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ON THE GROSS-PRASAD CONJECTURE WITH ITS REFINEMENT FOR $(\mathrm{SO}(5), \mathrm{SO}(2))$ AND THE GENERALIZED BÖCHERER CONJECTURE

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Abstract. We investigate the Gross-Prasad conjecture and its refinement for the Bessel periods in the case of $(\mathrm{SO}(5), \mathrm{SO}(2))$. In particular, by combining several theta correspondences, we prove the Ichino-Ikeda type formula for any tempered irreducible cuspidal automorphic representation. As a corollary of our formula, we prove an explicit formula relating certain weighted averages of Fourier coefficients of holomorphic Siegel cusp forms of degree two which are Hecke eigenforms to central special values of L-functions. The formula is regarded as a natural generalization of the Böcherer conjecture to the non-trivial toroidal character case.

1. Introduction

To investigate relations between periods of automorphic forms and special values of L-functions is one of the focal research subjects in number theory. The central special values are of keen interest in light of the Birch and Swinnerton-Dyer conjecture and its generalizations.

Gross and Prasad [44, 45] proclaimed a global conjecture relating non-vanishing of certain period integrals on special orthogonal groups to non-vanishing of central special values of certain tensor product L-functions, together with the local counterpart conjecture in the early 1990s. Later with Gan [32], they extended the conjecture to classical groups and metaplectic groups. Meanwhile a refinement of the Gross-Prasad conjecture, which is a precise formula for the central special values of the tensor product L-functions for tempered cuspidal automorphic representations, was formulated by Ichino and Ikeda [57] in the co-dimension one special orthogonal case. Subsequently Harris [48] formulated a refinement of the Gan-Gross-Prasad conjecture in the co-dimension one unitary case. Later an extension of the work of Ichino-Ikeda and Harris to the general Bessel period case

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was formulated by Liu [76] and the one to the general Fourier-Jacobi period case for symplectic-metaplectic groups was formulated by Xue [117].

In [27] we investigated the Gross-Prasad conjecture for Bessel periods for $\mathrm{SO}(2n+1) \times \mathrm{SO}(2)$ when the character on $\mathrm{SO}(2)$ is trivial, i.e. the special Bessel periods case and then, in the sequel [28], we proved its refinement, i.e the Ichino-Ikeda type precise L-value formula under the condition that the base field is totally real and all components at archimedean places are discrete series representations. As a corollary of our special value formula in [28], we obtained a proof of the long-standing conjecture by Böcherer in [13], concerning central critical values of imaginary quadratic twists of spinor L-functions for holomorphic Siegel cusp forms of degree two which are Hecke eigenforms, thanks to the explicit calculations of the local integrals by Dickson, Pitale, Saha and Schmidt [21].

In this paper, for $(\mathrm{SO}(5), \mathrm{SO}(2))$, we vastly generalize the main results in [27] and [28]. Namely we prove the Gross-Prasad conjecture and its refinement for any Bessel period in the case of $(\mathrm{SO}(5), \mathrm{SO}(2))$. As a corollary, we prove the generalized Böcherer conjecture in the square-free case formulated in [21].

Let us introduce some notation and then state our main results precisely.

1.1. **Notation.** Let F be a number field. We denote its ring of adeles by $\mathbb{A}_F$, which is mostly abbreviated as $\mathbb{A}$ for simplicity. Let ψ be a non-trivial character of $\mathbb{A}/F$. For $a \in F^\times$, we write by ψ^a the character of $\mathbb{A}/F$ defined by $\psi^a(x) = \psi(ax)$. For a place v of F, we denote by F_v the completion of F at v. When v is non-archimedean, we write by ϖ_v and q_v an uniformizer of F_v and the cardinality of the residue field of F_v, respectively.

Let E be a quadratic extension of F and $\mathbb{A}_E$ be its ring of adeles. We denote by $x \mapsto x^\sigma$ the unique non-trivial automorphism of E over F. Let us denote by $\mathrm{N}_{E/F}$ the norm map from E to F. We choose $\eta \in E^\times$ such that $\eta^\sigma = -\eta$ and fix. Let $d = \eta^2$. We denote by χ_E the quadratic character of $\mathbb{A}^\times$ corresponding to the quadratic extension E/F. We fix a character Λ of $\mathbb{A}_E^\times / E^\times$ whose restriction to $\mathbb{A}^\times$ is trivial once and for all.

1.2. **Measures.** Throughout the paper, for an algebraic group $\mathbf{G}$ defined over F, we write $\mathbf{G}_v$ for $\mathbf{G}(F_v)$, the group of rational points of $\mathbf{G}$ over F_v, and we always take the measure dg on $\mathbf{G}(\mathbb{A})$ to be the Tamagawa measure unless specified otherwise. For each v, we take the self-dual measure with respect to ψ_v on F_v. Then recall that the product measure on $\mathbb{A}$ is the self-dual measure with respect to ψ and is also the Tamagawa measure since $\mathrm{Vol}(\mathbb{A}/F) = 1$. For a unipotent algebraic group $\mathbf{U}$ defined over F, we also specify the local measure du_v on $\mathbf{U}(F_v)$ to be the measure corresponding to the gauge form defined over F, together with our choice of the measure on F_v, at each place v of F. Thus in particular we have

$$du = \prod_v du_v \qquad \text{and} \qquad \mathrm{Vol}\left(\mathbf{U}(F) \backslash \mathbf{U}(\mathbb{A}), du\right) = 1.$$

1.3. **Similitudes.** Various similitude groups appear in this article. Unless there exists a fear of confusion, we denote by $\lambda(g)$ the similitude of an element g of a similitude group for simplicity.

1.4. **Bessel periods.** First we recall that when V is a five dimensional vector space over F equipped with a non-degenerate symmetric bilinear form whose Witt index is at least one, there exists a quaternion algebra D over F such that

$$\mathrm{SO}(V) = \mathbb{G}_D \tag{1.4.1}$$

where $\mathbb{G}_D = G_D/Z_D$, G_D is a similitude quaternionic unitary group over F defined by

$$G_D(F) := \left\{ g \in \mathrm{GL}_2(D) : {}^t\overline{g} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} g = \lambda(g) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \lambda(g) \in F^\times \right\} \tag{1.4.2}$$

and Z_D is the center of G_D. Here

$$\overline{g} := \begin{pmatrix} \bar{t} & \overline{u} \\ \overline{w} & \overline{v} \end{pmatrix} \quad \text{for} \quad g = \begin{pmatrix} t & u \\ w & v \end{pmatrix} \in \mathrm{GL}_2(D)$$

where denoted by $x \mapsto \overline{x}$ for $x \in D$ is the canonical involution of D. Also, we define a quaternionic unitary group G_D^1 over F by

$$G_D^1 := \{ g \in G_D : \lambda(g) = 1 \}.$$

Let

$$D^- := \{ x \in D : \mathrm{tr}_D(x) = 0 \}$$

where tr_D denotes the reduced trace of D over F. We recall that when $D \simeq \mathrm{Mat}_{2\times 2}(F)$, G_D is isomorphic to the similitude symplectic group GSp_2 which we denote by G, i.e.

(1.4.3)

$$G(F) := \left\{ g \in \mathrm{GL}_4(F) : {}^t g \begin{pmatrix} 0 & 1_2 \\ -1_2 & 0 \end{pmatrix} g = \lambda(g) \begin{pmatrix} 0 & 1_2 \\ -1_2 & 0 \end{pmatrix}, \lambda(g) \in F^\times \right\}.$$

Also, we define the symplectic group Sp_2, which we denote by G^1, as

$$G^1 := \{ g \in G : \lambda(g) = 1 \}.$$

We denote $\mathrm{PGSp}_2 = G/Z_G$ by $\mathbb{G}$, where Z_G denotes the center of G. Thus when D is split, $G_D \simeq G = \mathrm{GSp}_2$, $G_D^1 \simeq G^1 = \mathrm{Sp}_2$ and $\mathbb{G}_D \simeq \mathbb{G} = \mathrm{PGSp}_2$.

The Siegel parabolic subgroup P_D of G_D has the Levi decomposition $P_D = M_D N_D$ where

$$M_D(F) := \left\{ \begin{pmatrix} x & 0 \\ 0 & \mu \cdot x \end{pmatrix} : x \in D^\times, \mu \in F^\times \right\}, \quad N_D(F) := \left\{ \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} : u \in D^- \right\}.$$

For $\xi \in D^-(F)$, let us define a character ψ_ξ on $N_D(\mathbb{A})$ by

$$\psi_\xi \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} := \psi\left(\mathrm{tr}_D(\xi u)\right). \tag{1.4.4}$$

We note that for $\begin{pmatrix} x & 0 \\ 0 & \mu \cdot x \end{pmatrix} \in M_D(F)$, we have

$$\psi_\xi \left[\begin{pmatrix} x & 0 \\ 0 & \mu \cdot x \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x & 0 \\ 0 & \mu \cdot x \end{pmatrix}^{-1} \right] = \psi_{\mu^{-1} \cdot x^{-1} \xi x} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}. \tag{1.4.5}$$

Suppose that $F(\xi) \simeq E$. Let us define a subgroup T_ξ of $D^\times$ by

$$T_\xi := \left\{ x \in D^\times : x\xi x^{-1} = \xi \right\}. \tag{1.4.6}$$

Then since $F(\xi)$ is a maximal commutative subfield of D, we have

$$T_\xi(F) = F(\xi)^\times \simeq E^\times. \tag{1.4.7}$$

We identify T_ξ with the subgroup of M_D given by

$$\left\{ \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} : x \in T_\xi \right\}. \tag{1.4.8}$$

We note that by (1.4.5), we have

$$\psi_\xi\left(tnt^{-1}\right) = \psi_\xi(n) \quad \text{for } t \in T_\xi(\mathbb{A}) \text{ and } n \in N_D(\mathbb{A}).$$

We define the Bessel subgroup R_ξ of G_D by

$$R_\xi := T_\xi N_D. \tag{1.4.9}$$

Then the Bessel periods defined below are indeed the periods in question in the Gross-Prasad conjecture for $(\mathrm{SO}(5), \mathrm{SO}(2))$.

Definition 1.1. *Let π be an irreducible cuspidal automorphic representation of $G_D(\mathbb{A})$ whose central character is trivial and V_π its space of automorphic forms. Let Λ be a character of $\mathbb{A}_E^\times / E^\times$ whose restriction to $\mathbb{A}^\times$ is trivial. Let $\xi \in D^-(F)$ such that $F(\xi) \simeq E$. Fix an F-isomorphism $T_\xi \simeq E^\times$ and regard Λ as a character of $T_\xi(\mathbb{A}) / T_\xi(F)$. We define a character $\chi^{\xi,\Lambda}$ on $R_\xi(\mathbb{A})$ by*

$$\chi^{\xi,\Lambda}(tn) := \Lambda(t)\,\psi_\xi(n) \quad \text{for } t \in T_\xi(\mathbb{A}) \text{ and } n \in N_D(\mathbb{A}). \tag{1.4.10}$$

Then for $f \in V_\pi$, we define $B_{\xi,\Lambda,\psi}(f)$, the (ξ,Λ,ψ)-Bessel period of f, by

$$B_{\xi,\Lambda,\psi}(f) := \int_{\mathbb{A}^\times R_\xi(F) \backslash R_\xi(\mathbb{A})} f(r)\,\chi^{\xi,\Lambda}(r)^{-1}\,dr. \tag{1.4.11}$$

We say that π has the (ξ,Λ,ψ)-Bessel period when the linear form $B_{\xi,\Lambda,\psi}$ is not identically zero on V_π.

Remark 1.1. *Here we record the dependency of $B_{\xi,\Lambda,\psi}$ on the choices of ξ and ψ. First we note that for $\xi' \in D^-(F)$, we have $F(\xi') \simeq E$ if and only if*

$$\xi' = \mu \cdot \alpha^{-1}\xi\alpha \quad \text{for some } \alpha \in D^\times(F) \text{ and } \mu \in F^\times \tag{1.4.12}$$

by the Skolem-Noether theorem. Suppose that $\xi' \in D^-(F)$ satisfies (1.4.12) and $\psi' = \psi^a$ where $a \in F^\times$. Let $m_0 = \begin{pmatrix} \alpha & 0 \\ 0 & a^{-1}\mu \cdot \alpha \end{pmatrix} \in M_D(F)$. Then by (1.4.5), we have

$$\begin{aligned} B_{\xi,\Lambda,\psi}(\pi(m_0)f) &= \int_{\mathbb{A}^\times T_{\xi'}(F) \backslash T_{\xi'}(\mathbb{A})} \int_{N_D(F) \backslash N_D(\mathbb{A})} \\ &\qquad f(t'n')\,\Lambda(t')^{-1}\,\psi'_{\xi'}(n')\,dt'\,dn' \\ &= B_{\xi',\Lambda,\psi'}(f) \end{aligned} \tag{1.4.13}$$

where we identify $T_{\xi'}(F)$ with $E^\times$ via the F-isomorphism $F(\xi') \ni x \mapsto \alpha x \alpha^{-1} \in F(\xi) \simeq E$.

Definition 1.2. *Let (π, V_π) be an irreducible cuspidal automorphic representation of $G_D(\mathbb{A})$ whose central character is trivial. Let Λ be a character of $\mathbb{A}_E^\times / E^\times$ whose restriction to $\mathbb{A}^\times$ is trivial. Then we say that π has the (E, Λ)-Bessel period if there exist $\xi \in D^-(F)$ such that $F(\xi) \simeq E$ and a non-trivial character ψ of $\mathbb{A}/F$ so that π has the (ξ, Λ, ψ)-Bessel period. This terminology is well-defined because of the relation* (1.4.13).

1.5. **Gross-Prasad conjecture.** First we introduce the following definition which is inspired by the notion of *locally G-equivalence* in Hiraga and Saito [51, p.23].

Definition 1.3. *Let (π, V_π) be an irreducible cuspidal automorphic representation of $G_D(\mathbb{A})$ whose central character is trivial. Let D' be a quaternion algebra over F and $(\pi', V_{\pi'})$ an irreducible cuspidal automorphic representation of $G_{D'}(\mathbb{A})$. Then we say that π is locally G^+-equivalent to π' if at almost all places v of F where $D(F_v) \simeq D'(F_v)$, there exists a character χ_v of $G_D(F_v)/G_D(F_v)^+$ such that $\pi_v \otimes \chi_v \simeq \pi'_v$. Here*

$$G_D(F)^+ := \left\{ g \in G_D(F) : \lambda(g) \in \mathrm{N}_{E/F}\left(E^\times\right) \right\}. \tag{1.5.1}$$

Remark 1.2. *When π and π' have weak functorial lifts to $\mathrm{GL}_4(\mathbb{A})$, say Π and Π', respectively, the notion of locally G^+-equivalence is described simply as the following. Suppose that π and π' are locally G^+-equivalent. Then there exists a character ω of $G_D(\mathbb{A})$ such that $\pi \otimes \omega$ is nearly equivalent to π', where ω may not be automorphic. Since ω_v is either χ_{E_v} or trivial at almost all places v of F, we have $\mathrm{BC}_{E/F}(\Pi) \simeq \mathrm{BC}_{E/F}(\Pi')$ where $\mathrm{BC}_{E/F}$ denotes the base change lift to $\mathrm{GL}_4(\mathbb{A}_E)$. Then by Arthur-Clozel* [2, Theorem 3.1]*, we have $\Pi \simeq \Pi'$ or $\Pi' \otimes \chi_E$. Hence π is nearly equivalent to either π' or $\pi' \otimes \chi_E$. The converse is clear.*

Then our first main result is on the Gross-Prasad conjecture for $(\mathrm{SO}(5), \mathrm{SO}(2))$.

Theorem 1.1. *Let E be a quadratic extension of F. Let (π, V_π) be an irreducible cuspidal automorphic representation of $G_D(\mathbb{A})$ with a trivial central character and Λ a character of $\mathbb{A}_E^\times / E^\times$ whose restriction to $\mathbb{A}^\times$ is trivial.*

(1) *Suppose that π has the (E, Λ)-Bessel period. Moreover assume that:*

(1.5.2) *there exists a finite place w of F such that*

π_w and its local theta lift to $\mathrm{GSO}_{4,2}(F_w)$ are generic.

Here $\mathrm{GSO}_{4,2}$ denote the identity component of $\mathrm{GO}_{4,2}$, the similitude orthogonal group associated to the six dimensional orthogonal space $(E, \mathrm{N}_{E/F}) \oplus \mathbb{H}^2$ over F where $\mathbb{H}$ denotes the hyperbolic plane over F.

Then there exists a finite set S_0 of places of F containing all archimedean places of F such that the partial L-function

$$L^S\left(\frac{1}{2}, \pi \times \mathcal{AI}(\Lambda)\right) \neq 0 \tag{1.5.3}$$

for any finite set S of places of F with $S \supset S_0$. Here, $\mathcal{AI}(\Lambda)$ denotes the automorphic induction of Λ from $\mathrm{GL}_1(\mathbb{A}_E)$ to $\mathrm{GL}_2(\mathbb{A})$. Moreover there exists a globally generic irreducible cuspidal automorphic representation π° of $G(\mathbb{A})$ which is locally G^+-equivalent to π.

(2) *Assume that:*

(1.5.4) *the endoscopic classification of Arthur,*

$$\textit{i.e.}\ [3,\ \text{Conjecture 9.4.2, Conjecture 9.5.4}]\ \textit{holds for}\ \mathbb{G}_{D_\circ}.$$

Here $D_\circ$ denotes an arbitrary quaternion algebra over F.

Suppose that π has a generic Arthur parameter, namely the parameter is of the form Π_0 or $\Pi_1 \boxplus \Pi_2$ where Π_i is an irreducible cuspidal automorphic representation of $\mathrm{GL}_4(\mathbb{A})$ for $i = 0$ and of $\mathrm{GL}_2(\mathbb{A})$ for $i = 1, 2$, respectively, such that $L(s, \Pi_i, \wedge^2)$ has a pole at $s = 1$.

Then we have

$$L\left(\frac{1}{2}, \pi \times \mathcal{AI}(\Lambda)\right) \neq 0 \tag{1.5.5}$$

if and only if there exists a pair (D', π') where D' is a quaternion algebra over F containing E and π' an irreducible cuspidal automorphic representation of $G_{D'}$ which is nearly equivalent to π such that π' has the (E, Λ)-Bessel period.

Moreover, when π is tempered, the pair (D', π') is uniquely determined.

Remark 1.3. *In (1.5.5), $L(s, \pi \times \mathcal{AI}(\Lambda))$ denotes the complete L-function defined as the following.*

When $\mathcal{AI}(\Lambda)$ is not cuspidal, i.e. $\Lambda = \Lambda_0 \circ \mathrm{N}_{E/F}$ for a character Λ_0 of $\mathbb{A}^\times / F^\times$, we define

$$L(s, \pi \times \mathcal{AI}(\Lambda)) := L(s, \pi \times \Lambda_0)\, L(s, \pi \times \Lambda_0 \chi_E)$$

where each factor on the right hand side is defined by the doubling method as in Lapid-Rallis [73] or Yamana [120].

When $\mathcal{AI}(\Lambda)$ is cuspidal, the partial L-function $L^S(s, \pi \times \mathcal{AI}(\Lambda))$ may be defined by Theorem C.1 in Appendix C for a finite set S of places of F such that π_v and $\Pi(\Lambda)_v$ are unramified at $v \notin S$. Further, we define the local L-factor at each place $v \in S$ by the local Langlands parameters for π_v and $\Pi(\Lambda)_v$, where the local Langlands parameters are given by Gan-Takeda [35] for $\mathbb{G}(F_v)$ (also Arthur [3]), Gan-Tantono [38] for $\mathbb{G}_D(F_v)$ and Kutzko [69] for $\mathrm{GL}_2(F_v)$ at finite places, and, by Langlands [70] at archimedean places.

We note that the condition (1.5.3) and the condition (1.5.5) are equivalent from the definition of local L-factors when π is tempered.

Remark 1.4. *Suppose that at a finite place w of F, the group $G_D(F_w)$ is split and the representation π_w is generic and tempered. Then by Gan and Ichino [40, Proposition C.4], the big theta lift of π_w and the local theta lift of π_w coincide. Thus the genericity of the local theta lift of π_w follows from Gan and Takeda [36, Corollary 4.4] for the dual pair $(G, \mathrm{GSO}_{3,3})$ and from a local analogue of the* *computations in* [83, Section 3.1] *for the dual pair* $\left(G^{+}, \mathrm{GSO}_{4,2}\right)$*, respectively. Here*

$$G(F)^{+} := \left\{g \in G : \lambda(g) \in \mathrm{N}_{E/F}\left(E^{\times}\right)\right\}. \tag{1.5.6}$$

When a local representation π_w *is unramified and tempered,* π_w *is generic as remarked in* [27, Remark 2]. *Hence the assumption* (1.5.2) *is fulfilled when* π *is tempered.*

In our previous paper [27], Theorem 1.1 for the pair $(\mathrm{SO}(2n+1), \mathrm{SO}(2))$ was proved when Λ is trivial. Meanwhile Jiang and Zhang [63] studied the Gross-Prasad conjecture in a very general setting assuming the endoscopic classification of Arthur in general by using the twisted automorphic descent. Though Theorem 1.1 is subsumed in [63] as a special case, we believe that our method, which is different from theirs, has its own merits because of its concreteness. We also note that because of the temperedness of π, the uniqueness of the pair (D', π') in Theorem 1.1 (2) follows from the local Gan-Gross-Prasad conjecture for $(\mathrm{SO}(5), \mathrm{SO}(2))$ by Prasad-Takloo–Bighash [92, Theorem 2] (see also Waldspurger [112] in general case) at finite places and by Luo [77] at archimedean places. We shall give another proof of this uniqueness by reducing it to a similar assertion in the unitary group case.

1.6. **Refined Gross-Prasad conjecture.** Let (π, V_π) be an irreducible cuspidal tempered automorphic representation of $G_D(\mathbb{A})$ with trivial central character. For $\phi_1, \phi_2 \in V_\pi$, we define the Petersson inner product $(\phi_1, \phi_2)_\pi$ on V_π by

$$(\phi_1, \phi_2)_\pi = \int_{Z_D(\mathbb{A}) G_D(F) \backslash G_D(\mathbb{A})} \phi_1(g) \overline{\phi_2(g)}\, dg$$

where dg denotes the Tamagawa measure. Then at each place v of F, we take a $G_D(F_v)$-invariant hermitian inner product on V_{π_v} so that we have a decomposition $(\ ,\ )_\pi = \prod_v (\ ,\ )_{\pi_v}$. In the definition of the Bessel period (1.4.11), we take $dr = dt\, du$ where dt and du are the Tamagawa measures on $T_\xi(\mathbb{A})$ and $N_D(\mathbb{Z})$, respectively. We take and fix the local measures du_v and dt_v so that $du = \prod_v du_v$ and

$$dt = C_\xi \prod dt_v \tag{1.6.1}$$

where C_ξ is a constant called the Haar measure constant in [57]. Then the local Bessel period $\alpha_v^{\xi,\Lambda} : V_{\pi_v} \times V_{\pi_v} \to \mathbb{C}$ and the local hermitian inner product $(\ ,\ )_{\pi_v}$ are defined as in Section 2.4.

Suppose that D is not split. Then by Li [74], there exists a pair (ξ', Λ') such that π has the (ξ', Λ', ψ)-Bessel period. Here $\xi' \in D^-(F)$ such that $E' := F(\xi')$ is a quadratic extension of F and Λ' is a character on $\mathbb{A}_{E'}^{\times}/\mathbb{A}^{\times} E'^{\times}$. Then by Proposition 4.1, which is a consequence of the proof of Theorem 1.1 (1), there exists an irreducible cuspidal automorphic representation π° of $G(\mathbb{A})$ which is generic and locally G^+-equivalent to π. We take the functorial lift of π° to $\mathrm{GL}_4(\mathbb{A})$ by Cogdell, Kim, Piatetski-Shapiro and Shahidi [19], which is of the form $\Pi_1 \boxplus \cdots \boxplus \Pi_{\ell_0}$ with Π_i an irreducible cuspidal automorphic representation of $\mathrm{GL}_{m_i}(\mathbb{A})$ for each i. Then we define an integer $\ell(\pi)$ by $\ell(\pi) = \ell_0$. We note that π° may not be unique, but $\ell(\pi)$ does not depend on the choice of the pair (ξ', Λ') by Proposition 4.1 and Lemma 4.2, 4.3, and thus it depends only on (π, V_π). When D is split, then π has the functorial lift to $\mathrm{GL}_4(\mathbb{A})$ by Arthur [3] (see also Cai-Friedberg-Kaplan [14]) and we define $\ell(\pi)$ in a similar way.

Our second main result is the refined Gross-Prasad conjecture formulated by Liu [76], i.e. the Ichino-Ikeda type explicit central value formula, in the case of $(\mathrm{SO}(5), \mathrm{SO}(2))$.

Theorem 1.2. *Let (π, V_π) be an irreducible cuspidal tempered automorphic representation of $G_D(\mathbb{A})$ with a trivial central character.*

Then for any non-zero decomposable cusp form $\phi = \otimes_v \phi_v \in V_\pi$, we have

$$\begin{aligned}(1.6.2)\quad & \frac{\left|B_{\xi,\Lambda,\psi}(\phi)\right|^2}{(\phi,\phi)_\pi} \\ & = 2^{-\ell(\pi)} C_\xi \cdot \left(\prod_{j=1}^{2} \zeta_F(2j)\right) \frac{L\left(\frac{1}{2}, \pi \times \mathcal{AI}(\Lambda)\right)}{L(1,\pi,\mathrm{Ad})\, L(1,\chi_E)} \cdot \prod_v \frac{\alpha_v^\natural(\phi_v)}{(\phi_v,\phi_v)_{\pi_v}}.\end{aligned}$$

Here $\zeta_F(s)$ denotes the complete zeta function of F and $\alpha_v^\natural(\phi_v)$ is defined by

$$\alpha_v^\natural(\phi_v) = \frac{L(1,\pi_v,\mathrm{Ad})\, L(1,\chi_{E,v})}{L(1/2, \pi_v \times \Pi(\Lambda)_v) \prod_{j=1}^2 \zeta_{F_v}(2j)} \cdot \alpha_{\Lambda_v,\psi_{\xi,v}}(\phi_v,\phi_v).$$

We note that $\dfrac{\alpha_v^\natural(\phi_v)}{(\phi_v,\phi_v)_{\pi_v}} = 1$ for almost all places v of F by [76].

Remark 1.5. *Under the assumption* (1.5.4)*, we have $|\mathcal{S}(\phi_\pi)| = 2^{\ell(\pi)}$, where ϕ_π denotes the Arthur parameter of π and $\mathcal{S}(\phi_\pi)$ the centralizer of ϕ_π in the complex dual group $\hat{G}$. Hence* (1.6.2) *coincides with the conjectural formula in Liu* [76, Conjecture 2.5 (3)]. *Thus when D is split, i.e. $G_D \simeq G$, our theorem proves Liu's conjecture since the assumption* (1.5.4) *is indeed fulfilled. After submitting this paper, Ishimoto posted a preprint* [59] *on arXiv, in which he gives the endoscopic classification of representations of non-quasi split orthogonal groups for generic Arthur parameters. Hence, our theorem proves* [76, Conjecture 2.5 (3)] *completely in the case of* $(\mathrm{SO}(5), \mathrm{SO}(2))$.

Remark 1.6. *Let π_{gen} denote the irreducible cuspidal globally generic automorphic representation of $G(\mathbb{A})$ which has the same L-parameter as π. When π_v is unramified at any finite place v of F, Chen and Ichino* [17] *proved an explicit formula of the ratio $L(1,\pi,\mathrm{Ad})/(\Phi_{\mathrm{gen}}, \Phi_{\mathrm{gen}})$ for a suitably normalized cusp form Φ_{gen} in the space of π_{gen}.*

Remark 1.7. *In the unitary case, a remarkable progress has been made in the Gan-Gross-Prasad conjecture and its refinement for Bessel periods, by studying the Jacquet-Rallis relative trace formula. In the striking paper* [10] *by Beuzart-Plessis, Liu, Zhang and Zhu, a proof in the co-dimension one case for irreducible cuspidal tempered automorphic representations of unitary groups such that their* *base change lifts are cuspidal was given by establishing an ingenious method to isolate the cuspidal spectrum. In yet another striking paper by Beuzart-Plessis, Chaudouard and Zydor* [9]*, a proof for all endoscopic cases in the co-dimension one setting was given by a precise study of the relative trace formula. Very recently, in a remarkable preprint by Beuzart-Plessis and Chaudouard* [8]*, the above results are extended to arbitrary co-dimension cases. Thus the Gan-Gross-Prasad conjecture and its refinement for Bessel periods on unitary groups are now proved in general.*

On the contrary, the orthogonal case in general is still open. We note that, in the $(\mathrm{SO}(5), \mathrm{SO}(2))$ *case, the first author has formulated relative trace formulas to approach the formula* (1.6.2) *and proved the fundamental lemmas in his joint work with Shalika* [30]*, Martin* [24] *and Matrin-Shalika* [25]*. In order to deduce the L-value formula from these relative trace formulas, several issues such as smooth transfer of test functions must be overcome. In the above mentioned co-dimension one unitary group case, reductions to Lie algebras played crucial roles to solve similar issues. However Bessel periods in our case involves integration over unipotent subgroups and it is not clear, at least to the first author, how to make the reduction to Lie algebras work.*

Remark 1.8. *In the co-dimension one orthogonal group case, the refined Gross-Prasad conjecture has been deduced from the Waldspurger formula* [112] *in the* $(\mathrm{SO}(3), \mathrm{SO}(2))$ *case and from the Ichino formula* [56] *in the* $(\mathrm{SO}(4), \mathrm{SO}(3))$ *case, respectively. Gan and Ichino* [39] *studied the* $(\mathrm{SO}(5), \mathrm{SO}(4))$*-case when the representation of* $\mathrm{SO}(5)$ *is a theta lift from* $\mathrm{GSO}(4)$ *by reduction to the* $(\mathrm{SO}(4), \mathrm{SO}(3))$ *case.*

Liu [76] *proved Theorem 1.2 when D is split and π is an endoscopic lift, i.e. a Yoshida lift, by reducing it to the Waldspurger formula* [112]*. The case when π is a non-endoscopic Yoshida lift was proved later by Corbett* [20] *in a similar manner.*

As a corollary of Theorem 1.2, we prove the $(\mathrm{SO}(5), \mathrm{SO}(2))$ case of the Gan-Gross-Prasad conjecture in the form as stated in [32, Conjecture 24.1].

Corollary 1.1. *Let (π, V_π) be an irreducible cuspidal tempered automorphic representation of $G_D(\mathbb{A})$ with a trivial central character. Then the following three conditions are equivalent.*

(1) *The (ξ, Λ, ψ)-Bessel period does not vanish on π.*
(2) $L\left(\frac{1}{2}, \pi \times \mathcal{AI}(\Lambda)\right) \neq 0$ *and the local Bessel period* $\alpha_{\Lambda_v, \psi_{\xi, v}} \not\equiv 0$ *on* π_v *at any place v of F.*
(3) $L\left(\frac{1}{2}, \pi \times \mathcal{AI}(\Lambda)\right) \neq 0$ *and* $\mathrm{Hom}_{R_{\xi,v}}\left(\pi_v, \chi_v^{\xi,\Lambda}\right) \neq \{0\}$ *at any place v of F.*

Remark 1.9. *The equivalence between the conditions (1) and (2) is immediate from Theorem 1.2. The equivalence*

$$\alpha_{\Lambda_v, \psi_{\xi, v}} \not\equiv 0 \Longleftrightarrow \mathrm{Hom}_{R_{\xi,v}}\left(\pi_v, \chi_v^{\xi,\Lambda}\right) \neq \{0\} \tag{1.6.3}$$

is proved by Waldspurger [115] *at any non-archimedean place v and by Luo* [77] *recently at any archimedean place v, respectively.*

1.7. **Method.** In [27] and [28] we used the theta correspondence for the dual pair $(\mathrm{SO}(2n+1), \mathrm{Mp}_n)$.

The main tool in [27] was the pull-back formula by the first author [23] for the Whittaker period on Mp_n, which is expressed by a certain integral involving the *Special* Bessel period on $\mathrm{SO}(2n+1)$. This forced us the restriction that the character Λ on $\mathrm{SO}(2)$ is trivial.

In [28], to prove the refined Gross-Prasad conjecture for $(\mathrm{SO}(2n+1), \mathrm{SO}(2))$ when Λ is trivial, the following additional restrictions were necessary:

1. (1) The base field F is totally real and at every archimedean place v of F, the representation π_v is a discrete series representation.
2. (2) The assumption (1.5.4).

