. In particular, $I_p(z_1)=0$ and $I_p(z_N)=z_N-\mathbb{E}_p[\tilde{z}]$.

2.2 Increasing in Risk: Second-Order Stochastic Dominance

Definition 1. Let p and q be simple monetary lotteries. Then, we say that q second-order stochastically dominates p if, for all $z\in\mathbb{R}$, $I_q(z)\leq I_p(z)$. We also say that q strictly second-order stochastically dominates p if q second-order stochastically dominates p and there exists $z\in\mathbb{R}$ such that $I_q(z)<I_p(z)$ (or equivalently $p\neq q$).

This states that q is smaller than p in terms of integrated cumulative probabilities. Since $I_q(z_n)\leq I_p(z_n)$ is equivalent to $\mathbb{E}_p[\mathrm{wins}(\tilde{z}\mid\tilde{z}\leq z_n)]\leq\mathbb{E}_q[\mathrm{wins}(\tilde{z}\mid\tilde{z}\leq z_n)]$ as seen before, it means that p is smaller than q in the sense that the expected value for p is lower than the one for q even if we winsorize on any upper side. In addition this, if we impose the so-called the same mean condition in order to minimize the effect that one lottery is more likely to produce higher prizes than the other and compare purely the size of risk, then we obtain the following formalization regarding increasing in risk.

Definition 2. We say that p has (strictly) more risk than q in mean-preserving second-order stochastic dominance when p and q have the same mean and q (strictly) second-order stochastically dominates p.

In what follows, we will mention about a simple characterization of second-order stochastic dominance. Note that for all lottery r whose support is included in $\{z_1, z_2, \ldots, z_N\}$, we have the following:

$$I_r(z)=\begin{cases}0 & \text{if } z\leq z_1,\\ I_r(z_n)+F_r(z_n)(z-z_n) & \text{if } z\in(z_n, z_{n+1}] \text{ and } n=1, \ldots, N-1,\\ I_r(z_N)+z-z_N & \text{if } z>z_N.\end{cases}$$

That is, I_r is an increasing piecewise linear (convex) function whose kink points are included in $\{z_1, z_2, \ldots, z_N\}$. It will then be easily verified that whether or not there is a second-order stochastic dominance relationship between two lotteries is simply confirmed by looking at the corresponding integrated cumulative probabilities for the prizes included in the union of their supports. We hence obtain the following.[2)]

Lemma 1. Let p and q be simple monetary lotteries with $\mathrm{supp}(p, q) = \{z_1, z_2, \ldots, z_N\}$, where $N \geq 2$ and for all $n = 1, \ldots, N-1$, $z_n < z_{n+1}$. Then:

(i) q second-order stochastically dominates p if and only if, for all $n = 2, \ldots, N$, $I_q(z_n) \leq I_p(z_n)$.[3)]

(ii) q strictly second-order stochastically dominates p if and only if, for all $n = 2, \ldots, N$, $I_q(z_n) \leq I_p(z_n)$ and there is an $m \geq 2$ such that $I_q(z_m) < I_p(z_m)$ (or equivalently, $p \neq q$).

Proof. This is a well-known fact, so proof is omitted. □

In the next subsection, according to Rothschild and Stiglitz (1970), we show that the fact that p and q are related as in Definition 2 can be paraphrased by saying that p is obtained from q by a simple procedure of shifting probabilities, which gives a critical part of Rothschild and Stiglitz's theorem.

2.3 Increasing in Risk: Mean Preserving Spread

Definition 3. Let p and q be simple monetary lotteries with $\mathrm{supp}(p, q) = \{z_1, z_2, \ldots, z_N\}$, where $N \geq 2$ and for all $n = 1, \ldots, N-1$, $z_n < z_{n+1}$.

(a) We say that p is an elementary mean-preserving spread of q if there exist three indexes l, m, h with $l < m < h$ and three numbers $s(z_l), s(z_m), s(z_h)$ such that:

(1) $s(z_l) \geq 0, s(z_m) \leq 0, s(z_h) \geq 0$ and $s(z_l) + s(z_m) + s(z_h) = 0$,

(2) $s(z_l)z_l + s(z_m)z_m + s(z_h)z_h = 0$,

(3) $p(z) = q(z) + s(z)$ for $z = z_l, z_m, z_h$, and otherwise $p(z) = q(z)$.

The three numbers $s(z_l), s(z_m), s(z_h)$ are called the probability shifts or transfers. If p is an elementary mean-preserving spread of q and the probability shifts are not zero vector, the spread is called non-null.[4)]

(b) We say that p is a simple mean-preserving spread of q if p can be obtained from q by a finite sequence of elementary mean-preserving spreads. If furthermore, there is a non-null elementary mean-preserving spread in the sequence, we say that p is a non-null simple mean-preserving spread of q.

(c) If p can be obtained from q by a finite sequence of elementary mean-preserving spreads and furthermore the support of any lottery comprising that sequence is contained in supp(p, q), then p is said to be a simple mean-preserving spread of q in their supports.

For non-null spreads, we define in the same manner as in (b).

This is a formalization regarding increasing in risk in the sense that p has more weight in the tails than q with the same mean.[5] Originally, Rothschild and Stiglitz (1970) defined elementary mean-preserving spread in a slightly different way. The above definition (a) was introduced by Rasmusen and Petrakis (1992), and also independently by Deb and Seo (2011), for the purpose of giving a correct proof for Rothschild and Stiglitz's theorem. See some related notes in Sections 2.4 and 3 for a few twists and turns until a correct proof of the theorem is given. The definition (b) was introduced by Rothschild and Stiglitz (1970). The definition (c) is a slightly strict version of the definition (b).

In what follows, according to Rothschild and Stiglitz (1970), we give a characterization of mean-preserving second-order stochastic dominance in terms of mean-preserving spread.

