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An optimal growth model in a declining population: Is a "smart contraction" of the Japanese economy possible?

Harutaka Takahashi[1]

Abstract

This study employs an optimal growth model to analyze an economy facing a declining birthrate, an aging population, and a shrinking population. In contrast with conventional models, the labor input variable in the production function is not the total population; rather, it is the working-age population, with the labor augmented technological progress. The construction of this model facilitates the derivation of an optimal steady-state solution that is saddle-point stable, thereby enabling the investigation of the conditions under which GDP per capita can exhibit growth in an economy confronted with population decline. Given that GDP per capita is widely regarded as a metric of a nation's well-being, an economy that experiences growth in GDP per capita despite population decline is designated as a "smart contraction" economy. The condition that enables such growth can be termed "smart contraction condition." The validity of this condition will be examined through an analysis of data from the Japanese economy. (154 words)

Keywords: Population decline, Working-age population share, Smart contraction

JEL codes: E2, J1

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1. Introduction

In recent years, many developed countries have confronted the issue of declining birth rates and an aging population, a demographic shift that has resulted in a decline in the working-age population and, consequently, impacted a nation's economic productivity and growth. This phenomenon has prompted numerous studies to investigate its implications. For instance, Faruqee and Muhleisen (2003) examined the ramifications of Japan's low birthrate and aging population on government financial stability. Additionally, Maestas et al. (2023) utilized state-level data from the United States to examine the impact of aging on economic growth, the labor force, and productivity. It should be noted that these are merely a few of the numerous studies that have been conducted on this subject.

The issues of low fertility and population aging are related, yet distinct. Even in situations where the population is growing, a declining birthrate can lead to population aging. The problem with low fertility and population aging is the decline in the working-age population, which has a direct impact on labor productivity. In the case of a declining population, there is no doubt that the working-age population is declining. It is imperative to examine the interplay between productivity and the working-age population, particularly its influence on GDP per capita, a crucial metric of a nation's well-being. According to the government's population estimates for Japan, as depicted in Figure 1, the population reached its peak in 2010 and has been steadily declining since then. Projections indicate that it will reach 87 million by 2070. The working-age population has experienced a decline due to population aging since 1995.

The subsequent relationship is yielded by differentiating the definition of GDP per capita with respect to time.

The growth rate of GDP per capita

= (i) the rate of change in the share of the working-age population in the total population

+ (ii) the growth rate of productivity per working-age population.

As illustrated in Figure 2, the share of the working-age population has been gradually declining. Consequently, should the increase in factor (ii) productivity prove inadequate, Japan's GDP per capita will decline. This will consequently result in a reduction in Japan's overall welfare level over the long term. In response to this pessimistic forecast, Fernandez-Villaverde et al. (2023) conducted a comparative analysis of Japan's productivity of the working-age population with that of OECD countries, identifying three key findings.

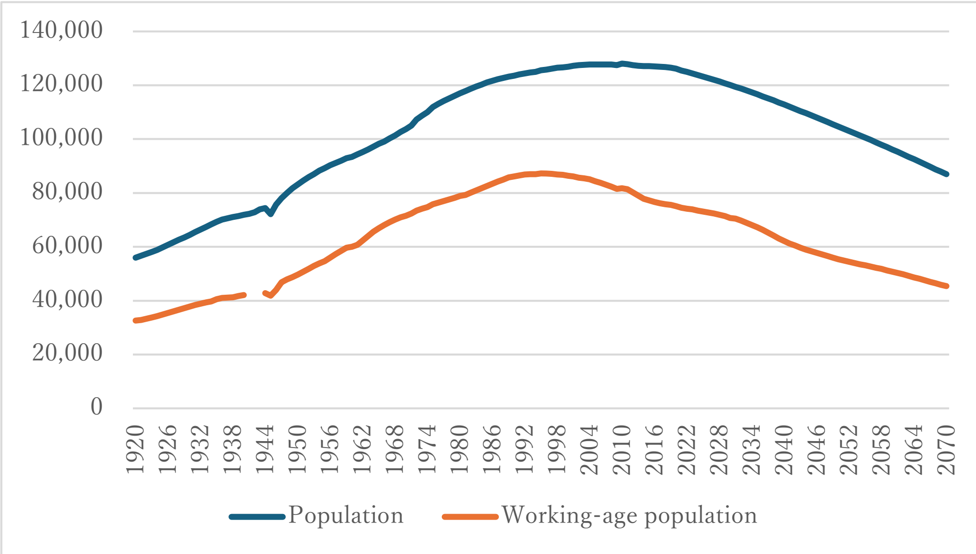

Figure 1. Japan's population estimation (Unit: Thousands of people)