Additional main tool needed in [28] was the Ichino-Ikeda type formula for the Whittaker periods on Mp_n by Lapid and Mao [72], which imposed on us the condition (1). In fact, their proof was to reduce the global identity to certain local identities. They proved the local identities in general at non-archimedean places. On the other hand, at archimedean places, their proof was to note the equivalence between their local identities and the formal degree conjecture by Hiraga-Ichino-Ikeda [49, 50] and then to prove the latter when π is a discrete series representation. Our proof in [28] was to reduce to the case when π has the special Bessel period by the assumption (1.5.4) and to combine these two main tools with the Siegel-Weil formula.

It does not seem plausible that a straightforward generalization of the method of [27] and [28] would allow us to remove these restrictions. Thus we need to adopt a new strategy in this paper.

Our main method here is again theta correspondence but we use it differently and in a more intricate way. First we consider the quaternionic dual pair $(G_D^+, \mathrm{GSU}_{3,D})$ where $\mathrm{GSU}_{3,D}$ denotes the identity component of the similitude quaternion unitary group $\mathrm{GU}_{3,D}$ defined by (2.1.9) and G_D^+ defined by (1.5.1). Then we recall the accidental isomorphism

$$\mathrm{PGSU}_{3,D} \simeq \mathrm{PGU}_{4,\varepsilon} \tag{1.7.1}$$

when $D \simeq D_\varepsilon$ given by (2.1.1) and $\mathrm{GU}_{4,\varepsilon}$ is the similitude unitary group defined by (2.1.14). Hence we have

$$\mathrm{GU}_{4,\varepsilon} \simeq \begin{cases} \mathrm{GU}_{2,2}, & \text{when } D \text{ is split, i.e. } \varepsilon \in \mathrm{N}_{E/F}(E^\times); \\ \mathrm{GU}_{3,1}, & \text{when } D \text{ is non-split, i.e. } \varepsilon \notin \mathrm{N}_{E/F}(E^\times). \end{cases} \tag{1.7.2}$$

Thus our theta correspondence for $(G_D^+, \mathrm{GSU}_{3,D})$ induces a correspondence for the pair $(\mathbb{G}_D, \mathrm{PGU}_{4,\varepsilon})$. Then we note that the pull-back of a certain Bessel period on $\mathrm{PGU}_{4,\varepsilon}$ is an integral involving the (ξ, Λ, ψ)-Bessel period on G_D.

Theorem 1.1 is reduced essentially to the Gan-Gross-Prasad conjecture for the Bessel periods on $\mathrm{GU}_{4,\varepsilon}$, which we proved in [29] using the theta correspondence for the pair $(\mathrm{GU}_{4,\varepsilon}, \mathrm{GU}_{2,2})$.

Similarly Theorem 1.2 is reduced to the refined Gan-Gross-Prasad conjecture for the Bessel periods on $\mathrm{GU}_{4,\varepsilon}$. For the reader's sake, here we present an outline of the proof when the (ξ, Λ, ψ)-Bessel period does not vanish. Note that in the following paragraph the notation used is provisionally and the argument is not rigorous since our intention here is to present a rough sketch of the main idea.

Let (π, V_π) be an irreducible cuspidal tempered automorphic representation of $G_D(\mathbb{A})$ with a trivial central character. Suppose that the (ξ, Λ, ψ)-Bessel period, which we denote by B, does not vanish on π. Let $\theta(\pi)$ be the theta lift of π to $\mathrm{GSU}_{3,D}$. When $G_D = G$ and the theta lift of π to $\mathrm{GSO}_{3,1}$ is non-zero, $\theta(\pi)$ is not cuspidal but the explicit formula (1.6.2) has been already proved by Corbett [20]. Thus suppose otherwise. Then $\theta(\pi)$ is a non-zero irreducible cuspidal tempered automorphic representation. The pull-back of a certain Bessel period, which we denote by $\mathcal{B}$ on $\mathrm{GSU}_{3,D}$ is written as an integral involving B. As in our previous paper [28], the explicit formula for B is reduced to the one for $\mathcal{B}$, which we obtain in the following steps.

(1) Via the isomorphism (1.7.1), regard $\theta(\pi)$ as an automorphic representation of $\mathrm{GU}_{4,\varepsilon}$ and then consider its theta lift $\theta_\Lambda(\theta(\pi))$, which depends on Λ, to $\mathrm{GU}_{2,2}$. The temperedness of π implies that $\theta_\Lambda(\theta(\pi))$ is an irreducible cuspidal automorphic representation of $\mathrm{GU}_{2,2}$. Then the pull-back of a certain Whittaker period $\mathcal{W}$ on $\mathrm{GU}_{2,2}$ is written as an integral involving the Bessel period $\mathcal{B}$. Then in [29], it is shown that the explicit formula for $\mathcal{B}$ follows from the one for $\mathcal{W}$. Thus we are reduced to show the explicit formula for $\mathcal{W}$.
(2) Via the isomorphism $\mathrm{PGU}_{2,2} \simeq \mathrm{PGSO}_{4,2}$, regard $\theta_\Lambda(\theta(\pi))$ as an automorphic representation of $\mathrm{GSO}_{4,2}$. Let π' be the theta lift of $\theta_\Lambda(\theta(\pi))$ to $G = \mathrm{GSp}_2$. Then it is shown that π' is a globally generic cuspidal automorphic representation of G and indeed the pull-back of the Whittaker period W on G is expressed as an integral involving $\mathcal{W}$. Hence we are reduced to the explicit formula for W.
(3) Since the theta lift of the globally generic cuspidal automorphic representation π' of G to either $\mathrm{GSO}_{2,2}$ or $\mathrm{GSO}_{3,3}$ is non-zero and cuspidal, we are further reduced to the explicit formulas for the Whittaker periods on $\mathrm{PGSO}_{2,2}$ and $\mathrm{PGSO}_{3,3}$ by the pull-back computation.
(4) Recall the accidental isomorphisms $\mathrm{PGSO}_{2,2} \simeq \mathrm{PGL}_2 \times \mathrm{PGL}_2$, $\mathrm{PGSO}_{3,3} \simeq \mathrm{PGL}_4$. Since the explicit formula for the Whittaker period on PGL_n is already proved by Lapid and Mao [71], we are done.

Remark 1.10. *Though we only consider the case when* $\mathrm{SO}(2)$ *is non-split in this paper, the split case is proved by a similar argument as follows. First we note that* D *is necessarily split when* $\mathrm{SO}(2)$ *is split and hence* $G_D \simeq G$*. If the theta lift to* $\mathrm{GSO}_{2,2}$ *is non-zero, it is a Yoshida lift and Liu* [76] *proved the explicit formula. Suppose otherwise. Then the theta lift to* $\mathrm{GSO}_{3,3}$ *is non-zero and cuspidal. The pull-back of a certain Bessel period on* $\mathrm{GSO}_{3,3}$ *is an integral involving the split Bessel period on* G *(see Section 3.1.2). We recall the accidental isomorphism* $\mathrm{PGSO}_{3,3} \simeq \mathrm{PGL}_4$*. We consider the theta correspondence for the pair* $(\mathrm{GL}_4, \mathrm{GL}_4)$ *instead of* $(\mathrm{GU}_{4,\varepsilon}, \mathrm{GU}_{4,\varepsilon})$ *in the non-split case. Then the pull-back computation may be interpreted as expressing the pull-back of the Whittaker period on* GL_4 *as* *an integral involving the Bessel period on* $\mathrm{GSO}_{3,3}$*, which is given in* [29]*. Thus as in the non-split case, we are reduced to the Ichino-Ikeda type explicit formula for the Whittaker period on* GL_4.

Here is the statement of the theorem in the split case.

Theorem 1.3. *Let* (π, V_π) *be an irreducible cuspidal automorphic representation of* $G(\mathbb{A})$ *with trivial central character. Suppose that* D *is split and the Arthur parameter of* π *is generic.*

Let $\xi \in D^-(F)$ *such that* $F(\xi) \simeq F \oplus F$ *and fix an* F*-isomorphism* $T_\xi \simeq F^\times \times F^\times$*. For a character* Λ *of* $\mathbb{A}^\times / F^\times$*, we also denote by* Λ *the character of* $T_\xi(\mathbb{A})$ *defined by* $\Lambda(a, b) := \Lambda\left(ab^{-1}\right)$.

The following assertions hold.

(1) *The* (ξ, Λ, ψ)*-Bessel period does not vanish on* V_π *if and only if* π *is generic and* $L\left(\frac{1}{2}, \pi \times \Lambda\right) \neq 0$*. Here we note that* $L\left(\frac{1}{2}, \pi \times \Lambda^{-1}\right)$ *is the complex conjugate of* $L\left(\frac{1}{2}, \pi \times \Lambda\right)$ *since* π *is self-dual.*
(2) *Further assume that* π *is tempered. Then for any non-zero decomposable cusp form* $\phi = \otimes_v \phi_v \in V_\pi$*, we have*

$$\frac{\left|B_{\xi,\Lambda,\psi}(\phi)\right|^2}{(\phi,\phi)_\pi} = 2^{-\ell(\pi)}\, C_\xi \cdot \left(\prod_{j=1}^{2} \zeta_F(2j)\right) \times \frac{L\left(\frac{1}{2}, \pi \times \Lambda\right) L\left(\frac{1}{2}, \pi \times \Lambda^{-1}\right)}{L(1, \pi, \mathrm{Ad})\, \zeta_F(1)} \cdot \prod_v \frac{\alpha_v^\natural(\phi_v)}{(\phi_v, \phi_v)_{\pi_v}}$$

where $\zeta_F(1)$ *stands for* $\mathrm{Res}_{s=1}\, \zeta_F(s)$.

1.8. **Generalized Böcherer conjecture.** Thanks to the meticulous local computation by Dickson, Pitale, Saha and Schmidt [21], Theorem 1.2 implies the generalized Böcherer conjecture. For brevity we only state the scalar valued full modular case here in the introduction. Indeed a more general version shall be proved in 8.3 as Theorem 8.1.

Theorem 1.4. *Let* Φ *be a holomorphic Siegel cusp form of degree two and weight* k *with respect to* $\mathrm{Sp}_2(\mathbb{Z})$ *which is a Hecke eigenform and* $\pi(\Phi)$ *the associated automorphic representation of* $\mathbb{G}\left(\mathbb{A}_\mathbb{Q}\right)$*. Let*

$$\Phi(Z) = \sum_{T>0} a(\Phi, T) \exp\left[2\pi\sqrt{-1}\,\mathrm{tr}(TZ)\right], \quad Z \in \mathfrak{H}_2, \tag{1.8.1}$$

be the Fourier expansion of Φ *where* T *runs over semi-integral positive definite two by two symmetric matrices and* $\mathfrak{H}_2$ *denotes the Siegel upper half space of degree two.*

Let E *be an imaginary quadratic extension of* $\mathbb{Q}$*. We denote by* $-D_E$ *its discriminant,* Cl_E *its ideal class group and* $w(E)$ *the number of distinct roots of unity in* E*. In* (1.8.1)*, when* $T' = {}^t\gamma T \gamma$ *for some* $\gamma \in \mathrm{SL}_2(\mathbb{Z})$*, we have* $a(\Phi, T') = a(\Phi, T)$*. By the Gauss composition law, we may naturally identify the* $\mathrm{SL}_2(\mathbb{Z})$*-equivalence* *classes of binary quadratic forms of discriminant* $-D_E$ *with the elements of* Cl_E*. Thus the notation* $a(\Phi, c)$ *for* $c \in \mathrm{Cl}_E$ *makes sense. For a character* Λ *of* Cl_E*, we define* $\mathcal{B}_\Lambda(\Phi, E)$ *by*

$$\mathcal{B}_\Lambda(\Phi, E) := w(E)^{-1} \cdot \sum_{c \in \mathrm{Cl}_E} a(\Phi, c)\, \Lambda^{-1}(c).$$

Suppose that Φ *is not a Saito-Kurokawa lift. Then we have*

$$\frac{|\mathcal{B}_\Lambda(\Phi, E)|^2}{\langle \Phi, \Phi \rangle} = 2^{2k-4} \cdot D_E^{k-1} \cdot \frac{L\left(\frac{1}{2}, \pi(\Phi) \times \mathcal{AI}(\Lambda)\right)}{L(1, \pi(\Phi), \mathrm{Ad})}. \tag{1.8.2}$$

Here

$$\langle \Phi, \Phi \rangle = \int_{\mathrm{Sp}_2(\mathbb{Z}) \backslash \mathfrak{H}_2} |\Phi(Z)|^2 \det(Y)^{k-3}\, dX\, dY \quad \textit{where } Z = X + \sqrt{-1}\, Y.$$

Remark 1.11. *In Theorem 8.1, we prove* (1.8.2) *allowing* Φ *to have a square-free level and to be vector-valued. Moreover, assuming the temperedness of* $\pi(\Phi)$*, the weight* 2 *case, which is of significant interest because of the modularity conjecture for abelian surfaces, is also included.*

The formula (1.8.2) *and its generalization* (8.3.1) *are expected to have a broad spectrum of interesting applications both arithmetic and analytic. Some of the examples are* [12]*,* [21, Section 3]*,* [22]*,*[55]*,* [97] *and* [111].

1.9. **Organization of the paper.** This paper is organized as follows. In Section 2, we introduce some more notation and define local and global Bessel periods. In Section 3, we carry out the pull-back computation of Bessel periods. In Section 4, we shall prove Theorem 1.1 using the results in Section 3. We also note some consequences of our proof of Theorem 1.1 (1), which will be used in the proof of Theorem 1.2 later. In Section 5, we recall the Rallis inner product formula for similitude groups. In Section 6, we will give an explicit formula for Bessel periods on $\mathrm{GU}_{4,\varepsilon}$ in certain cases as explained in our strategy for the proof of Theorem 1.2. In Section 7, we complete our proof of Theorem 1.2. In Section 8, we prove the generalized Böcherer conjecture, including the vector valued case. In Appendix A, we will give an explicit formula of Whittaker periods for irreducible cuspidal tempered automorphic representations of G. In Appendix B, we compute the local Bessel periods explicitly for representation of $G(\mathbb{R})$ corresponding to vector valued holomorphic Siegel modular forms. This result is used in Section 8. In Appendix C, we consider the meromorphic continuation of the L-function for $\mathrm{SO}(5) \times \mathrm{SO}(2)$.

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2. Preliminaries

2.1. **Groups.**

2.1.1. *Quaternion algebras.* Let $X(E : F)$ denote the set of F-isomorphism classes of central simple algebras over F containing E. Then we recall that the map $\varepsilon \mapsto D_\varepsilon$ gives a bijection between $F^\times / \mathrm{N}_{E/F}(E^\times)$ and $X(E : F)$ (see [30, Lemma 1.3]) where

$$D_\varepsilon := \left\{ \begin{pmatrix} a & \varepsilon b \\ b^\sigma & a^\sigma \end{pmatrix} : a, b \in E \right\} \quad \text{for } \varepsilon \in F^\times. \tag{2.1.1}$$

Here we regard E as a subalgebra of D_ε by

$$E \ni a \mapsto \begin{pmatrix} a & 0 \\ 0 & a^\sigma \end{pmatrix} \in D_\varepsilon.$$

We also note that $D_\varepsilon \simeq \mathrm{Mat}_{2\times 2}(F)$ when $\varepsilon \in \mathrm{N}_{E/F}(E^\times)$. The canonical involution $D_\varepsilon \ni x \mapsto \bar{x} \in D_\varepsilon$ is given by

$$\bar{x} = \begin{pmatrix} a^\sigma & -\varepsilon b \\ -b^\sigma & a \end{pmatrix} \quad \text{for } x = \begin{pmatrix} a & \varepsilon b \\ b^\sigma & a^\sigma \end{pmatrix}.$$

We denote the reduced trace of D by tr_D.

2.1.2. *Orthogonal groups.* For a non-negative integer n, a symmetric matrix $S_n \in \mathrm{Mat}_{(2n+2)\times(2n+2)}(F)$ is defined inductively by

$$S_0 := \begin{pmatrix} 2 & 0 \\ 0 & -2d \end{pmatrix} \quad \text{and} \quad S_n := \begin{pmatrix} 0 & 0 & 1 \\ 0 & S_{n-1} & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \text{for } n \geq 1. \tag{2.1.2}$$

We recall that $E = F(\eta)$ where $\eta^2 = d$. Then we write the corresponding orthogonal group, the special orthogonal group and the similitude orthogonal group by

$$\mathrm{O}(S_n) = \mathrm{O}_{n+2,n}, \quad \mathrm{SO}(S_n) = \mathrm{SO}_{n+2,n} \quad \text{and} \quad \mathrm{GO}(S_n) = \mathrm{GO}_{n+2,n}, \tag{2.1.3}$$

respectively. Let $\mathrm{GSO}_{n+2,n}$ denote the identity component of $\mathrm{GO}_{n+2,n}$. Thus

$$\mathrm{GSO}_{n+2,n}(F) = \{ g \in \mathrm{GO}_{n+2,n}(F) : \det(g) = \lambda(g)^{n+1} \} \tag{2.1.4}$$

where

$$\mathrm{GO}_{n+2,n}(F) = \left\{ g \in \mathrm{GL}_{2n+2}(F) : {}^t g\, S_n\, g = \lambda(g) S_n,\ \lambda(g) \in F^\times \right\}. \tag{2.1.5}$$

For a positive integer n, we denote by J_{2n} the $2n \times 2n$ symmetric matrix with ones on the non-principal diagonal and zeros elsewhere, i.e.

$$J_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad J_{2(n+1)} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & J_{2n} & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \text{for } n \geq 1. \tag{2.1.6}$$

Then the similitude orthogonal group $\mathrm{GO}_{n,n}$ is defined by

$$\mathrm{GO}_{n,n}(F) := \left\{ g \in \mathrm{GL}_{2n}(F) : {}^t g\, J_{2n}\, g = \lambda(g) J_{2n},\ \lambda(g) \in F^\times \right\} \tag{2.1.7}$$

and we denote by $\mathrm{GSO}_{n,n}$ its identity component, which is given by

$$\mathrm{GSO}_{n,n}(F) = \{g \in \mathrm{GO}_{n,n}(F) : \det(g) = \lambda(g)^n\}. \tag{2.1.8}$$

2.1.3. *Quaternionic unitary groups.* Let D be a quaternion algebra over F containing E. Recall that G_D denotes the similitude quaternionic unitary group of degree 2 defined by (1.4.2).

We define a similitude quaternionic unitary group $\mathrm{GU}_{3,D}$ of degree 3 by

$$\mathrm{GU}_{3,D}(F) := \left\{g \in \mathrm{GL}_3(D) : {}^t\bar{g}\,\mathbf{J}_\eta\, g = \lambda(g)\mathbf{J}_\eta,\ \lambda(g) \in F^\times\right\} \tag{2.1.9}$$

where we define a skew-hermitian matrix $\mathbf{J}_\eta$ by

$$\mathbf{J}_\eta := \begin{pmatrix} 0 & 0 & \eta \\ 0 & \eta & 0 \\ \eta & 0 & 0 \end{pmatrix}. \tag{2.1.10}$$

Here $\bar{A} = (\bar{a}_{ij})$ for $A = (a_{ij}) \in \mathrm{Mat}_{m\times n}(D)$. Let us denote by $\mathrm{GSU}_{3,D}$ the identity component of $\mathrm{GU}_{3,D}$. Then unlike the orthogonal case, as noted in [81, p.21–22], we have

$$\mathrm{GSU}_{3,D}(F) = \mathrm{GU}_{3,D}(F)$$

and

$$\mathrm{GSU}_{3,D}(F_v) = \mathrm{GU}_{3,D}(F_v) \text{ when } D \otimes_F F_v \text{ is not split.}$$

Moreover when $D \otimes_F F_v$ is split at a place v of F, we have

$$\mathrm{GU}_{3,D}(F_v) \simeq \begin{cases} \mathrm{GO}_{4,2}(F_v) & \text{if } E \otimes F_v \text{ is a quadratic extension of } F_v; \\ \mathrm{GO}_{3,3}(F_v) & \text{if } E \otimes F_v \simeq F_v \oplus F_v. \end{cases} \tag{2.1.11}$$

We also define $\mathrm{GU}_{1,D}$ by

$$\mathrm{GU}_{1,D}(F) := \left\{\alpha \in D^\times : \bar{\alpha}\eta\alpha = \lambda(\alpha)\,\eta,\ \lambda(\alpha) \in F^\times\right\} \tag{2.1.12}$$

and denote its identity component by $\mathrm{GSU}_{1,D}$. Then we note that

$$\begin{aligned} \mathrm{GSU}_{1,D}(F) &= \left\{\alpha \in D^\times : \bar{\alpha}\eta\alpha = \mathrm{n}_D(\alpha)\,\eta\right\} \\ &= \left\{x \in D^\times \mid x\eta = \eta x\right\} = T_\eta \end{aligned} \tag{2.1.13}$$

where T_η is defined by (1.4.6) with $\xi = \eta$ and n_D denotes the reduced norm of D.

2.1.4. *Unitary groups.* Suppose that $D = D_\varepsilon$ defined by (2.1.1). Then we define $\mathrm{GU}_{4,\varepsilon}$ a similitude unitary group of degree 4 by

$$\mathrm{GU}_{4,\varepsilon}(F) := \left\{g \in \mathrm{GL}_4(E) : {}^t g^\sigma \mathcal{J}_\varepsilon g = \lambda(g)\mathcal{J}_\varepsilon,\ \lambda(g) \in F^\times\right\} \tag{2.1.14}$$

where we define a hermitian matrix $\mathcal{J}_\varepsilon$ by

$$\mathcal{J}_\varepsilon := \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & \varepsilon & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Here $A^\sigma = \left(a_{ij}^\sigma\right)$ for $A = \left(a_{ij}\right) \in \mathrm{Mat}_{m\times n}(E)$. Then we have

$$\mathrm{GU}_{4,\varepsilon} \simeq \begin{cases} \mathrm{GU}_{2,2}, & \text{when } D \text{ is split, i.e. } \varepsilon \in \mathrm{N}_{E/F}(E^\times); \\ \mathrm{GU}_{3,1}, & \text{when } D \text{ is non-split, i.e. } \varepsilon \notin \mathrm{N}_{E/F}(E^\times). \end{cases} \tag{2.1.15}$$

We also define $\mathrm{GU}_{2,\varepsilon}$ a similitude unitary group of degree 2 by

$$\mathrm{GU}_{2,\varepsilon}(F) := \left\{g \in \mathrm{GL}_2(E) : {}^t g^\sigma J_\varepsilon g = \lambda(g) J_\varepsilon,\ \lambda(g) \in F^\times\right\} \tag{2.1.16}$$

$$\text{where} \quad J_\varepsilon = \begin{pmatrix} -1 & 0 \\ 0 & \varepsilon \end{pmatrix}.$$

2.2. **Accidental isomorphisms.** We need to explicate the accidental isomorphisms of our concern, since we use them in a crucial way to transfer an automorphic period on one group to the one on the other group. The reader may consult, for example, Satake [102] and Tsukamoto [108] about the details of the material here.

2.2.1. $\mathrm{PGSU}_{3,D} \simeq \mathrm{PGU}_{4,\varepsilon}$. Suppose that $D = D_\varepsilon$. Then we may naturally realize $\mathrm{GSU}_{3,D}(F)$ as a subgroup of $\mathrm{GL}_6(E)$. We note that

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\varepsilon & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\varepsilon & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\varepsilon \end{pmatrix} {}^t\bar{g} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\varepsilon & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\varepsilon & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\varepsilon \end{pmatrix}^{-1} = {}^t g^\sigma$$

and

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix} {}^t\bar{g} \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}^{-1} = {}^t g.$$

Thus in this realization, we have

$$\mathrm{GSU}_{3,D}(F) = \{g \in \mathrm{GSO}_{3,3}(E) : {}^t g^\sigma \mathcal{J}_\varepsilon^\circ g = \lambda(g)\mathcal{J}_\varepsilon^\circ,\ \lambda(g) \in F^\times\} \tag{2.2.1}$$

$$\text{where} \quad \mathcal{J}_\varepsilon^\circ = -\begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \varepsilon \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \varepsilon & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \varepsilon & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Here we recall that

$$\mathrm{GSO}_{3,3}(E) \simeq \mathrm{GL}_4(E) \times \mathrm{GL}_1(E) / \{(z, z^{-2}) : z \in E^\times\}. \tag{2.2.2}$$

In fact the isomorphism (2.2.2) is realized as follows. Let us take the standard basis

$$b_1 = {}^t(1,0,0,0), \quad b_2 = {}^t(0,1,0,0), \quad b_3 = {}^t(0,0,1,0), \quad b_4 = {}^t(0,0,0,1),$$

of E^4. Then we may consider $V := \wedge^2 E^4$ as an orthogonal space over E with a quadratic form $(\ ,\ )_V$ defined by

$$v_1 \wedge v_2 = (v_1, v_2)_V \cdot b_1 \wedge b_2 \wedge b_3 \wedge b_4$$

for $v_1, v_2 \in V$. As a basis of V over E, we take $\{\varepsilon_i : 1 \le i \le 6\}$ given by

$$\varepsilon_1 = b_1 \wedge b_2, \varepsilon_2 = b_1 \wedge b_3, \varepsilon_3 = b_1 \wedge b_4, \varepsilon_4 = b_2 \wedge b_3, \varepsilon_5 = b_4 \wedge b_2, \varepsilon_6 = b_3 \wedge b_4.$$

Let the group $\mathrm{GL}_4(E) \times \mathrm{GL}_1(E)$ act on V by $(g, a)(w_1 \wedge w_2) = a \cdot (gw_1 \wedge gw_2)$ where $w_1, w_2 \in E^4$. This action defines a homomorphism

$$\mathrm{GL}_4(E) \times \mathrm{GL}_1(E) \to \mathrm{GSO}_{3,3}(E) \tag{2.2.3}$$

where we take $\{\varepsilon_i : 1 \le i \le 6\}$ as a basis of V and the homomorphism (2.2.3) induces the isomorphism (2.2.2). By a direct computation we observe that $(-\mathcal{J}_\varepsilon, 1)$ is mapped to $\mathcal{J}_\varepsilon^\circ$ under (2.2.3) and the restriction of the homomorphism (2.2.3) gives a homomorphism

$$\mathrm{GU}_{4,\varepsilon}(F) \to \mathrm{GSU}_{3,D}(F). \tag{2.2.4}$$

Then it is easily seen that the isomorphism

$$\Phi_D : \mathrm{PGU}_{4,\varepsilon}(F) \xrightarrow{\sim} \mathrm{PGSU}_{3,D}(F) \tag{2.2.5}$$

is induced.

2.2.2. $\mathrm{PGU}_{2,2} \simeq \mathrm{PGSO}_{4,2}$. When $\varepsilon \in \mathrm{N}_{E/F}(E^\times)$, the quaternion algebra $D = D_\varepsilon$ is split and the isomorphism (2.2.5) gives an isomorphism $\mathrm{PGU}_{2,2} \simeq \mathrm{PGSO}_{4,2}$. We recall the concrete realization of this isomorphism. First we define $\mathrm{GU}_{2,2}$ by

$$\mathrm{GU}_{2,2} := \left\{ g \in \mathrm{GL}_4(E) : {}^t g^\sigma J_4\, g = \lambda(g) J_4,\ \lambda(g) \in F^\times \right\}$$

$$\text{where } J_4 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

as (2.1.6). Let

$$\mathcal{V} := \left\{ B\left((x_i)_{1 \le i \le 6}\right) := \begin{pmatrix} 0 & \eta x_1 & x_3 + \eta x_4 & x_2 \\ -\eta x_1 & 0 & x_5 & -x_3 + \eta x_4 \\ -x_3 - \eta x_4 & -x_5 & 0 & \eta^{-1} x_6 \\ -x_2 & x_3 - \eta x_4 & -\eta^{-1} x_6 & 0 \end{pmatrix} : x_i \in F\ (1 \le i \le 6) \right\}.$$

We define $\Psi : \mathcal{V} \to F$ by

$$\Psi(B) := \mathrm{Tr}\left(B \begin{pmatrix} 0 & 1_2 \\ 1_2 & 0 \end{pmatrix} {}^t B^\sigma \begin{pmatrix} 0 & 1_2 \\ 1_2 & 0 \end{pmatrix} \right).$$

Then we have

$$\Psi\left(B\left((x_i)_{1 \le i \le 6}\right)\right) = -4 \left\{ x_1 x_6 + x_2 x_5 - \left(x_3^2 - d x_4^2\right) \right\}.$$

Let $\mathrm{GSU}_{2,2}$ denote the identity component of $\mathrm{GU}_{2,2}$, i.e.

$$\mathrm{GSU}_{2,2} = \{ g \in \mathrm{GU}_{2,2} : \det(g) = \lambda(g)^2 \}.$$

We let $\mathrm{GSU}_{2,2}$ act on $\mathcal{V}$ by

$$\mathrm{GSU}_{2,2} \times \mathcal{V} \ni (g, B) \mapsto (wgw)\, B \left(w\, {}^t g w\right) \in \mathcal{V} \quad \text{where } w = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Then this action induces a homomorphism $\phi : \mathrm{GSU}_{2,2} \to \mathrm{GO}(\mathcal{V})$. We note that

$$\lambda(\phi(g)) = \det(g) \quad \text{for } g \in \mathrm{GSU}_{2,2}$$

and this implies that the image of ϕ is contained in $\mathrm{GSO}(\mathcal{V})$. As a basis of $\mathcal{V}$, we may take

$$f_1 = \begin{pmatrix} 0 & \eta & 0 & 0 \\ -\eta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad f_2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad f_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

$$f_4 = \begin{pmatrix} 0 & 0 & \eta & 0 \\ 0 & 0 & 0 & \eta \\ -\eta & 0 & 0 & 0 \\ 0 & -\eta & 0 & 0 \end{pmatrix}, \quad f_5 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad f_6 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \eta^{-1} \\ 0 & 0 & -\eta^{-1} & 0 \end{pmatrix}.$$

With respect to this basis, we may regard ϕ as a homomorphism from $\mathrm{GSU}_{2,2}$ to $\mathrm{GO}_{4,2}$, where the group $\mathrm{GO}_{4,2}$ is given by (2.1.5) for $n = 2$. Let us consider $\mathrm{GSU}_{2,2} \rtimes E^\times$ where the action of $\alpha \in E^\times$ on $g \in \mathrm{GSU}_{2,2}$ is given by

$$\alpha \cdot g = \begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & (\alpha^\sigma)^{-1} \end{pmatrix} g \begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & (\alpha^\sigma)^{-1} \end{pmatrix}^{-1}.$$

Then as in [83, p.32–34], ϕ may be extended to $\mathrm{GSU}_{2,2} \rtimes E^\times$ and we have a homomorphism $\mathrm{GSU}_{2,2} \rtimes E^\times \to \mathrm{PGSO}_{4,2}$ which induces the isomorphism

$$\Phi : \mathrm{PGU}_{2,2} \xrightarrow{\sim} \mathrm{PGSO}_{4,2}. \tag{2.2.6}$$

2.3. **Bessel periods.** Let us introduce Bessel periods on various groups.

2.3.1. *Bessel periods on* $G = \mathrm{GSp}_2$. Though we already introduced Bessel periods on G_D in general as (1.4.11), we would like to describe them concretely in the case of G here for our explicit pull-back computations in the next section.

Let P be the Siegel parabolic subgroup of G with the Levi decomposition $P = MN$ where

$$M(F) = \left\{ \begin{pmatrix} g & 0 \\ 0 & \lambda \cdot {}^t g^{-1} \end{pmatrix} : \begin{matrix} g \in \mathrm{GL}_2(F), \\ \lambda \in F^\times \end{matrix} \right\}, \quad N(F) = \left\{ \begin{pmatrix} 1 & X \\ 0 & 1 \end{pmatrix} : X \in \mathrm{Sym}_2(F) \right\}.$$

Here $\mathrm{Sym}_n(F)$ denotes the set of n by n symmetric matrices with entries in F for a positive integer n. For $S \in \mathrm{Sym}_2(F)$, let us define a character ψ_S of $N(\mathbb{A})$ by

$$\psi_S \begin{pmatrix} 1 & X \\ 0 & 1 \end{pmatrix} = \psi\left[\mathrm{tr}(SX)\right].$$

For $S \in \mathrm{Sym}_2(F)$ such that $\det S \neq 0$, let

$$T_S := \left\{ g \in \mathrm{GL}_2 : {}^t g S g = \det(g) S \right\}.$$

We identify T_S with the subgroup of G given by

$$\left\{ \begin{pmatrix} g & 0 \\ 0 & \det(g) \cdot {}^t g^{-1} \end{pmatrix} : g \in T_S \right\}.$$

Definition 2.1. *Let us take $S \in \mathrm{Sym}_2(F)$ such that $T_S(F)$ is isomorphic to $E^\times$. Let π be an irreducible cuspidal automorphic representation of $G(\mathbb{A})$ whose central character is trivial and V_π its space of automorphic forms. Fix an F-isomorphism $T_S(F) \simeq E^\times$. Let Λ be a character of $\mathbb{A}_E^\times / E^\times$ such that $\Lambda|_{\mathbb{A}^\times}$ is trivial. We regard Λ as a character of $T_S(\mathbb{A}) / \mathbb{A}^\times T_S(F)$.*

Then for $\varphi \in V_\pi$, we define $B_{S,\Lambda,\psi}(\varphi)$, the (S,Λ,ψ)-Bessel period of φ by

$$B_{S,\Lambda,\psi}(\varphi) = \int_{\mathbb{A}^\times T_S(F) \backslash T_S(\mathbb{A})} \int_{N(F)\backslash N(\mathbb{A})} \varphi(uh) \Lambda^{-1}(h) \psi_S^{-1}(u)\, du\, dh. \tag{2.3.1}$$

We say that π has the (S,Λ,ψ)-Bessel period when $B_{S,\Lambda,\psi} \not\equiv 0$ on V_π. Then we also say that π has the (E,Λ)-Bessel period as in Definition 1.2.