Suppose that p is an elementary mean-preserving spread of q with the probability shifts $s(z_l)$, $s(z_m)$, $s(z_h)$. Then, we can easily verify from the conditions (2) and (3) in Definition 3 (a) that p and q have the same mean. In addition this, by imposing the condition (1), we reach the fact that q second-order stochastically dominates p. To see it, consider a nontrivial case where the spread is non-null. We show that q strictly second-order stochastically dominates p. Remember the functional form of the integrated cumulative probability function, and observe the following:

$$I_p(z_n) = \begin{cases} I_q(z_n) & \text{if } n \leq l, \\ I_q(z_n) + s(z_l)(z_n - z_l) & \text{if } l < n \leq m, \\ I_q(z_n) + s(z_l)(z_n - z_l) + s(z_m)(z_n - z_m) & \text{if } m < n \leq h, \\ I_q(z_n) + s(z_l)(z_n - z_l) + s(z_m)(z_n - z_m) + s(z_h)(z_n - z_h) & \text{if } n > h. \end{cases}$$

Then, by carefully comparing $I_p(z_n)$ and $I_q(z_n)$, we have $I_q(z_n) \leq I_p(z_n)$ in all cases and a strict inequality in some case.[6] Therefore, from Lemma 1, q strictly second-order stochastically dominates p. To sum up, if p is an (non-null) elementary mean-preserving spread of q, then q (strictly) mean-preserving second-order stochastically dominates p. The next theorem shows a kind of converse result, which forms a stronger version of a part of Rothschild and Stiglitz's theorem and becomes a critical part of the theorem.[7]

Theorem 1. Let p and q be simple monetary lotteries. If p has (strictly) more risk than q in mean-preserving second-order stochastic dominance, then p is a (non-null) simple mean-preserving spread of q in their supports.

Proof. See Appendix. □

2.4 Relation to Lemma 1 in Rothschild and Stiglitz (1970)

Consider the following.

Theorem 2. Let p and q be simple monetary lotteries. Then:

(A) If p has (strictly) more risk than q in mean-preserving second-order stochastic dominance, then p is a (non-null) simple mean-preserving spread of q in the sense of Rothschild and Stiglitz (1970).

(B) If p has (strictly) more risk than q in mean-preserving second-order stochastic dominance, then p is a (non-null) simple mean-preserving spread of q.

Theorem 2(A) is a part of the original Rothschild and Stiglitz's theorem (Rothschild and Stiglitz, 1970, Lemma 1) and Theorem 2(B) is Lemma 1* in Deb and Seo (2011). We briefly compare Theorem 2(A) and Theorem 2(B), followed by Theorem 2(B) and Theorem 1.

First, the difference between Theorems 2(A) and 2(B) is the adopted definitions of elementary mean-preserving spread. The one in Definition 3 is adopted in Theorem 2(B), while another definition is adopted in Theorem 2(A). Rasmusen and Petrakis (1992) refer to the former spread as 3pt mean-preserving spread because it is defined using probability shifts for three prizes, and in their terminology, the original spread by Rothschild and Stiglitz (1970) is 4pt mean-preserving spread.[8] It is known from Theorem 1 in Rasmusen and Petrakis (1992) and also Proposition 1 in Deb and Seo (2011) that if p is a 4pt mean-preserving spread of q, then p is a simple mean-preserving spread of q in the sense of 3pt mean-preserving spread, on the other hand, it is also known from Example 1 in Deb and Seo (2011) that just because p is a 3pt mean-preserving spread of q does not guarantee that p is a simple mean-preserving spread in the sense of 4pt mean-preserving spread.[9] Thus, if Theorem 2(A) is true, then Theorem 2 (B) is also true, but unfortunately, Theorem 2(A) is not true in general as demonstrated with Example 1 in Deb and Seo (2011).

Second, Theorem 1 and Theorem 2(B) differ in that, in the former, one can successfully get a sequence of elementary mean-preserving spreads within the support of p and q, while the latter does not preclude taking a sequence from outside the support.[10] Since Theorem 1 makes a stronger claim, we obtain Theorem 2(B) as its corollary. The stronger point of Theorem 1 appears in the way of construction of a sequence of elementary mean-preserving spreads. In the proof of the theorem, we construct a desired sequence as follows. We set $r_0=q$ first and then construct r_m using the next inductive procedure until it becomes p. For each m, we just take the three smallest prizes $z<z'<z''$ that satisfy $p(z)<r_m(z)$, $p(z')>r_m(z')$, $p(z'')<r_m(z'')$ and make the probability transfers among them according to the definition of elementary mean-pre-

serving spread. This procedure is indeed extremely simple and intuitive.

3 Multiple equivalence result of Rothschild and Stiglitz (1970)

We state the so-called Rothschild and Stiglitz's Theorem on increasing in risk formally.[11)]

Theorem 3 (Rothschild and Stiglitz's Theorem). Let p and q be simple monetary lotteries. Then, the following are equivalent:

(1) Every risk averse individual (and at least one strictly) prefers q to p.

(2) p has (strictly) more risk than q in mean-preserving second-order stochastic dominance.

(3) p is a (non-null) simple mean-preserving spread of q *in their supports*.

(4) p is a (non-null) simple mean-preserving spread of q.

(5) p is obtained by applying an (non-null) unbiased stochastic replacement of prizes to q.

(6) p is obtained from q by adding (nontrivial) zero-conditional-mean noise.

Each statement gives a formalization regarding p being riskier than q. The first one is a choice-based formalization. The second is based on a dominance relationship between the distributions. The third and fourth are based on procedural aspects of intuitive probability transfers. The fifth is based on the manipulative perspective of replacing prizes. The sixth is based on the perspective that noise creates risk. Here, we briefly mention about the fifth and the sixth. They are formally stated as follows.