Source: *'Population Estimates' by Ministry of Internal Affairs and Communications.*

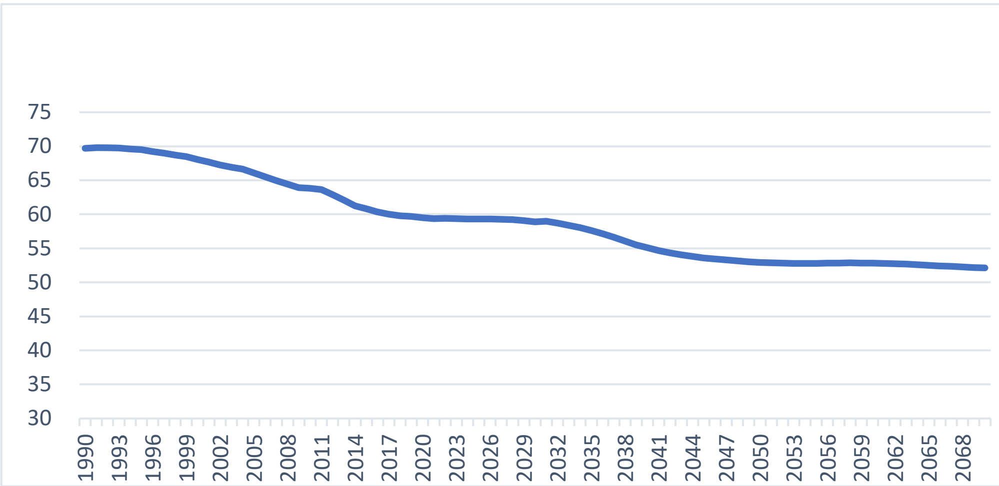

Figure 2. Estimated working-age population share (%) from 1990 to 2070

Source: *'Population Estimates' by Ministry of Internal Affairs and Communications.*

Fact 1: Japan's real GDP growth is nearly 1% a year lower than the U.S. (1.78% vs. 2.71%) from 1981 to 2019, while in the same period, Japan has grown at an annual rate of 1.96% and has grown slightly faster than the U.S. (1.78%) per working-age population.

Fact 2: From 1990 to 2019, Japan has the highest growth rate in terms of per working-age population among G7 countries plus Spain. No lost decades?

Fact 3: Even after the Japanese bubble bust, Japan's GDP per working-age population has grown at a rate of 1.44%, only slightly lower than the U.S. (1.56%) and ahead of Canada, France, Italy and Spain.

The presented findings do not inherently ensure positive GDP per capita growth in Japan, despite the decline in population. This is due to the presence of other pertinent factors, such as Factor (ii) indicated above. A salient question that emerges is whether a nation's GDP per capita can exhibit sustained growth in the face of a declining population. For the sake of clarity, an economy that achieves this kind of economic situation may be designated as a "smart contraction[2]."

The objective of this study is to examine the feasibility of achieving a "smart contraction" in the Japanese economy in the long run, given the ongoing demographic decline. To this end, an optimal growth model incorporating population decline has been developed. Futagami and Hori's (2010) theoretical investigation examined the impact of low fertility on the rate of technological progress and GDP per capita growth in a semi-endogenous R&D growth model that originated in Jones (1995). In contrast, Sasaki and Hoshida's (2017) analysis focused on the same model with a negative population growth rate. Their findings indicate that, under certain conditions, a zero endogenous rate of technological progress can be counterbalanced by a sufficiently large capital deepening effect, resulting in a positive real GDP per capita growth rate in the long run. A related study was conducted by Sasaki (2019). The study examined the Solow-type economic growth model with external technical progress in the negative population growth rate economy. The study showed that in circumstances where the rate of decline in population exceeds the rate of decline in the capital stock due to the decline in savings associated with the decline in population, real GDP per capita will continue to rise due to continued capital deepening. It is noteworthy that the validity of these findings hinges on the assumption that a substantial decline in population is necessary to trigger the process of

[2] Takao Komine of Taisho University called this economy the "smart shrink" in an article published in the Nikkei.

capital deepening.

Assuming a conventional capital depreciation rate of 15% per annum, it can be deduced that the rate of population decline must exceed this figure. However, an analysis of Japan's data reveals that the annual population decline ranges from -0.02% to -0.05%, a figure that is relatively negligible. Consequently, the assumption that a substantial population decline would generate capital deepening appears to be counterfactual.