2.3.2. *Bessel periods on* $\mathrm{GSU}_{3,D}$. Let us introduce Bessel periods on the group $\mathrm{GSU}_{3,D}$ defined in 2.1.3. Let $P_{3,D}$ be a maximal parabolic subgroup of $\mathrm{GSU}_{3,D}$ with the Levi decomposition $P_{3,D} = M_{3,D} N_{3,D}$ where

$$M_{3,D} = \left\{ \begin{pmatrix} g & 0 & 0 \\ 0 & h & 0 \\ 0 & 0 & g \end{pmatrix} : \begin{array}{l} g \in D^\times, \\ h \in T_\eta, \\ \mathrm{n}_D(g) = \mathrm{n}_D(h) \end{array} \right\}, \quad N_{3,D} = \left\{ \begin{pmatrix} 1 & A' & B \\ 0 & 1 & A \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{GSU}_{3,D} \right\}.$$

As for T_η, we recall (2.1.13) and $T_\eta \simeq E^\times$. For $X \in D^\times$, we define a character $\psi_{X,D}$ of $N_{3,D}(\mathbb{A})$ by

$$\psi_{X,D} \begin{pmatrix} 1 & A' & B \\ 0 & 1 & A \\ 0 & 0 & 1 \end{pmatrix} = \psi\left[\mathrm{tr}_D(XA)\right].$$

Then the identity component of the stabilizer of $\psi_{X,D}$ in $M_{3,D}$ is

$$M_{X,D} = \left\{ \begin{pmatrix} h^X & 0 & 0 \\ 0 & h & 0 \\ 0 & 0 & h^X \end{pmatrix} : h \in T_\eta \right\} \quad \text{where} \quad h^X = XhX^{-1}.$$

We identify M_X with T_η by

$$M_{X,D} \ni \begin{pmatrix} h^X & 0 & 0 \\ 0 & h & 0 \\ 0 & 0 & h^X \end{pmatrix} \mapsto h \in T_\eta \tag{2.3.2}$$

and we fix an F-isomorphism $T_\eta \simeq E^\times$.

Definition 2.2. *Let σ_D be an irreducible cuspidal automorphic representation of $\mathrm{GSU}_{3,D}(\mathbb{A})$ and V_{σ_D} its space of automorphic forms. Let χ be a character of $\mathbb{A}_E^\times/E^\times$ and we regard χ as a character of $M_{X,D}(\mathbb{A})/M_{X,D}(F)$. Suppose that $\chi|_{\mathbb{A}^\times} = \omega_{\sigma_D}$, the central character of σ_D.*

Then for $\varphi \in V_{\sigma_D}$, we define $\mathcal{B}^D_{X,\chi,\psi}(\varphi)$, the (X,χ,ψ)-Bessel period of φ by

$$(2.3.3)\quad \mathcal{B}^D_{X,\chi,\psi}(\varphi) = \int_{\mathbb{A}^\times M_{X,D}(F)\backslash M_{X,D}(\mathbb{A})} \int_{N_{3,D}(F)\backslash N_{3,D}(\mathbb{A})} \varphi(uh) \times \chi(h)^{-1}\psi_{X,D}(u)^{-1}\,du\,dh.$$

2.3.3. *Bessel periods on* $\mathrm{GU}_{4,\varepsilon}$. In light of the accidental isomorphism (2.2.5), Bessel periods on the group $\mathrm{GU}_{4,\varepsilon}$ is defined as follows.

Let $P_{4,\varepsilon}$ be a maximal parabolic subgroup of $\mathrm{GU}_{4,\varepsilon}$ with the Levi decomposition $M_{4,\varepsilon}N_{4,\varepsilon}$ where

$$M_{4,\varepsilon}(F) = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & g & 0 \\ 0 & 0 & \lambda(g)(a^\sigma)^{-1} \end{pmatrix} : a \in E^\times, g \in \mathrm{GU}_{2,\varepsilon}(F) \right\},$$

$$N_{4,\varepsilon}(F) = \left\{ \begin{pmatrix} 1 & A & B \\ 0 & 1_2 & A' \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{GU}_{4,\varepsilon}(F) \right\}.$$

Let us take an anisotropic vector $e \in E^4$ of the form ${}^t(0,*,*,0)$. Then we define a character χ_e of $N_{4,\varepsilon}(\mathbb{A})$ by

$$\chi_e(u) = \psi((ue, b_1)_\varepsilon) \quad \text{where } (x,y)_\varepsilon = {}^t x^\sigma J_\varepsilon y.$$

Here we recall that J_ε is as given in (2.1.16) and $b_1 = {}^t(1,0,0,0)$. Let D_e denote the subgroup of $M_{4,\varepsilon}$ given by

$$D_e := \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & h & 0 \\ 0 & 0 & 1 \end{pmatrix} : h \in \mathrm{U}_{2,\varepsilon},\ he = e \right\}.$$

Then the group $D_e(\mathbb{A})$ stabilizes the character χ_e by conjugation. We note that

$$D_e(F) \simeq \mathrm{U}_1(F) := \{a \in E^\times : \bar{a}a = 1\}.$$

Hence for a character Λ of $\mathbb{A}_E^\times$ which is trivial on $\mathbb{A}^\times$, we may regard Λ as a character of $D_e(\mathbb{A})$ by $d \mapsto \Lambda(\det d)$. Then we define a character $\chi_{e,\Lambda}$ of $R_e(\mathbb{A})$ where $R_e := D_e N_{4,\varepsilon}$ by

$$(2.3.4)\qquad \chi_{e,\Lambda}(ts) := \Lambda(t)\chi_e(s) \quad \text{for} \quad t \in D_e(\mathbb{A}),\ s \in N_{4,\varepsilon}(\mathbb{A}).$$

Definition 2.3. *For a cusp form φ on $\mathrm{GU}_{4,\varepsilon}(\mathbb{A}_F)$ with a trivial central character, we define $B_{e,\Lambda,\psi}(\varphi)$, the (e,Λ,ψ)-Bessel period of φ, by*

$$(2.3.5)\qquad B_{e,\Lambda,\psi}(\varphi) = \int_{D_e(F)\backslash D_e(\mathbb{A}_F)} \int_{N_{4,\varepsilon}(F)\backslash N_{4,\varepsilon}(\mathbb{A}_F)} \chi_{e,\Lambda}(ts)^{-1}\,\varphi(ts)\,ds\,dt.$$

2.3.4. *Bessel periods on* $\mathrm{GSO}_{4,2}$ *and* $\mathrm{GSO}_{3,3}$. By combining the accidental isomorphisms (2.2.5) and (2.2.6) in the split case, we shall define Bessel periods on $\mathrm{GSO}_{4,2}$ and $\mathrm{GSO}_{3,3}$ as the following.

Let $P_{4,2}$ denote a maximal parabolic subgroup of $\mathrm{GSO}_{4,2}$ with the Levi decomposition $P_{4,2} = M_{4,2}N_{4,2}$ where

$$M_{4,2} = \left\{ \begin{pmatrix} g & 0 & 0 \\ 0 & h & 0 \\ 0 & 0 & g^* \cdot \det h \end{pmatrix} : \begin{matrix} g \in \mathrm{GL}_2, \\ h \in \mathrm{GSO}_{2,0} \end{matrix} \right\}, \quad N_{4,2} = \left\{ \begin{pmatrix} 1_2 & A' & B \\ 0 & 1_2 & A \\ 0 & 0 & 1_2 \end{pmatrix} \in \mathrm{GSO}_{4,2} \right\}.$$

Here

$$g^* = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} {}^t g^{-1} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{for} \quad g \in \mathrm{GL}_2.$$

Then for $X \in \mathrm{Mat}_{2\times 2}(F)$, we define a character ψ_X of $N_{4,2}(\mathbb{A})$ by

$$\psi_X \begin{pmatrix} 1_2 & A' & B \\ 0 & 1_2 & A \\ 0 & 0 & 1_2 \end{pmatrix} = \psi\left[\mathrm{tr}(XA)\right].$$

Suppose that $\det X \neq 0$ and let

$$M_X := \left\{ \begin{pmatrix} (\det h) \cdot (h^X)^* & 0 & 0 \\ 0 & h & 0 \\ 0 & 0 & h^X \end{pmatrix} : h \in \mathrm{GSO}_{2,0} \right\}$$

where $h^X = XhX^{-1}$. Then $M_X(\mathbb{A})$ stabilizes the character ψ_X and M_X is isomorphic to $\mathrm{GSO}_{2,0}$. We fix an isomorphism $\mathrm{GSO}_{2,0}(F) \simeq E^\times$ and we regard a character of $\mathbb{A}_E^\times$ as a character of $M_X(\mathbb{A})$.

Definition 2.4. *Let σ be an irreducible cuspidal automorphic representation of $\mathrm{GSO}_{4,2}(\mathbb{A})$ with its space of automorphic forms V_σ and the central character ω_σ. For a character χ of $\mathbb{A}_E^\times$ such that $\chi|_{\mathbb{A}^\times} = \omega_\sigma$, we define $\mathcal{B}_{X,\chi,\psi}(\varphi)$, the (X,χ,ψ)-Bessel period of $\varphi \in V_\sigma$ by*

(2.3.6)

$$\mathcal{B}_{X,\chi,\psi}(\varphi) = \int_{N_{4,2}(F)\backslash N_{4,2}(\mathbb{A})} \int_{M_X(F)\mathbb{A}^\times\backslash M_X(\mathbb{A})} \varphi(uh)\chi(h)^{-1}\psi_X(u)^{-1}\, du\, dh.$$

When $d \in (F^\times)^2$, we know that $\mathrm{GSO}(S_2) \simeq \mathrm{GSO}_{3,3}$. Hence, as above, for a cusp form φ on $\mathrm{GSO}_{3,3}$ with central character ω and characters Λ_1, Λ_2 of $\mathbb{A}^\times/F^\times$ such that $\Lambda_1\Lambda_2 = \omega$, we define $(X, \Lambda_1, \Lambda_2, \psi)$-Bessel period by

$$\mathcal{B}_{X,\Lambda,\psi}(\varphi) = \int_{N_{4,2}(F)\backslash N_{4,2}(\mathbb{A})} \int_{M_X(F)\mathbb{A}^\times\backslash M_X(\mathbb{A})} \varphi(uh)\chi_{\Lambda_1,\Lambda_2}(h)^{-1}\psi_X(u)^{-1}\, du\, dh.$$

Here, since $M_{4,2} \simeq \mathrm{GL}_2 \times \mathrm{GSO}_{1,1}$ and $\mathrm{GSO}_{1,1}(F) = \{\left(\begin{smallmatrix} a & \\ & b \end{smallmatrix}\right) : a, b \in F^\times\}$, we define a character $\chi_{\Lambda_1,\Lambda_2}$ of $\mathrm{GSO}_{1,1}(\mathbb{A})$ by

$$\chi_{\Lambda_1,\Lambda_2} \begin{pmatrix} a & \\ & b \end{pmatrix} = \Lambda_1(a)\Lambda_2(b).$$

When ω is trivial, we have $\Lambda_2 = \Lambda_1^{-1}$. In this case, we simply call $(X, \Lambda_1, \Lambda_1^{-1}, \psi)$-Bessel period as (X, Λ_1, ψ)-Bessel period and simply write $\chi_{\Lambda_1,\Lambda_1^{-1}} = \Lambda_1$.

2.4. **Local Bessel periods.** Let us introduce local counterparts to the global Bessel periods. Let k be a local field of characteristic zero and D a quaternion algebra over k.

Since the local Bessel periods are deduced from the global ones in a uniform way, by abuse of notation, let a quintuple (H, T, N, χ, ψ_N) stand for one of

$$\left(G_D, T_\xi, N_D, \Lambda, \psi_\xi\right) \text{ in (1.4.11)},$$
$$\left(\mathrm{GSp}_2, T_S, N, \Lambda, \psi_S\right) \text{ in (2.3.1), or,}$$
$$\left(\mathrm{GSU}_{3,D}, M_X, N_{4,2}, \chi, \psi_X\right) \text{ in (2.3.3)}.$$

Let (π, V_π) be an irreducible tempered representation of $H = H(k)$ with trivial central character and $[\ ,\ ]$ a H-invariant hermitian pairing on V_π, the space of π. Let us denote by V_π^∞ the space of smooth vectors in V_π. When k is non-archimedean, clearly $V_\pi^\infty = V_\pi$. Let χ be a character of $T = T(k)$ which is trivial on $Z_H = Z_H(k)$, where Z_H denotes the center of H.

Suppose that k is non-archimedean. Then for $\phi, \phi' \in V_\pi$, we define the local Bessel period $\alpha^H_{\chi,\psi_N}(\phi, \phi') = \alpha_{\chi,\psi_N}(\phi, \phi') = \alpha(\phi, \phi')$ by

$$\alpha(\phi, \phi') := \int_{T/Z_H} \int_N^{\mathrm{st}} [\pi(ut)\phi, \phi']\, \chi(t)^{-1}\, \psi_N(u)^{-1}\, du\, dt. \tag{2.4.1}$$

Here the inner integral of (2.4.1) is the stable integral in the sense of Lapid and Mao [71, Definition 2.1, Remark 2.2]. Indeed it is shown that for any $t \in T$ the inner integral stabilizes at a certain compact open subgroup of $N = N(k)$ and the outer integral converges by Liu [76, Proposition 3.1, Theorem 2.1]. We note that it is also shown in Waldspurger [114, Section 5.1, Lemme] that (2.4.1) is well-defined. We often simply write $\alpha(\phi) = \alpha(\phi, \phi)$.

Now suppose that k is archimedean. Then the local Bessel period is defined as a regularized integral whose regularization is achieved by the Fourier transform as in Liu [76, 3.4]. Let us briefly recall the definition. We define a subgroup $N_{-\infty}$ of $N = N(k)$ by:

$$N_{-\infty} := \left\{ \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \in N_D : \mathrm{tr}_D(\xi u) = 0 \right\} \quad \text{in the } G_D\text{-case;}$$
$$N_{-\infty} := \left\{ \begin{pmatrix} 1 & Y \\ 0 & 1 \end{pmatrix} \in N : \mathrm{tr}(SY) = 0 \right\} \quad \text{in the } \mathrm{GSp}_2\text{-case;}$$
$$N_{-\infty} := \left\{ \begin{pmatrix} 1 & A' & B \\ 0 & 1 & A \\ 0 & 0 & 1 \end{pmatrix} \in N_{3,D} : \mathrm{tr}_D(XA) = 0 \right\} \quad \text{in the } \mathrm{GSU}_{3,D}\text{-case,}$$

respectively. Then it is shown in Liu [76, Corollary 3.13] that for $u \in N$,

$$\alpha_{\phi,\phi'}(u) := \int_{T/Z_G} \int_{N_{-\infty}} [\pi(ust)\phi, \phi']\, \chi(t)^{-1}\, ds\, dt$$

converges absolutely for $\varphi, \varphi' \in V_\pi^\infty$ and it gives a tempered distribution on $N/N_{-\infty}$.

For an abelian Lie group $\mathcal{N}$, we denote by $\mathcal{D}(\mathcal{N})$ (resp. $\mathcal{S}(\mathcal{N})$) the space of tempered distributions (resp. Schwartz functions) on $\mathcal{N}$. Then we recall that the

Fourier transform $\hat{}: \mathcal{D}(\mathcal{N}) \to \mathcal{D}(\mathcal{N})$ is defined by the formula

$$\left(\hat{\mathfrak{a}}, \phi\right) = \left(\mathfrak{a}, \hat{\phi}\right) \quad \text{for } \mathfrak{a} \in \mathcal{D}\left(\mathcal{N}\right) \text{ and } \phi \in \mathcal{S}\left(\mathcal{N}\right),$$

where $(\ ,\ )$ denotes the natural pairing $\mathcal{D}(\mathcal{N}) \times \mathcal{S}(\mathcal{N}) \to \mathbb{C}$ and $\hat{\phi}$ is the Fourier transform of $\phi \in \mathcal{S}\left(\mathcal{N}\right)$.

Then by Liu [76, Proposition 3.14], the Fourier transform $\widehat{\alpha_{\phi,\phi'}}$ is smooth on the regular locus $(\widehat{N/N_{-\infty}})^{\mathrm{reg}}$ of the Pontryagin dual $\widehat{N/N_{-\infty}}$ and we define the local Bessel period $\alpha\left(\phi, \phi'\right)$ by

$$\alpha^{H}_{\chi,\psi_N}\left(\phi, \phi'\right) = \alpha_{\chi,\psi_N}\left(\phi, \phi'\right) = \alpha\left(\phi, \phi'\right) := \widehat{\alpha_{\phi,\phi'}}\left(\psi_N\right). \tag{2.4.2}$$

As in the non-archimedean case, we often simply write $\alpha(\phi) = \alpha(\phi, \phi)$.

3. Pull-back of Bessel periods

In this section, we establish the pull-back formulas of the global Bessel periods with respect to the dual pairs, $\left(\mathrm{GSp}_2, \mathrm{GSO}_{4,2}\right)$, $\left(\mathrm{GSp}_2, \mathrm{GSO}_{3,3}\right)$ and $\left(G_D, \mathrm{GSU}_{3,D}\right)$. We recall that the first two cases may be regarded as the special case when D is split of the last one, by the accidental isomorphisms explained in 2.2.

3.1. $\left(\mathrm{GSp}_2, \mathrm{GSO}_{4,2}\right)$ and $\left(\mathrm{GSp}_2, \mathrm{GSO}_{3,3}\right)$ case.

3.1.1. *Symplectic-orthogonal theta correspondence with similitudes.* Let X (resp. Y) be a finite dimensional vector space over F equipped with a non-degenerate alternating (resp. symmetric) bilinear form. Assume that $\dim_F Y$ is even. We denote their similitude groups by $\mathrm{GSp}(X)$ and $\mathrm{GO}(Y)$, and, their isometry groups by $\mathrm{Sp}(X)$ and $\mathrm{O}(Y)$, respectively. We denote the identity component of $\mathrm{GO}(Y)$ and $\mathrm{O}(Y)$ by $\mathrm{GSO}(Y)$ and $\mathrm{SO}(Y)$, respectively. We let $\mathrm{GSp}(X)$ (resp. $\mathrm{GO}(Y)$) act on X from right (resp. left). The space $Z = X \otimes Y$ has a natural non-degenerate alternating form $\langle\ ,\ \rangle$, and we have an embedding $\mathrm{Sp}(X) \times \mathrm{O}(Y) \to \mathrm{Sp}(Z)$ defined by

$$(x \otimes y)(g, h) = xg \otimes h^{-1}y, \quad \text{for } x \in X, y \in Y, h \in \mathrm{O}(Y), g \in \mathrm{Sp}(X). \tag{3.1.1}$$

Fix a polarization $Z = Z_+ \oplus Z_-$. Let us denote by $(\omega_\psi, \mathcal{S}(Z_+(\mathbb{A})))$ the Schrödinger model of the Weil representation of $\widetilde{\mathrm{Sp}}(Z)$ corresponding to this polarization with the Schwartz-Bruhat space $\mathcal{S}(Z_+)$ on Z_+. We write a typical element of $\mathrm{Sp}(Z)$ by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad \text{where} \quad \begin{cases} A \in \mathrm{Hom}(Z_+, Z_+), & B \in \mathrm{Hom}(Z_+, Z_-), \\ C \in \mathrm{Hom}(Z_-, Z_+), & D \in \mathrm{Hom}(Z_-, Z_-). \end{cases}$$

Then the action of ω_ψ on $\phi \in \mathcal{S}(Z_+)$ is given by the following formulas:

$$\omega_\psi\left(\begin{pmatrix} A & B \\ 0 & {}^tA^{-1} \end{pmatrix}, \varepsilon\right)\phi(z_+) = \varepsilon\frac{\gamma_\psi(1)}{\gamma_\psi(\det A)}|\det(A)|^{\frac{1}{2}}\psi\left(\frac{1}{2}\langle z_+A, z_+B\rangle\right)\phi(z_+A) \tag{3.1.2}$$

$$\omega_\psi\left(\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \varepsilon\right)\phi(z_+) = \varepsilon(\gamma_\psi(1))^{-\dim Z_+}\int_{Z_+}\psi\left(\left\langle z', z\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}\right\rangle\right)\phi(z')\,dz', \tag{3.1.3}$$

where $\gamma_\psi(t)$ is a certain eighth root of unity called the Weil factor. Moreover, since the embedding given by (3.1.1) splits in the metaplectic group $\mathrm{Mp}(Z)$, we obtain the Weil representation of $\mathrm{Sp}(X,\mathbb{A})\times\mathrm{O}(Y,\mathbb{A})$ by restriction. We also denote this representation by ω_ψ.

We have a natural homomorphism

$$i:\mathrm{GSp}(X)\times\mathrm{GO}(Y)\to\mathrm{GSp}(Z)$$

given by the action (3.1.1). Then we note that $\lambda(i(g,h))=\lambda(g)\lambda(h)^{-1}$. Let

$$R:=\{(g,h)\in\mathrm{GSp}(X)\times\mathrm{GO}(Y)\mid\lambda(g)=\lambda(h)\}\supset\mathrm{Sp}(X)\times\mathrm{O}(Y).$$

We may define an extension of the Weil representation of $\mathrm{Sp}(X,\mathbb{A})\times\mathrm{O}(Y,\mathbb{A})$ to $R(\mathbb{A})$ as follows. Let $X=X_+\oplus X_-$ be a polarization of X and use the polarization $Z_\pm=X_\pm\otimes Y$ of Z to realize the Weil representation ω_ψ. Then we note that

$$\omega_\psi(1,h)\phi(z)=\phi\left(i\left(h\right)^{-1}z\right)\quad\text{for } h\in\mathrm{O}\left(\mathbb{A}\right)\text{ and }\phi\in\mathcal{S}(Z_+(\mathbb{A})).$$

Thus we define an action L of $\mathrm{GO}\left(Y,\mathbb{A}\right)$ on $\mathcal{S}(Z_+(\mathbb{A}))$ by

$$L\left(h\right)\phi\left(z\right)=|\lambda(h)|^{-\frac{1}{8}\dim\,X\cdot\dim\,Y}\phi\left(i\left(h\right)^{-1}z\right).$$

Then we may extend the Weil representation ω_ψ of $\mathrm{Sp}(X,\mathbb{A})\times\mathrm{O}(Y,\mathbb{A})$ to $R(\mathbb{A})$ by

$$\omega_\psi(g,h)\phi=\omega_\psi(g_1,1)L\left(h\right)\phi\quad\text{for }\phi\in\mathcal{S}(Z_+(\mathbb{A}))\text{ and }(g,h)\in R\left(\mathbb{A}\right),$$

where

$$g_1=g\begin{pmatrix}\lambda(g)^{-1}&0\\0&1\end{pmatrix}\in\mathrm{Sp}(X,\mathbb{A}).$$

In general, for any polarization $Z=Z_+'\oplus Z_-'$, there exists an $\mathrm{Sp}(X,\mathbb{A})\times\mathrm{O}(Y)(\mathbb{A})$-isomorphism $p:\mathcal{S}(Z_+(\mathbb{A}))\to\mathcal{S}(Z_+'(\mathbb{A}))$ given by an integral transform (see Ichino-Prasanna [58, Lemma 3.3]). Let us denote the realization of the Weil representation of $\mathrm{Sp}(X,\mathbb{A})\times\mathrm{O}(Y)(\mathbb{A})$ on $\mathcal{S}(Z_+'(\mathbb{A}))$ by ω_ψ'. Then we may extend ω_ψ' to $R\left(\mathbb{A}\right)$ by

$$\omega_\psi'\left(g,h\right)=p\circ\omega_\psi\left(g,h\right)\circ p^{-1}\quad\text{for }(g,h)\in R(\mathbb{A}).$$

For $\phi\in\mathcal{S}(Z_+(\mathbb{A}))$, we define the theta kernel θ^ϕ by

$$\theta_\psi^\phi(g,h)=\theta^\phi(g,h):=\sum_{z_+\in Z_+(F)}\omega_\psi\left(g,h\right)\phi(z_+)\quad\text{for }(g,h)\in R(\mathbb{A}).$$

Let

(3.1.4) $\mathrm{GSp}(X,\mathbb{A})^+=\{g\in\mathrm{GSp}(X,\mathbb{A})\mid\lambda(g)=\lambda(h)\text{ for some }h\in\mathrm{GO}(Y,\mathbb{A})\}$

and $\mathrm{GSp}(X,F)^+=\mathrm{GSp}(X,\mathbb{A})^+\cap\mathrm{GSp}(X,F)$.

As in [46, Section 5.1], for a cusp form f on $\mathrm{GSp}(X,\mathbb{A})^+$, we define its theta lift to $\mathrm{GO}(Y,\mathbb{A})$ by

$$\Theta_\psi^{X,Y}(f,\phi)(h)=\Theta(f,\phi)(h):=\int_{\mathrm{Sp}(X,F)\backslash\mathrm{Sp}(X,\mathbb{A})}\theta^\phi(g_1g,h)f(g_1g)\,dg_1$$

for $h \in \mathrm{GO}(Y, \mathbb{A})$, where $g \in \mathrm{GSp}(X, \mathbb{A})^+$ is chosen so that $\lambda(g) = \lambda(h)$. It defines an automorphic form on $\mathrm{GO}(Y, \mathbb{A})$. For a cuspidal automorphic representation (π_+, V_{π_+}) of $\mathrm{GSp}(X, \mathbb{A})^+$, we denote by $\Theta_\psi(\pi_+)$ the theta lift of π_+ to $\mathrm{GO}(Y, \mathbb{A})$. Namely

$$\Theta_\psi^{X,Y}(\pi_+) = \Theta_\psi(\pi_+) := \left\{\Theta(f, \phi) : f \in V_{\pi_+},\ \phi \in \mathcal{S}(Z_+(\mathbb{A}))\right\}.$$

Furthermore, for an irreducible cuspidal automorphic representation (π, V_π) of $\mathrm{GSp}(X, \mathbb{A})$, we define

$$\Theta_\psi(\pi) := \Theta_\psi(\pi|_{\mathrm{GSp}(X,\mathbb{A})^+})$$

where $\pi|_{\mathrm{GSp}(X,\mathbb{A})^+}$ denotes the automorphic representation of $\mathrm{GSp}(X, \mathbb{A})^+$ with its space of automorphic forms $\left\{\varphi|_{\mathrm{GSp}(X,\mathbb{A})^+} : \varphi \in V_\pi\right\}$.

As for the opposite direction, for a cusp form f' on $\mathrm{GO}(Y, \mathbb{A})$, we define its theta lift $\Theta(f', \phi)$ to $\mathrm{GSp}(X, \mathbb{A})^+$ by

$$\Theta(f', \phi)(g) := \int_{\mathrm{O}(Y,F)\backslash \mathrm{O}(Y,\mathbb{A})} \theta^\phi(g, h_1 h) f(h_1 h)\, dh_1 \quad \text{for } g \in \mathrm{GSp}\,(X, \mathbb{A})^+,$$

where $h \in \mathrm{GO}(Y, \mathbb{A})$ is chosen so that $\lambda(g) = \lambda(h)$. For an irreducible cuspidal automorphic representation (σ, V_σ) of $\mathrm{GO}(Y, \mathbb{A})$, we define the theta lift $\Theta_\psi(\sigma)$ of σ to $\mathrm{GSp}(X, \mathbb{A})^+$ by

$$\Theta_\psi(\sigma) := \{\Theta(f', \phi) : f' \in V_\sigma,\ \phi \in \mathcal{S}(Z_+(\mathbb{A}))\}.$$

Moreover we extend $\theta(f', \phi)$ to an automorphic form on $\mathrm{GSp}(X, \mathbb{A})$ by the natural embedding

$$\mathrm{GSp}(X, F)^+\backslash \mathrm{GSp}(X, \mathbb{A})^+ \longrightarrow \mathrm{GSp}(X, F)\backslash \mathrm{GSp}(X, \mathbb{A})$$

and extension by zero. Then we define the theta lift $\Theta_\psi(\sigma)$ of σ to $\mathrm{GSp}(X, \mathbb{A})$ as the $\mathrm{GSp}\,(X, \mathbb{A})$ representation generated by such $\theta\,(f', \phi)$ for $f' \in V_\sigma$ and $\phi \in \mathcal{S}\,(Z_+\,(\mathbb{A}))$.

For some X and Y, theta correspondence for the dual pair $(\mathrm{GSp}(X)^+, \mathrm{GO}(Y))$ gives theta correspondence between $\mathrm{GSp}(X)^+$ and $\mathrm{GSO}(Y)$ by the restriction of representations of $\mathrm{GO}(Y)$ to $\mathrm{GSO}(Y)$. Indeed, when $\dim X = 4$ and $\dim Y = 6$, we may consider theta correspondence for the pair $(\mathrm{GSp}(X)^+, \mathrm{GSO}(Y))$. In Gan-Takeda [34, 36], they study the case when $\mathrm{GSO}(Y) \simeq \mathrm{GSO}_{3,3}$ or $\mathrm{GSO}_{5,1}$, and, in [83], the case when $\mathrm{GSO}(Y) \simeq \mathrm{GSO}_{4,2}$ is studied. In these cases, for a cusp form f on $\mathrm{GSp}(X, \mathbb{A})^+$, we denote by $\theta(f, \phi)$ the restriction of $\Theta(f, \phi)$ to $\mathrm{GSO}(Y, \mathbb{A})$. Moreover, for a cuspidal automorphic representation (π_+, V_{π_+}) of $\mathrm{GSp}(X, \mathbb{A})^+$, we define the theta lift $\theta_\psi\,(\pi_+)$ of π_+ to $\mathrm{GSO}(Y, \mathbb{A})$ by

$$\theta_\psi^{X,Y}(\pi_+) = \theta_\psi(\pi_+) := \left\{\theta(f, \phi) : f \in V_{\pi_+}, \phi \in \mathcal{S}(Z_+(\mathbb{A}))\right\}.$$

Similarly, for a cusp form f' on $\mathrm{GSO}(Y, \mathbb{A})$, we define its theta lift $\theta(f', \phi)$ to $\mathrm{GSp}(X, \mathbb{A})^+$ by

$$\theta(f', \phi)(g) := \int_{\mathrm{SO}(Y,F)\backslash \mathrm{SO}(Y,\mathbb{A})} \theta^\phi(g, h_1 h) f(h_1 h)\, dh_1 \quad \text{for } g \in \mathrm{GSp}\,(X, \mathbb{A})^+,$$

where $h \in \mathrm{GSO}(Y, \mathbb{A})$ is chosen so that $\lambda(g) = \lambda(h)$. We extend it to an automorphic from on $\mathrm{GSp}(X, \mathbb{A})$ as above. For a cuspidal automorphic representation (σ, V_σ) of $\mathrm{GSO}(Y, \mathbb{A})$, we define the theta lift $\theta_\psi(\sigma)$ of σ to $\mathrm{GSp}(X, \mathbb{A})^+$ by

$$\theta_\psi(\sigma) := \{\theta(f', \phi) : f' \in V_\sigma,\ \phi \in \mathcal{S}(Z_+(\mathbb{A}))\}.$$

Remark 3.1. *Suppose that $\Theta_\psi(\pi_+)$ (resp. $\theta_\psi(\sigma)$) is non-zero and cuspidal where (π_+, V_{π_+}) (resp. (σ, V_σ)) is an irreducible cuspidal automorphic representation of* $\mathrm{GSp}(X, \mathbb{A})^+$ *(resp.* $\mathrm{GO}(Y, \mathbb{A})$*). Then Gan* [31, Proposition 2.12] *has shown that the Howe duality, which was proved by Howe* [52] *at archimedean places, by Waldspurger* [113] *at odd finite places and finally by Gan and Takeda* [37] *at all finite places, implies that $\Theta_\psi(\pi_+)$ (resp. $\theta_\psi(\sigma)$) is irreducible and cuspidal. Moreover in the case of our concern, namely when* $\dim_F X = 4$ *and* $\dim_F Y = 6$*, the irreducibility of $\Theta_\psi(\pi_+)$ implies that of $\theta_\psi(\pi_+)$ by the conservation relation due to Sun and Zhu* [105].