Definition 4. Let p and q be simple monetary lotteries. We say that p is obtained by applying an unbiased stochastic replacement of prizes to q if there exists a function $M: \mathrm{supp}(q) \times \mathrm{supp}(p) \to [0, 1]$ such that (1) for all $z \in \mathrm{supp}(q)$, $M(\cdot \mid z) \in \Delta(\mathrm{supp}(p))$, (2) for all $z' \in \mathrm{supp}(p)$, $\sum_{z \in \mathrm{supp}(q)} M(z' \mid z) q(z) = p(z')$, and (3) for all $z \in \mathrm{supp}(q)$, $\sum_{z' \in \mathrm{supp}(p)} z' M(z' \mid z) = z$. If further there is $z \in \mathrm{supp}(q)$ such that for all $z' \in \mathrm{supp}(p)$, $M(z' \mid z) < 1$, then we say that the replacement is non-null.

Definition 5. Let p and q be simple monetary lotteries. We say that p is obtained from q by adding zero-conditional-mean noise if F_q and F_p are the cumulative distribution functions corresponding to the random variables $\tilde{z}$ and $\tilde{z} + \tilde{\varepsilon}$ such that the conditional expectation of $\tilde{\varepsilon}$ given $\tilde{z}$ is equal to zero with probability 1. If further there exists a positive probability event of $\tilde{z}$ such that $\tilde{\varepsilon}$ is a non-degenerate random variable conditional on the event, then we say that noise is non-trivial.

Definition 4(1) says that the function M is a $\mathrm{supp}(q) \times \mathrm{supp}(p)$ row-stochastic transition matrix and Definition 4(2) says that p is obtained by applying the transition to q. That is, we can interpret M as it generates p by stochastically replacing each prize z of q to another prize according to the probability distribution $M(\cdot \mid z)$. Definition 4(3) then says that the replacement is unbiased in the sense that it does not change prizes as expected values.[12] Definition 5 is a paraphrase of Definition 4 as shown by Strassen (1965) and independently Rothschild and Stiglitz (1970), which characterizes the stochastic replacement by adding zero-conditional-mean noise.[13]

Now we give a comment on Theorem 3. Rothschild and Stiglitz's original theorem (Rothschild and Stiglitz, 1970, Theorem 2) is the theorem stating the equivalence among the above (1), (2), and (6). However, if we examine the theorem in detail to its proof, we can see that it is the above theorem where the conditions (3) and (4) are replaced with the statement in Theorem 2(A), just say (A). Rothschild and Stiglitz offer a self-contained proof of the theorem that proves the cycle of implications that (1) ⇒ (2), (2) ⇒ (A), (A) ⇒ (5), (5) ⇒ (6), and (6) ⇒ (1). Unfortunately, the proof of (2) ⇒ (A) (Rothschild and Stiglitz, 1970, Lemma 1) has an error as pointed out in Rasmusen and Petrakis (1992) and Leshno, Levy and Spector (1997). Rasmusen and Petrakis (1992) and Leshno, Levy and Spector (1997) argue that the original theorem could still be proved correct with appropriate modifications to the proof, but Deb and Seo (2011) disprove this and instead argue that Rothschild and Stiglitz's theorem is valid if the condition (A) is replaced with the statement in Theorem 2(B), that is, (4). They argue that one can prove (2) ⇒ (4) by appropriately and slightly modifying the technique that Leshno, Levy and Spector (1997) develop to try to prove (2) ⇒ (A) and further show that (4) indeed implies (5) in a similar fashion to the proof of (A) ⇒ (5) in Rothschild and Stiglitz (1970), which recover Rothschild and Stiglitz's cycle of implications.

We instead recover Rothschild and Stiglitz's cycle of implications by proving the above form of theorem. We prove (2) ⇒ (3) in Theorem 1 and, as mentioned in Section 2.4, provide an extremely simple and intuitive method of constructing a sequence of elementary mean-preserving spreads from p to q. Then, since it is obvious that (3) ⇒ (4), we provide a self-contained alternative proof for Rothschild and Stiglitz's Theorem.

4 Comparison to "Slicing" Procedures

In this section, we consider simple monetary lotteries whose supports are included in a closed interval $[\underline{z}, \bar{z}]$. Machina and Pratt (1997) say that p is a mean-preserving spread of q if p and q have the same mean and there exist $z', z'' \in [\underline{z}, \bar{z}]$ such that $z' \leq z''$ and p has no greater probability than q to any open subinterval of (z', z'') and p has no smaller probability than q to any open interval either to the left or the right of (z', z'').[14] We can easily verify that if p is an elementary mean-preserving spread of q, then p is a mean-preserving spread of q in the sense of Machina and Pratt (1997).[15] Hence, this is a general formalization of mean-preserving spread that can be applied regardless of whether lotteries are discrete or not.

Machina and Pratt (1997) show that if p has more risk than q in mean-preserving second-order stochastic dominance, then p is a simple mean-preserving spread of q in the sense of the above-mentioned general definition. The key to construct a sequence of mean-preserving spreads is to "slice" an integrated cumulative probability function in order to generate a new integrated cumulative probability function that is mean-preserving spread of the original function. Recall that $I_q(z)$ is an increasing convex function. Hence, we can "slice" this using an affine function L with a positive slope, in the sense of setting for all z, $\hat{I}(z)=\max\{I_q(z), L(z)\}$, and generate a new increasing convex function $\hat{I}$ that satisfies $\hat{I}(\underline{z})=I_q(\underline{z})$ and $\hat{I}(\bar{z})=I_q(\bar{z})$. Assume that L slices I_q at prizes z' and z'' with $z'<z''$. Then, we can easily verify that $\hat{I}$ corresponds to an integrated cumulative probability function for a lottery $\hat{q}$ that is generated from q by shifting all the probability mass from the interval (z', z'') out to its end points while keeping the mean. In other wards, $\hat{I}$ is an integrated cumulative probability function for $\hat{q}$ that is a mean-preserving spread of q in the sense of Machina and Pratt (1997). Based on this observation, Machina and Pratt (1997) takes L described above as a tangent line supporting I_p from below and sets $q_1=\hat{q}$. Repeating this "slicing" procedure, we obtain a desired sequence of mean-preserving spreads that converges to p under the assumption that p has more risk than q in mean-preserving second-order stochastic dominance.