This issue arises because the total population is involved in production and is incorporated as a variable into the production function. To circumvent this issue, we adhere to the approach proposed by Fernandez-Villaverde et al. (2023), which posits that it is the working-age population that is directly engaged in production. It is important to note that the total population encompasses individuals beyond the working age group, including the elderly, children, and infants who are not engaged in the workforce. Consequently, the conventional assumption in growth theory that the total population is involved in production is inherently problematic.

The following steps will be undertaken: In Section 2, an optimal growth model will be constructed, integrating the decline in population and the concomitant decline in the working-age population. The optimal steady-state solutions and saddle point stability will be proven. In Section 3, the conditions for establishing "smart contraction" will be derived from the optimal steady-state solution. The final section will examine the conditions derived in Section 3 using data from the Japanese economy to ascertain their satisfaction. The detailed mathematical arguments are relegated to the appendix.

2. Optimal growth model with population decline

In this section, we establish the following optimal growth model. The model incorporates the working-age population as a variable within the Cobb-Douglass production function, thereby including the labor augmented technical progress. It is imperative to note that this model exhibits a substantial deviation from the labor input variable employed in the conventional growth model. The conventional model is represented by the total population.

$$(1)\ \underset{\{c_t\}}{Max}\sum_{t=0}^{\infty}\beta^t N_t \log\left(\frac{C_t}{N_t}\right),$$

$$s.t.$$

$$(2)\ \ Y_t = K_t^{\theta}(A_t L_t)^{1-\theta}, (3)\ C_t + I_t = Y_t \text{ and } (4)\ K_{t+1} = I_t + (1-\delta)K_t.$$

The variables have the following meaning.

$\beta \in (0,1)$: discount factor,
C_t: consumption,
L_t : production age population,
$A_t = A_0(1+g)^t$: labor argumented technical progress,where $g \in (0,1)$,
$N_t = N_0(1+n)^t$: total population, $n \in (-1,1)$: rate of population change,
Y_t : GDP, K_t : capital stock,
I_t : gross investment,
$\delta \in (0,1)$: depreciation rate.

To solve the model, we rewrite the above model by (1)' through (5) in terms of efficiency per capita unit as follows:

$$(1)'\ \underset{\{\hat{c}_t\}}{Max}\sum_{t=0}^{\infty}\beta^t N_t \log \hat{c}_t;\ \hat{c}_t = \frac{C_t}{A_t N_t},$$

$$(2)'\ \hat{y}_t = \hat{k}_t^{\theta}\ell_t^{1-\theta}\ ;\ \hat{y}_t = \frac{Y_t}{A_t N_t}, \hat{k}_t = \frac{K_t}{A_t N_t}\ ,\ \ell_t = \frac{A_t L_t}{A_t N_t} = \frac{L_t}{N_t}$$: production age population share,

$$(3)'\ \hat{c}_t + \hat{i}_t = \hat{y}_t,$$

$$(4)'\ (1+g)(1+n)\hat{k}_{t+1} = \hat{i}_t + (1-\delta)\hat{k}_t;\ \hat{k}_{t+1} = \frac{K_{t+1}}{A_{t+1}N_{t+1}} \Rightarrow \frac{K_{t+1}}{A_t N_t} = \frac{K_{t+1}}{A_{t+1}N_{t+1}}\frac{A_{t+1}N_{t+1}}{A_t N_t} \Rightarrow \hat{k}_{t+1}(1+g)(1+n),$$

From (4)',

$$(5)\ \hat{i}_t = (1+g)(1+n)\hat{k}_{t+1} - (1-\delta)\hat{k}_t.$$

We will make the following important assumption here.

Assumption. The production-age population share (ℓ_t) declines at a constant rate $\lambda \in (-1, 0)$, i.e. $\ell_t = \ell_0 (1+\lambda)^t$.

Remark 1. *We use the long-run value of λ by estimating the slope of the regression line of Fig.2*[3]*. The estimated slope coefficient is about -0.003.*

In addition, we will develop a revised efficient unit model to achieve an optimal steady state, based on the aforementioned assumption. Dividing all variables by $(1+\lambda)^t$ yields.