3.1.2. *Pull-back of the global Bessel periods for the dual pairs* $(\mathrm{GSp}_2, \mathrm{GSO}_{4,2})$ *and* $(\mathrm{GSp}_2, \mathrm{GSO}_{3,3})$. Our goal here is to prove the pull-back formula (3.1.6).

First we introduce the set-up. Let X be the space of 4 dimensional row vectors over F equipped with the symplectic form

$$\langle w_1, w_2\rangle = w_1 \begin{pmatrix} 0 & 1_2 \\ -1_2 & 0 \end{pmatrix} {}^t w_2.$$

Let us take the standard basis of X and name the basis vectors as

(3.1.5)
$$x_1 = (1,0,0,0), \quad x_2 = (0,1,0,0), \quad x_{-1} = (0,0,1,0), \quad x_{-2} = (0,0,0,1).$$

Then the matrix representation of $\mathrm{GSp}(X)$ with respect to the standard basis is $G = \mathrm{GSp}_2$ defined by (1.4.3). We let G act on X from the right.

Let Y be the space of 6 dimensional column vectors over F equipped with the non-degenerate symmetric bilinear form

$$(v_1, v_2) = {}^t v_1 S_2 v_2$$

where the symmetric matrix S_2 is given by (2.1.2). Let us take the standard basis of Y and name the basis vectors as

$$\begin{aligned} y_{-2} &= {}^t(1,0,0,0,0,0), & y_{-1} &= {}^t(0,1,0,0,0,0), \\ e_1 &= {}^t(0,0,1,0,0,0), & e_2 &= {}^t(0,0,0,1,0,0), \\ y_1 &= {}^t(0,0,0,0,1,0), & y_2 &= {}^t(0,0,0,0,0,1). \end{aligned}$$

We note that $(y_i, y_j) = \delta_{ij}$, $(e_1, e_1) = 2$ and $(e_2, e_2) = -2d$. Since $d \in F^\times \setminus (F^\times)^2$, with respect to the standard basis, the matrix representations of $\mathrm{GO}(Y)$ and $\mathrm{GSO}(Y)$ are $\mathrm{GO}_{4,2}$ defined by (2.1.5) and $\mathrm{GSO}_{4,2}$ defined by (2.1.4), respectively. In this section, we also study the theta correspondence for the dual pair $(\mathrm{GSp}(X), \mathrm{GSO}_{3,3})$, for which, we may use the above matrix representation with $d \in (F^\times)^2$. Hence, in the remaining of this section, we study theta correspondence for $(\mathrm{GSp}(X), \mathrm{GSO}(Y))$ for an arbitrary $d \in F^\times$.

We shall denote $\mathrm{GSp}(X,\mathbb{A})^+$ as $G(\mathbb{A})^+$ and also $\mathrm{GSp}(X,F)^+$ as $G(F)^+$. We note that when $d \in (F^\times)^2$, $\mathrm{GSp}(X)^+ = \mathrm{GSp}(X)$.

Let $Z = X \otimes Y$ and we take a polarization $Z = Z_+ \oplus Z_-$ as follows. First we take $X = X_+ \oplus X_-$ where

$$X_+ = F \cdot x_1 + F \cdot x_2 \quad \text{and} \quad X_- = F \cdot x_{-1} + F \cdot x_{-2}$$

as the polarization of X. Then we decompose Y as $Y = Y_+ \oplus Y_0 \oplus Y_-$ where

$$Y_+ = F \cdot y_1 + F \cdot y_2, \quad Y_0 = F \cdot e_1 + F \cdot e_2 \quad \text{and } Y_- = F \cdot y_{-1} + F \cdot y_{-2}.$$

Then let

$$Z_\pm = (X \otimes Y_\pm) \oplus (X_\pm \otimes Y_0)$$

where the double sign corresponds. To simplify the notation, we sometimes write $z_+ \in Z_+$ as $z_+ = (a_1, a_2; b_1, b_2)$ when

$$z_+ = a_1 \otimes y_1 + a_2 \otimes y_2 + b_1 \otimes e_1 + b_2 \otimes e_2 \in Z_+, \quad \text{where } a_i \in X,\ b_i \in X_+\ (i = 1, 2).$$

Let us compute the pull-back of (X, χ, ψ)-Bessel periods on $\mathrm{GSO}(Y)$ defined by (2.3.6) with respect to the theta lift from G.

Proposition 3.1. *Let (π, V_π) be an irreducible cuspidal automorphic representation of $G(\mathbb{A})$ whose central character is ω_π and χ a character of $\mathbb{A}_E^\times$ such that $\chi\,|_{\mathbb{A}^\times} = \omega_\pi^{-1}$. Let $X \in \mathrm{Mat}_{2\times 2}(F)$ such that $\det X \neq 0$.*

Then for $f \in V_\pi$ and $\phi \in \mathcal{S}(Z_+(\mathbb{A}))$, we have

(3.1.6)
$$\mathcal{B}_{X,\chi,\psi}(\theta(f:\phi)) = \int_{N(\mathbb{A})\backslash G^1(\mathbb{A})} B_{S_X,\chi^{-1},\psi}(\pi(g)f)\left(\omega_\psi(g,1)\phi\right)(v_X)\,dg$$

where $B_{S_X,\chi^{-1},\psi}$ is the $\left(S_X,\chi^{-1},\psi\right)$-Bessel period on G defined by (2.3.1).

Here, for $X = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}$, we define a vector $v_X \in Z_+$ by

$$v_X := \left(x_{-2}, x_{-1}; \frac{x_{21}}{2}x_1 + \frac{x_{11}}{2}x_2, -\frac{x_{22}}{2d}x_1 - \frac{x_{12}}{2d}x_2\right) \tag{3.1.7}$$

and a 2 by 2 symmetric matrix S_X by

$$S_X := \frac{1}{4d}\,{}^t(J_2\,{}^tXJ_2)S_0(J_2\,{}^tXJ_2). \tag{3.1.8}$$

We regard χ as a character of $\mathrm{GSO}(S_X)(\mathbb{A})$ by

$$\mathrm{GSO}(S_X) \ni k \mapsto \chi((J_2{}^tXJ_2)k(J_2{}^tXJ_2)^{-1}) \in \mathbb{C}^\times. \tag{3.1.9}$$

In particular, the (S_X,χ^{-1},ψ)-Bessel period does not vanish on V_π if and only if the (X,χ,ψ)-Bessel period does not vanish on $\theta_\psi(\pi)$.

Proof. We compute the (X, χ, ψ)-Bessel period defined by (2.3.6) in stages. We consider subgroups of $N_{4,2}$ given by:

(3.1.10)
$$N_0(F) = \left\{ u_0(x) := \begin{pmatrix} 1 & -{}^t X_0 S_1 & 0 \\ 0 & 1_4 & X_0 \\ 0 & 0 & 1 \end{pmatrix} \mid X_0 = \begin{pmatrix} x \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\};$$

(3.1.11)
$$N_1(F) = \left\{ u_1(s_1, t_1) := \begin{pmatrix} 1 & -{}^t X_1 S_1 & -\frac{1}{2} {}^t X_1 S_1 X_1 \\ 0 & 1_4 & X_1 \\ 0 & 0 & 1 \end{pmatrix} \mid X_1 = \begin{pmatrix} 0 \\ s_1 \\ t_1 \\ 0 \end{pmatrix} \right\};$$

(3.1.12)
$$N_2(F) = \left\{ u_2(s_2, t_2) := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -{}^t X_2 S_0 & -\frac{1}{2} {}^t X_2 S_0 X_2 & 0 \\ 0 & 0 & 1_2 & X_2 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \mid X_2 = \begin{pmatrix} s_2 \\ t_2 \end{pmatrix} \right\}$$

where S_0 and S_1 are given by (2.1.2). Then we have
$$N_0 \lhd N_0 N_1 \lhd N_0 N_1 N_2 = N_{4.2}.$$

Thus we may write

$$\mathcal{B}_{X,\chi,\psi}(\theta(f : \phi)) = \int_{\mathbb{A}^\times M_X(F) \backslash M_X(\mathbb{A})} \int_{(F \backslash \mathbb{A}_F)^2} \int_{(F \backslash \mathbb{A}_F)^2} \int_{F \backslash \mathbb{A}_F} \theta(f, \phi)(u_0(x) u_1(s_1, t_1) u_2(s_2, t_2) h) \times \psi(x_{21} s_1 + x_{22} t_1 + x_{11} s_2 + x_{12} t_2)^{-1} \chi(h)^{-1} \, dx \, ds_1 \, dt_1 \, ds_2 \, dt_2 \, dh. \tag{3.1.13}$$

For $h \in \mathrm{GSO}(Y, \mathbb{A})$, let us define
$$W_0(\theta(f : \phi))(h) := \int_{F \backslash \mathbb{A}_F} \theta(f, \phi)(u_0(x) h) \, dx.$$

From the definition of the theta lift, we have

(3.1.14) $W_0(\theta(f, \phi))(h)$
$$= \int_{F \backslash \mathbb{A}_F} \int_{G^1(F) \backslash G^1(\mathbb{A}_F)} \sum_{a_i \in X, b_i \in X_+} \left(\omega_\psi(g_1 \lambda_s(\nu(h)), u_0(x) h) \phi \right) (a_1, a_2; b_1, b_2) \times f(g_1 \lambda_s(\lambda(h))) \, dg_1 \, dx.$$

Here, for $a \in \mathbb{A}^\times$, we write
$$\lambda_s(a) = \begin{pmatrix} 1_2 & 0 \\ 0 & a \cdot 1_2 \end{pmatrix}.$$

Since $Z_-(1, u_0(x)) = Z_-$ and we have

$$z_+(1, u_0(x)) = z_+ + (x \cdot a_1 \otimes y_{-2} - x \cdot a_2 \otimes y_{-1}),$$

we observe that

$$\begin{aligned}(\omega_\psi(1, u_0(x))\phi)(z_+) &= \psi\left(\frac{1}{2}\langle z_+, x \cdot a_1 \otimes y_{-2} - x \cdot a_2 \otimes y_{-1}\rangle\right)\phi(z_+) \\ &= \psi(-x\langle a_1, a_2\rangle)\,\phi(z_+).\end{aligned} \tag{3.1.15}$$

Thus in the summation of the right-hand side of (3.1.14), only a_i such that $\langle a_1, a_2\rangle = 0$ contributes to the integral $W_0(\theta(f, \phi))$, and we obtain

$$W_0(\theta(f,\phi))(h) = \int_{G^1(F)\backslash G^1(\mathbb{A}_F)} \sum_{\substack{a_i \in X, \langle a_1, a_2\rangle = 0,\\ b_i \in X_+}} (\omega_\psi(g_1\lambda_s(\lambda(h)), h)\phi)(a_1, a_2; b_1, b_2) f(g_1\lambda_s(\lambda(h)))\, dg_1.$$

Since the space spanned by a_1 and a_2 is isotropic, there exists $\gamma \in G^1(F)$ such that $a_1\gamma^{-1}, a_2\gamma^{-1} \in X_-$. Let us define an equivalence relation $\sim$ on $(X_-)^2$ by

$$(a_1, a_2) \sim (a'_1, a'_2) \underset{\text{def.}}{\iff} \text{there exists } \gamma \in G^1(F) \text{ such that } a'_i = a_i\gamma \text{ for } i = 1, 2.$$

Let us denote by $\mathcal{X}_-$ the set of equivalence classes $(X_-)^2/\sim$ and by $\overline{(a_1, a_2)}$ the equivalence class containing $(a_1, a_2) \in (X_-)^2$. Then we may write $W_0(\theta(f, \phi))(h)$ as

$$\int_{G^1(F)\backslash G^1(\mathbb{A}_F)} \sum_{\overline{(a_1,a_2)} \in \mathcal{X}_-} \sum_{\gamma \in V(a_1,a_2)\backslash G^1(F)} \sum_{b_i \in X_+} (\omega_\psi(g_1\lambda_s(\lambda(h)), h)\phi)(a_1\gamma, a_2\gamma; b_1, b_2) \times f(g_1\lambda_s(\lambda(h)))\, dg_1.$$

Here

$$V(a_1, a_2) = \{g \in G^1(F) \mid a_i g = a_i \text{ for } i = 1, 2\}.$$

Lemma 3.1. *For any $g \in G(\mathbb{A})^+$ and $h \in \mathrm{GSO}(Y, \mathbb{A})$ such that $\lambda(g) = \lambda(h)$,*

$$\sum_{b_i \in X_+} (\omega_\psi(g, h)\phi)(a_1\gamma, a_2\gamma, b_1, b_2) = \sum_{b_i \in X_+} (\omega_\psi(\gamma g, h)\phi)(a_1, a_2, b_1, b_2).$$

Proof. This is proved by an argument similar to the one for [23, Lemma 2]. □

Further, by an argument similar to the one for $W_0(\theta(f, \phi))(h)$, we shall prove the following lemma.

Lemma 3.2. *For any $g \in G(\mathbb{A})^+$ and $h \in \mathrm{GSO}(Y, \mathbb{A})$ such that $\lambda(g) = \lambda(h)$,*

$$\int_{(F\backslash\mathbb{A}_F)^2} \psi^{-1}(x_{21}s_1 + x_{22}t_1) \left(\omega_\psi(g, u_1(s_1,t_1)h)\phi\right)(a_1,a_2,b_1,b_2)\, ds_1\, dt_1$$
$$= \begin{cases} \left(\omega_\psi(g,h)\phi\right)(a_1,a_2,b_1,b_2) & \text{if } \langle a_2, b_1\rangle = -\frac{x_{21}}{2} \text{ and } \langle a_2, b_2\rangle = \frac{x_{22}}{2d}; \\ 0 & \text{otherwise} \end{cases}$$

and

$$\int_{(F\backslash\mathbb{A}_F)^2} \psi^{-1}(x_{11}s_2 + x_{12}t_2) \left(\omega_\psi(g, u_2(s_2,t_2)h)\phi\right)(a_1,a_2,b_1,b_2)\, ds_2\, dt_2$$
$$= \begin{cases} \left(\omega_\psi(g,h)\phi\right)(a_1,a_2,b_1,b_2) & \text{if } \langle a_1, b_1\rangle = -\frac{x_{11}}{2} \text{ and } \langle a_1, b_2\rangle = \frac{x_{12}}{2d}; \\ 0 & \text{otherwise}. \end{cases}$$

Proof. Since $Z_-\left(1, u_1(s_1,t_1)\right) = Z_-$ and we have

$$z_+\left(1, u_1(s_1,t_1)\right) = z_+ + 2s_1(b_1 \otimes y_{-2}) - 2dt_1(b_2 \otimes y_{-2}) + (-s_1^2 + 2dt_1^2)a_2 \otimes y_{-2} - s_1 a_2 \otimes e_1 - t_1 a_2 \otimes e_2,$$

we obtain

$$\left(\omega_\psi(1, u_1(s_1,t_1))\phi\right)(z_+) = \psi\left(\frac{1}{2}\left(2s_1\langle a_2,b_1\rangle - 2dt_1\langle a_2,b_2\rangle\right)\right) \times \psi\left(\frac{1}{2}\left((-s_1^2+2dt_1^2)\langle a_2,a_2\rangle - 2s_1\langle b_1,a_2\rangle + 2dt_1\langle b_2,a_2\rangle\right)\right)\phi(z_+)$$
$$= \psi\left(2s_1\langle a_2,b_1\rangle - 2dt_1\langle a_2,b_2\rangle\right)\phi(z_+).$$

Then the first assertion readily follows.

Similarly, since $Z_-\left(1, u_2(s_2,t_2)\right) = Z_-$ and we have

$$z_+\left(1, u_2(s_2,t_2)\right) = z_+ + a_1 \otimes ((s_2^2 - dt_2^2)y_{-1} - s_2 e_1 - t_2 e_2) + 2s_2 b_1 \otimes y_{-1} - 2dt_2 b_2 \otimes y_{-1},$$

we obtain

$$\omega(1, u_2(s_2,t_2))\phi(z_+) = \psi\left(\frac{1}{2}\left(-2s_2\langle b_1,a_1\rangle + 2dt_2\langle b_2,a_1\rangle\right)\right) \times \psi\left(\frac{1}{2}\left(2s_2\langle a_1,b_1\rangle - 2dt_2\langle a_1,b_2\rangle\right)\right)\phi(z_+)$$
$$= \psi\left(2s_2\langle a_1,b_1\rangle - 2dt_2\langle a_1,b_2\rangle\right)\phi(z_+)$$

and the second assertion follows. ∎

Lemma 3.2 implies that

$$\mathcal{B}_{X,\chi,\psi}(\theta(f:\phi)) = \int_{\mathbb{A}^\times M_X(F)\backslash M_X(\mathbb{A})} \int_{G^1(F)\backslash G^1(\mathbb{A}_F)} \chi(h)^{-1} \\ \times \sum_{\overline{(a_1,a_2)} \in \mathcal{X}_-} \sum_{\gamma \in V(a_1,a_2)\backslash G^1(F)} \sum_{\substack{b_i \in X_+, \langle a_i, b_1\rangle = \frac{x_{i1}}{2}, \\ \langle a_i, b_2 \rangle = -\frac{x_{i2}}{2d}}} \\ \left(\omega_\psi(\gamma g_1 \lambda_s(\lambda(h)), h)\phi\right)(a_1,a_2,b_1,b_2)\, f(g_1\lambda_s(\lambda(h)))\, dg_1\, dh.$$

We note that a_1 and a_2 are linearly independent from the conditions on a_i and $\det(X) \neq 0$. Since $a_i \in X_-$ and $\dim X_- = 2$, we may take $(a_1,a_2) = (x_{-2}, x_{-1})$ as a representative. Then we should have

$$b_1 = \frac{x_{21}}{2}x_1 + \frac{x_{11}}{2}x_2, \quad b_2 = -\frac{x_{22}}{2d}x_1 - \frac{x_{12}}{2d}x_2.$$

Hence we get

$$\begin{aligned} (3.1.16)\quad \mathcal{B}_{X,\chi,\psi}(\theta(f:\phi)) &= \int_{\mathbb{A}^\times M_X(F)\backslash M_X(\mathbb{A})} \int_{G^1(F)\backslash G^1(\mathbb{A}_F)} \chi(h)^{-1} \\ &\times \sum_{\gamma \in N(F)\backslash G^1(F)} \left(\omega_\psi(\gamma g_1\lambda_s(\lambda(h)), h)\phi\right)(v_X)\, f(g_1\lambda_s(\lambda(h)))\, dg_1\, dh \\ &= \int_{N(\mathbb{A})\backslash G^1(\mathbb{A}_F)} \int_{\mathbb{A}^\times M_X(F)\backslash M_X(\mathbb{A})} \int_{N(F)\backslash N(\mathbb{A})} \\ &\qquad \chi(h)^{-1} \omega(vg_1\lambda_s(\lambda(h)), h)\phi(v_X) f(vg_1\lambda_s(\lambda(h)))\, dv\, dg_1\, dh \end{aligned}$$

where we put $v_X = (x_{-2}, x_{-1}; \frac{x_{21}}{2}x_1 + \frac{x_{11}}{2}x_2, -\frac{x_{22}}{2d}x_1 - \frac{x_{12}}{2d}x_2)$.

For $u = \begin{pmatrix} 1_2 & A \\ 0 & 1_2 \end{pmatrix}$ where $A = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \in \mathrm{Sym}^2$, we have

$$\begin{aligned} &\left(x_{-2}\otimes y_1 + x_{-1}\otimes y_2 + \left(\frac{x_{21}}{2}x_1 + \frac{x_{11}}{2}x_2\right)\otimes e_1 + \left(-\frac{x_{22}}{2d}x_1 - \frac{x_{12}}{2d}x_2\right)\otimes e_2\right)(u,1) \\ =& x_{-2}\otimes y_1 + x_{-1}\otimes y_2 + \left(\frac{x_{21}}{2}(x_1 + ax_{-1} + bx_{-2}) + \frac{x_{11}}{2}(x_2 + bx_{-1} + cx_{-2})\right)\otimes e_1 \\ &+ \left(-\frac{x_{22}}{2d}(x_1 + ax_{-1} + bx_{-2}) - \frac{x_{12}}{2d}(x_2 + bx_{-1} + cx_{-2})\right)\otimes e_2. \end{aligned}$$

Hence, when we put

$$\begin{aligned} S_X &= \frac{1}{4d}{}^t(J_2{}^tXJ_2)S_0(J_2{}^tXJ_2) \\ &= \frac{1}{2d}\begin{pmatrix} x_{22}^2 - dx_{21}^2 & x_{22}x_{12} - dx_{21}x_{11} \\ x_{22}x_{12} - dx_{21}x_{11} & x_{12}^2 - dx_{11}^2 \end{pmatrix} \in \mathrm{Sym}^2(F), \end{aligned}$$

for $u \in N(\mathbb{A})$, we have

$$\left(\omega_\psi(ug\lambda_s(\lambda(h)), h)\phi\right)(v_X) = \psi_{S_X}(u)^{-1}\, \omega_\psi(g\lambda_s(\lambda(h)), h)\phi(v_X).$$

Therefore, we get

$$\int_{N(\mathbb{A})\backslash G^1(\mathbb{A}_F)}\int_{\mathbb{A}^\times M_X(F)\backslash M_X(\mathbb{A})}\int_{N(F)\backslash N(\mathbb{A})}\chi(h)^{-1}$$
$$\times\left(\omega_\psi(g_1\lambda_s(\lambda(h)),h)\phi\right)(v_X)\, f(ug_1\lambda_s(\lambda(h)))\psi_{S_X}(u)^{-1}\,du\,dh\,dg_1$$
$$=\int_{N(\mathbb{A})\backslash G^1(\mathbb{A}_F)}\int_{\mathbb{A}^\times M_X(F)\backslash M_X(\mathbb{A})}\int_{N(F)\backslash N(\mathbb{A})}\chi(h)^{-1}$$
$$\times\,\omega_\psi(\lambda_s(\lambda(h))g_1,h)\phi(v_X)\,|\lambda(h)|^3 f(u\lambda_s(\lambda(h))g_1)\psi_{S_X}(u)^{-1}\,du\,dh\,dg_1.$$

By a direct computation, we see that

$$\left(\omega_\psi(\lambda_s(\lambda(h))g_1,h)\phi\right)(v_X)=|\lambda(h)|^{-3}\left(\omega_\psi(h_0\lambda_s(\lambda(h))g_1,1)\phi\right)(v_X)$$

when we write

$$h=\begin{pmatrix}(\det h)(h^X)^* & 0 & 0\\ 0 & h & 0\\ 0 & 0 & h^X\end{pmatrix},\; h_0=\begin{pmatrix}({}^tXJ_2)^{-1\,t}h({}^tXJ_2) & 0\\ 0 & (J_2X)h^{-1}(J_2X)^{-1}\end{pmatrix}.$$

For $g\in \mathrm{GSO}(S_0)$, we have ${}^tg=wgw$ and we may write

$$h_0=\begin{pmatrix}(J_2{}^tXJ_2)^{-1\,t}h(J_2{}^tXJ_2) & 0\\ 0 & {}^t\left((J_2{}^tXJ_2)^{-1\,t}h(J_2{}^tXJ_2)\right)^{-1}\end{pmatrix}.$$

Since we have

$$\mathrm{GSO}(S_X)=(J_2{}^tXJ_2)^{-1}\mathrm{GSO}(S_0)(J_2{}^tXJ_2),$$

we get

$$\int_{N(\mathbb{A})\backslash G^1(\mathbb{A}_F)}\int_{\mathbb{A}^\times T_{S_X}(F)\backslash T_{S_X}(\mathbb{A})}\int_{N(F)\backslash N(\mathbb{A})}\chi(h)\times\left(\omega_\psi(g_1,1)\phi\right)(v_X)f(uhg_1)\psi_{S_X}(u)^{-1}\,du\,dh\,dg_1$$
$$=\int_{N(\mathbb{A})\backslash G^1(\mathbb{A}_F)}B_{S_X,\chi^{-1}}(\pi(g_1)f)\left(\omega_\psi(g_1,1)\phi\right)(v_X)dg_1 \tag{3.1.17}$$

where we regard χ as a character of $\mathrm{GSO}(S_X)(\mathbb{A})$ by (3.1.9).

Finally the last statement concerning the equivalence of the non-vanishing conditions on the (S_X,χ^{-1},ψ)-Bessel period and the (X,χ)-Bessel period follows from the pull-back formula (3.1.6) by an argument similar to the one in the proof of Proposition 2 in [27]. □

3.2. $(G_D, \mathrm{GSU}_{3,D})$ case.

3.2.1. *Theta correspondence for quaternionic dual pair with similitudes.* Let D be a quaternion division algebra over F. Let X_D (resp. Y_D) be a right (resp. left) D-vector space of finite rank equipped with a non-degenerate hermitian bilinear form $(\ ,\ )_{X_D}$ (resp. non-degenerate skew-hermitian bilinear form $\langle\ ,\ \rangle_{Y_D}$). Hence $(\ ,\ )_{X_D}$ and $\langle\ ,\ \rangle_{Y_D}$ are D-valued F-bilinear form on X_D and Y_D satisfying:

$$\overline{(x,x')_{X_D}}=(x',x)_{X_D},\quad (xa,x'b)_{X_D}=\bar a(x,x')_{X_D}b,$$
$$\overline{\langle y,y'\rangle_{Y_D}}=-\langle y',y\rangle_{Y_D},\quad \langle ay,y'b\rangle_{Y_D}=a\langle y,y'\rangle_{Y_D}\bar b,$$

for $x, x' \in X_D$, $y, y' \in Y_D$ and $a, b \in D$. We denote the isometry group of X_D and Y_D by $\mathrm{U}(X_D)$ and $\mathrm{U}(Y_D)$, respectively. Then the space $Z_D = X_D \otimes_D Y_D$ is regarded as a symplectic space over F with the non-degenerate alternating form $\langle \, , \, \rangle$ defined by

$$\langle x_1 \otimes y_1, x_2 \otimes y_2 \rangle = \mathrm{tr}_D \left((x_1, x_2)_{X_D} \overline{\langle y_1, y_2 \rangle_{Y_D}} \right) \in F \tag{3.2.1}$$

and we have a homomorphism $\mathrm{U}(X_D) \times \mathrm{U}(Y_D) \to \mathrm{Sp}(Z_D)$ defined by

$$(x \otimes y)(g, h) = xg \otimes h^{-1} y \quad \text{for } x \in X, y \in Y, h \in \mathrm{U}(Y_D) \text{ and } g \in \mathrm{U}(X_D). \tag{3.2.2}$$

As in the case when $D \simeq \mathrm{Mat}_{2\times 2}$, this mapping splits in the metaplectic group $\mathrm{Mp}(Z_D)$. Hence we have the Weil representation ω_ψ of $\mathrm{U}(X_D, \mathbb{A}) \times \mathrm{U}(Y_D, \mathbb{A})$ by restriction.

From now on, we suppose that the rank of X_D is $2k$ and X_D is maximally split, in the sense that its maximal isotropic subspace has rank k.

Let us denote by $\mathrm{GU}(X_D)$ (resp. $\mathrm{GU}(Y_D)$) the similitude unitary group of X_D (resp. Y_D) with the similitude character λ_D (resp. ν_D). Also we write the identity component of $\mathrm{GU}(Y_D)$ by $\mathrm{GSU}(Y_D)$. Then the action (3.2.2) extends to a homomorphism

$$i_D : \mathrm{GU}(X_D) \times \mathrm{GU}(Y_D) \to \mathrm{GSp}(Z_D)$$

with the property $\lambda(i_D(g, h)) = \lambda_D(g) \nu_D(h)^{-1}$. Let

$$R_D := \{(g, h) \in \mathrm{GU}(X_D) \times \mathrm{GU}(Y_D) \mid \lambda_D(g) = \nu_D(h)\} \supset \mathrm{U}(X_D) \times \mathrm{U}(Y_D).$$

Since X_D is maximally split, we have a Witt decomposition $X_D = X_D^+ \oplus X_D^-$ with maximal isotropic subspaces $X_D^\pm$. Then as in Section 3.1.1, we may realize the Weil representation ω_ψ of $\mathrm{U}(X_D) \times \mathrm{U}(Y_D)$ on $\mathcal{S}((X_D^+ \otimes Y_D)(\mathbb{A}))$. In this realization, for $h \in \mathrm{U}(Y_D)$ and $\phi \in \mathcal{S}((X_D^+ \otimes Y_D)(\mathbb{A}))$, we have

$$\omega_\psi(1, h)\phi(z) = \phi(i_D(h)^{-1} z).$$

Hence, as in Section 3.1.1, we may extend ω_ψ to $R_D(\mathbb{A})$ by

$$\omega_\psi(g, h)\phi(z) = |\lambda(h)|^{-2\mathrm{rank}\, X_D \cdot \mathrm{rank}\, Y_D} \omega_\psi(g_1, 1)\phi\left(i_D(h)^{-1} z\right)$$

for $(g, h) \in R_D(\mathbb{A})$, where

$$g_1 = g \begin{pmatrix} \lambda_D(g)^{-1} & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{U}(X_D).$$

Then as in Section 3.1.1, we may extend the Weil representation ω_ψ of $\mathrm{U}(X_D) \times \mathrm{U}(Y_D)$ on $\mathcal{S}(Z_+(\mathbb{A}_F))$, where $Z_D = Z_D^+ \oplus Z_D^-$ is an arbitrary polarization, to $R_D(\mathbb{A})$, by using the $\mathrm{U}(X_D) \times \mathrm{U}(Y_D)$-isomorphism $p : \mathcal{S}((X_D^+ \otimes Y_D)(\mathbb{A})) \to \mathcal{S}(Z_+(\mathbb{A}_F))$. Thus for $\phi \in \mathcal{S}(Z_+(\mathbb{A}_F))$, the theta kernel $\theta_\psi^\phi = \theta^\phi$ on $R_D(\mathbb{A})$ is defined by

$$\theta_\psi^\phi(g, h) = \theta^\phi(g, h) = \sum_{z_+ \in Z_D^+(F)} \omega_\psi(g, h)\phi(z_+) \quad \text{for } (g, h) \in R_D(\mathbb{A}).$$

Let us define

$$\mathrm{GU}(X_D, \mathbb{A})^+ = \{h \in \mathrm{GU}(X_D, \mathbb{A}) : \lambda_D(h) \in \nu_D(\mathrm{GU}(Y_D, \mathbb{A}))\}.$$

and

$$\mathrm{GU}(X_D, F)^+ = \mathrm{GU}(X_D, \mathbb{A})^+ \cap \mathrm{GU}(X_D, F).$$

We note that $\nu_D\,(\mathrm{GU}(Y_D, F_v))$ contains $N_D(D(F_v)^\times)$ for any place v. Thus, if v is non-archimedean or complex, we have $\mathrm{GU}(X_D, F_v)^+ = \mathrm{GU}(X_D, F_v)$, and if v is real, $|\mathrm{GU}(X_D, F_v)/\mathrm{GU}(X_D, F_v)^+| \le 2$.

For a cusp form f on $\mathrm{GU}(X_D, \mathbb{A})^+$, as in 3.1.1, we define the theta lift of f to $\mathrm{GU}(Y_D, \mathbb{A})$ by

$$\Theta(f, \phi)(h) := \int_{\mathrm{U}(X_D,F)\backslash \mathrm{U}(X_D,\mathbb{A})} \theta^\phi(g_1 g, h) f(g_1 g)\, dg_1$$

where $g \in \mathrm{GU}(X_D, \mathbb{A})^+$ is chosen so that $\lambda_D(g) = \nu_D(h)$. It defines an automorphic form on $\mathrm{GU}(Y_D, \mathbb{A})$. When we regard $\Theta(f, \phi)(h)$ as an automorphic form on $\mathrm{GSU}(Y_D, \mathbb{A})$ by the restriction, we denote it as $\theta(f, \phi)(h)$. For an irreducible cuspidal automorphic representation (π_+, V_{π_+}) of $\mathrm{GU}(X_D, \mathbb{A})^+$, we denote by $\Theta_\psi(\pi_+)$ (resp. $\theta_\psi(\pi_+)$) the theta lift of π_+ to $\mathrm{GU}(Y_D, \mathbb{A})$ (resp. $\mathrm{GSU}(Y_D, \mathbb{A})$), namely

$$\Theta_\psi(\pi) := \left\{\Theta(f, \phi) : f \in V_{\pi_+}, \phi \in \mathcal{S}(Z_D^+(\mathbb{A}))\right\},$$
$$\theta_\psi(\pi) := \left\{\theta(f, \phi) : f \in V_{\pi_+}, \phi \in \mathcal{S}(Z_D^+(\mathbb{A}))\right\},$$

respectively. Moreover, for an irreducible cuspidal automorphic representation (π, V_π) of $\mathrm{GU}(X_D, \mathbb{A})$, we define the theta lift $\Theta_\psi(\pi)$ (resp. $\theta_\psi(\pi)$) of π to $\mathrm{GU}(Y_D, \mathbb{A})$ (resp. $\mathrm{GSU}(Y_D, \mathbb{A})$) by $\Theta_\psi(\pi) := \Theta_\psi(\pi|_{\mathrm{GU}(X_D,\mathbb{A})^+})$ (resp. $\theta_\psi(\pi) := \theta_\psi(\pi|_{\mathrm{GU}(X_D,\mathbb{A})^+})$).