The next example shows an outcome of mean-preserving spreads obtained by a particular way of "slicing" procedure.[16] Let p^* be a simple monetary lottery such that $p^*(0.1)=0.1$, $p^*(0.6)=0.1$, $p^*(0.9)=0.1$, and $p^*(1.0)=0.7$. Let also q^* be a simple monetary lottery such that $q^*(0.3)=0.1$, $q^*(0.5)=0.1$, $q^*(0.8)=0.1$, and $q^*(1.0)=0.7$. We can easily verify that p^* has more risk than q^* in mean-preserving second-order stochastic dominance.[17] Observe then that q_1^{MP} with $q_1^{MP}(0.1)=0.1$, $q_1^{MP}(0.7)=0.1$, $q_1^{MP}(0.8)=0.1$, and $q_1^{MP}(1.0)=0.7$ is obtained by a particular way of "slicing" procedure described below. We first find the lowest prize such that $p^*(z)>q^*(z)$. That is, $z=0.1$. We then take a supporting hyperplane L_1 of I_{p^*} at $z=0.1$ such that its slope equals to the right derivative of I_{p^*} at $z=0.1$. Since L_1 slices I_{q^*} at prizes $z=0.1$ and $z=0.7$, we shift all the probability mass $q^*(0.3)+q^*(0.5)$ in the interval $(0.1, 0.7)$ from q^* to the end points so as not to change the mean and obtain q_1^{MP}. It is then easily verified that the same procedure generates q_1^{MP} to p^*.[18]

Müller and Stoyan (2002) introduce a concept of local mean-preserving spread, which is a special form of elementary mean-preserving spread, and show that if p has more risk than q in mean-preserving second-order stochastic dominance, then p is a simple local mean-preserving spread of q. Let $\mathrm{supp}(p, q) = \{z_1, z_2, \ldots, z_N\}$, where $N \geq 2$ and for all $n = 1, \ldots, N-1$, $z_n < z_{n+1}$. Suppose that p is an elementary mean-preserving spread of q with the probability shifts $s(z_l)$, $s(z_m)$, $s(z_h)$ in the sense of Definition 3. According to Müller and Stoyan (2002), if the probability shifts satisfy $l = m-1$, $h = m+1$, and $q(z_m) + s(z_m) = 0$, then we say that p is a local mean-preserving spread of q. That is, the fact that p is a local mean-preserving spread of q corresponds to the fact that p is obtained from q by shifting all the probability mass on z_m to the neighboring prizes on both sides in the support without changing the mean.

We demonstrate the result of Müller and Stoyan (2002) by using p^* and q^*, which appeared in the third paragraph in this section. Consider three lotteries q_1^{MS}, q_2^{MS}, and q_3^{MS} described below. First, q_1^{MS} is such that $q_1^{MS}(0.1) = 0.05$, $q_1^{MS}(0.5) = 0.15$, $q_1^{MS}(0.8) = 0.1$, $q_1^{MS}(1.0) = 0.7$. Second, q_2^{MS} is such that $q_2^{MS}(0.1) = 0.1$, $q_2^{MS}(0.7) = 0.1$, $q_2^{MS}(0.8) = 0.1$, $q_2^{MS}(1.0) = 0.7$. Third, q_3^{MS} is such that $q_3^{MS}(0.1) = 0.1$, $q_3^{MS}(0.6) = 0.05$, $q_3^{MS}(0.8) = 0.15$, $q_3^{MS}(1.0) = 0.7$. Then, we can observe that q_n^{MS} is a local mean-preserving spread of q_{n-1}^{MS} for all $n = 1, \ldots, 4$, where $q_0^{MS} = q^*$ and $q_4^{MS} = p^*$. In fact, this is nothing more than a sequence of lotteries that is obtained by decomposing the slicing procedure implemented in the third paragraph in this section into multiple steps to fit the local mean-preserving spread conditions.[19)]

Now we demonstrate Theorem 1 in this note with p^* and q^* and give a final comment. Recall that this note proposes a simple inductive procedure to obtain a sequence of mean-preserving spreads. We set $q_0 = q$ first and then construct q_m using the next inductive procedure until it becomes p. For each m, we just take the three smallest prizes $z < z' < z''$ that satisfy $p(z) < q_m(z)$, $p(z') > q_m(z')$, $p(z'') < q_m(z'')$ and make the probability transfers among them according to the definition of elementary mean-preserving spread. The sequence of lotteries obtained in this manner is as follows. First, we obtain q_1 with $q_1(0.1) = \frac{3}{50}$, $q_1(0.5) = 0.1$, $q_1(0.6) = \frac{2}{50}$, $q_1(0.8) = 0.1$, $q_1(1.0) = 0.7$. Second, we obtain q_2 with $q_2(0.1) = \frac{3}{40}$, $q_2(0.5) = \frac{1}{40}$, $q_2(0.6) = 0.1$, $q_2(0.8) = 0.1$, $q_2(1.0) = 0.7$. Third, we obtain q_3 with $q_3(0.1) = \frac{7}{80}$, $q_3(0.6) = 0.1$, $q_3(0.8) = 0.1$, $q_3(0.9) = \frac{1}{80}$, $q_3(1.0) = 0.7$. And, finally, we obtain $q_4 = p^*$. The crucial difference between $(q_n)_{n=0}^4$ and $(q_n^{MP})_{n=0}^2$ or $(q_n^{MS})_{n=0}^4$ is that $(q_n)_{n=0}^4$ does not rely on a slicing procedure and we construct them by just adjusting the probabilities within the support of p and q. As a result, one can successfully get a sequence of elementary mean-preserving spreads within the support of p and q. This gives a simple and alternative algorithm for generating a sequence of mean-preserving spreads.