(6) $$c_t = \frac{\hat{c}_t}{(1+\lambda)^t},\ y_t = \frac{\hat{y}_t}{(1+\lambda)^t},\ i_t = \frac{\hat{i}_t}{(1+\lambda)^t},\ \text{and } k_t = \frac{\hat{k}_t}{(1+\lambda)^t}.$$

We can rewrite the efficiency model by the above variables as follows.

$$\underset{\{c_t\}}{Max} \sum_{t=0}^{\infty} \beta^t N_t \log c_t,$$

$s.t.$

k_0 and ℓ_0 given,

$y_t = k_t^{\theta} \ell_0^{1-\theta}$, $c_t + i_t = y_t$ and $i_t = (1+\lambda)(1+g)(1+n)k_{t+1} - (1-\delta)k_t$.

To yield the Euler equation, we will solve the following two-period problem:

[3] We may apply the following regression model to the Japan's working-age population share data from 1995 to 2070 illustrated in Fig.2. The estimated model is as follows: $Y_t = \underset{(0.0003)}{0.7048} \underset{(0.0001)}{-0.003} T_t$ $\overline{R}^2 = 0.95$, where the values in parentheses are the standard deviations, Y_t: Working-age population share in year t and T_t: year t.

$$\underset{k_t}{Max}\ \Upsilon \equiv \beta(1+n)\log c_t + \log c_{t-1}$$
$$\equiv \beta(1+n)V(k_t, k_{t+1}) + V(k_{t-1}, k_t)$$
where
$$c_t = y_t - i_t = k_t^{\theta}\ell_0^{1-\theta} - (1+\lambda)(1+g)(1+n)k_{t+1} + (1-\delta)k_t,$$
$$c_{t-1} = y_{t-1} - i_{t-1} = k_{t-1}^{\theta}\ell_0^{1-\theta} - (1+\lambda)(1+g)(1+n)k_t + (1-\delta)k_{t-1}.$$

Solving this two-period problem[4] yields the following Euler equation (7):

$$(7)\quad \beta(1+n)\left(\frac{1}{c_t}\right)\left[\theta k_t^{\theta-1}\ell_0^{1-\theta} + (1-\delta)\right] - (1+\lambda)(1+g)(1+n)\left(\frac{1}{c_{t-1}}\right) = 0.$$

The optimal steady state is the solution of (7) such that when ℓ_0 is given,

$$\mathrm{y}_t = y_{t-1} = y_*,\ c_t = c_{t-1} = c_*,\ k_t = k_{t-1} = k_* \quad \text{holds for all } t.$$

Substituting y_*, k_* and c_* into the Euler equation (7) yields,

$$\beta(1+n)\left(\frac{1}{c_*}\right)\left[\theta k_*^{\theta-1}\ell_0^{1-\theta} + (1-\delta)\right] - (1+\lambda)(1+g)(1+n)\left(\frac{1}{c_*}\right) = 0$$
$$\Rightarrow \beta\left[\theta k_*^{\theta-1}\ell_0^{1-\theta} + (1-\delta)\right] - (1+\lambda)(1+g) = 0$$

Then we yield,

$$(8)\quad \theta k_*^{\theta-1}\ell_0^{1-\theta} + (1-\delta) = \theta\left(\frac{y_*}{k_*}\right) + (1-\delta) = \frac{(1+\lambda)(1+g)}{\beta}$$

Solving (8) with respect to k_* yields

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[4] See the Mathematical Appendix for the detailed derivation.

$$k_* = \left(\frac{\theta}{\frac{(1+\lambda)(1+g)}{\beta} - (1-\delta)} \right)^{\frac{1}{1-\theta}} \ell_0 = \Delta^{\frac{1}{1-\theta}} \ell_0 .$$

Substituting this into (7) also gives

(9) $y_* = \Delta^{\frac{\theta}{1-\theta}} \ell_0.$

Remark 2. It is noteworthy that the population growth rate is excluded from the Euler equation, thereby precluding its contribution to the determination of the optimal steady state. Additionally, it can be observed that a similar formula, analogous to Eq. (9), can be derived from the CES production function;

$$Y_t = \left[\theta K_t^\rho + (1-\theta)\left(A_t L_t\right)^\rho \right]^{\frac{1}{\rho}} \Rightarrow y_t = \left[\theta k_t^\rho + (1-\theta)\ell_0^\rho \right]^{\frac{1}{\rho}} .$$