As for the opposite direction, as in 3.1.1, for a cusp form f' on $\mathrm{GSU}(Y_D, \mathbb{A})$, we define the theta lift of f' to $\mathrm{GU}(X_D, \mathbb{A})^+$ by

$$\theta(f', \phi)(g) := \int_{\mathrm{SU}(Y_D,F)\backslash \mathrm{SU}(Y_D,\mathbb{A})} \theta^\phi(g, h_1 h) f(h_1 h)\, dh_1$$

where $h \in \mathrm{GSU}(Y_D, \mathbb{A})$ is chosen so that $\lambda_D(g) = \nu_D(h)$. For an irreducible cuspidal automorphic representation (σ, V_σ) of $\mathrm{GSU}(Y_D, \mathbb{A})$, we denote by $\theta_\psi(\sigma)$ the theta lift of σ to $\mathrm{GU}(X_D, \mathbb{A})^+$. Moreover, we extend $\theta(f', \phi)$ to an automorphic form on $\mathrm{GU}(X_D, \mathbb{A})$ by the natural embedding

$$\mathrm{GU}(X_D, F)^+\backslash \mathrm{GU}(X_D, \mathbb{A})^+ \to \mathrm{GU}(X_D, F)\backslash \mathrm{GU}(X_D, \mathbb{A})$$

and extension by zero. Then we define the theta lift $\Theta_\psi\,(\sigma)$ of σ to $\mathrm{GU}(X_D, \mathbb{A})$ as the $\mathrm{GU}(X_D, \mathbb{A})$ representation generated by such $\theta\,(f', \phi)$ for $f' \in V_\sigma$ and $\phi \in \mathcal{S}\,(Z_+\,(\mathbb{A}))$.

Remark 3.2. *Suppose that (π_+, V_{π_+}) (resp. (σ, V_σ)) is an irreducible cuspidal automorphic representation of $\mathrm{GU}(X_D, \mathbb{A})^+$ (resp. $\mathrm{GSU}(Y_D, \mathbb{A})$). Suppose moreover that the theta lift $\Theta_\psi(\pi_+)$ (resp. $\theta_\psi(\sigma)$) is non-zero and cuspidal. Then by Gan [31,* Proposition 2.12], *$\Theta_\psi(\pi_+)$ (resp. $\theta_\psi(\sigma)$) is an irreducible cuspidal automorphic representation because of the Howe duality for quaternionic dual pairs proved by Gan and Sun [33] and Gan and Takeda [37]. We shall study the case $\dim_D X_D = 2$ and $\dim_D Y_D = 3$. In this case, by the conservation relation proved by Sun and Zhu [105], the irreducibility of $\Theta_\psi(\pi_+)$ implies that of $\theta_\psi(\pi_+)$.*

3.2.2. *Pull-back of the global Bessel periods for the dual pair* $(\mathrm{G}_D, \mathrm{GSU}_{3,D})$. The set-up is as follows.

Let X_D be the space of 2 dimensional row vectors over D equipped with the hermitian form

$$(x, x')_{X_D} = \bar{x} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} {}^t x'.$$

Let us take the standard basis of X_D and name the basis vectors as

$$x_+ = (1, 0), \quad x_- = (0, 1).$$

Then G_D defined by (1.4.2) is the matrix representation of the similitude unitary group $\mathrm{GU}(X_D)$ for X_D with respect to the standard basis.

Let Y_D be the space of 3 dimensional column vectors over D equipped with the skew-hermitian form

$$\langle y, y' \rangle_{Y_D} = {}^t y \begin{pmatrix} 0 & 0 & \eta \\ 0 & \eta & 0 \\ \eta & 0 & 0 \end{pmatrix} \overline{y'}.$$

Let us take the standard basis of Y_D and name the basis vectors as

$$y_- = {}^t (1, 0, 0), \quad e = {}^t (0, 1, 0), \quad y_- = {}^t (0, 0, 1).$$

Then $\mathrm{GSU}_{3,D}$ defined in 2.1.3 is the matrix representation of the group $\mathrm{GSU}(Y_D)$ for Y_D with respect to the standard basis.

We take a polarization $Z_D = Z_{D,+} \oplus Z_{D,-}$ of $Z_D = X_D \otimes_D Y_D$ defined as follows. Let

$$X_{D,\pm} = x_\pm \cdot D$$

where the double sign corresponds. We decompose Y_D as $Y_D = Y_{D,+} \oplus Y_{D,0} \oplus Y_{D,-}$ where

$$Y_{D,+} = D \cdot y_+, \quad Y_{D,0} = D \cdot y_0, \quad Y_{D,-} = D \cdot y_-.$$

Then let

$$Z_{D,\pm} = (X_D \otimes Y_{D,\pm}) \oplus (X_{D,\pm} \otimes Y_{D,0}) \tag{3.2.3}$$

where the double sign corresponds. To simplify the notation, we write $z_+ \in Z_{D,+}(\mathbb{A})$ as $z_+ = (a, b)$ when

$$z_+ = a \otimes y_+ + b \otimes e \quad \text{where } a \in X_D(\mathbb{A}) \text{ and } b \in X_{D,+}(\mathbb{A})$$

and $\phi(z_+)$ as $\phi(a, b)$ for $\phi \in \mathcal{S}(Z_{D,+}(\mathbb{A}))$.

Let us compute the pull-back of the (X, χ, ψ)-Bessel periods on $\mathrm{GSU}_{3,D}$ defined by (2.3.3) with respect to the theta lift from G_D.

Proposition 3.2. *Let* (π_D, V_{π_D}) *be an irreducible cuspidal automorphic representation of* $G_D(\mathbb{A})$ *whose central character is* ω_π *and* χ *a character of* $\mathbb{A}_E^\times$ *such that* $\chi|_{\mathbb{A}^\times} = \omega_\pi^{-1}$. *Let* $X \in D^\times$.

Then for $f \in V_{\pi_D}$ *and* $\phi \in \mathcal{S}(Z_{D,+}(\mathbb{A}))$, *we have*

(3.2.4)

$$\mathcal{B}^D_{X,\chi,\psi}(\theta(f : \phi)) = \int_{N_D(\mathbb{A}) \backslash G^1_D(\mathbb{A})} B_{\xi_X, \chi^{-1}, \psi}(\pi(g) f)\, (\omega(g, 1)\phi)\, (v_{D,X})\, dg$$

where

$$\xi_X := X\eta\bar{X} \in D^-(F), \quad \nu_{D,X} := (x_-, -\eta^{-1}Xx_+) \in Z_{D,+}, \tag{3.2.5}$$

and $B_{\xi_X,\chi^{-1},\psi}$ *denotes the* (ξ_X, χ^{-1}, ψ)*-Bessel period on* G_D *defined by* (1.4.11).

In particular, the (ξ_X, χ^{-1}, ψ)*-Bessel period does not vanish on* V_{π_D} *if and only if the* (X, χ, ψ)*-Bessel period does not vanish on* $\theta_\psi(\pi_D)$.

Proof. The proof of this proposition is similar to the one for Proposition 3.1.

Let $N_{0,D}$ be a subgroup of $N_{3,D}$ given by

$$N_{0,D}(F) = \left\{ u_D(x) := \begin{pmatrix} 1 & 0 & \eta x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : x \in F \right\}.$$

Then we note that $N_{0,D}$ is a normal subgroup of $N_{3,D}$ and $\psi_{X,D}$ is trivial on $N_{0,D}(\mathbb{A})$. Since

$Z_{D,-}(\mathbb{A})(1, u_D(x)) = Z_{D,-}(\mathbb{A})$ and $z_+(1, u_D(x)) = z_+ + a \otimes (-\eta x)y_-$ for $x \in \mathbb{A}$,

we have

$$(\omega(1, u_D(x))\phi)(z_+) = \psi\left(-\frac{1}{2}\operatorname{tr}_D\left(\langle a, a\rangle \eta^2 x\right)\right)\phi(z_+).$$

Thus by an argument similar to the one in the proof of Proposition 3.1, one may show that

$$\begin{aligned} &\int_{N_{3,D}(F)\backslash N_{3,D}(\mathbb{A})} \theta(f;\phi)(hu)\psi_{X,D}^{-1}(u)\,du \\ &= \int_{N_{3,D}(F)\backslash N_{3,D}(\mathbb{A})} \int_{G_D^1(F)\backslash G_D^1(\mathbb{A})} \sum_{\bar{a}\in \mathcal{X}_{D,-}} \sum_{\gamma\in V_D(a)\backslash G_D^1(F)} \sum_{b\in X_{D,+}} \\ &\qquad \left(\omega(\gamma g_1 \lambda_s^D(\nu(h)), uh)\phi\right)(a,b) f(g_1\lambda_s(\nu(h)))\,dg_1\,du. \end{aligned} \tag{3.2.6}$$

Here $\mathcal{X}_{D,-}$ is the set of equivalence classes $X_{D,-}/\!\sim$ where $a \sim a'$ if and only if there exists a $\gamma \in G_D^1(F)$ such that $a' = a\gamma$, $\bar{a}$ denotes the equivalence class of $\mathcal{X}_{D,-}$ containing $a \in X_{D,-}$, and, $V(a) = \left\{\gamma \in G_D^1(F) \mid a\gamma = a\right\}$. Then we may rewrite (3.2.6) as

$$\begin{aligned} &\int_{N_{3,D}(F)\backslash N_{3,D}(\mathbb{A})} \theta(f;\phi)(hu)\psi_{X,D}^{-1}(u)\,du = \int_{N_{3,D}(F)\backslash N_{3,D}(\mathbb{A})} \int_{G_D^1(F)\backslash G_D^1(\mathbb{A})} \\ &\sum_{N_D(F)\backslash G_D^1(F)} \sum_{b\in X_{D,+}} \left(\omega(\gamma g_1 \lambda_s^D(\nu(h)), uh)\phi\right)(x_-,b) f(g_1\lambda_s(\nu(h)))\,dg_1\,du. \end{aligned} \tag{3.2.7}$$

Since, for $u = \begin{pmatrix} 1 & -\eta^{-1}\bar{A}\eta & B \\ 0 & 1 & A \\ 0 & 0 & 1 \end{pmatrix} \in N_{3,D}(\mathbb{A})$, we have $Z_{D,-}(\mathbb{A})(1,u) = Z_{D,-}(\mathbb{A})$ and

$$\begin{aligned} z_+(1,u) &= z_+ + x_- \otimes (B'y_- - Ae + y_+) + b \otimes (\eta^{-1}\bar{A}\eta y_- + e) \\ &= z_+ + x_- \otimes (B'y_- - Ae) + b \otimes (\eta^{-1}\bar{A}\eta y_-), \end{aligned}$$

we obtain

$$(\omega(1,u)\phi)(z_+) = \psi\left(\mathrm{tr}_D\left(\langle b, x_-\rangle\overline{(e, -Ae)}\right)\right)\phi(z_+) = \psi\left(\mathrm{tr}_D\left(\eta\langle b, x_-\rangle A\right)\right)\phi(z_+).$$

Hence in (3.2.7), only $b \in X_{D,-}$ satisfying $\eta\langle b, x_-\rangle = X$, i.e. $b = x_+(-\overline{X}\eta^{-1})$ contributes. Thus our integral is equal to

$$\begin{aligned} &\int_{N_D(F)\backslash G_D^1(\mathbb{A})} \left(\omega(g_1\lambda_s^D(\nu(h)), uh)\phi\right)(v_{D,X}) f(g_1\lambda_s^D(\nu(h)))\, dg_1\, du \\ &= \int_{N_D(\mathbb{A})\backslash G_D^1(\mathbb{A})} \int_{N_D(F)\backslash N_D(\mathbb{A})} \omega(ug_1\lambda_s^D(\nu(h)), uh)\phi(v_{D,X}) \\ &\qquad \times f(ug_1\lambda_s^D(\nu(h)))\, dg_1\, du \end{aligned}$$

where $v_{D,X} = (x_-, x_+(-\overline{X}\eta^{-1}))$. Further for $u = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \in N_D(\mathbb{A})$, we have

$$(\omega(ug,h)\phi)(v_{D,X}) = \psi_{\xi_X}(u)^{-1}(\omega(g,h)\phi)(v_{D,X})$$

where we put $\xi_X = X\eta\overline{X}$. Thus our integral becomes

$$\begin{aligned} \int_{N_D(\mathbb{A})\backslash G_D^1(\mathbb{A})} \int_{N_D(F)\backslash N_D(\mathbb{A})} &\psi_{\xi_X}(u)^{-1}\omega(g_1\lambda_s^D(\nu(h)), h)\phi(v_{D,X}) \\ &\times f(ug_1\lambda_s^D(\nu(h)))\, du\, dg_1. \end{aligned}$$

As for the integration over $\mathbb{A}^\times M_{X,D}(F)\backslash M_{X,D}(\mathbb{A})$ in (2.3.3), by a direct computation, we see that

$$\omega(\lambda_s^D(\nu(h))g_1, h)\phi(v_{D,X}) = |\nu(h)|^{-3}\omega(h_0\lambda_s(\nu(h))g_1, 1)\phi(v_{D,X})$$

where

$$h = \begin{pmatrix} n_D(h)\cdot(h^X)^* & 0 & 0 \\ 0 & h & 0 \\ 0 & 0 & h^X \end{pmatrix} \quad \text{and} \quad h_0 = \begin{pmatrix} \overline{h^X} & 0 \\ 0 & (h^X)^{-1} \end{pmatrix}.$$

Therefore, as in the previous case, we obtain

$$\mathcal{B}_{X,\chi,\psi}^D(\theta(f:\phi)) = \int_{N_D(\mathbb{A})\backslash G_D^1(\mathbb{A})} B_{\xi_X,\chi^{-1},\psi}(\pi(g_1)f)(\omega(g_1,1)\phi)(v_{D,X})dg_1.$$

The equivalence of the non-vanishing conditions follows from the pull-back formula (3.2.4) as Proposition 3.1. □

3.3. **Theta correspondence for similitude unitary groups.** In our proof of Theorem 1.1 and 1.2, we shall use theta correspondence for similitude unitary groups besides theta correspondences for dual pairs $(\mathrm{GSp}_2, \mathrm{GSO}_{4,2})$ and $(G_D, \mathrm{GSU}_{3,D})$. Let us recall the definition of the theta lifts in this case.

Let $(X, (\ ,\ )_X)$ be an m-dimensional hermitian space over E, and let $(Y, (\ ,\ )_Y)$ be an n-dimensional skew-hermitian space over E. Then we may define the quadratic space

$$(W_{X,Y}, (\ ,\ )_{X,Y}) := \left(\mathrm{Res}_{E/F} X \otimes Y, \mathrm{Tr}_{E/F}\left((\ ,\ )_X \otimes \overline{(\ ,\ )_Y}\right)\right).$$

This is a $2mn$-dimensional symplectic space over F. Then we denote its isometry group by $\mathrm{Sp}\left(W_{X,Y}\right)$. For each place v of F, we denote the metaplectic extension of $\mathrm{Sp}\left(W_{X,Y}\right)(F_v)$ by $\mathrm{Mp}\left(W_{X,Y}\right)(F_v)$. Also, $\mathrm{Mp}\left(W_{X,Y}\right)(\mathbb{A})$ denotes the metaplectic extension of $\mathrm{Sp}\left(W_{X,Y}\right)(\mathbb{A})$.

Let χ_X and χ_Y be characters of $\mathbb{A}_E^\times/E^\times$ such that $\chi_X|_{\mathbb{A}^\times} = \chi_E^m$ and $\chi_Y|_{\mathbb{A}^\times} = \chi_E^n$. For each place v of F, let

$$\iota_{\chi_v} : \mathrm{U}(X)(F_v) \times \mathrm{U}(Y)(F_v) \longrightarrow \mathrm{Mp}(W_{X,Y})(F_v)$$

be the local splitting given by Kudla [67] depending on the choice of a pair of characters $\chi_v = (\chi_{X,v}, \chi_{Y,v})$. Using this local splitting, we get a splitting

$$\iota_\chi : \mathrm{U}(X)(\mathbb{A}) \times \mathrm{U}(Y)(\mathbb{A}) \longrightarrow \mathrm{Mp}(W_{X,Y})(\mathbb{A}),$$

depending on $\chi = (\chi_X, \chi_Y)$. Then by the pull-back, we obtain the Weil representation $\omega_{\psi,\chi}$ of $\mathrm{U}(X)(\mathbb{A}) \times \mathrm{U}(Y)(\mathbb{A})$. When we fix a polarization $W_{X,Y} = W_{X,Y}^+ \oplus W_{X,Y}^-$, we may realize $\omega_{\psi,\chi}$ so that its space of smooth vectors is given by $\mathcal{S}(W_{X,Y}^+(\mathbb{A}))$, the space of Schwartz-Bruhat functions on $W_{X,Y}^+(\mathbb{A})$. We define

$$R := \{(g, h) \in \mathrm{GU}(X) \times \mathrm{GU}(Y) : \lambda(g) = \lambda(h)\} \supset \mathrm{U}(X) \times \mathrm{U}(Y).$$

Suppose that $\dim Y$ is even and Y is maximally split, in the sense that Y has a maximal isotropic subspace of dimension $\frac{1}{2}\dim Y$. In this case, as in Section 3.1.1 and 3.2.1, we may extend $\omega_{\psi,\chi}$ to $R(\mathbb{A})$. On the other hand, in this case, we have an explicit local splitting of $R(F_v) \to \mathrm{Sp}(W_{X,Y})(F_v)$ by Zhang [121] and we may extend $\omega_{\psi,\chi}$ to $R(\mathbb{A})$ using this splitting. These two extensions of $\omega_{\psi,\chi}$ to $R(\mathbb{A})$ coincide.

Then for $\phi \in \mathcal{S}(W_{X,Y}^+(\mathbb{A}))$, we define the theta function $\theta_{\psi,\chi}^\phi$ on $R(\mathbb{A})$ by

$$\theta_{\psi,\chi}^\phi(g, h) = \sum_{w \in W_{X,Y}^+(F)} \omega_{\psi,\chi}(g, h)\phi(w). \tag{3.3.1}$$

Let us define

$$\mathrm{GU}(X)(\mathbb{A})^+ := \{g \in \mathrm{GU}(X)(\mathbb{A}) : \lambda(g) \in \lambda(\mathrm{GU}(Y)(\mathbb{A}))\},$$
$$\mathrm{GU}(X)(F)^+ := \mathrm{GU}(X)(\mathbb{A})^+ \cap \mathrm{GU}(X)(F).$$

We define $\mathrm{GU}(Y)(\mathbb{A})^+$ and $\mathrm{GU}(Y)(F)^+$ in a similar manner. Let (σ, V_σ) be an irreducible cuspidal automorphic representation of $\mathrm{GU}(X)(\mathbb{A})^+$. Then for $\varphi \in V_\sigma$

and $\phi \in \mathcal{S}(W^+_{X,Y}(\mathbb{A}))$, we define the theta lift of φ by

$$\theta^{\phi}_{\psi,\chi}(\varphi)(h) = \int_{\mathrm{U}(X)(F)\backslash \mathrm{U}(X)(\mathbb{A})} \varphi(g_1 g)\theta^{\phi}_{\psi,\chi}(g_1 g, h)\, dg_1$$

where $g_1 \in \mathrm{GU}(X)(\mathbb{A})^+$ is chosen so that $\lambda(g) = \lambda(h)$. Further, we define the theta lift of σ by

$$\Theta^{X,Y}_{\psi,\chi}(\sigma) = \langle \theta^{\phi}_{\psi,\chi}(\varphi); \varphi \in \sigma, \phi \in \mathcal{S}(W^+_{X,Y}(\mathbb{A}))\rangle.$$

When the space we consider is clear, we simply write $\Theta^{X,Y}_{\psi,\chi}(\sigma) = \Theta_{\psi,\chi}(\sigma)$. Similarly, for an irreducible cuspidal automorphic representation τ of $\mathrm{U}(Y)(\mathbb{A})$, we define $\Theta^{Y,X}_{\psi,\chi}(\tau)$ and we simply write it by $\Theta_{\psi,\chi}(\tau)$.

4. Proof of the Gross-Prasad conjecture for $(\mathrm{SO}(5), \mathrm{SO}(2))$

In this section we prove Theorem 1.1, i.e. the Gross-Prasad conjecture for $(\mathrm{SO}(5), \mathrm{SO}(2))$, based on the pull-back formulas obtained in the previous section.

4.1. **Proof of the statement (1) in Theorem 1.1.**

Let (π, V_π) be as in Theorem 1.1 (1). By the uniqueness of the Bessel model due to Gan, Gross and Prasad [32, Corollary 15.3] at finite places and to Jiang, Sun and Zhu [62, Theorem A] at archimedean places, there exists uniquely an irreducible constituent π^B_+ of $\pi\mid_{G_D(\mathbb{A})^+}$ that has the (ξ, Λ, ψ)-Bessel period.

When D is split and π^B_+ is a theta lift from an irreducible cuspidal automorphic representation of $\mathrm{GSO}_{3,1}(\mathbb{A})$, our assertion has been proved by Corbett [20]. Hence in the remainder of this subsection, we assume that:

(4.1.1) *when D is split, π is not a theta lift from* $\mathrm{GSO}_{3,1}$ *of an irreducible cuspidal automorphic representation*.

Let us proceed under the assumption (4.1.1). By Proposition 3.1 and 3.2, the theta lift $\theta_\psi(\pi^B_+)$ of π^B_+ to $\mathrm{GSU}_{3,D}(\mathbb{A})$ has the $(X_\xi, \Lambda^{-1}, \psi)$-Bessel period and, in particular, $\theta_\psi(\pi^B_+) \ne 0$ where we take $X_\xi \in D^-(F)$ so that $\xi_{X_\xi} = \xi$. For example, when we take $\xi = \eta$, we may take $X_\xi = 1$.

Lemma 4.1. *$\theta_\psi(\pi^B_+)$ is an irreducible cuspidal automorphic representation of* $\mathrm{GSU}_{3,D}(\mathbb{A})$.

Proof. First we note that the irreducibility follows from the cuspidality by Remark 3.1 and 3.2.

Let us show the cuspidality. Suppose on the contrary that $\theta_\psi(\pi^B_+)$ is not cuspidal.

When D is not split, the Rallis tower property implies that the the theta lift $\theta_{D,\psi}(\pi^B_+)$ of π^B_+ to $\mathrm{GSU}_{1,D}(\mathbb{A})$ is non-zero and cuspidal. Let w be a finite place of F such that $D(F_w)$ is split and $\pi^B_{+,w}$ is a generic representation of $G(F_w)^+$. Since $\pi^B_{+,w}$ is generic, the theta lift of $\pi^B_{+,w}$ to $\mathrm{GSO}_2(F_w)$ vanishes by the same argument as the one for [42, Proposition 2.4]. We note that $\mathrm{GSU}_{1,D}(F_w) \simeq \mathrm{GSO}_2(F_w)$ and hence the theta lift of π^B_+ to $\mathrm{GSU}_{1,D}(\mathbb{A})$ must vanish. This is a contradiction.

Suppose that D is split. Then the theta lift of π^B_+ to $\mathrm{GSO}_{3,1}$ is non-zero by the Rallis tower property. Moreover, it is not cuspidal by our assumption on π. Thus the theta lift of π_+^B to $\mathrm{GSO}_{2,0}$ is non-zero, again by the Rallis tower property. Then we reach a contradiction by the same argument as in the non-split case. □

We may regard $\theta_\psi(\pi_+^B)$ as an irreducible cuspidal automorphic representation of $\mathrm{PGU}_{2,2}$ or $\mathrm{PGU}_{3,1}$ according to whether D is split or not, under the isomorphism Φ in (2.2.6) or Φ_D in (2.2.5). Recall our assumption that $\theta_{\psi_w}(\pi_{+,w}^B)$ is generic at a finite place w. Then the non-vanishing of $(X_\xi, \Lambda^{-1}, \psi)$-Bessel period on $\theta_\psi(\pi_+^B)$ implies the non-vanishing of the central value of the standard L-function for $\theta_\psi(\pi_+^B)$ of PGU_4 twisted by Λ^{-1}, namely

$$L^S\left(\frac{1}{2}, \theta_\psi(\pi_+^B) \times \Lambda^{-1}\right) \neq 0$$

for any finite set S of places of F containing all archimedean places because of the unitary group case of the Gan-Gross-Prasad conjecture for $\theta_\psi(\pi_+^B)$ proved by Proposition A.2 and Remark A.1 in [29]. Moreover, from the explicit computation of local theta correspondence in [36] and [83], we see that

$$L(s, \pi_v \times \mathcal{AI}(\Lambda)_v) = L\left(s, \theta_\psi(\pi_+^B)_v \times \Lambda_v^{-1}\right)$$

at a finite place v where all data are unramified. Thus when we take S_0, a finite set of places of F containing all archimedean places, so that all data are unramified at $v \notin S_0$, we have

$$L^S\left(\frac{1}{2}, \pi \times \mathcal{AI}(\Lambda)\right) = L^S\left(\frac{1}{2}, \theta_\psi(\pi_+^B) \times \Lambda\right) \neq 0$$

for any finite set S of places of F with $S \supset S_0$.

Let us show an existence of π°. We denote $\theta_\psi(\pi_+^B)$ by σ. Then the theta lift $\Sigma := \Theta_{\psi,(\Lambda^{-1},\Lambda^{-1})}(\sigma)$ of σ to $\mathrm{GU}_{2,2}$ which we may regard as an automorphic representation of $\mathrm{GSO}_{4,2}$ by the accidental isomorphism (2.2.6), is an irreducible cuspidal globally generic automorphic representation with trivial central character by the proof of [29, Proposition A.2] since $\theta_\psi(\pi_+^B)$ has the $(X_\xi, \Lambda^{-1}, \psi)$-Bessel period.

Here we recall that, by the conservation relation due to Sun and Zhu [105, Theorem 1.10, Theorem 7.6], for any irreducible admissible representations τ of $\mathrm{GO}_{4,2}(k)$ (resp. $\mathrm{GO}_{3,3}(k)$) over a local field k of characteristic zero, theta lifts of either τ or $\tau \otimes \det$ to $\mathrm{GSp}_3(k)^+$ (resp. $\mathrm{GSp}_3(k)$) is non-zero. Thus we may extend Σ to an automorphic representation of $\mathrm{GO}_{4,2}(\mathbb{A})$ as in Harris–Soudry–Taylor [47, Proposition 2] so that its local theta lift to $\mathrm{GSp}_3(F_v)^+$ is non-zero at every place v.

On the other hand, since Σ is nearly equivalent to σ, we have

$$L^S(s, \Sigma, \mathrm{std}) = L^S(s, \pi, \mathrm{std} \otimes \chi_E)\zeta_F^S(s) \tag{4.1.2}$$

for a sufficiently large finite set S of places of F containing all archimedean places by the explicit computation of local theta correspondences in [36] and [83]. Here

$$L^S(1, \pi, \mathrm{std} \otimes \chi_E) \neq 0$$

by Yamana [120, proof of Theorem 10.2, Theorem 10.3], since the theta lift $\theta_\psi(\pi_+^B)$ of π_+^B to $\mathrm{GSU}_{3,D}(\mathbb{A})$ is non-zero and cuspidal. Hence the left hand side of (4.1.2) has a pole at $s = 1$. In particular, it is non-zero and the theta lift of Σ to $\mathrm{GSp}_3(\mathbb{A})^+$ is non-zero by Takeda [106, Theorem 1.1 (1)]. Further, again by Takeda [106, Theorem 1.1 (1)], this theta lift actually descends to $\mathrm{GSp}_2(\mathbb{A})^+ = G(\mathbb{A})^+$. Namely, the theta lift $\pi'_+ := \theta_{\psi^{-1}}(\Sigma)$ of Σ to $G(\mathbb{A})^+$ is non-zero since $L^S(s, \Sigma, \mathrm{std})$ actually has a pole at $s = 1$.

Suppose that π'_+ is not cuspidal. Then by the Rallis tower property, the theta lift of Σ to $\mathrm{GL}_2(\mathbb{A})^+$ is non-zero and cuspidal. Meanwhile the local theta lift of Σ_v to $\mathrm{GL}_2(F_v)^+$ vanishes by a computation similar to the one for [42, Proposition 3.3] since Σ_v is generic. This is a contradiction and hence π'_+ is cuspidal.

Since Σ is generic, so is π'_+ by [83, Proposition 3.3]. Let us take an extension π° of π'_+ to $G(\mathbb{A})$. Since $|G(F_v)/G(F_v)^+| = 2$, we have $\pi'_v \simeq \pi_v$ or $\pi'_v \simeq \pi_v \otimes \chi_{E_v}$ at almost all places v such that $\pi'_{+,v} \simeq \pi^B_{+,v}$. Hence π is locally G^+-nearly equivalent to π°. □

4.2. **Some consequences of the proof of Theorem 1.1 (1).** As preliminaries for our further considerations, we would like to discuss some consequence of the proof of Theorem 1.1 (1) and related results.

First we note the following result concerning the functorial transfer.

Proposition 4.1. *Let (π, V_π) be an irreducible cuspidal automorphic representation of $G_D(\mathbb{A})$ with a trivial central character. Assume that there exists a finite place w at which π_w is generic and tempered.*

Then there exists a globally generic irreducible cuspidal automorphic representation π° of $G(\mathbb{A})$ and an étale quadratic extension E° of F such that π° is G^{+,E°-nearly equivalent to π. In particular we have a weak functorial lift of π to $\mathrm{GL}_4(\mathbb{A}_{E^\circ})$ *with respect to* $\mathrm{BC} \circ \mathrm{spin}$.

Moreover, π is tempered if and only if π° is tempered.

Remark 4.1. *When D is split, our assumption implies that π has a generic Arthur parameter. Though our assertion thus follows from the global descent method by Ginzburg, Rallis and Soudry [43] and Arthur [3], we shall present another proof which does not refer to these papers.*

Proof. Suppose that D is split. When π participates in the theta correspondence with $\mathrm{GSO}_{3,1}$, our assertion follows from [96]. Thus we now assume that the theta lift of π to $\mathrm{GSO}_{3,1}$ is zero. By [74], π has $(S_\circ, \Lambda_\circ, \psi)$-Bessel period for some $S_\circ$ and $\Lambda_\circ$. When $\mathrm{GSO}(S_\circ)$ is not split, the existence of a globally generic irreducible cuspidal automorphic representation follows from Theorem 1.1 (1). Suppose that $\mathrm{GSO}(S_\circ)$ is split. Then by Proposition 3.1, the theta lift of π to $\mathrm{GSO}_{3,3}$ is non-zero. Since π_w is generic, the local theta lift of π_w to $\mathrm{GSO}_{1,1}$ is zero as in the proof of Theorem 1.1 (1) and hence the theta lift of π to $\mathrm{GSO}_{1,1}$ is zero. Hence by the Rallis tower property, either the theta lift of π to $\mathrm{GSO}_{2,2}$ or the one to $\mathrm{GSO}_{3,3}$ is non-zero and cuspidal. Then π itself is globally generic by Proposition A.1 in the former case. In the latter case, the global genericity of π readily follows from the proof of Soudry [103, Proposition 1.1] (see also Theorem in p.264 of [103]).

In any case when D is split, we have a globally generic irreducible cuspidal automorphic representation π° of $G(\mathbb{A})$ which is nearly equivalent to π. Thus when we take the strong lift of π° to $\mathrm{GL}_4(\mathbb{A})$ by [19], it is a weak lift of π to $\mathrm{GL}_4(\mathbb{A})$.

Suppose that D is not split. Then by Li [74], there exist an $\eta_\circ \in D^-(F)$ where $E_\circ := F(\eta_\circ)$ is a quadratic extension of F, and a character $\Lambda_\circ$ of $\mathbb{A}_{E_\circ}^\times / E_\circ^\times \mathbb{A}^\times$ such that π has the $(\eta_\circ, \Lambda_\circ)$-Bessel period. Then there exists a desired automorphic representation π° of $G(\mathbb{A})$ by Theorem 1.1 (1).