Appendix: Proof of Theorem 1

We below prove the nontrivial case that if p has strictly more risk than q in second-order stochastic dominance, then p is a non-null simple mean-preserving spread of q in their supports. Let $\mathrm{supp}(p, q)=\{z_1, z_2, \ldots, z_N\}$, where $N\geq 2$ and for all $n=1, \ldots, N-1$, $z_n<z_{n+1}$.

We below show that there exists a number $M\in\mathbb{N}$ and a sequence of simple monetary lotteries $\{r_m\}_{m=0}^{M}$ such that (1) $r_m\in\Delta(\mathrm{supp}(p, q))$ for all $m=0, 1, \ldots, M$, (2) $r_0=q$, (3) $r_M=p$, and (4) r_{m+1} is a non-null elementary mean-preserving spread of r_m for all $m=0, 1, \ldots, M-1$.

Let $r_0=q$. Below, we will construct a sequence of simple monetary lotteries $\{r_m\}_{m=1}^{M}$ and define a sequence of sets consisting of triplets of natural numbers, T_m for $m=0, 1, \ldots, M$, as follows:

$$T_m=\{(i, j, k)\mid i<j<k,\ r_m(z_i)<p(z_i),\ r_m(z_j)>p(z_j),\ r_m(z_k)<p(z_k)\}.$$

Obviously, whatever $\{r_m\}_{m=0}^{M}$ was, T_m is a finite set for all m.

We first claim the following:

Claim 1. $T_0\neq\emptyset$ and furthermore $n<\min\{i\mid r_0(z_i)<p(z_i)\}$ implies $r_0(z_n)=p(z_n)$.

Proof. Step 1: We show that there exists an index $i\in\{1, \ldots, N-2\}$ such that $r_0(z_i)<p(z_i)$. If $r_0(z_i)\geq p(z_i)$ for all $i\in\{1, \ldots, N-2\}$, then we have $F_{r_0}(z_i)\geq F_p(z_i)$ for all $i\in\{1, \ldots, N-2\}$, which in turn implies $I_{r_0}(z_{i+1})\geq I_p(z_{i+1})$ for all $i\in\{1, \ldots, N-2\}$ and hence, together with $I_{r_0}(z_1)=I_p(z_1)=0$ and $I_{r_0}(z_N)=I_p(z_N)$, we have $I_{r_0}(z_j)\geq I_p(z_j)$ for all $j\in\{1, \ldots, N\}$, which contradicts to r_0 dominating p in strict second-order stochastic dominance.

Step 2: By Step 1, we can take $i^*=\min\{i\mid r_0(z_i)<p(z_i)\}$. Then, we show that $i<i^*$ implies $r_0(z_i)=p(z_i)$. First, by definition of i^*, we have $r_0(z_i)\geq p(z_i)$ for all $i<i^*$. Second, if there exists $i<i^*$ such that $r_0(z_i)>p(z_i)$, then we have $F_{r_0}(z_j)\geq F_p(z_j)$ for all $j<i^*$ and $F_{r_0}(z_i)>F_p(z_i)$. However, we then have $I_{r_0}(z_{i+1})>I_p(z_{i+1})$, which contradicts to r_0 dominating p in strict second-order stochastic dominance.

Step 3: We show that there exists $j\in\{i^*+1, \ldots, N\}$ such that $r_0(z_j)>p(z_j)$. If $r_0(z_j)\leq p(z_j)$ for all $j\in\{i^*+1, \ldots, N\}$, then by Step 1 and 2, we have $1=\sum_{m=1}^{N}r_0(z_i)<\sum_{m=1}^{N}p(z_i)=1$, a contradiction.

Step 4: We show $r_0(z_N)\leq p(z_N)$. Note that $F_{r_0}(z_{N-1})(z_N-z_{N-1})=I_{r_0}(z_N)-I_{r_0}(z_{N-1})\geq I_p(z_N)-I_p(z_{N-1})=F_p(z_{N-1})(z_N-z_{N-1})$, where the inequality follows from the fact that r_0 and p have the same mean and r_0 second-order stochastically dominates p. We hence have $F_{r_0}(z_{N-1})\geq F_p(z_{N-1})$, which is equivalent to $r_0(z_N)\leq p(z_N)$.

Step 5: We can take i^* (from Step 2) and j (from Step 3) such that $i^*<j<N$ (from Step 4), $r_0(z_{i^*})<p(z_{i^*})$, and $r_0(z_j)>p(z_j)$. Then, we show that there exists $k\in\{j+1, \ldots, N\}$ such that $r_0(z_k)<p(z_k)$. If $r_0(z_k)\geq p(z_k)$ for all $k\in\{j+1, \ldots, N\}$, then by Step 4 $r_0(z_N)=p(z_N)$. Then, in the same manner as Step 4, we can show $r_0(z_{N-1})\leq p(z_{N-1})$ and hence $r_0(z_{N-1})=p(z_{N-1})$. Repeating this procedure, we have $r_0(z_j)=p(z_j)$, which contradicts to $r_0(z_j)>p(z_j)$.