Differentiate the Euler equation (7) around the OSS yields,

$$\beta(1+n) \frac{\partial^2 V(k_t, k_{t+1})}{\partial k_t \partial k_{t+1}} \bigg|_{(k_*,k_*)} (k_{t+1} - k_*)$$

$$+ \left\{ \beta(1+n) \frac{\partial^2 V(k_t, k_{t+1})}{\partial k_t^2} \bigg|_{(k_*,k_*)} + \frac{\partial^2 V(k_{t-1}, k_t)}{\partial k_t^2} \bigg|_{(k_*,k_*)} \right\} (k_t - k_*) + \frac{\partial^2 V(k_{t-1}, k_t)}{\partial k_t \partial k_{t-1}} \bigg|_{(k_*,k_*)} (k_{t-1} - k_*) = 0.$$

Defining $\lambda_t = k_t - k_*$,

$$\Rightarrow \beta(1+n) \frac{\partial^2 V(k_t, k_{t+1})}{\partial k_t \partial k_{t+1}} \bigg|_{(k_*,k_*)} \lambda^2 + \left\{ \beta(1+n) \frac{\partial^2 V(k_t, k_{t+1})}{\partial k_t^2} \bigg|_{(k_*,k_*)} + \frac{\partial^2 V(k_{t-1}, k_t)}{\partial k_t^2} \bigg|_{(k_*,k_*)} \right\} \lambda$$

$$+ \frac{\partial^2 V(k_{t-1}, k_t)}{\partial k_t \partial k_{t-1}} \bigg|_{(k_*,k_*)} = 0.$$

Accordingly, the following characteristic equation is derived[5];

$$f(\lambda) \equiv \beta(1+n)c_*^{-2}\frac{(1+\lambda)^2(1+g)^2(1+n)}{\beta}\lambda^2$$
$$+\left\{\beta(1+n_*)\left[-c_*^{-2}\left(\frac{1+g}{\beta}\right)^2 - c_*^{-1}\theta(1-\theta)\left(\frac{y_*}{k_*^2}\right)\right]+\left[-\left[(1+\lambda)(1+g)(1+n)\right]^2 c_*^{-2}\right]\right\}\lambda$$
$$+c_*^{-2}\frac{(1+\lambda)^2(1+g)^2(1+n)}{\beta}=0.$$

In accordance with the characteristic equation, the following proposition can be demonstrated.

Proposition 1. *If* $f(\lambda)$ *stisfies* $f(0)>0$ *and* $f(1)<0$*, then the chracteristic roots* λ_1 *and* λ_2 *satisfy* $0<\lambda_1<1<\lambda_2$*. The optimal steady state satisfies the saddle-point stability.*

Proof. See the detailed proof in A2 of the Mathematical Appendix.■

In accordance with this proposition, the optimal steady state of GDP per capita in terms of efficiency unit; y_* is defined as a saddle-point stable equilibrium. Consequently, any optimal path will converge to this equilibrium in the long run. We analyze the properties of this equilibrium in the following section.

3. Smart contraction

By analyzing the optimal steady state path, it is possible to analyze the nature of long-run economic growth. To do this, it is necessary to convert variables in efficient units back to variables in their original units. This can be implemented as follows:

From the definition of y_*,

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[5] For a more thorough derivation of the equation, please refer to Appendix A2.

$$y_* = \frac{Y_t}{N_t A_t (1+\lambda)^t} \Rightarrow \frac{Y_t}{N_t} = y_* A_t (1+\lambda)^t = y_* A_0 (1+g)^t (1+\lambda)^t .$$

Substituting $y_* = \Delta^{\frac{\theta}{1-\theta}} \ell_0$ into the above equation, we obtain

$$\frac{Y_t}{N_t} = A_0 (1+g)^t (1+\gamma)^t \Delta^{\frac{\theta}{1-\theta}} \ell_0 = A_0 \ell_0 \Delta^{\frac{\theta}{1-\theta}} \left[(1+g)(1+\gamma)\right]^t$$

$$\Rightarrow \frac{Y_t}{N_t} = A_0 \ell_0 \Delta^{\frac{\theta}{1-\theta}} \left[1+(g+\gamma)+g\gamma\right]^t .$$

Finally, since the term: $g\gamma$ is negligible, it follows that

$$(10)\ \frac{Y_t}{N_t} \simeq A_0 \ell_0 \Delta^{\frac{\theta}{1-\theta}} \left[1+(g+\gamma)\right]^t .$$

From Eq. (10), we have shown the following proposition.

Proposition 2. *If* $g+\gamma > 0$ *is valid, then GDP per capita in the original unit,* $\left(Y_t/N_t\right)$ *will grow at the rate* $g+\gamma$.

Proposition 2 shows that the goal of the "smart contraction" is achieved. In other words, we have derived the following condition for the "smart contraction":

Condition of the smart contraction: $g+\gamma > 0$.