Let us discuss the temperedness. Let σ, Σ and π'_+ denote the same as in the proof of Theorem 1.1 (1). Suppose that π is tempered. Then the temperedness of σ follows from a similar argument as in Atobe-Gan [5, Proposition 5.5] (see also [40, Proposition C.1]) at finite places, from Paul [85, Theorem 15, Theorem 30], [87, Theorem 15, Theorem 18, Corollary 24] and Li-Paul-Tan-Zhu [75, Theorem 4.20, Theorem 5.1] at real places and from Adams-Barbasch [1, Theorem 2.7] at complex places. Then the temperedness of σ implies that of Σ by Atobe-Gan [5, Proposition 5.5] at finite places, by Paul [86, Theorem 3.4] at non-split real places, by Mœglin [80, Proposition III.9] at split real places and by Adams-Barbasch [1, Theorem 2.6] at complex places. As we obtained the temperedness of σ from that of π, the temperedness of Σ implies that of π'_+ and hence π° is tempered. The opposite direction, i.e., the temperedness of π° implies that of π, follows by the same argument. □

Lemma 4.2. *Let π be as in Theorem 1.1 (1). Suppose that $\sigma = \theta_\psi\left(\pi_+^B\right)$ is an irreducible cuspidal automorphic representation of $\mathrm{GSU}_{3,D}(\mathbb{A})$. Here π_+^B denotes the unique irreducible constituent of $\pi|_{G_D(\mathbb{A})^+}$ such that π_+^B has the (E,Λ)-Bessel period. We regard σ as an automorphic representation of $\mathrm{GU}_{4,\varepsilon}(\mathbb{A})$ via (2.2.5) or (2.2.6) and let Π_σ denote the base change lift of $\sigma|_{\mathrm{U}_{4,\varepsilon}(\mathbb{A})}$ to $\mathrm{GL}_4(\mathbb{A}_E)$. Let π° be a globally generic irreducible cuspidal automorphic representation of $G(\mathbb{A})$ whose existence is proved in Theorem 1.1 (1). We denote the functorial lift of π° to $\mathrm{GL}_4(\mathbb{A})$ by Π_{π°.*

Suppose that

$$\Pi_{\pi^\circ} = \Pi_1 \boxplus \cdots \boxplus \Pi_\ell \tag{4.2.1}$$

where Π_i are irreducible cuspidal automorphic representations of $\mathrm{GL}_{n_i}(\mathbb{A})$ and

$$\Pi_\sigma = \Pi'_1 \boxplus \cdots \boxplus \Pi'_k \tag{4.2.2}$$

where Π'_j are irreducible cuspidal automorphic representations of $\mathrm{GL}_{m_j}(\mathbb{A}_E)$.

Then we have $\Pi_\sigma = \mathrm{BC}(\Pi_{\pi^\circ})$, $\Pi_{\pi^\circ} \not\simeq \Pi_{\pi^\circ} \otimes \chi_E$ and $\mathrm{BC}(\Pi_i)$ is cuspidal for each i. In particular, we have $\ell = k$. Here BC denotes the base change from F to E.

Proof. By the explicit computation of local theta correspondences in [36] and [83], we see that $(\Pi_\sigma)_v \simeq \mathrm{BC}(\Pi_{\pi^\circ})_v$ at almost all finite places v of E. Thus, $\Pi_\sigma = \mathrm{BC}(\Pi_{\pi^\circ})$ by the strong multiplicity one theorem. Also, by [19], we know that $\ell = 1$ or 2.

Suppose that $\ell = 1$. We note that the cuspidality of $\mathrm{BC}(\Pi_{\pi^\circ})$ is equivalent to $\Pi_{\pi^\circ} \otimes \chi_E \not\simeq \Pi_{\pi^\circ}$. Suppose otherwise, i.e. $\Pi_{\pi^\circ} \simeq \Pi_{\pi^\circ} \otimes \chi_E$. Then $\Pi_{\pi^\circ} = \mathcal{AI}(\tau)$ for some irreducible cuspidal automorphic representation τ of $\mathrm{GL}_2(\mathbb{A}_E)$. Since Π_{π° is a lift from PGSp_2, the central character of τ needs to be trivial and hence $\tau \simeq \tau^\vee$. On the other hand, we have

$$\Pi_\sigma = \mathrm{BC}\left(\mathcal{AI}(\tau)\right) = \tau \boxplus \tau^\sigma.$$

Since this is a base change lift of $\sigma|_{\mathrm{U}_{4,\varepsilon}(\mathbb{A})}$, we have $\tau = (\tau^\sigma)^\vee$ and $\tau \not\simeq \tau^\sigma$ by [2] (see also [91, Proposition 3.1]). In particular, $\tau \not\simeq \tau^\vee$ and we have a contradiction. Thus $\mathrm{BC}\left(\Pi_{\pi^\circ}\right)$ is cuspidal and $k = 1$.

Suppose that $\ell = 2$. First we show that $\Pi_{\pi^\circ} \not\simeq \Pi_{\pi^\circ} \otimes \chi_E$. Suppose otherwise, i.e. $\Pi_{\pi^\circ} \simeq \Pi_{\pi^\circ} \otimes \chi_E$. Then either $\Pi_i \simeq \Pi_i \otimes \chi_E$ for $i = 1, 2$, or, $\Pi_2 \simeq \Pi_1 \otimes \chi_E$. In the former case, we have $\Pi_i = \mathcal{AI}\left(\chi_i\right)$ with a character χ_i of $\mathbb{A}_E^\times / E^\times$ for $i = 1, 2$. Then we have $\Pi_{\pi^\circ} = \mathcal{AI}\left(\chi_1\right) \boxplus \mathcal{AI}\left(\chi_2\right)$ and $\Pi_\sigma = \chi_1 \boxplus \chi_1^\sigma \boxplus \chi_2 \boxplus \chi_2^\sigma$. Since Π_{π° is a lift from PGSp_2, the central character of $\mathcal{AI}\left(\chi_i\right)$ is trivial and hence $\chi_i \mid_{\mathbb{A}^\times} = \chi_E$. On the other hand, since Π_σ is a base change lift of $\sigma|_{\mathrm{U}_{4,\varepsilon}(\mathbb{A})}$, we see that $\chi_i \mid_{\mathbb{A}^\times}$ is trivial. This is a contradiction. In the latter case, we have $\mathrm{BC}\left(\Pi_2\right) = \mathrm{BC}\left(\Pi_1 \otimes \chi_E\right) = \mathrm{BC}\left(\Pi_1\right)$ and hence $\Pi_\sigma = \mathrm{BC}\left(\Pi_1\right) \boxplus \mathrm{BC}\left(\Pi_1\right)$. This implies that Π_σ is not in the image of the base change lift from the unitary group and again we have a contradiction. Thus we have $\Pi_{\pi^\circ} \not\simeq \Pi_{\pi^\circ} \otimes \chi_E$. Then $\Pi_i \not\simeq \Pi_i \otimes \chi_E$ at least one of $i = 1, 2$. Suppose that this is so only for one of the two, say $i = 2$. Then $\Pi_1 = \mathcal{AI}\left(\chi\right)$ for some character χ of $\mathbb{A}_E^\times / E^\times$ and $\mathrm{BC}\left(\Pi_2\right)$ is cuspidal. We have $\Pi_{\pi^\circ} = \mathcal{AI}\left(\chi\right) \boxplus \Pi_2$ and $\Pi_\sigma = \chi \boxplus \chi^\sigma \boxplus \mathrm{BC}\left(\Pi_2\right)$. Then $\chi \mid_{\mathbb{A}^\times}$ is trivial from the former equality and $\chi \mid_{\mathbb{A}^\times} = \chi_E$ from the latter equality as above. Hence we have a contradiction. Thus $\mathrm{BC}\left(\Pi_i\right)$ for $i = 1, 2$ are both cuspidal, $\Pi_\sigma = \mathrm{BC}\left(\Pi_1\right) \boxplus \mathrm{BC}\left(\Pi_2\right)$ and $k = 2$. □

The following lemma gives the uniqueness of the constant $\ell(\pi)$ defined before Theorem 1.2.

Lemma 4.3. *Let π be as in Theorem 1.1 (1). For $i = 1, 2$, let E_i be a quadratic extension of F and π_i° an irreducible cuspidal automorphic representation of $G\left(\mathbb{A}\right)$ which is G^{+,E_i}-locally near equivalent to π. Let $\Pi_{\pi_i^\circ}$ be the functorial lift of π_i° to $\mathrm{GL}_4\left(\mathbb{A}\right)$ and consider the decomposition*

$$\Pi_{\pi_i^\circ} = \Pi_{i,1} \boxplus \cdots \boxplus \Pi_{i,\ell_i} \quad \text{for } i = 1, 2$$

as (4.2.1). Then we have $\ell_1 = \ell_2$.

Proof. Since the case when $E_1 = E_2$ is trivial, suppose that $E_1 \neq E_2$. Let $K = E_1 E_2$. From the definition of the base change, we have

$$\mathrm{BC}_{K/E_1}\left(\mathrm{BC}_{E_1/F}(\Pi_{\pi_1^\circ})\right) = \mathrm{BC}_{K/E_1}\left(\mathrm{BC}_{E_1/F}(\Pi_{\pi_2^\circ})\right).$$

Hence

$$\mathrm{BC}_{E_1/F}(\Pi_{\pi_1^\circ}) = \mathrm{BC}_{E_1/F}(\Pi_{\pi_2^\circ}) \quad \text{or} \quad \mathrm{BC}_{E_1/F}(\Pi_{\pi_1^\circ}) = \mathrm{BC}_{E_1/F}(\Pi_{\pi_2^\circ}) \otimes \chi_{K/E_1}$$

where χ_{K/E_1} denotes the character of $\mathbb{A}_E^\times$ corresponding to K/E_1. In the former case, we have

$$\Pi_{\pi_1^\circ} = \Pi_{\pi_2^\circ} \quad \text{or} \quad \Pi_{\pi_1^\circ} = \Pi_{\pi_2^\circ} \otimes \chi_{E_1}$$

and our claim follows. In the latter case, since $\chi_{K/E_1} = \chi_{E_2} \circ N_{E_1/F}$, we have

$$\Pi_{\pi_1^\circ} = \Pi_{\pi_2^\circ} \otimes \chi_{E_2} \quad \text{or} \quad \Pi_{\pi_1^\circ} = \Pi_{\pi_2^\circ} \otimes \chi_{E_2}\chi_{E_1}$$

and our claim follows. □

Definition 4.1. *Let π be as in Theorem 1.1 (1). Then we say that π is of Type I if π and $\pi \otimes \chi_E$ are nearly equivalent. Moreover, we say that π is of type I-A if π participates in the theta correspondence with* $\mathrm{GSO}(S_1) = \mathrm{GSO}_{3,1}$ *and that π is of type I-B if π participates in the theta correspondence with* $\mathrm{GSO}(X_\circ)$ *for some four dimensional anisotropic orthogonal space $X_\circ$ over F with discriminant algebra E.*

Remark 4.2. *From the proof of Theorem 1.1 (1), if π is not of type I-A, then the theta lift of π to* $\mathrm{GSU}_{3,D}$ *is cuspidal. Further, we note that D is necessarily split when π is of type I-A or I-B, by definition.*

In order to study an explicit formula using theta lifts from $G_D(\mathbb{A})$, the following lemma will be important later.

Lemma 4.4. *Let π be as in Theorem 1.1 (1). Then π is either type I-A or I-B if and only if π is nearly equivalent to $\pi \otimes \chi_E$. In particular, when π is neither of type I-A nor I-B, $\pi|_{\mathcal{G}_D}$ is irreducible where*

$$\mathcal{G}_D = Z_{G_D}(\mathbb{A})\, G_D(\mathbb{A})^+\, G_D(F)\,. \tag{4.2.3}$$

Proof. Suppose that π is nearly equivalent to $\pi \otimes \chi_E$. Then at almost all places v of F, $\mathrm{Ind}_{G_D(F_v)^+}^{G_D(F_v)}(\pi_{+,v})$ is irreducible where $\pi_{+,v}$ is an irreducible constituent of $\pi_v\,|_{G_D(F_v)^+}$. This implies that π and π° are nearly equivalent and hence π° is nearly equivalent to $\pi^\circ \otimes \chi_E$. Thus Π_{π° is nearly equivalent to $\Pi_{\pi^\circ} \otimes \chi_E$ and hence $\Pi_{\pi^\circ} = \Pi_{\pi^\circ} \otimes \chi_E$ by the strong multiplicity one theorem. When π is neither of type I-A nor I-B, this does not happen by Lemma 4.1 and Lemma 4.2.

Suppose that π is either of type I-A or I-B. Then D is split and the functorial lift Π_π of π to $\mathrm{GL}_4(\mathbb{A})$ is of the form $\mathcal{AI}(\tau)$ for an irreducible automorphic representation τ of $\mathrm{GL}_2(\mathbb{A}_E)$ by Roberts [96]. Then we have $\Pi_\pi = \Pi_\pi \otimes \chi_E$. Hence π is nearly equivalent to $\pi \otimes \chi_E$.

When π is not nearly equivalent to $\pi \otimes \chi_E$, $\pi\,|_{\mathcal{G}_D}$ is irreducible since $\mathcal{G}_D$ is of index 2 in $G_D(\mathbb{A})$. □

Remark 4.3. *This lemma give a classification of π such that the twist $\pi \otimes \chi_E$ of π by χ_E has the same Arthur parameter as π. A classification of π such that π and $\pi \otimes \chi_E$ are isomorphic when $G_D \simeq G$ is given in Chan* [16].

4.3. **Proof of the statement (2) in Theorem 1.1.** Suppose that π has a generic Arthur parameter.

When there exists a pair (D', π') as described in Theorem 1.1 (2), π and π' share the same generic Arthur parameter since they are nearly equivalent to each other.

Hence by Theorem 1.1 (1), we have

$$L^S\left(\frac{1}{2}, \pi \times \mathcal{AI}(\Lambda)\right) = L^S\left(\frac{1}{2}, \pi' \times \mathcal{AI}(\Lambda)\right) \neq 0$$

when S is a sufficiently large finite set of places of F. Then by Remark 1.3, we have

$$L\left(\frac{1}{2}, \pi \times \mathcal{AI}(\Lambda)\right) \neq 0,$$

i.e. (1.5.5) holds.

Conversely suppose that $L\left(\frac{1}{2}, \pi \times \mathcal{AI}(\Lambda)\right) \neq 0$. There exists an irreducible cuspidal globally generic automorphic representation π° of $G(\mathbb{A})$ which is nearly equivalent to π since π has a generic Arthur parameter. Let U be a maximal unipotent subgroup of $\mathrm{GSO}_{4,2}$ and ψ_U be a non-degenerate character of $U(\mathbb{A})$ defined below by (6.1.2) and (6.1.3), which are the same as [83, (2.4)] and [83, (3.1)], respectively. Let U_G be the maximal unipotent subgroup of GSp_2 defined by (6.2.1) and ψ_{U_G} the non-degenerate character of $U_G(\mathbb{A})$ defined by (6.2.2) in 6.2. Note that in [83], U_G is denoted by N and ψ_{U_G} is denoted by ψ_N in [83, p.34] and [83, (3.2)], respectively. Then we note that the restriction of π° to $G(\mathbb{A})^+$ contains a unique ψ_{U_G}-generic irreducible constituent and we denote it by π°_+. Let us consider the theta lift $\Sigma := \theta_\psi(\pi^\circ_+)$ of π°_+ to $\mathrm{GSO}_{4,2}(\mathbb{A})$. Then by [83, Proposition 3.3], we know that Σ is ψ_U-globally generic and hence non-zero. We divide into two cases according to the cuspidality of Σ.

Suppose that Σ is not cuspidal. Then by Rallis tower property, π°_+ participates in the theta correspondence with $\mathrm{GSO}_{3,1}$. As in the proof of Lemma 4.1, the theta lift of π°_+ to GSO_2 is zero since π°_+ is generic. Hence the theta lift $\tau := \theta_\psi^{X,S_1}(\pi^\circ_+)$ of π°_+ to $\mathrm{GSO}_{3,1}$ is cuspidal and non-zero. By Remark 3.1, τ is also irreducible.

Recall that

$$\mathrm{GSO}_{3,1}(F) \simeq \mathrm{GL}_2(E) \times F^\times / \{(z, \mathrm{N}_{E/F}(z)) : z \in E^\times\}, \ \mathrm{PGSO}_{3,1}(F) \simeq \mathrm{PGL}_2(E).$$

Then we may regard τ as an irreducible cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_E)$ with a trivial central character since the central character of π°_+ is trivial.

Let Π denote the strong functorial lift of π° to $\mathrm{GL}_4(\mathbb{A})$ by [19]. Then at almost all finite places v of F, we have $\Pi_v \simeq \mathcal{AI}(\tau)_v$, and thus by the strong multiplicity one theorem, $\Pi = \mathcal{AI}(\tau)$ holds. Since π is nearly equivalent to π°, Remark 1.3 and our assumption imply that for a sufficiently large finite set S of places of F, we have

$$\begin{aligned} &L^S\left(\frac{1}{2}, \tau \times \Lambda\right) L^S\left(\frac{1}{2}, \tau \times \Lambda^{-1}\right) = L^S\left(\frac{1}{2}, \pi^\circ \times \mathcal{AI}(\Lambda)\right) \\ =&L^S\left(\frac{1}{2}, \pi \times \mathcal{AI}(\Lambda)\right) \neq 0. \end{aligned}$$

Then by Waldspurger [112], τ has the split torus model with respect to the character (Λ, Λ^{-1}). Hence, the equation in Corbett [20, p.78] implies that π° has the (E, Λ)-Bessel period. Hence we may take $D' = \mathrm{Mat}_{2\times 2}$ and $\pi' = \pi^\circ$. Thus the case when Σ is not cuspidal is settled.

Suppose that Σ is cuspidal. We may regard Σ as an irreducible cuspidal globally generic automorphic representation of $\mathrm{GU}(2,2)$ with trivial central character because of the accidental isomorphism (2.2.6). As in the proof of Theorem 1.1 (1), our assumption implies that $L\left(\frac{1}{2}, \Sigma \times \Lambda\right) \neq 0$. Then by [29, Proposition A.2], there exists an irreducible cuspidal automorphic representation Σ' of $\mathrm{GU}(V)$ such that Σ' is locally $\mathrm{U}(V)$-nearly equivalent to Σ and Σ' has the (e, Λ, ψ)-Bessel period where V is a 4-dimensional hermitian space over E whose Witt index is at least 1. Then we note that $\mathrm{PGU}(V) \simeq \mathrm{PGSO}_{4,2}$ or $\mathrm{PGU}_{3,D'}$ for some quaternion division algebra D' over F.

In the first case, we consider the theta lift $\pi'_+ := \theta_{\psi^{-1}}(\Sigma')$ of Σ' to $G(\mathbb{A})^+$. Then by the same argument as the one in the proof of Theorem 1.1 (1), we see that $\pi'_+ \neq 0$ by Takeda [106, Theorem 1.1 (1)] and that it is an irreducible cuspidal automorphic representation of $G(\mathbb{A})^+$. Since Σ' has the (e, Λ, ψ)-Bessel period, π'_+ has the (E, Λ)-Bessel period by Proposition 3.1. From the definition, π'_+ is nearly equivalent to π°_+. Let us take an irreducible cuspidal automorphic representation $(\pi', V_{\pi'})$ of $G(\mathbb{A})$ such that $\pi' \mid_{G(\mathbb{A})^+} \supset \pi'_+$. Then π' is locally G^+-nearly equivalent, and thus either π' or $\pi' \otimes \chi_E$ is nearly equivalent to π by Remark 1.2. Since both π' and $\pi' \otimes \chi_E$ have the (E, Λ)-Bessel period, our claim follows.

In the second case, we consider the theta lift of Σ' to $G_{D'}(\mathbb{A})$. Then by an argument similar to the one in the first case, we may show that the theta lift of Σ' to $G_{D'}(\mathbb{A})$ contains an irreducible constituent which is cuspidal, locally G^+-nearly equivalent to π and has the (E, Λ)-Bessel period. Here we use [120, Lemma 10.2] and its proof in the case of (I_1) with $n = 3, m = 2$, noting Remark 4.5. This completes our proof of the existence of a pair (D', π').

Let us show the uniqueness of a pair (D', π') under the assumption that π is tempered. Suppose that for $i = 1, 2$ there exists a pair(D_i, π_i) where D_i is a quaternion algebra over F and π_i is an irreducible cuspidal automorphic representation of $G_{D_i}(\mathbb{A})$ which is nearly equivalent to π such that π_i has the (E, Λ)-Bessel period.

Suppose that π_i is nearly equivalent to $\pi_i \otimes \chi_E$ for $i = 1, 2$. Then by Proposition 4.4, π_1, π_2 are of type I-A or I-B and in particular $D_1 \simeq D_2 \simeq \mathrm{Mat}_{2\times2}$. Hence for $i = 1, 2$, there exist a four dimensional orthogonal space X_i over F with discriminant algebra E and an irreducible cuspidal automorphic representation σ_i of $\mathrm{GSO}(X_i, \mathbb{A})$ such that $\pi_i = \theta_\psi(\sigma_i)$. Since $\mathrm{PGSO}(X_i, F) \simeq (D'_i)^\times(E)/E^\times$ for some quaternion algebra D'_i over F, we may regard σ_i as an automorphic representation of $(D'_i)^\times(\mathbb{A}_E)$ with the trivial central character. Since π_i has the (E, Λ)-Bessel period, σ_i has the split torus period with respect to a character $\left(\Lambda, \Lambda^{-1}\right)$ by [20, p.78]. Hence $D'_i(E) \simeq \mathrm{Mat}_{2\times2}(E)$ by [112]. Since σ_1 is nearly equivalent to σ_2, we have $\sigma_1 = \sigma_2$ by the strong multiplicity one. Thus $\pi_1 \simeq \pi_2$.

Suppose that π_i is neither type I-A nor I-B for $i = 1, 2$. For each i, let us take a unique irreducible constituent $\pi^B_{i,+}$ of $\pi_i|_{G_{D_i}(\mathbb{A})^+}$ that has the (ξ_i, Λ, ψ)-Bessel period. Note that $\pi^B_{1,+}$ and $\pi^B_{2,+}$ are nearly equivalent to each other.

Now let σ_i denote the theta lift $\theta_\psi(\pi^B_{i,+})$ of $\pi^B_{i,+}$ to GSU_{3,D_i}. Then we regard σ_i as an automorphic representation of $\mathrm{GU}_{4,\varepsilon}$ via (2.2.5), (2.2.6) and let $\Sigma_i := \Theta_{\psi,(\Lambda^{-1},\Lambda^{-1})}$ denote the theta lift of σ_i to $\mathrm{GU}_{2,2}$. In turn, we regard Σ_i as an automorphic representation of $\mathrm{GSO}_{4,2}$ via (2.2.6) and we denote by $\pi'_{i,+}$ its theta lift to $G(\mathbb{A})^+$. Then from the proof of Theorem 1.1 (1), σ_i, Σ_i and $\pi'_{i,+}$ are irreducible and cuspidal. Moreover $\pi'_{1,+}$ and $\pi'_{2,+}$ are both globally generic and nearly equivalent to each other. Furthermore, since π_i is tempered, $\sigma_i = \theta_\psi(\pi^B_{i,+})$ is tempered at finite places by an argument similar to the one in Atobe-Gan [5, Proposition 5.5] (see also [40, Proposition C.1]) and similarly at real and complex places by Paul [85, Theorem 15, Theorem 30] and Li-Paul-Tan-Zhu [75, Theorem 4.20, Theorem 5.1], and, by Adams-Barbasch [1, Theorem 2.7], respectively. Similarly Σ_i and $\pi'_{i,+}$ are also tempered.

By Proposition 3.1 and Proposition 3.2, we know that σ_i has the $(X_{\xi_i}, \Lambda, \psi)$-Bessel period. Let GU_i denote the similitude unitary group which modulo center is isomorphic to PGSU_{3,D_i} by (2.2.5). Then $\sigma_i\mid_{\mathrm{U}_i}$ has a unique irreducible constituent ν_i which has the $(X_{\xi_i}, \Lambda, \psi)$-Bessel period. Then by Beuzart-Plessis [6, 7] (also by Xue [118] at the real place), we see that $\mathrm{U}_1 \simeq \mathrm{U}_2$ since ν_1 and ν_2 are equivalent to each other. This implies that $D_1 \simeq D_2$ and hence $G_{D_1} \simeq G_{D_2}$. Let us denote $D' \simeq D_i$ for $i = 1, 2$.

We take an irreducible cuspidal automorphic representation π'_i of $G(\mathbb{A})$ such that $\pi'_i|_{G(\mathbb{A})^+}$ contains $\pi'_{i,+}$. Then by Remark 1.2, we may suppose that π'_1 is nearly equivalent to π'_2 or $\pi'_2 \otimes \chi_E$. Thus replacing π'_2 by $\pi'_2 \otimes \chi_E$ if necessary, we may assume that π'_1 and π'_2 are nearly equivalent to each other. Then since π'_1 and π'_2 are generic and they have the same L-parameter because of the temperedness of π'_i, we have $\pi'_1 \simeq \pi'_2$ by the uniqueness of the generic member in the L-packet by Atobe [4] or Varma [109] at finite places and by Vogan [110] at archimedean places. Hence in particular, $\pi'_{1,+} \simeq \pi'_{2,+}$.

From the definition of $\pi'_{+,i}$, we get $\pi^B_{1,+} \simeq \pi^B_{2,+}$. Then, we see that $\pi_1 \simeq \pi_2 \otimes \omega$ for some character ω of $G_{D'}(\mathbb{A})$ such that ω_v is trivial or $\chi_{E,v}$ at each place v of F. Since π_1 and π_2 have the same L-parameter, $\pi_{1,v}$ and $\pi_{1,v} \otimes \omega_v$ are in the same L-packet for every place v of F.

Let us take a place v of F, and write the L-parameter of $\pi_{1,v}$ as $\phi_v : WD_{F_v} \to G^1(\mathbb{C})$. If ϕ_v is an irreducible four dimensional representation, the L-packet of ϕ_v is singleton, and thus $\pi_{1,v} \simeq \pi_{2,v}$. So let us suppose that $\phi_v = \phi_1 \oplus \phi_2$ with two dimensional irreducible representations ϕ_i. Further, we may suppose that $\omega_v = \chi_{E,v}$ since there is nothing to prove when ω_v is trivial. This implies that $\phi_v \otimes \chi_{E,v} \simeq \phi_v$. Then, by [91, Proposition 3.1], we have $\phi_i = \pi(\chi_i)$ for some character χ_i of $E_v^\times$ for $i = 1, 2$. Moreover, any member of the L-packet of π_1 is given by the theta lift from an irreducible representation $\mathrm{JL}(\pi(\chi_1)) \boxtimes \pi(\chi_2)$ of $D'(F_v)^\times \times \mathrm{GL}_2(F_v)$ where JL denotes the Jacquet-Langlands transfer. Since the theta lift preserves the character twist, we see that

$$\theta(\mathrm{JL}(\pi(\chi_i)) \boxtimes \pi(\chi_j)) \otimes \chi_{E,v} \simeq \theta(\mathrm{JL}(\pi(\chi_i)) \boxtimes \pi(\chi_j))$$

by $\pi(\chi_i) \otimes \chi_{E,v} \simeq \pi(\chi_i)$. This shows that in this case, any element in the L-packet is invariant under the twist by $\chi_{E,v}$. Thus $\pi_{1,v} \otimes \chi_{E,v} \simeq \pi_{2,v}$ and hence $\pi_{1,v} \simeq \pi_{2,v}$. □

Remark 4.4. *As we remarked in the end of Section 1.5, the uniqueness of* (D', π') *follows from the local Gan-Gross-Prasad conjecture for* $(\mathrm{SO}(5), \mathrm{SO}(2))$*, which is proved by Luo* [77] *at archimedean places and by Prasad–Takloo-Bighash* [92] *(see also Waldspurger* [115] *in general case) at finite places. Our proof gives another proof of the uniqueness.*

Remark 4.5. *There is a typo in the statement of* [120, Lemma 10.2]*. The first condition stated there should be the holomorphy at* $s = -s_m + \frac{1}{2}$.

5. Rallis inner product formula for similitude groups

In this section, as a preliminary for the proof of Theorem 1.2, we recall Rallis inner product formulas for similitude dual pairs.

5.1. **For the theta lift from** G **to** $\mathrm{GSO}_{4,2}$**.** In this section, we shall recall the Rallis inner product formula for the theta lift from G to $\mathrm{GSO}_{4,2}$. It is derived from the isometry case in a manner similar to the one in Gan-Ichino [39, Section 6], where the case of the theta lift from GL_2 to $\mathrm{GSO}_{3,1}$ is treated.

Let (π, V_π) be an irreducible cuspidal automorphic representation of $G(\mathbb{A})$ with a trivial central character. Let us define a subgroup $\mathcal{G}$ of $G(\mathbb{A})$ by

$$\mathcal{G} := Z_G(\mathbb{A}) G(\mathbb{A})^+ G(F) \tag{5.1.1}$$

and in this section we assume that:

(5.1.2) *the restriction of* π *to* $\mathcal{G}$ *is irreducible, i.e.* $\pi \otimes \chi_E \not\simeq \pi$

for our later use.

Let us recall the notation in 3.1.2. Thus X denotes the four dimensional symplectic space on which G acts on the right and Y denotes the six dimensional orthogonal space on which $\mathrm{GSO}_{4,2}$ acts on the left. Then $Z = X \otimes Y$ is a symplectic space over F. Here we take $X_\pm \otimes Y$ as the polarization and we realize the Weil representation ω_ψ of $\mathrm{Mp}(Z)(\mathbb{A})$ on $V_\omega := \mathcal{S}((X_+ \otimes Y)(\mathbb{A}))$.

Put $X^\square = X \oplus (-X)$. Then $X^\square$ is naturally a symplectic space. Let $\widetilde{G} := \mathrm{GSp}(X^\square)$ and we denote by $\mathbf{G}$ a subgroup of $G \times G$ given by

$$\mathbf{G} := \{(g_1, g_2) \in G \times G : \lambda(g_1) = \lambda(g_2)\},$$

which has a natural embedding $\iota : \mathbf{G} \to \widetilde{G}$. We define the canonical pairing $\mathcal{B}_\omega : V_\omega \otimes V_\omega \to \mathbb{C}$ by

$$\mathcal{B}_\omega(\varphi_1, \varphi_2) := \int_{(X_+ \otimes Y)(\mathbb{A})} \varphi_1(x)\, \overline{\varphi_2(x)}\, dx \quad \text{for } \varphi_1, \varphi_2 \in V_\omega$$

where dx denotes the Tamagawa measure on $(X_+ \otimes Y)(\mathbb{A})$.