Step 6: We can take i^* (from Step 2), j (from Step 3), and k (from Step 5) such that $r_0(z_{i^*})<p(z_{i^*})$, $r_0(z_j)>p(z_j)$, and $r_0(z_k)<p(z_k)$. This guarantees $T_0\neq\emptyset$. □

We construct $\{r_m\}_{m=1}^{M}$. Thanks to Claim 1, T_0 is a nonempty set. Let $i^*=\min\{i\in\{1, \ldots, N\}\mid r_0(z_i)<p(z_i)\}$, $j^*=\min\{j\in\{1, \ldots, N\}\mid r_0(z_j)>p(z_j)\}$, and $k^*=\min\{k\in\{j^*+1, \ldots, N\}\mid r_0(z_k)<p(z_k)\}$. Then, we have $r_0(z_n)=p(z_n)$ if $n<i^*$, $r_0(z_n)\leq p(z_n)$ if $i^*<n<j^*$, and $r_0(z_n)\geq p(z_n)$ if $j^*<n<k^*$.

Now we construct r_1. First, we define r_1 by $r_1(z_n)=r_0(z_n)$ for all $n\neq i^*, j^*, k^*$. Second, we let $\alpha^*=\min\{\alpha_{i^*}, \alpha_{k^*}, 1\}$, where α_{i^*} and α_{k^*} are real numbers that satisfy the following equations, respectively:

$$r_0(z_{i^*})+\alpha_{i^*}\{r_0(z_{j^*})-p(z_{j^*})\}\frac{z_{k^*}-z_{j^*}}{z_{k^*}-z_{i^*}}=p(z_{i^*}),$$

$$r_0(z_{k^*})+\alpha_{k^*}\{r_0(z_{j^*})-p(z_{j^*})\}\frac{z_{j^*}-z_{i^*}}{z_{k^*}-z_{i^*}}=p(z_{k^*}).$$

We then define r_1 on $\{i^*, j^*, k^*\}$ as follows:

$$r_1(z_{i^*})=r_0(z_{i^*})+\alpha^*\{r_0(z_{j^*})-p(z_{j^*})\}\frac{z_{k^*}-z_{j^*}}{z_{k^*}-z_{i^*}},$$

$$r_1(z_{j^*})=r_0(z_{j^*})-\alpha^*\{r_0(z_{j^*})-p(z_{j^*})\},$$

$$r_1(z_{k^*})=r_0(z_{k^*})+\alpha^*\{r_0(z_{j^*})-p(z_{j^*})\}\frac{z_{j^*}-z_{i^*}}{z_{k^*}-z_{i^*}}.$$

By construction, $r_1(z_{i^*})\leq p(z_{i^*})$, $r_1(z_{j^*})\geq p(z_{j^*})$, and $r_1(z_{k^*})\leq p(z_{k^*})$, and we find that at least one inequality is, in fact, equality. We hence find that $T_1\subsetneq T_0$. Moreover, it is easily verified that r_1 is a non-null elementary mean-preserving spread of r_0.

We claim the following:

Claim 2. r_1 still second-order stochastically dominates p.

Proof. Suppose negatively that r_1 does not dominate p. That is, we assume that there is an n^* such that $I_{r_1}(z_{n^*})>I_p(z_{n^*})$. Take such an n^* arbitrarily. First, note from the construction of r_1 that $r_1(z_n)\leq p(z_n)$ for all $n<j^*$. Hence $F_{r_1}(z_n)\leq F_p(z_n)$ for all $n<j^*$. Then, since $I_{r_1}(z_n)-I_p(z_n)=\sum_{i=1}^{n-1}\{F_{r_1}(z_i)-F_p(z_i)\}(z_{i+1}-z_i)$ for all n, we have $I_{r_1}(z_n)-I_p(z_n)\leq 0$ for all $n\leq j^*$, which in tune implies $n^*>j^*$. Second, note that $I_{r_1}(z_n)=I_{r_0}(z_n)\leq I_p(z_n)$ for all $n\geq k^*$. This immediately implies $n^*<k^*$. We hence find that r_1 second-order stochastically dominates p for the case that $k^*=j^*+1$. In what follows, assume $k^*>j^*+1$.

Let $n^*=\min\arg\max_{n\in\{j^*+1,\dots,k^*-1\}}\{I_{r_1}(z_n)-I_p(z_n)\}$. Then, we have $I_{r_1}(z_{n^*})-I_p(z_{n^*})>I_{r_1}(z_{n^*-1})-I_p(z_{n^*-1})$. This implies $F_{r_1}(z_{n^*-1})(z_{n^*}-z_{n^*-1})=I_{r_1}(z_{n^*})-I_{r_1}(z_{n^*-1})>I_p(z_{n^*})-I_p(z_{n^*-1})=F_p(z_{n^*-1})(z_{n^*}-z_{n^*-1})$. We hence have $F_{r_1}(z_{n^*-1})>F_p(z_{n^*-1})$. Similarly, since $I_{r_1}(z_{n^*})-I_p(z_{n^*})\geq I_{r_1}(z_{n^*+1})-I_p(z_{n^*+1})$, we have $F_{r_1}(z_{n^*})\leq F_p(z_{n^*})$. Then, we have $p(z_{n^*})>r_1(z_{n^*})$, which contradicts to $n^*<k^*$. Therefore, we conclude that for all $n=1,\dots,N$, $I_{r_1}(z_n)\leq I_p(z_n)$. From Lemma 1, r_1 second-order stochastically dominates p. □

From Claim 2, we find that r_1 strictly second-order stochastically dominates p or $r_1=p$. If the latter holds, we set $M=1$ and the proof is done. Suppose that r_1 strictly second-order stochastically dominates p. Then, from Claim 1, T_1 is a nonempty set. We can do the same procedure to T_1 and obtain $r_2\in\Delta(\mathrm{supp}(p, r_1))$, and hence $r_2\in\Delta(\mathrm{supp}(p, q))$, such that r_2 is a non-null elementary mean-preserving spread of r_1 and still second-order stochastically dominates p. Continue this procedure until the resulting lottery is equal to p. Since $\{T_m\}_m$ is a decreasing sequence of finite sets, this is completed in a finite number of steps. We set M by the final number of steps and the proof is completed.