The condition suggests that smart contraction will be achieved if the rate of labor productivity growth exceeds the rate of decline in the working-age population share. Section 4 examines whether this is the case for the Japanese economy on the basis of the available data.

Remark 3. *It is worth noting that although our condition is derived as the long-run equilibrium on the basis of the optimal growth model, it is approximately the same as the direct condition derived directly from the definition of GDP per capita as discussed in*

Section 1: the GDP per capita growth rate = (g) the rate of change in the share of the working-age population +(γ) the rate of change in labor productivity.

4. Possibility of the smart contraction in Japan

Table 1 indicates the data for g and γ . The rate of change in technological progress (g) is calculated as the rate of change in labor productivity, which is determined by dividing the value-added by the number of hours worked. The value of γ is defined as the rate of change in the ratio of the total population to working-age population and is estimated as the slope coefficient of the regression line in Fig.2. As reported in Remark 2, it is -0.003. Note that g is the average calculated over a period of 2, 5, or 9 years from 1995 to 2020.

	1995-2000	2000-2005	2005-2010	2015-2020	2000-2010	2010-2018	2018-2020
γ	-0.003	-0.003	-0.003	-0.003	-0.003	-0.003	-0.003
g	0.013	0.019	0.006	0.006	0.013	0.01	0.006
γ +g	0.010	0.016	0.003	0.003	0.010	0.007	0.003

Table 1. Japanese data for the period 1995-2020

Source: *National Institute of Population and Social Security Research, Population Projections for Japan*: https://www.ipss.go.jp/pp-zenkoku/j/zenkoku2023/pp_zenkoku2023.asp and *JIP Data*: https://www.rieti.go.jp/jp/database/jip.html

It is clear from Table 1 that the condition for a smart contraction has been met over the entire period.

5. Conclusion

Section 4 has shown that the smart contraction condition derived in Section 3 has been fulfilled in the Japanese economy in recent years. The impact of the decline in the working-age population has been mitigated to some extent by the rise in the labor force participation rate of those aged 65 and over. This increase can be attributed to factors such as the extension of the retirement age and the reemployment system. The hypothesis of a "smart contraction" will be viable in the future if two conditions are met: first, the labor productivity growth rate must exceed 0.3%; second, the decline in the working-age population ratio must continue at a rate of -0.003. The viability of this contraction depends on the maintenance of a positive labor productivity growth rate, provided that the trend in the rate of decline of the working-age population ratio remains at -0.003. Achieving a labor productivity growth rate in excess of 0.3 percent appears to be a feasible policy objective. The implementation of "smart contraction" has the potential to increase GDP per capita and promote greater welfare among the population. However, it is important to note that the decline in the total population leads to a reduction in GDP, which in turn may limit the feasibility of implementing public investment and social security policies.

In the foreseeable future, should the replacement of workers with robots emerge as a viable solution to the diminishing workforce, the issues previously discussed may be resolved instantaneously. Consequently, in an economy experiencing a declining population, such as Japan's, the recent discourse surrounding the unemployment of workers due to the substitution of human labor by robots may not be regarded as a significant concern.

Mathematical Appendix

A1. Derivation of the Euler equation

$$\frac{\partial \Upsilon}{\partial k_t} = 0 \Rightarrow \beta(1+n)\frac{\partial V(k_t, k_{t+1})}{\partial k_t} + \frac{\partial V(k_{t-1}, k_t)}{\partial k_t} = 0$$

$$\Rightarrow \beta(1+n)\left(\frac{1}{c_t}\right)\left(\frac{\partial c_t}{\partial k_t}\right) + \left(\frac{1}{c_{t-1}}\right)\left(\frac{\partial c_{t-1}}{\partial k_t}\right) = 0.$$

$$\Rightarrow \beta(1+n)\left(\frac{1}{c_t}\right)\left[\theta k_t^{\theta-1}\hat{\ell}^{1-\theta} + (1-\delta)\right] - (1+\lambda)(1+g)(1+n)\left(\frac{1}{c_{t-1}}\right) = 0$$

Note that

$$\left(\frac{\partial c_t}{\partial k_t}\right) = \theta\left(\frac{y_t}{k_t}\right)^{1-\rho} + (1-\delta) \text{ and } \left(\frac{\partial c_{t-1}}{\partial k_t}\right) = -(1+\lambda)(1+g)(1+n).$$

where

$$\frac{\partial V(k_t, k_{t+1})}{\partial k_t} = c_t^{-1}\left[\theta\left(\frac{y_t}{k_t}\right)^{1-\rho} + (1-\delta)\right] \text{ and } \frac{\partial V(k_{t-1}, k_t)}{\partial k_t} = -c_{t-1}^{-1}(1+\lambda)(1+g)(1+n).$$