Let $\widetilde{Z} = X^\square \otimes Y$ and we take a polarization $\widetilde{Z} = \widetilde{Z}_+ \oplus \widetilde{Z}_-$ with

$$\widetilde{Z}_\pm := (X_\pm \oplus (-X_\pm)) \otimes Y$$

where the double sign corresponds. Let us denote by $\widetilde{\omega}_\psi$ the Weil representation of $\mathrm{Mp}(\widetilde{Z}(\mathbb{A}))$ on $\mathcal{S}(\widetilde{Z}^+(\mathbb{A}))$. On the other hand, let

$$X^\nabla := \{(x, -x) : x \in X\} \quad \text{and} \quad \widetilde{X}^\nabla := X^\nabla \otimes Y.$$

Then we have a natural isomorphism

$$V_\omega \otimes V_\omega \simeq \mathcal{S}(\widetilde{X}^\nabla(\mathbb{A}))$$

by which we regard $\mathcal{S}(\widetilde{X}^\nabla(\mathbb{A}))$ as a representation of $\mathrm{Mp}(Z)(\mathbb{A}) \times \mathrm{Mp}(Z)(\mathbb{A})$. Meanwhile we may realize $\widetilde{\omega}_\psi$ on $\mathcal{S}(\widetilde{X}^\nabla(\mathbb{A}))$ and indeed we have an isomorphism

$$\delta : \mathcal{S}(\widetilde{Z}_+(\mathbb{A})) \to \mathcal{S}(\widetilde{X}^\nabla(\mathbb{A}))$$

as representations of $\mathrm{Mp}(\widetilde{Z})(\mathbb{A})$ such that

$$\delta(\varphi_1 \otimes \overline{\varphi}_2)(0) = \mathcal{B}_\omega(\varphi_1, \varphi_2) \quad \text{for } \varphi_1, \varphi_2 \in V_\omega.$$

Let us define Petersson inner products on $G(\mathbb{A})$ and $G(\mathbb{A})^+$ as follows. For $f_1, f_2 \in V_\pi$, we define the Petersson inner product $(\ ,\ )_\pi$ on $G(\mathbb{A})$ by

$$(f_1, f_2)_\pi := \int_{\mathbb{A}^\times G(F)\backslash G(\mathbb{A})} f_1(g)\, \overline{f_2(g)}\, dg$$

where dg denotes the Tamagawa measure. Then regarding f_1, f_2 as automorphic forms on $G(\mathbb{A})^+$, we define

$$(f_1, f_2)_\pi^+ := \int_{\mathbb{A}^\times G(F)^+\backslash G(\mathbb{A})^+} f_1(h)\, \overline{f_2(h)}\, dh$$

where the measure dh is normalized so that

$$\mathrm{vol}(\mathbb{A}^\times G(F)^+\backslash G(\mathbb{A})^+) = 1.$$

Then from our assumption (5.1.2) on π, as in [39, Lemma 6.3], we see that

$$(f_1, f_2)_\pi^+ = \frac{1}{2}\, (f_1, f_2)_\pi$$

since $\mathrm{Vol}(\mathbb{A}^\times G(F)\backslash G(\mathbb{A})) = 2$. For each place v of F, we take a hermitian $G(F_v)$-invariant local pairing $(\ ,\ )_{\pi_v}$ of π_v so that

$$(f_1, f_2)_\pi = \prod_v \left(f_{1,v}, f_{2,v}\right)_{\pi_v} \quad \text{for } f_i = \otimes_v f_{i,v} \in V_\pi \ (i = 1, 2). \tag{5.1.3}$$

We also choose a local Haar measure dg_v on $G(F_v)$ for each place v of F so that $\mathrm{Vol}(K_{G,v}, d_{g_v}) = 1$ at almost all v, where $K_{G,v}$ is a maximal compact subgroup of $G(F_v)$. We define positive constants C_G by

$$dg = C_G \cdot \prod_v dg_v$$

Local doubling zeta integrals are defined as follows. Let $I(s)$ denote the degenerate principal series representation of $\widetilde{G}(\mathbb{A})$ defined by

$$I(s) := \mathrm{Ind}_{\widetilde{P}(\mathbb{A})}^{\widetilde{G}(\mathbb{A})} \left(\chi_E\, \delta_{\widetilde{P}}^{s/9}\right)$$

where $\widetilde{P}$ denotes the Siegel parabolic subgroup of $\widetilde{G}$. Then for each place v, we define a local zeta integral by

$$Z_v(s, \Phi_v, f_{1,v}, f_{2,v}) := \int_{G^1(F_v)} \Phi_v(\iota(g_v, 1), s)\, \left(\pi_v(g_v) f_{1,v}, f_{2,v}\right)_{\pi_v}\, dg_v$$

for $\Phi_v \in I(s)$, $f_{1,v}, f_{2,v} \in V_{\pi_v}$, where $G^1 = \{g \in G : \lambda(g) = 1\}$. The integral converges absolutely at $s = \frac{1}{2}$ when $\Phi_v \in I_v(s)$ is a holomorphic section by [88, Proposition 6.4] (see also [39, Lemma 6.5]). Moreover, when we define a map $\mathcal{S}(\widetilde{X}^{\nabla}(\mathbb{A})) \ni \varphi \mapsto [\varphi] \in I\left(\frac{1}{2}\right)$ by

$$[\varphi]\left(g, \frac{1}{2}\right) := |\nu(g)|^{-4}\left(\widetilde{\omega}_{\psi}(\begin{pmatrix} 1_4 & \\ & \lambda(g)^{-1} 1_4 \end{pmatrix} g))\varphi\right)(0),$$

we may naturally extend $[\varphi]$ to a holomorphic section in $I(s)$.

By an argument similar to the one in the proof of [39, Proposition 6.10], we may derive the following Rallis inner product formula in the similitude groups case from the one [41, Theorem 8.1] in the isometry groups case.

Proposition 5.1. *Keep the above notation.*

Then for decomposable vectors $f = \otimes f_v \in V_\pi$ and $\phi = \otimes \phi_v \in V_\omega$, we have

$$\frac{\langle \Theta(f;\phi), \Theta(f;\phi)\rangle}{(f,f)_\pi} = C_G \cdot \frac{1}{2} \cdot \frac{L(1, \pi, \mathrm{std} \otimes \chi_E)}{L(3, \chi_E) L(2, \mathbf{1}) L(4, \mathbf{1})} \times \prod_v Z_v^{\sharp}\left(\frac{1}{2}, [\delta(\phi_v \otimes \phi_v)], f_v, f_v\right).$$

Here we recall that $\Theta_\psi(f;\phi)$ is the theta lift of f to $\mathrm{GO}_{4,2}$, $\langle\ ,\ \rangle$ denotes the Petersson inner product with respect to the Tamagawa measure and we define

$$Z_v^{\sharp}\left(\frac{1}{2}, [\delta(\phi_v \otimes \phi_v)], f_v, f_v\right) := \frac{1}{(f_v, f_v)_{\pi_v}} \frac{L(3, \chi_{E_v/F_v}) L(2, \mathbf{1}_v) L(4, \mathbf{1}_v)}{L(1, \pi_v, \mathrm{std} \otimes \chi_{E_v/F_v})} \times Z_v\left(\frac{1}{2}, [\delta(\phi_v \otimes \phi_v)], f_v, f_v\right),$$

which is equal to 1 at almost all places v of F by [88].

Recall that $\theta(f;\phi)$ denotes the restriction of $\Theta_\psi(f;\phi)$ to $\mathrm{GSO}_{4,2}(\mathbb{A})$, namely the theta lift of f to $\mathrm{GSO}_{4,2}$. Then as in [39, Lemma 2.1], we see that

$$2\langle \Theta(f;\phi), \Theta(f;\phi)\rangle = \langle \theta(f;\phi), \theta(f;\phi)\rangle$$

where the right hand side denotes the Petersson inner product on $\mathrm{GSO}_{4,2}$ with respect to the Tamagawa measure. Hence, Proposition 5.1 yields

$$\frac{\langle \theta(f;\phi), \theta(f;\phi)\rangle}{(f,f)_\pi} = C_G \cdot \frac{L(1, \pi, \mathrm{std} \otimes \chi_E)}{L(3, \chi_E) L(2, \mathbf{1}) L(4, \mathbf{1})} \times \prod_v Z_v^{\sharp}\left(\frac{1}{2}, [\delta(\phi_v \otimes \phi_v)], f_v, f_v\right). \tag{5.1.4}$$

5.2. **Theta lift from G_D to $\mathrm{GSU}_{3,D}$.** In this subsection, we shall consider the Rallis inner product formula for the theta lift from G_D to $\mathrm{GSU}_{3,D}$ as in the previous section. We recall that the formula in the case of isometry groups is proved by Yamana [120, Lemma 10.1] where our case corresponds to (I_3) with $m = 3, n = 2$.

Let (π, V_π) be an irreducible cuspidal automorphic representation of $G_D(\mathbb{A})$ with a trivial central character. Recall that $\mathcal{G}_D$ denotes the subgroup of $G_D(\mathbb{A})$ given by (4.2.3). In this section, assume that:

(5.2.1) *the restriction of π to $\mathcal{G}_D$ is irreducible*

for our later use.

Let us recall the notation in 3.2.2. Thus X_D denotes the hermitian space of degree two over D on which G_D acts on the right and Y_D denotes the skew-hermitian space of degree three over D on which $\mathrm{GSU}_{3,D}$ acts on the left. Then $Z_D = X_D \otimes_D Y_D$ is a symplectic space over F by (3.2.1). Here we take $X_{D,\pm} \otimes_D Y_D$ as the polarization and we realize the Weil representation ω_ψ of $\mathrm{Mp}(Z_D)(\mathbb{A})$ on $V_{\omega,D} := \mathcal{S}\left(\left(X_{D,+} \otimes_D Y_D\right)(\mathbb{A})\right)$.

Put $X_D^\square = X_D \oplus \overline{X_D}$. Then $X_D^\square$ is naturally a hermitian space over D. Let $\widetilde{G}_D := \mathrm{GU}\left(X_D^\square\right)$ and we denote by $\mathbf{G}_D$ a subgroup of $G_D \times G_D$ given by

$$\mathbf{G}_D := \{(g_1, g_2) \in G_D \times G_D : \lambda(g_1) = \lambda(g_2)\}$$

which has a natural embedding $\iota : \mathbf{G}_D \to \widetilde{G}_D$. We define the canonical pairing $\mathcal{B}_\omega : V_{\omega,D} \otimes V_{\omega,D} \to \mathbb{C}$ by

$$\mathcal{B}_\omega(\varphi_1, \varphi_2) := \int_{(X_{D,+}\otimes Y_D)(\mathbb{A})} \varphi_1(x)\, \overline{\varphi_2(x)}\, dx \quad \text{for } \varphi_1, \varphi_2 \in V_{\omega,D}$$

where dx denotes the Tamagawa measure on $(X_{D,+} \otimes Y_D)(\mathbb{A})$.

Let $\widetilde{Z}_D = X_D^\square \otimes Y_D$ and we take a polarization $\widetilde{Z}_D = \widetilde{Z}_{D,+} \oplus \widetilde{Z}_{D,-}$ with

$$\widetilde{Z}_{D,\pm} = \left(X_{D,\pm} \oplus \overline{-X_{D,\pm}}\right) \otimes Y_D$$

where the double sign corresponds. Let us denote by $\widetilde{\omega}_\psi$ the Weil representation of $\mathrm{Mp}(\widetilde{Z}_D)(\mathbb{A})$ on $\mathcal{S}(\widetilde{Z}_{D,+}(\mathbb{A}))$. On the other hand, let

$$X_D^\nabla := \{(x, \bar{x}) : x \in X_D\} \quad \text{and} \quad \widetilde{X}_D^\nabla := X_D^\nabla \otimes Y_D.$$

Then we have a natural isomorphism

$$V_{\omega,D} \otimes V_{\omega,D} \simeq \mathcal{S}(\widetilde{X}_D^\nabla(\mathbb{A}))$$

by which we regard $\mathcal{S}(\widetilde{X}_D^\nabla(\mathbb{A}))$ as a representation of $\mathrm{Mp}(Z_D)(\mathbb{A}) \times \mathrm{Mp}(Z_D)(\mathbb{A})$. Meanwhile we may realize $\widetilde{\omega}_\psi$ on $\mathcal{S}(\widetilde{X}_D^\nabla(\mathbb{A}))$ and indeed we have an isomorphism

$$\delta : \mathcal{S}(\widetilde{Z}_{D,+}(\mathbb{A})) \to \mathcal{S}(\widetilde{X}_D^\nabla(\mathbb{A}))$$

as representations of $\mathrm{Mp}(\widetilde{Z}_D)(\mathbb{A})$ such that

$$\delta(\varphi_1 \otimes \overline{\varphi}_2)(0) = \mathcal{B}_\omega(\varphi_1, \varphi_2) \quad \text{for } \varphi_1, \varphi_2 \in V_{\omega,D}.$$

Let us define Petersson inner products on $G_D(\mathbb{A})$ and $G_D(\mathbb{A})^+$ as follows. For $f_1, f_2 \in V_{\pi_D}$, we define the Petersson inner product $(\,,\,)_{\pi_D}$ on $G_D(\mathbb{A})$ by

$$(f_1, f_2)_{\pi_D} := \int_{\mathbb{A}^\times G_D(F)\backslash G_D(\mathbb{A})} f_1(g)\, \overline{f_2(g)}\, dg$$

where dg denotes the Tamagawa measure. Then regarding f_1, f_2 as automorphic forms on $G_D(\mathbb{A})^+$, we define

$$(f_1, f_2)^+_{\pi_D} := \int_{\mathbb{A}^\times G_D(F)^+ \backslash G_D(\mathbb{A})^+} f_1(h)\, \overline{f_2(h)}\, dh$$

where the measure dh is normalized so that

$$\mathrm{vol}(\mathbb{A}^\times G_D(F)^+ \backslash G_D(\mathbb{A})^+) = 1.$$

Then from our assumption (5.2.1) on π_D, as in [39, Lemma 6.3], we see that

$$(f_1, f_2)^+_{\pi_D} = \frac{1}{2}\,(f_1, f_2)_{\pi_D}$$

since $\mathrm{Vol}(\mathbb{A}^\times G_D(F) \backslash G_D(\mathbb{A})) = 2$. For each place v of F, we take a hermitian $G_D(F_v)$-invariant local pairing $(\ ,\ )_{\pi_{D,v}}$ of $\pi_{D,v}$ so that

$$(f_1, f_2)_{\pi_D} = \prod_v \left(f_{1,v}, f_{2,v}\right)_{\pi_{D,v}} \quad \text{for } f_i = \otimes_v f_{i,v} \in V_{\pi_D}\ (i = 1, 2). \tag{5.2.2}$$

As in the previous section, we choose local Haar measures dg_v on $G_D(F_v)$ at each place v of F and we have

$$dg = C_{G_D} \cdot \prod_v dg_v$$

for some positive constant C_{G_D}.

Local doubling zeta integrals are defined as follows. Let $I_D(s)$ denote the degenerate principal series representation of $\widetilde{G}_D(\mathbb{A})$ defined by

$$I_D(s) := \mathrm{Ind}^{\widetilde{G}_D(\mathbb{A})}_{\widetilde{P}_D(\mathbb{A})}\left(\chi_E\, \delta^{s/9}_{\widetilde{P}_D}\right)$$

where $\widetilde{P}_D$ denotes the Siegel parabolic subgroup of $\widetilde{G}_D$. Then for each place v, we define a local zeta integral for $\Phi_v \in I_{D,v}(s)$, $f_{1,v}, f_{2,v} \in V_{\pi_{D,v}}$ by

$$Z_v(s, \Phi_v, f_{1,v}, f_{2,v}) := \int_{G^1_D(F_v)} \Phi_v(\iota(g_v, 1), s)\left(\pi_{D,v}(g_v) f_{1,v}, f_{2,v}\right)_{\pi_v} dg_v$$

where $G^1_D = \{g \in G_D : \lambda(g) = 1\}$. The integral converges absolutely at $s = \frac{1}{2}$ when $\Phi_v \in I_{D,v}(s)$ is a holomorphic section by [88, Proposition 6.4] (see also [39, Lemma 6.5]). Moreover, when we define a map $\mathcal{S}(\widetilde{X}^\nabla_D(\mathbb{A})) \ni \varphi \mapsto [\varphi] \in I_D\left(\frac{1}{2}\right)$ by

$$[\varphi]\left(g, \frac{1}{2}\right) := |\lambda(g)|^{-4}\left(\widetilde{\omega}_\psi(\begin{pmatrix} 1_4 & \\ & \lambda(g)^{-1}\, 1_4 \end{pmatrix} g))\varphi\right)(0),$$

we may naturally extend $[\varphi]$ to a holomorphic section in $I_D(s)$.

By an argument similar to the one in the proof of [39, Proposition 6.10], we may derive the following Rallis inner product formula in the similitude groups case from the one [119, Theorem 2] in the isometry groups case.

Proposition 5.2. *Keep the above notation.*

Then for decomposable vectors $f = \otimes f_v \in V_{\pi_D}$ and $\phi = \otimes \phi_v \in V_{\omega,D}$, we have

$$\frac{\langle \theta(f;\phi), \theta(f;\phi)\rangle}{(f_{\pi_D}, f_{\pi_D})} = \frac{L(1,\pi,\mathrm{std}\otimes\chi_E)}{L(3,\chi_E)L(2,\mathbf{1})L(4,\mathbf{1})} \prod_v Z_v^\sharp\left(\frac{1}{2}, [\delta(\phi_v\otimes\phi_v)], f_v, f_v\right).$$

Here recall that $\theta_\psi(f;\phi)$ is the theta lift of f to $\mathrm{GSU}_{3,D}$, $\langle\ ,\ \rangle$ denotes the Petersson inner product with respect to the Tamagawa measure and we define

$$Z_v^\sharp\left(\frac{1}{2}, [\delta(\phi_v\otimes\phi_v)], f_v, f_v\right) := \frac{1}{(f_v, f_v)_{\pi_{D,v}}} \frac{L(3,\chi_{E_v/F_v})L(2,\mathbf{1}_v)L(4,\mathbf{1}_v)}{L\left(1,\pi_v,\mathrm{std}\otimes\chi_{E_v/F_v}\right)} \times Z_v\left(\frac{1}{2}, [\delta(\phi_v\otimes\phi_v)], f_v, f_v\right),$$

which is equal to 1 at almost all places v of F by [88].

6. Explicit formula for Bessel periods on GU(4)

Let GU (4) stand for one of $\mathrm{GU}_{2,2}$ or $\mathrm{GU}_{3,1}$. In [29], the explicit formula for the Bessel periods on GU (4) is proved under the assumption that the explicit formula for the Whittaker periods on $\mathrm{GU}_{2,2}$ holds. In this section we shall show that this assumption is indeed satisfied in the cases we need, from the explicit formula for the Whittaker periods on $G = \mathrm{GSp}_2$, which in turn will be proved in Appendix A. Thus the explicit formula for the Bessel periods on GU(4) holds by [29], in the cases which we need for the proof of Theorem 1.2.

6.1. **Explicit formulas.** Let (π, V_π) be an irreducible cuspidal tempered globally generic automorphic representation of $G(\mathbb{A})$ such that $\pi|_{\mathcal{G}}$ is irreducible. We recall that the subgroup $\mathcal{G}$ of $G(\mathbb{A})$ is defined by (5.1.1). Let π° denote the unique generic irreducible constituent of $\pi|_{G(\mathbb{A})^+}$. Let (Σ, V_Σ) denote the theta lift of π° to $\mathrm{GSO}_{4,2}(\mathbb{A})$. Then as in [83, Proposition 3.3], we know that Σ is an irreducible globally generic cuspidal tempered automorphic representation. Here we prove the explicit formula for the Whittaker periods for Σ assuming the explicit formula for the Whittaker periods for π.

Let us recall some notation. Let X, Y, Y_0 and Z be as in Section 3.1.2 and we use a polarization $Z = Z_+ \oplus Z_-$ with

$$Z_\pm = (X\otimes Y_\pm)\oplus(X_\pm\otimes Y_0)$$

where the double sign corresponds. We write $z_+ = (a_1, a_2; b_1, b_2)$ when

$$z_+ = a_1\otimes y_1 + a_2\otimes y_2 + b_1\otimes e_1 + b_2\otimes e_2 \in Z_+ \quad \text{with } a_i \in X,\ b_i\in X_+.$$

Recall that the unipotent subgroups N_0, N_1 and N_2 of $\mathrm{GSO}_{4,2}$ are defined by (3.1.10), (3.1.11) and (3.1.12), respectively. Let us define an unipotent subgroup $\tilde{U}$ of $\mathrm{GSO}_{4,2}$ by

$$\tilde{U} := \left\{ \tilde{u}(b) := \begin{pmatrix} 1 & -{}^t\widetilde{X}S_1 & 0 \\ 0 & 1_4 & \widetilde{X} \\ 0 & 0 & 1\end{pmatrix} : \widetilde{X} = \begin{pmatrix} 0\\0\\0\\-b\end{pmatrix}\right\} \tag{6.1.1}$$

where S_1 is given by (2.1.2). Let

$$U := N_{4,2}\,\tilde{U}. \tag{6.1.2}$$

Then U is a maximal unipotent subgroup of $\mathrm{GSO}_{4,2}$ and we have

$$N_0 \triangleleft N_0N_1 \triangleleft N_0N_1N_2 = N_{4,2} \triangleleft N_{4,2}\,\tilde{U} = U.$$

Then we define a non-degenerate character ψ_U of $U(\mathbb{A})$ by

$$\psi_U(u_0(x)u_1(s_1,t_1)u_2(s_2,t_2)\tilde{u}(b)) := \psi(2dt_2 + b). \tag{6.1.3}$$

By [83, Proposition 3.3], Σ is ψ_U-generic. Namely

$$W^{\psi_U}(\varphi) := \int_{U(F)\backslash U(\mathbb{A})} \varphi(u)\,\psi_U^{-1}(u)\,du \quad \text{for } \varphi \in V_\Sigma,$$

is not identically zero on V_Σ. Now we regard Σ as an automorphic representation of $\mathrm{GU}_{2,2}$ by the accidental isomorphism (2.2.6) and let $\Pi_\Sigma = \Pi_1' \boxplus \cdots \boxplus \Pi_\ell'$ denote the base change lift of $\Sigma\mid_{\mathrm{U}_{2,2}}$ to $\mathrm{GL}_4(\mathbb{A}_E)$ where Π_i' is an irreducible cuspidal automorphic representations of $\mathrm{GL}_{m_i}(\mathbb{A}_E)$. Here the existence of Π_Σ follows from [65].

Recall that in Section 5.1, the Petersson inner products on $G(\mathbb{A})$ and $\mathrm{GSO}_{4,2}(\mathbb{A})$ using the Tamagawa measures, denoted respectively as $(\ ,\ )$ and $\langle\ ,\ \rangle$, are introduced. Moreover at each place v of F, we choose and fix an $G(F_v)$-invariant hermitian inner product $(\ ,\ )_v$ on $V_{\pi_v^\circ}$ so that the decomposition formula (5.1.3) holds. Similarly at each place v, we choose and fix a $\mathrm{GSO}_{4,2}(F_v)$-invariant hermitian inner product $\langle\ ,\ \rangle_v$ on V_{Σ_v} so that the decomposition formula

$$\langle \phi_1, \phi_2\rangle = \prod_v \langle \phi_{1,v}, \phi_{2,v}\rangle_v \quad \text{for } \phi_i = \otimes_v\,\phi_{i,v} \in V_\Sigma \quad (i = 1,2) \tag{6.1.4}$$

holds.

Then as in Section 2.4, at each place v of F, we may define a local period $\mathcal{W}_v(\varphi_v)$ for $\varphi_v \in V_{\Sigma_v}$ by the stable integral

$$\mathcal{W}_v(\varphi_v) := \int_{U(F_v)}^{\mathrm{st}} \frac{\langle \Sigma_v\,(n_v)\,\varphi_v, \varphi_v\rangle_v}{\langle \varphi_v, \varphi_v\rangle_v} \cdot \psi_U^{-1}\,(n_v)\;dn_v \tag{6.1.5}$$

when v is finite. When v is archimedean, we use the Fourier transform to define $\mathcal{W}_v(\varphi_v)$. See [76, Proposition 3.5, Proposition 3.15] for the details.

We shall prove the following theorem, namely the explicit formula for the Whittaker periods on V_Σ, in 6.2.

Theorem 6.1. *For a non-zero decomposable vector $\varphi = \otimes\varphi_v \in V_\Sigma$, we have*

$$\frac{|W^{\psi_U}(\varphi)|^2}{\langle \varphi, \varphi\rangle} = \frac{1}{2^\ell} \cdot \frac{\prod_{j=1}^4 L\left(j, \chi_E^j\right)}{L\left(1, \Pi_\Sigma, \mathrm{As}^+\right)} \cdot \prod_v \mathcal{W}_v^\circ(\varphi_v) \tag{6.1.6}$$

where

$$\mathcal{W}_v^\circ(\varphi_v) := \frac{L\left(1, \Pi_{\Sigma_v}, \mathrm{As}^+\right)}{\prod_{j=1}^4 L\left(j, \chi_{E_v}^j\right)} \cdot \mathcal{W}_v(\varphi_v).$$

Here we note that $\mathcal{W}_v^\circ(\varphi_v) = 1$ at almost all places v by Lapid and Mao [71].

Before proceeding to the proof of Theorem 6.1, by assuming it, we prove the following theorem, namely the explicit formula for the Bessel periods on GU (4).

Theorem 6.2. *Let (π, V_π) be an irreducible cuspidal tempered automorphic representation of $G_D(\mathbb{A})$ with trivial central character. Suppose that π has the (ξ, Λ, ψ)-Bessel period and that π is neither of type I-A nor type I-B. Let π_+^B denote the unique irreducible constituent of $\pi|_{G_D(\mathbb{A})^+}$ which has the (ξ, Λ, ψ)-Bessel period. We denote by (σ, V_σ) the theta lift of π_+^B to $\mathrm{GSU}_{3,D}$, which is an irreducible cuspidal automorphic representation by Lemma 4.1 and Lemma 4.4.*

Then for a non-zero decomposable vector $\varphi = \otimes \varphi_v \in V_\sigma$, we have

$$\frac{|\mathcal{B}_{X,\psi.\Lambda}(\varphi)|^2}{(\varphi, \varphi)} = \frac{1}{2^\ell}\left(\prod_{j=1}^{4} L(1, \chi_E^j)\right)\frac{L\left(\frac{1}{2}, \sigma \times \Lambda^{-1}\right)}{L(1, \pi, \mathrm{std} \otimes \chi_E)L(1, \chi_E)}\prod_v \alpha^{\natural}_{\Lambda_v, \psi_{X,v}}(\varphi_v)$$

where

$$\alpha^{\natural}_{\Lambda_v, \psi_{X,v}}(\varphi_v) = \left(\prod_{j=1}^{4} L(1, \chi_{E,v}^j)\right)^{-1}\frac{L(1, \pi_v, \mathrm{std} \otimes \chi_{E,v})L(1, \chi_{E_v})}{L\left(\frac{1}{2}, \sigma_v \times \Lambda_v^{-1}\right)} \cdot \frac{\alpha_{\Lambda_v, \psi_{X,v}}(\varphi_v)}{(\varphi_v, \varphi_v)_v}$$

and $X \in D^\times$ is taken so that $\xi = S_X$ in (3.2.5).

Proof. Let us regard σ as an automorphic representation of $\mathrm{GU}(4)$ with trivial central character via the accidental isomorphisms Φ (2.2.6) or Φ_D (2.2.5), depending whether D is split or not. Let $\theta(\sigma) = \Theta_{\psi,(\Lambda^{-1},\Lambda^{-1})}(\sigma)$ denote the theta lift of σ to $\mathrm{GU}_{2,2}$ with respect to ψ and $(\Lambda^{-1}, \Lambda^{-1})$. By [29, Proposition 3.1], $\theta(\sigma)$ is globally generic and, in particular, non-zero. By the same argument as in the proof of [29, Theorem 1], we see that $\theta(\sigma)$ is cuspidal and hence irreducible by Remark 3.1 and 3.2. Moreover by the unramified computations in [68] and [83, (3.6)], we see that $L^S(s, \Sigma, \wedge_t^2)$ has a pole at $s = 1$ when S is a sufficiently large finite set of places of F containing all archimedean places, where $L^S(s, \Sigma, \wedge_t^2)$ denotes the twisted exterior square L-function of Σ (see [26, Section 2.1.1] for the definition). Since $\theta(\sigma)$ is generic, [26, Theorem 4.1] implies that it has the unitary Shalika period defined in [26, (2.5)]. Then, by [83, Theorem B], the theta lift of $\theta(\sigma)$ to $G(\mathbb{A})^+$, which we denote by $(\pi'_+, V_{\pi'_+})$, is an irreducible cuspidal globally generic automorphic representation of $G(\mathbb{A})^+$. We note that π_+^B is nearly equivalent to π'_+.

Let us take an irreducible cuspidal automorphic representation $(\pi', V_{\pi'})$ of $G(\mathbb{A})$ such that $V_{\pi'}|_{G(\mathbb{A})^+} \supset V_{\pi'_+}$. Then π' is globally generic. Moreover $\pi' \otimes \chi_E$ is not nearly equivalent to π' by our assumption on π. Hence $\pi'|_{\mathcal{G}}$ is irreducible. Thus we may apply Theorem 6.1, taking $\pi^\circ = \pi'$ and $\Sigma = \theta(\sigma)$, and we obtain the explicit formula for the Whittaker periods on $\theta(\sigma)$. Then by [29, Theorem A.1], the required explicit formula for the Bessel periods follows. □

6.2. **Proof of Theorem 6.1.** We reduce Theorem 6.1 to a certain local identity in 6.2.2 and then prove the local identity in 6.2.3.

As we stated in the beginning of this section, what we do essentially is to deduce the explicit formula (6.1.6) for the Whittaker periods on $\mathrm{GSO}_{4,2}$ from (6.2.3) below, the one for the Whittaker periods on G.

6.2.1. *Explicit formula for the Whittaker periods on $G = \mathrm{GSp}_2$.* Let U_G denote the maximal unipotent subgroup of G. Namely

$$U_G := \left\{ m(n) \begin{pmatrix} 1_2 & X \\ 0 & 1_2 \end{pmatrix} : X \in \mathrm{Sym}_2, n \in N_2 \right\} \tag{6.2.1}$$

where $m(h) = \begin{pmatrix} h & 0 \\ 0 & {}^t h^{-1} \end{pmatrix}$ for $h \in \mathrm{GL}_2$ and N_2 denotes the group of upper unipotent matrices in GL_2. Then we define a non-degenerate character ψ_{U_G} of $U_G(\mathbb{A})$ by

$$\psi_{U_G}(u) := \psi(u_{12} + d\, u_{24}) \quad \text{for } u = (u_{ij}) \in U_G(\mathbb{A}). \tag{6.2.2}$$

Then for an automorphic form ϕ on $G(\mathbb{A})$, we define the Whittaker period $W_{\psi_{U_G}}(\phi)$ of ϕ by

$$W_{\psi_{U_G}}(\phi) = \int_{U_G(F)\backslash U_G(\mathbb{A})} \phi(n)\, \psi_{U_G}^{-1}(n)\, dn.$$

The following theorem shall be proved in Appendix A.

Theorem 6.3. *Suppose that (π, V_π) is an irreducible cuspidal tempered globally generic automorphic representation of $G(\mathbb{A})$. Let $\Pi_\pi = \Pi_1 \boxplus \cdots \boxplus \Pi_k$ denote the functorial lift of π to $\mathrm{GL}_4(\mathbb{A})$.*

Then for any non-zero decomposable vector $\varphi = \otimes \varphi_v \in V_\pi$, we have

$$\frac{|W_{\psi_{U_G}}(\varphi)|^2}{(\varphi, \varphi)} = \frac{1}{2^k} \cdot \frac{\prod_{j=1}^{2} \xi_F(2j)}{L\left(1, \Pi_\pi, \mathrm{Sym}^2\right)} \cdot \prod_v \mathcal{W}_{G,v}^{\circ}(\varphi_v). \tag{6.2.3}$$

Here $\mathcal{W}_{G,v}^{\circ}(\varphi_v)$ is defined by

$$\mathcal{W}_{G,v}^{\circ}(\varphi_v) = \frac{L\left(1, \Pi_{\pi,v}, \mathrm{Sym}^2\right)}{\prod_{j=1}^{2} \zeta_{F_v}(2j)} \mathcal{W}_{G,v}(\varphi_v)$$

and $\mathcal{W}_{G,v}(\varphi_v)$ is defined by

$$\mathcal{W}_{G,v}(\varphi_v) = \int_{U_G(F_v)}^{st} \frac{\left(\pi_v^{\circ}(n)\varphi_v, \varphi_v\right)}{(\varphi_v, \varphi_v)}\, \psi_{U_G}^{-1}(n)\, dn$$

when v is finite and by the Fourier transform when v is archimedean.

6.2.2. *Reduction to a local identity.* Let us go back to the situation stated in the beginning of 6.1.

First we note that the unramified computation in [68] implies the following lemma.

Lemma 6.1. *There exists a finite set S_0 of places of F containing all archimedean places such that for a place $v \notin S_0$, we have*

$$L\left(1, \Pi_{\Sigma_v}, \mathrm{As}^+\right) = L\left(1, \pi_v, \mathrm{std} \otimes \chi_E\right) L\left(1, \Pi_{\pi,v}, \mathrm{Sym}^2\right) L\left(1, \chi_{E_v}\right).$$

Let us recall the following pull-back formula for the Whittaker period on $\Sigma = \theta_\psi(\pi^\circ)$.

Proposition 6.1. [83, p. 40] *Let $f \in V_{\pi^\circ}$ and $\phi \in \mathcal{S}(Z_+(\mathbb{A}))$. Then*

$$W^{\psi_U}(\theta(\phi; f)) = \int_{N(\mathbb{A})\backslash G^1(\mathbb{A})} (\omega_\psi(g_1, 1)\phi)((x_{-2}, x_{-1}, 0, x_2)) \times W_{\psi_{U_G}}(\pi^\circ(g_1)f)\, dg_1. \tag{6.2.4}$$

Suppose that $f = \otimes f_v$ and $\phi = \otimes \phi_v$. Then by an argument similar to the one in obtaining [28, (2.27)], when $W_{\psi_{U_G}}(f) \neq 0$, we have

$$W^{\psi_U}(\theta(\phi; f)) = C_{G^1} \cdot W_{\psi_{U_G}}(f) \cdot \prod_v \mathcal{L}_v^\circ(\phi_v, f_v)$$

where

$$\mathcal{L}_v^\circ(\phi_v, f_v) := \int_{N(F_v)\backslash G^1(F_v)} (\omega_{\psi_v}(g_1, 1)\phi_v)((x_{-2}, x_{-1}, 0, x_2)) \times \mathcal{W}_{G,v}^\circ(\pi_v^\circ(g_1)f_v)\, dg_{1,v}$$

when $\phi = \otimes_v \phi_v$ and $f = \otimes_v f_v$. We also define

$$\mathcal{L}_v(\phi_v, f_v) := \frac{\prod_{j=1}^2 \xi_F(2j)}{L\left(1, \Pi_{\pi,v}, \mathrm{Sym}^2\right)} \frac{\mathcal{L}_v^\circ(\phi_v, f_v)}{\mathcal{W}_{G,v}(f_v)}.$$

Here the measures are taken as the following. Let dg_v be the measure on $G^1(F_v)$ defined by the gauge form and dn_v the measure on $N(F_v)$ defined in the manner stated in 1.2. Then we take the measure $dg_{1,v}$ on $N(F_v)\backslash G^1(F_v)$ so that $dg_v = dn_v\, dg_{1,v}$.