Notes

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1) More precisely, the claim to be proven is a slightly stronger one than what I have written here. See Section 2.4 for a comment about this.

2) For example, Hermalin and Katz (2001) adopt this characterization as its definition of second-order stochastic dominance.

3) Since $I_p(z_N)=z_N-\mathbb{E}_p[\tilde{z}]$ and $I_q(z_N)=z_N-E_q[\tilde{z}]$, the same mean condition can be rephrased as $I_p(z_N)=I_q(z_N)$. Hence, the fact that p has more risk than q in mean-preserving second-order stochastic dominance simply means for all $n=2,\ldots,N-1$, $I_q(z_n)\leq I_p(z_n)$ and $\mathrm{I}_q(z_N)=I_p(z_N)$, which is often called an integral condition.

4) Note that, under conditions (1) and (2), either all of $s(z_l)\geq 0$, $s(z_m)\leq 0$, $s(z_h)\geq 0$ are equal or all of them are strict inequalities. To see it, observe from (1) that at least one of $s(z_l)$ and $s(z_h)$ is positive if and only if $s(z_m)$ is negative. Assume hence $s(z_l)>0$ and $s(z_m)<0$ without loss of generality. If further $s(z_h)=0$, then by (1) we have $s(z_l)=-s(z_m)$ and this with (2) in turn implies $s(z_l)(z_l-z_m)=0$, which contradicts to $z_l<z_m$. Therefore, if any of $s(z_l)$, $s(z_m)$, $s(z_h)$ is positive, we have all of them are positive.

5) It is, however, important to note that p is not simply obtained by shifting a probability mass of the center of q to the tails because the condition of keeping the same mean has a strong influence on the shapes of the distributions. See the discussion about this in Section 3 in Landsberger and Meilijson (1993).

6) We have $I_q(z_n)=I_p(z_n)$ for $n\leq l$ and $I_q(z_n)<I_p(z_n)$ for n with $l<n\leq m$ because $s(z_l)(z_n-z_l)>0$. In the case that $m<n\leq h$, we have $s(z_l)(z_n-z_l)+s(z_m)(z_n-z_m)=\{s(z_l)+s(z_m)\}z_n-\{s(z_l)z_l+s(z_m)z_m\}=-s_hz_n+s(z_h)z_h=s(z_h)(z_h-z_n)>0$ and hence $I_q(z_n)\leq I_p(z_n)$ with equality if and only if $n=h$. Finally, in the case that $n>h$, we have $s(z_l)(z_n-z_l)+s(z_m)(z_n-z_m)+s(z_h)(z_n-z_h)=\{s(z_l)+s(z_m)+s(z_h)\}z_n-\{s(z_l)z_l+s(z_m)z_m+s(z_h)z_h\}=0$ and hence $I_q(z_n)=I_p(z_n)$.

7) At the time of writing this note, I found that Huang (2018) independently asserted a proposition equivalent to this theorem, while I could not find a proof in that. Here, I give a formal constructive proof and, in the proof, a simple and intuitive algorithm that makes a sequence of mean-preserving spreads from a lottery to another lottery.

8) p is said to be a 4pt mean-preserving spread of q if there exist four indexes l, L, h, H with $l<L<h<H$ and four numbers $s(z_l)$, $s(z_L)$, $s(z_h)$, $s(z_H)$ such that:

(1) $s(z_l)=-s(z_L)\geq 0$ and $s(z_H)=-s(z_h)\geq 0$,

(2) $s(z_l)z_l+s(z_L)z_L+s(z_h)z_h+s(z_H)z_H=0$,

(3) $p(z)=q(z)+s(z)$ for $z=z_l, z_L, z_h, z_H$, and otherwise $p(z)=q(z)$.

9) This difference can be resolved by generalizing the mean-preserving spread concepts in a natural way. For example, we can consider a concept of simple convex transfer in Müller (2013) as a more general spread concept that can treat 3pt and 4pt mean-preserving spreads in a unified manner. Actually, using the concepts that have appeared so far, the fact that p is obtained from q by a simple

convex transfer is restated as that p is a 3pt or 4pt mean-preserving spread of q.

10) This feature of Theorem 1 will also be discussed in Section 4.

11) Regarding with the equivalence of three or more multiple conditions on risk as in this theorem, there are two related studies that should be mentioned. First, Elton and Hill (1992) prove the theorem without (3) as their Theorem 4.7 but with condition (4) replaced by another condition that q is a fusion of p, where fusion is a roughly reverse concept of mean-preserving spread. This reflects that the fact that p can be obtained from q by mean-preserving spread means that q can be obtained from p by its reverse procedure. See also Definition 2.1 in Elton and Hill (1998) and Sherman (1951) in this respect. Elton and Hill (1992) introduce its theorem as a special case of a more general theorem in a general prize space. Thus, its proof is not as self-contained as the one given here. Second, Müller (2013) proves in a multi-dimensional prize setting the equivalence of (1), (3), and (6) by replacing the concept of mean-preserving spread of (3) with a more general spread concept called a general convex transfer. Furthermore, one can read the paper to suggest that in a one-dimensional prize setting, the general spread concept could be replaced by a weaker spread concept (but more general than the concept of elementary mean-preserving spread in this paper) called a simple convex transfer (see footnote 9). However, the proof given in Müller (2013) requires a considerably higher level of mathematical background than the proof treated here, since it makes use of duality in linear spaces.

12) Bayesian statistical decision theory with a mathematically more general structured setup treats the prizes in our setup as signals (or posterior beliefs) about some uncertainty. See, for example, Blackwell (1951, 1953) and Blackwell and Girshick (1979) as representative references on this.