A2. Proof of Proposition 1

Totally differentiating the Euler equation (*) around the OSS yields

$$\beta(1+n_*)\frac{\partial^2 V(k_t, k_{t+1})}{\partial k_t \partial k_{t+1}}\bigg|_{(k_*, k_*)}(k_{t+1} - k_*) + \left\{\beta(1+n_*)\frac{\partial^2 V(k_t, k_{t+1})}{\partial k_t^2}\bigg|_{(k_*, k_*)} + \frac{\partial^2 V(k_{t-1}, k_t)}{\partial k_t^2}\bigg|_{(k_*, k_*)}\right\}(k_t - k_*)$$
$$+\frac{\partial^2 V(k_{t-1}, k_t)}{\partial k_t \partial k_{t-1}}\bigg|_{(k_*, k_*)}(k_{t-1} - k_*) = 0.$$

Substituting the second order partial derivatives evaluated at the OSS ,(A1) to (A4) into the above equation yields,

$$\beta(1+n_*)\frac{\partial^2 V(k_t, k_{t+1})}{\partial k_t \partial k_{t+1}}\bigg|_{(k_*, k_*)}\lambda^2 + \left\{\beta(1+n_*)\frac{\partial^2 V(k_t, k_{t+1})}{\partial k_t^2}\bigg|_{(k_*, k_*)} + \frac{\partial^2 V(k_{t-1}, k_t)}{\partial k_t^2}\bigg|_{(k_*, k_*)}\right\}\lambda + \frac{\partial^2 V(k_{t-1}, k_t)}{\partial k_t \partial k_{t-1}}\bigg|_{(k_*, k_*)} = 0.$$

Note that the second order derivatives evaluated at the OSS are given as (11) through (14) as below.

$$(11)\quad \frac{\partial^2 V(k_t, k_{t+1})}{\partial k_t \partial k_{t+1}}\bigg|_{(k_*, k_*)} = c_t^{-2}\left(\theta\left(\frac{y_t}{k_t}\right)^{1-\rho} + (1-\delta)\right)(1+\lambda)(1+g)(1+n_t)\bigg|_{(k_*, k_*)}$$
$$= c_*^{-2}\left(\theta\left(\frac{y_*}{k_*}\right)^{1-\rho} + (1-\delta)\right)(1+\lambda)(1+g)(1+n_*)$$
$$= c_*^{-2}\frac{(1+\lambda)^2(1+g)^2(1+n_*)}{\beta} > 0.$$

$$(12)\ \left.\frac{\partial^2 V(k_t,k_{t+1})}{\partial k_t^2}\right|_{(k_*,k_*)} = -c_t^{-2}\left(\theta\frac{y_t}{k_t}+(1-\delta)\right)^2 + c_t^{-1}\theta(\theta-1)\left(\frac{y_t}{k_t^2}\right)\Bigg|_{(k_*,k_*)}$$

$$= -c_*^{-2}\left(\theta\frac{y_*}{k_*}+(1-\delta)\right)^2 + c_*^{-1}\theta(\theta-1)\left(\frac{y_*}{k_*^2}\right)$$

$$= -c_*^{-2}\left(\frac{1+g}{\beta}\right)^2 - c_*^{-1}\theta(1-\theta)\left(\frac{y_*}{k_*^2}\right) < 0.$$

$$(13)\ \left.\frac{\partial^2 V(k_{t-1},k_t)}{\partial k_t \partial k_{t-1}}\right|_{(k_*,k_*)} = c_{t-1}^{-2}\left(\theta\frac{y_{t-1}}{k_{t-1}}+(1-\delta)\right)(1+\lambda)(1+g)(1+n_t)\Bigg|_{(k_*,k_*)}$$

$$= c_*^{-2}\left(\theta\frac{y_*}{k_*}+(1-\delta)\right)(1+\lambda)(1+g)(1+n_*)$$

$$= c_*^{-2}\frac{(1+\lambda)^2(1+g)^2(1+n_*)}{\beta} > 0.$$

$$(14)\ \left.\frac{\partial^2 V(k_{t-1},k_t)}{\partial k_t^2}\right|_{(k_*,k_*)} = (1+\lambda)(1+g)(1+n_t)c_{t-1}^{-2}\left[-(1+\lambda)(1+g)(1+n_t)\right]\Big|_{(k_*,k_*)}$$

$$= -\left[(1+\lambda)(1+g)(1+n_*)\right]^2 c_*^{-2} < 0.$$

Finally, we obtain the following characteristic equation $f(\lambda)$.