13) When p is obtained from q by adding zero-conditional-mean noise, the ordered pair (p, q) is also said to admit a martingale coupling or simply be martingalizable. See Leskelä and Vihola (2017) and Elton and Hill (1992) for these terminologies.

14) Equivalently, $F_p(z)-F_q(z)$ is nondecreasing on $(-\infty, z')$ and (z'', ∞) and nonincreasing on (z', z'').

15) Suppose that p is a non-null elementary mean-preserving spread of q with the probability shifts $s(z_l)$, $s(z_m)$, $s(z_h)$. Then, $z<z_l$ or $z\geq z_h$ implies $F_p(z)-F_q(z)=0$, $z\in[z_l, z_m]$ implies for $F_p(z)-F_q(z)=s(z_l)>0$, and $z\in[z_m, z_h]$ implies $F_p(z)-F_q(z)=s(z_l)+s(z_m)<0$. Hence, $F_p(z)-F_q(z)$ is nondecreasing on $(-\infty, z')$ and (z'', ∞) and nonincreasing on (z', z'') for some $z'\in(z_l, z_m)$ and $z''\in(z_m, z_h)$. Similarly, we can easily verify that if p is a mean-preserving spread of q in the sense of Rothschild and Stiglitz (1970), then p is a mean-preserving spread in the sense of Machina and Pratt (1997).

16) Note that there are many ways to slice an integrated cumulative probability function.

17) In fact, this example is used in Leshno, Levy and Spector (1997) to illustrate Rothschild and Stiglitz's mean-preserving spread.

18) The particular way of "slicing" procedure employed here differs from the specific way employed by Machina and Pratt (1997) to prove their theorem for general distributions. We adopt this particular way in order to take full advantage of discrete distributions and create a shorter sequence of

mean-preserving spreads.

19) Note that $q_2^{MS}=q_1^{MP}$.

References

Blackwell, David (1951) "Comparison of Experiments," in Neyman, Jerzy ed. *Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability*, pp. 93–102, Berkeley, CA: University of California Press.

——— (1953) "Equivalent Comparisons of Experiments," *Annals of Mathematical Statistics*, Vol. 24, No. 2, pp. 265–272.

Blackwell, David and Meyer A. Girshick (1979) *Theory of Games and Statistical Decisions*, New York, NY: Dover Publications, (Originally Published by John Wiley & Sons in 1954.).

Deb, Rajat and Tae Kun Seo (2011) "Rothschild and Stiglitz's Mean Preserving: Revisited," *Social Choice and Welfare*, Vol. 37, No. 4, pp. 659–668.

Elton, John and Theodore P. Hill (1992) "Fusions of a Probability Distribution," *Annals of Probability*, Vol. 20, No. 1, pp. 421–454.

——— (1998) "On the Basic Representation Theorem for Convex Domination of Measures," *Journal of Mathematical Analysis and Applications*, Vol. 228, No. 2, pp. 449–466.

Hermalin, Benjamin E. and Michael Katz (2001) "Corporate Diversification and Agency," in Hammond, Peter J. and Gareth Myles eds. *Incentives, Organization, and Public Economics: Papers in Honour of Sir James Mirrlees*, Oxford, UK: Oxford University Press, Chap. 2, pp. 17–39.

Huang, James (2018) "Mean-Preserving Increases in Risk and Three-Atom Spreads," Unpublished Paper. Available at SSRN: https//ssrn.com/abstract=4033670 (accessed 2024-08-01).

Landsberger, Michael and Isaac Meilijson (1993) "Mean-Preserving Portfolio Dominance," *Review of Economic Studies*, Vol. 60, No. 2, pp. 479–485.

Leshno, Moshe, Haim Levy, and Yishay Spector (1997) "A Comment on Rothschild and Stiglitz's "Increasing Risk: I. A Definition"," *Journal of Economic Theory*, Vol. 77, No.1, pp. 223–228.

Leskelä, Lasse and Matti Vihola (2017) "Conditional Convex Orders and Measurable Martingale Couplings," *Bernoulli*, Vol. 23, No. 4A, pp. 2784–2807.

Machina, Mark J. and John W. Pratt (1997) "Increasing Risk: Some Direct Constructions," *Journal of Risk and Uncertainty*, Vol. 14, No. 2, pp. 103–127.

Müller, Alfred (2013) "Duality Theory and Transfers for Stochastic Order Relations," in Li, Haijun and Xiaohu Li eds. *Stochastic Orders in Reliability and Risk*, New York, NY: Springer, Chap. 2, pp. 41–57.

Müller, Alfred and Dietrich Stoyan (2002) *Comparison Methods for Stochastic Models and Risks*, Wiley Series in Probability and Statistics, Chichester: John Wiley & Sons, Inc.

Rasmusen, Eric and Emmanuel Petrakis (1992) "Defining the Mean-Preserving Spread: 3-PT Versus 4-PT," in Geweke, John ed. *Decision Making Under Risk and Uncertainty: New Models and Empirical Findings*, Dordrecht: Springer, pp. 53–58.

Rothschild, Michael and Joseph E. Stiglitz (1970) "Increasing Risk: I. A Definition," *Journal of Economic Theory*, Vol. 2, No. 3, pp. 225–243.

——— (1972) "Addendum to "Increasing Risk: I. A Definition"," *Journal of Economic Theory*, Vol. 5, No. 2, p. 306.

Sherman, Seymour (1951) "On a Theorem of Hardy, Littlewood, Polya, and Blackwell," *Proceedings of the National Academy of Sciences*, Vol. 37, No. 12, pp. 826–831.

Strassen, Volker (1965) "The Existence of Probability Measures with Given Marginals," *Annals of Mathematical Statistics*, Vol. 36, No. 2, pp. 423–439.