$$f(\lambda) \equiv \beta(1+n_*)c_*^{-2}\frac{(1+\lambda)^2(1+g)^2(1+n_*)}{\beta}\lambda^2$$

$$+\left\{\beta(1+n_*)\left[-c_*^{-2}\frac{\left[(1+\lambda)(1+g)\right]^2}{\beta^2} - c_*^{-1}\theta(1-\theta)\left(\frac{y_*}{k_*^2}\right)\right] + \left[-\left[(1+\lambda)(1+g)(1+n_*)\right]^2 c_*^{-2}\right]\right\}\lambda$$

$$+c_*^{-2}\frac{\left[(1+\lambda)(1+g)\right]^2(1+n_*)}{\beta} = 0$$

To prove Proposition 1, we need to show that $f(1)<0$ and $f(0)>0$ as follows;

$$f(0)=c_*^{-2}\frac{(1+\lambda)^2(1+g)^2(1+n_*)}{\beta}>0$$

and

$$f(1)=\beta(1+n_*)c_*^{-2}\frac{(1+\lambda)^2(1+g)^2(1+n_*)}{\beta}$$

$$+\left\{\beta(1+n_*)\left[-c_*^{-2}\left(\frac{1+g}{\beta}\right)^2-c_*^{-1}\theta(1-\theta)\left(\frac{y_*}{k_*^2}\right)\right]+\left[-\left[(1+\lambda)(1+g)(1+n_*)\right]^2c_*^{-2}\right]\right\}$$

$$+c_*^{-2}\frac{(1+\lambda)^2(1+g)^2(1+n_*)}{\beta}=c_*^{-2}\left[(1+\lambda)(1+g)(1+n_*)\right]^2-c_*^{-2}\frac{(1+\lambda)^2(1+g)^2(1+n_*)}{\beta}$$

$$-\beta(1+n_*)c_*^{-1}\theta(1-\theta)\left(\frac{y_*}{k_*^2}\right)-\left[(1+\lambda)(1+g)(1+n_*)\right]^2c_*^{-2}+c_*^{-2}\frac{(1+\lambda)^2(1+g)^2(1+n_*)}{\beta}$$

$$=-\beta(1+n_*)c_*^{-1}\theta(1-\theta)\left(\frac{y_*}{k_*^2}\right)<0.$$ This completes the proof.■

A3. The Solow growth model with population decline

In this section, we will examine a growth model for a continuous-time system. This model will serve to contrast it with the discrete-time optimal growth model that has been previously studied. The continuous-time Solow model with the Cobb-Douglas production function can be presented in terms of the variables defined in Eq. (6) as follows. Note that "s" stands for the savings rate and "$\dot{x}$" for the time derivative of the variable x.

$$\dot{k}=sy-(\delta+\gamma+n)k \text{ and } y=k^{\alpha}\ell_0^{1-\alpha}, \text{ where } N=N_0e^{nt}, A=A_0e^{\gamma t} \text{ and } \ell=\ell_0e^{\lambda t}.$$

Solving the above system to obtain a steady state solution yields the following.

$$sk^{\alpha-1}\ell_0^{1-\alpha}-(\delta+\gamma+n)=0\Rightarrow k_*=\left(\frac{s}{\delta+\gamma+n}\right)^{\frac{1}{1-\alpha}}\ell_0$$

Substituting the above result into the production function y yields.

$$y = \frac{Y}{NA_0 e^{\gamma t} e^{\lambda t}} = k_*^{\alpha} \ell_0^{1-\alpha} = \left(\left(\frac{s}{\delta + \gamma + n} \right)^{\frac{1}{1-\alpha}} \ell_0 \right)^{\alpha} \ell_0^{1-\alpha} = \left(\frac{s}{\delta + \gamma + n} \right)^{\frac{\alpha}{1-\alpha}} \ell_0$$

$$\Rightarrow \frac{Y}{N} = \left(\frac{s}{\delta + \gamma + n} \right)^{\frac{\alpha}{1-\alpha}} \ell_0 e^{(\gamma+\lambda)t}.$$

Consequently, the result obtained from this study is consistent with that of Proposition 2.

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