Let $\Theta\left(\pi_v^\circ, \psi_v\right) := \mathrm{Hom}_{G(F_v)^+}\left(\Omega_{\psi_v}, \bar{\pi}_v^\circ\right)$ where Ω_{ψ_v} is the extended local Weil representation of $G(F_v)^+ \times \mathrm{GSO}_{4,2}(F_v)$ realized on $\mathcal{S}(Z_+(F_v))$, the space of Schwartz-Bruhat functions on $Z_+(F_v)$. We recall that the action of $G(F_v)^+ \times \mathrm{GSO}_{4,2}(F_v)$ on $\mathcal{S}(Z_+(F_v))$ via Ω_{ψ_v} is defined as in the global case (e.g. see [83, 2.2]). We also recall that for $\Sigma = \theta_\psi(\pi^\circ)$, we have $\Sigma = \otimes_v \Sigma_v$ where $\Sigma_v = \theta_{\psi_v}\left(\pi_v^\circ\right)$ is the local theta lift of π_v°.

Let

$$\theta_v : \mathcal{S}(Z_+(F_v)) \otimes V_{\pi_v^\circ} \longrightarrow V_{\Sigma_v}$$

be a $G(F_v)^+ \times \mathrm{GSO}_{4,2}(F_v)$-equivariant linear map, which is unique up to a scalar multiplication. Since the global mapping

$$\mathcal{S}(Z_+(\mathbb{A})) \otimes V_{\pi^\circ} \ni (\phi', f') \mapsto \theta_\psi(\phi'; f') \in V_\Sigma$$

is $G(F_v)^+ \times \mathrm{GSO}_{4,2}(F_v)$-equivariant at any place v, by the uniqueness of θ_v, we may adjust $\{\theta_v\}_v$ so that

$$\theta_\psi(\phi'; f') = \otimes_v \theta_v\left(\phi_v' \otimes f_v'\right) \quad \text{for } f' = \otimes_v f_v' \in V_{\pi^\circ},\ \phi' = \otimes_v \phi_v' \in \mathcal{S}(Z_+(\mathbb{A})).$$

Then as in [28, Section 2.4], combining Theorem 6.3, the Rallis inner product formula (5.1.4), Lemma 6.1, Lemma 4.2 and Proposition 6.1, we see that a proof of Theorem 6.1 is reduced to a proof of the following local identity (6.2.5).

Proposition 6.2. *Let v be an arbitrary place of F. For a given $f_v \in V_{\pi_v^\circ}^\infty$ satisfying $\mathcal{W}_{G,v}(f_v) \neq 0$, there exists $\phi_v \in \mathcal{S}(Z_+(F_v))$ such that the local integral $\mathcal{L}_v(\phi_v, f_v)$ converges absolutely, $\mathcal{L}_v(\phi_v, f_v) \neq 0$ and the equality*

$$\frac{Z_v(\phi_v, f_v, \pi_v) \cdot \mathcal{W}_v(\theta(\phi_v \otimes f_v))}{|\mathcal{L}_v(\phi_v, f_v)|^2} = \mathcal{W}_{G,v}(f_v) \tag{6.2.5}$$

holds with respect to the specified local measures.

Let us define a hermitian inner product $\mathcal{B}_{\omega_v}$ on $\mathcal{S}(Z_+(F_v))$ by

$$\mathcal{B}_{\omega_v}(\phi, \phi') = \int_{Z_+(F_v)} \phi(x)\, \overline{\phi'}(x)\, dx \quad \text{for} \quad \phi, \phi' \in \mathcal{S}(Z_+(F_v)).$$

Here on $Z_+(F_v) \simeq (F_v)^{12}$, we take the product measure of the one on F_v. Then we consider the integral

(6.2.6)

$$Z^\flat(f, f'; \phi, \phi') = \int_{G^1(F_v)} \langle \pi_v^\circ(g) f, f' \rangle_v\, \mathcal{B}_{\omega_v}(\omega_\psi(g)\phi, \phi')\, dg$$

$$= \int_{G^1(F_v)} \int_{Z_+(F_v)} \langle \pi_v^\circ(g) f, f' \rangle_v \, \left(\omega_{\psi_v}(g,1)\phi\right)(z)\overline{\phi'(z)}\, dx\, dg \quad \text{for } f, f' \in V_{\pi_v^\circ}.$$

The integral (6.2.6) converges absolutely by Yamana [120, Lemma 7.2]. As in Gan and Ichino [40, 16.5], we may define a $\mathrm{GSO}_{4,2}(F_v)$-invariant hermitian inner product $\mathcal{B}_{\Sigma_v} : V_{\Sigma_v} \times V_{\Sigma_v} \to \mathbb{C}$ by

$$\mathcal{B}_\Sigma(\theta(\phi \otimes f), \theta(\phi' \otimes f')) := Z^\flat(f, f'; \phi, \phi').$$

Here we note that for $h \in \mathrm{SO}_{4,2}(F_v)$, we have

$$\mathcal{B}_\Sigma(\Sigma(h)\theta(\phi \otimes f), \theta(\phi' \otimes f')) = \mathcal{B}_\Sigma(\theta(\omega_\psi(1,h)\phi \otimes f), \theta(\phi' \otimes f')).$$

As in the definition of W_v, we define

$$\mathcal{W}^{\psi_U}(\tilde{\phi}_1, \tilde{\phi}_2) := \int_{U(F_v)}^{st} \mathcal{B}_\Sigma(\Sigma(n)\tilde{\phi}_1, \tilde{\phi}_2)\psi_U(n)^{-1}\, dn \quad \text{for } \widetilde{\phi}_i \in \Sigma_v \ \ (i = 1, 2).$$

Then by an argument similar to the one in [28, 3.2–3.3], indeed by word for word, Proposition 6.2 is reduced to the following another local identity, which is regarded as a local pull-back computation of the Whittaker periods with respect to the theta lift.

Proposition 6.3. *For any $f, f' \in V_{\pi_v^\circ}$ and any $\phi, \phi' \in C_c^\infty(Z_+(F_v))$, we have*

$$\mathcal{W}_{\psi_U}\left(\theta(\phi \otimes f), \theta(\phi' \otimes f')\right) = \int_{N(F_v)\backslash G^1(F_v)} \int_{N(F_v)\backslash G^1(F_v)} \mathcal{W}_{G,v}\left(\pi_v^\circ(g) f, \pi_v^\circ(g') f'\right) \left(\omega_{\psi_v}(g,1)\phi\right)(x_0)\, \overline{\left(\omega_{\psi_v}(g',1)\phi'\right)(x_0)}\, dg\, dg'. \tag{6.2.7}$$

Remark 6.1. *Since $\{g \cdot x_0 : g \in G^1(F_v)\}$ is locally closed in $Z_+(F_v)$, the mappings*

$$N(F_v)\backslash G^1(F_v) \ni g \mapsto \phi(g^{-1} \cdot x_0) \in \mathbb{C}, \quad N(F_v)\backslash G^1(F_v) \ni g' \mapsto \phi'(g^{-1} \cdot x_0) \in \mathbb{C}$$

are compactly supported, and thus the right-hand side of (6.2.7) converges absolutely for $\phi, \phi' \in C_c^\infty(Z_+(F_v))$.

6.2.3. *Local pull-back computation.* Here we shall prove Proposition 6.3 and thus complete our proof of Theorem 6.1.

Since we work over a fixed place v of F, we shall suppress v from the notation in this subsection, e.g. F means F_v. Further, for any algebraic group K over F, we denote its group of F-rational points $K(F)$ by K for simplicity.

The case when F is non-archimedean. Suppose that F is non-archimedean. From the definition, the local Whittaker period is equal to

$$\int_U^{st} \int_{G^1} \int_{Z_+} (\omega_\psi(g,n)\phi)(x)\overline{\phi'(x)}\langle \pi^\circ(g)f, f'\rangle \psi_U(n)^{-1}\, dx\, dg\, dn.$$

Recall that we have defined subgroups N_0, N_1, N_2 and $\widetilde{U}$ of U in (3.1.10), (3.1.11), (3.1.12) and (6.1.1), respectively. Then because of the absolute convergence of the integral (6.2.6), the above local integral can be written as

$$\begin{aligned}(6.2.8)\quad \int_{\widetilde{U}}^{st}\int_N^{st}\int_{N_1}\int_{N_0}\int_{Z_+}\int_{G^1} &(\omega_\psi(g,u_0u_1u_2\tilde{u})\phi)(x)\overline{\phi'(x)} \\ &\times \langle \pi^\circ(g)f, f'\rangle \psi_U(u_2\tilde{u})^{-1}\, dx\, dg\, du_0\, du_1\, du_2\, d\tilde{u}.\end{aligned}$$

Let us define $Z_{+,\circ} := \{(a_1,a_2;0,0) \in Z_+ : a_1 \text{ and } a_2 \text{ are linearly independent}\}$. Then since $Z_{+,\circ} \oplus (X_+ \otimes Y_0)$ is open and dense in Z_+, we have

$$\int_{Z_+} \Phi(z)\, dz = \int_{Z_{+,\circ}} \int_{X_+\otimes Y_0} \Phi(z_1+z_2)\, dz_2\, dz_1$$

for any $\Phi \in L^1(Z_+)$. We consider a map $p : Z_{+,\circ} \to F$ defined by $p((a_1,a_2;0,0)) = \langle a_1, a_2\rangle$. This is clearly surjective. For each $t \in F$, we fix $x_t \in Z_{+,\circ}$ such that $p(x_t) = t$. Then by Witt's theorem, the fiber $p^{-1}(x_t)$ of $x_t := (a_1^t, a_2^t; 0, 0)$ is given by

$$p^{-1}(x_t) = \left\{\gamma \cdot x_t := (\gamma a_1^t, \gamma a_2^t : 0,0) : \gamma \in G^1\right\}.$$

We may identify this space with G^1/R_t as a G^1-homogeneous space. Here R_t denotes the stabilizer of x_t in G^1. From this observation, the following lemma readily follows (cf. [28, Lemma 3]).

Lemma 6.2. *For each $x_t \in Z_{+,\circ}$, there exists a Haar measure dr_t on R_t such that*

$$\int_{Z_+} \Phi(z)\, dz = \int_F \int_{R_t\backslash G^1} \int_{X_+\otimes Y_0} \Phi(g^{-1}\cdot x_t + z)\, dz\, dg_t\, dt.$$

Here dg_t denotes the quotient measure $dr_t\backslash dg$ on $R_t\backslash G^1$.

Further, we note that the following lemma, which is proved by an argument similar to the one for [76, Lemma 3.20]. (cf. [28, Lemma 3]).

Lemma 6.3. *For $\phi_1, \phi_2 \in C_c^\infty(Z_+)$ and $f_1, f_2 \in V_{\pi^\circ}$, let*

$$\mathcal{G}_{\phi_1,\phi_2,f_1,f_2}(t) = \int_{G^1}\int_{R_t\backslash G^1} \phi_1\left((gg')^{-1}\cdot x_t\right)\phi_2\left(g^{-1}\cdot x_t\right)\langle \pi^\circ(g')f_1, f_2\rangle\, dg\, dg'$$

for $t \in F$. Then the integral is absolutely convergent and is locally constant.

Remark 6.2. *When F is archimedean, by an argument similar to the one for* [76, Proposition 3.22], *we see that this integral is absolutely convergent and is a continuous function on F not only for $C_c^\infty(Z_+)$ but also for $\mathcal{S}(Z_+)$.*

By Lemma 6.2, the integral (6.2.8) can be written as

$$\int_{N_0}\int_F\int_{R_t\backslash G^1}\int_{X_+\otimes Y_0}\int_{G^1}(\omega_\psi(g,u_0h)\phi)(\gamma^{-1}\cdot x_t+z)\overline{\phi'(\gamma^{-1}\cdot x_t+z)} \\ \times\langle\pi^\circ(g)f,f'\rangle\,dg\,dz\,d\gamma_t\,dt\,du_0.$$

Moreover, by the computation in [83, Section 3.1], we have

$$(\omega_\psi(g,u_0(x)h)\phi)(\gamma^{-1}\cdot x_t+z)=\psi(-xt)\phi(\gamma^{-1}\cdot x_t+z).$$

Then because of Lemma 6.3, we may apply the Fourier inversion with respect to x and t, and thus the above integral is equal to

$$\begin{aligned}(6.2.9)\quad &\int_{R_0\backslash G^1}\int_{X_+\otimes Y_0}\int_{G^1}(\omega_\psi(g,h)\phi)(\gamma^{-1}\cdot x_0+z)\overline{\phi'(\gamma^{-1}\cdot x_0+z)}\\ &\qquad\times\langle\pi^\circ(g)f,f'\rangle\,dg\,dz\,d\gamma_0\,dt\,du_0\\ &=\int_{R_0\backslash G^1}\int_{X_+\otimes Y_0}\int_{G^1}(\omega_\psi(\gamma g,h)\phi)(x_0+z)\overline{(\omega_\psi(\gamma,1)\phi')(x_0+z)}\\ &\qquad\times\langle\pi^\circ(g)f,f'\rangle\,dg\,dz\,d\gamma_0.\end{aligned}$$

The support of $\phi'(\gamma^{-1}\cdot x_0+z)$ as a function of $X_+\otimes Y_0$ is compact since $\phi'\in C_c^\infty(Z_+)$. Therefore this integral converges absolutely and is equal to

$$\int_{X_+\otimes Y_0}\int_{R_0\backslash G^1}\int_{G^1}(\omega_\psi(\gamma g,h)\phi)(x_0+z)\overline{(\omega_\psi(\gamma,1)\phi')(x_0+z)} \\ \times\langle\pi^\circ(g)f,f'\rangle\,dg\,d\gamma_0\,dz.$$

Now, let us take $(x_{-2},x_{-1}:0,0)$ as x_0. Then we have

$$R_0=N.$$

Let us define a map $q: X_+\otimes Y_0\longrightarrow \mathrm{Mat}_{2\times 2}$ by

$$q(b_1\otimes e_1+b_2\otimes e_2)=\begin{pmatrix}\langle x_{-2},b_1\rangle & \langle x_{-2},b_2\rangle\\ \langle x_{-1},b_1\rangle & \langle x_{-1},b_2\rangle\end{pmatrix}$$

with $b_i\in X_+$. Clearly this map is bijective. Hence, there exists a measure dT on $\mathrm{Mat}_{2\times 2}$ such that we have

$$\int_{X_+\otimes Y_0}\Phi(x_{-2},x_{-1}:z)\,dz=\int_{\mathrm{Mat}_{2\times 2}}\Phi(x_{-2},x_{-1}:x_T)\,dT$$

with $x_T=q^{-1}(T)$. Here we note that the measure dz on $X_+\otimes Y_0$ is taken to be the Tamagawa measure and hence we have the Fourier inversion

$$\int_{\mathrm{Mat}_{2\times 2}}\int_{\mathrm{Mat}_{2\times 2}}\Phi(T)\psi\left(\mathrm{tr}\,(TS_0T')\right)\,dT\,dT'=\Phi(0)$$

with the above Haar measures dT, dT' on $\mathrm{Mat}_{2\times 2}$ if the integral converges. Thus we have

$$\int_{N_2}^{st}\int_{N_1}\int_{X_+\otimes Y_0}\int_{N\backslash G^1}\int_{G^1}(\omega_\psi(\gamma g, u_1u_2h)\phi)(x_0+z)\overline{(\omega_\psi(\gamma,1)\phi')(x_0+z)} \times \langle \pi^\circ(g)f, f'\rangle\, dg\, d\gamma_0\, dz\, du_1\, du_2$$
$$= \int_{N}^{st}\int_{N_1}\int_{\mathrm{Mat}_{2\times 2}}\int_{N\backslash G^1}\int_{G^1}(\omega_\psi(\gamma g, u_1u_2h)\phi)(x_0+x_T)\overline{(\omega_\psi(\gamma,1)\phi')(x_0+x_T)} \times \langle \pi^\circ(g)f, f'\rangle\, dg\, d\gamma_0\, dT\, du_1\, du_2.$$

Moreover, similarly to the global computation in [83, Section 3.1], we may write this integral as

$$\text{(6.2.10)}\quad \int_{N_2}^{st}\int_{N_1}\int_{\mathrm{Mat}_{2\times 2}}\int_{N\backslash G^1}\int_{G^1}\psi\left(\mathrm{tr}\left(\begin{pmatrix} s_1 & s_2\\ t_1 & t_2\end{pmatrix} S_0(x_T - x_{T_0})\right)\right)$$
$$\times(\omega_\psi(\gamma g, h)\phi)(x_0+x_T)\overline{(\omega_\psi(\gamma,1)\phi')(x_0+x_T)}\langle \pi^\circ(g)f, f'\rangle\, dg\, d\gamma_0\, dT\, du_1\, du_2$$

where we write $u_1 = u_1(s_1,t_1)$ and $u_2 = u_2(s_2,t_2)$, and we put $T_0 = \begin{pmatrix} 0 & 0\\ 0 & 1\end{pmatrix}$. By an argument similar to the proof to show (6.2.9), we may apply the Fourier inversion to this integral, and we see that this is equal to

$$\int_{N_H\backslash G^1}\int_{G^1}(\omega_\psi(\gamma g, h)\phi)(x_0+x_{T_0})\overline{(\omega_\psi(\gamma,1)\phi')(x_0+x_{T_0})}\langle \pi^\circ(g)f, f'\rangle\, dg\, d\gamma_0.$$

Now we note that from the argument to obtain (6.2.9), this integral converges absolutely. Then by telescoping the G^1-integration, we obtain

$$\int_{N\backslash G^1}\int_{N\backslash G^1}\int_{N}(\omega_\psi(rg, h)\phi)(x_0+x_{T_0})\overline{(\omega_\psi(\gamma,1)\phi')(x_0+x_{T_0})} \times \langle \pi^\circ(rg)f, \pi^\circ(\gamma)f'\rangle\, dr\, dg\, d\gamma_0.$$

Put $z_0 = x_0 + x_{T_0} = (x_{-2}, x_{-1}, 0, x_2)$. Recall that from the computation in [83, Section 3.1], we have

$$\text{(6.2.11)}\qquad \omega_\psi(v(A)g, \tilde{u}(b)h)\phi(z_0) = \psi(-da_{22})\omega_\psi(g, \tilde{u}(b)h)\phi(z_0)$$

when we write $A = \begin{pmatrix} a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}$, and we have

$$\text{(6.2.12)}\qquad z_0(1, \tilde{u}(b)) = z_0(w(b), 1).$$

Therefore, $\mathcal{W}_{\psi_U}(\theta(\phi\otimes f),\theta(\phi'\otimes f'))$ is equal to

$$\int_F^{st}\int_{N\backslash G^1}\int_{N\backslash G^1}\int_N \psi(-b)(\omega_\psi(rg,\tilde{u}(b))\phi)(z_0)\overline{(\omega_\psi(\gamma,1)\phi')(z_0)} \times\langle\pi^\circ(rg)f,\pi^\circ(\gamma)f'\rangle\,dr\,dg\,d\gamma_0\,db$$
$$=\int_F^{st}\int_{N\backslash G^1}\int_{N\backslash G^1}\int_{\mathrm{Sym}^2}\psi(-b-da_{22})(\omega_\psi(w(b)g,1)\phi)(z_0) \times\overline{(\omega_\psi(\gamma,1)\phi')(z_0)}\langle\pi^\circ(v(A)g)f,\pi^\circ(\gamma)f'\rangle\,dA\,dg\,d\gamma_0\,db.$$

By an argument similar to the one in [28] showing that [28, (3.30)] is equal to $\alpha(\pi(g)\phi,\pi(h)\phi')$ there, indeed, by word for word, we see that this integral is equal to

$$\int_{N\backslash G^1}\int_{N\backslash G^1}\int_{U_G}^{st}\psi_{U_G}^{-1}(n)(\omega_\psi(g,1)\phi)(z_0)\overline{(\omega_\psi(\gamma,1)\phi')(z_0)} \times\langle\pi^\circ(ng)f,\pi^\circ(\gamma)f'\rangle\,dn\,dg\,d\gamma_0.$$

Thus Proposition 6.3 in the non-archimedean case is proved.

The case when F is archimedean. Suppose that F is archimedean. Recall that

$$\mathcal{W}^{\psi_U}(\tilde{\phi}_1,\tilde{\phi}_2)=\widehat{\mathcal{W}_{\tilde{\phi}_1,\tilde{\phi}_2}}(\psi_U)\quad\text{for }\tilde{\phi}_i\in\Sigma^\infty\quad(i=1,2),$$

where we set

$$\mathcal{W}_{\tilde{\phi}_1,\tilde{\phi}_2}(n)=\int_{U_{-\infty}}\mathcal{B}_\Sigma(\Sigma(nu)\tilde{\phi}_1,\tilde{\phi}_2)\psi_U^{-1}(nu)\,du\quad\text{for }n\in U,$$

which converges absolutely and gives a tempered distribution on $U/U_{-\infty}$ by [76, Corollary 3.13]. Let us define $U'=N_0N_1N_2$. Then $U'_{-\infty}=U_{-\infty}$. Moreover, for any $\tilde{u}\in\widetilde{U}$ and $u'\in U'$, we have $\tilde{u}u'\tilde{u}^{-1}(u')^{-1}\in U'_{-\infty}$ and we obtain $\mathcal{W}_{\tilde{\phi}_1,\tilde{\phi}_2}(\tilde{u}u')=\mathcal{W}_{\tilde{\phi}_1,\tilde{\phi}_2}(u'\tilde{u})$. Hence, we may regard it as a tempered distribution on $\tilde{U}\times\left(U'/U'_{-\infty}\right)$. Then for a tempered distribution I on $\tilde{U}\times\left(U'/U'_{-\infty}\right)$, we define partial Fourier transforms $\widehat{I}^j$ of I for $j=1,2$ by

$$\langle I,\widehat{f_1}\otimes f_2\rangle=\langle\widehat{I}^1,f_1\otimes f_2\rangle\quad\text{and}\quad\langle I,f_1\otimes\widehat{f_2}\rangle=\langle\widehat{I}^2,f_1\otimes f_2\rangle$$

where $f_1\in\mathcal{S}(\tilde{U})$ and $f_2\in\mathcal{S}\left(U'/U'_{-\infty}\right)$, respectively. Then we have

$$\widehat{\widehat{I}^2}^1(\psi_U)=\widehat{\widehat{I}^1}^2(\psi_U)=\widehat{I}(\psi_U).$$

From the definition of $\mathcal{B}_\Sigma$, we have

$$\mathcal{W}_{\theta(\phi\otimes f),\theta(\phi'\otimes f')}(n)=\int_{U_{-\infty}}\int_{G^1}\int_{Z_+}(\omega_\psi(g,nu)\phi)(x)\overline{\phi'(x)} \times\langle\pi^\circ(g)f,f'\rangle\psi_U^{-1}(nu)\,dx\,dg\,du$$
$$=\int_{U_{-\infty}/N_0}\int_{N_0}\int_{G^1}\int_{Z_+}(\omega_\psi(g,nu_0u)\phi)(x)\overline{\phi'(x)} \times\langle\pi^\circ(g)f,f'\rangle\psi_U^{-1}(nu)\,dx\,dg\,du_0\,du,$$

for $\phi, \phi' \in \mathcal{S}(Z_+)$ and $f, f' \in V_{\pi^\circ}^\infty$. Clearly, Lemma 6.2 holds in the archimedean case also. Then as in (6.2.9), because of Remark 6.2 and the Fourier inversion, the above integral is equal to

$$\int_{U_{-\infty}/N_0}\int_{N\backslash G^1}\int_{X_+\otimes Y_0}\int_{G^1}(\omega_\psi(\gamma g, nu)\phi)(x_0+z)\overline{(\omega_\psi(\gamma,1)\phi')(x_0+z)} \times \langle \pi^\circ(g)f, f'\rangle \, dg\, dz\, d\gamma_0\, du.$$

As (6.2.9), this integral converges absolutely. Let us denote this integral by $J_{\phi,\phi',f,f'}(n)$. Then from the definition,

$$\widehat{J_{\phi,\phi',f,f'}} = \widehat{\mathcal{W}_{\theta(\phi\otimes f),\theta(\phi'\otimes f')}}.$$

Again, from the definition, for $\varphi \in \mathcal{S}(U'/U'_{-\infty})$, we have

$$(\widehat{J_{\phi,\phi',f,f'}}^2, \psi_U\cdot\varphi) = (J_{\phi,\phi',f,f'}, \widehat{\psi_U\cdot\varphi}) = \int_{U'/U'_{-\infty}}\int_{U'_{-\infty}/N_0}\int_{N\backslash G^1}\int_{X_+\otimes Y_0}\int_{G^1} \times (\omega_\psi(\gamma g, nu)\phi)(x_0+z)\overline{(\omega_\psi(\gamma,1)\phi')(x_0+z)} \times \langle\pi^\circ(g)f, f'\rangle\widehat{\varphi}(n)\psi_U^{-1}(n)\, dg\, dz\, d\gamma_0\, du\, dn.$$

By a computation similar to the one to obtain (6.2.10), this integral is equal to

$$\begin{aligned}&\int_{N_1}\int_{N_2}\int_{N\backslash G^1}\int_{X_+\otimes Y_0}\int_{G^1}(\omega_\psi(\gamma g, u_1u_2u)\phi)(x_0+z)\overline{(\omega_\psi(\gamma,1)\phi')(x_0+z)}\\ &\quad\times\langle\pi^\circ(g)f, f'\rangle\widehat{\varphi}(u_1u_2)\psi_U^{-1}(u_1u_2)\, dg\, dz\, d\gamma_0\, du\, du_1\, du_2\\ &=\int_{N_1}\int_{N_2}\int_{\mathrm{Mat}_{2\times 2}}\int_{N\backslash G^1}\int_{G^1}\psi\left(\mathrm{tr}\left(\begin{pmatrix}s_1 & s_2\\ t_1 & t_2\end{pmatrix}S_0(x_T-x_{T_0})\right)\right)\\ &\quad\times(\omega_\psi(\gamma g, h)\phi)(x_0+x_T)\overline{(\omega_\psi(\gamma,1)\phi')(x_0+x_T)}\\ &\quad\times\langle\pi^\circ(g)f, f'\rangle\widehat{\varphi}(u_1u_2)\, dg\, d\gamma_0\, dT\, du_1\, du_2.\end{aligned}$$

As above, we may apply the Fourier inversion, and thus this is equal to

$$\widehat{\varphi}(1)\cdot\int_{N\backslash G^1}\int_{G^1}(\omega_\psi(\gamma g,1)\phi)(x_0+x_{T_0})\overline{(\omega_\psi(\gamma,1)\phi')(x_0+x_{T_0})} \times\langle\pi^\circ(g)f, f'\rangle\, dg\, d\gamma_0.$$

Hence,

$$\widehat{J_{\phi,\phi',f,f'}}^2(\psi_U) = \int_{N\backslash G^1}\int_{G^1}(\omega_\psi(\gamma g,1)\phi)(x_0+x_{T_0})\overline{(\omega_\psi(\gamma,1)\phi')(x_0+x_{T_0})} \times\langle\pi^\circ(g)f, f'\rangle\, dg\, d\gamma_0.$$

Here, we note that by Remark 6.2, this integral converges absolutely. Then this identity shows that we have

(6.2.13)
$$\widehat{\widehat{J_{\phi,\phi',f,f'}}^{2}}^{1}(\varphi) = \int_{\tilde{U}} \int_{N\backslash G^1} \int_{G^1} (\omega_\psi(\gamma g, b)\phi)(x_0 + x_{T_0}) \overline{(\omega_\psi(\gamma, 1)\phi')(x_0 + x_{T_0})} \times \langle \pi^\circ(g) f, f' \rangle \varphi(b)\, dg\, d\gamma_0\, db$$

for $\varphi \in \mathcal{S}(\tilde{U})$. As in the non-archimedean case, by (6.2.11) and (6.2.12), we may easily show that this is equal to

$$\int_{N\backslash G^1} \int_{N\backslash G^1} \int_N \int_F \psi_{U_G}^{-1}(v(x)n)(\omega_\psi(g,1)\phi)(z_0)\overline{(\omega_\psi(\gamma,1)\phi')(z_0)} \times \langle \pi^\circ(v(x)ng) f, \pi^\circ(\gamma) f' \rangle \varphi(\tilde{u}(x))\, dx\, dn\, dg\, d\gamma_0$$

since the integral in (6.2.13) converges absolutely. Thus Proposition 6.3 is proved in the archimedean case also.

7. Proof of Theorem 1.2

In this section, we complete our proof of Theorem 1.2. Let (π, V_π) be an irreducible cuspidal tempered automorphic representation of $G_D(\mathbb{A})$ with a trivial central character. Throughout this section, we suppose that π is neither of type I-A nor type I-B. When π is one of these types, our theorem is already proved in [20, Theorem 7.5].

The case when $B_{\xi,\Lambda,\psi} \not\equiv 0$ on V_π is treated in 7.1 and the case when $B_{\xi,\Lambda,\psi} \equiv 0$ on V_π is treated in 7.2, respectively.

7.1. Proof of Theorem 1.2 when $B_{\xi,\Lambda,\psi} \not\equiv 0$.

7.1.1. *Reduction to a local identity.* Suppose that $B_{\xi,\Lambda,\psi} \not\equiv 0$ on V_π. Let (σ, V_σ) denote the theta lift of π to $\mathrm{GSU}_{3,D}(\mathbb{A})$, which is an irreducible cuspidal automorphic representation. As in the proof of Theorem 6.1, our theorem may be reduced to a certain local identity. Let us set some notation to explain our local identity.

As in Section 5.1 and Section 5.2, we fix the Petersson inner product $(\ ,\ )$ on V_π and the local hermitian pairing $(\ ,\ )_v$ on π_v. As in (3.2.3), we define the maximal isotropic subspaces $Z_{D,\pm}$. Let

$$\theta_{D,v} : \mathcal{S}\left(Z_{D,+}(F_v)\right) \otimes V_{\pi_v} \to V_{\sigma_v}$$

be the $G_D(F_v)^+ \times \mathrm{GSU}_{3,D}(F_v)$-equivariant linear map, which is unique up to multiplication by a scalar. As in Section 6.1, let us adjust $\left\{\theta_{D,v}\right\}_v$ so that

$$\theta_{D,\psi}(\phi'; f') = \otimes_v\, \theta_{D,v}\left(\phi'_v \otimes f'_v\right)$$

for $f' = \otimes_v f'_v \in V_\pi$ and $\phi' = \otimes_v \phi'_v \in \mathcal{S}\left(Z_{D,+}(\mathbb{A})\right)$. Let us choose $X \in D^\times(F)$ so that $S_X = \xi$. Then by Proposition 3.2, we have

$$\mathcal{B}_{X,\Lambda^{-1}}(\theta(f : \phi)) = B_{\xi,\Lambda}(f) \cdot \prod_v \mathcal{K}_v(f_v; \phi_v) \tag{7.1.1}$$

where $f = \otimes f_v \in V_{\pi_D}$ and $\phi = \otimes \phi_v \in \mathcal{S}(Z_{D,+}(\mathbb{A}))$, and we define

$$\mathcal{K}_v(f_v; \phi_v) = \int_{N_D(F_v)\backslash G_D^1(F_v)} \alpha_{\Lambda_v, \psi_{\xi,v}}(\pi_v(g) f_v) \phi_v(g^{-1} \cdot v_{D,X})\, dg.$$

Here, we take the measure dh_v on $G_D^1(F_v)$ defined by the gauge form, the measure dn_v on $N_{G_D}(F_v)$ defined in 1.2 under the identification $D(F_v) \simeq F_v^4$ and the measure $dg_{1,v}$ on $N_{G_D}(F_v)\backslash G_D^1(F_v)$ such that $dh_v = dn_v\, dg_{1,v}$. Then by combining the explicit formula of the Bessel periods on σ given in Theorem 6.2, the Rallis inner product formulas (5.1.4) and Proposition 5.2, Lemma 6.1 and Lemma 4.2, and the above pull-back formula (7.1.1), we see that Theorem 1.2 is reduced to the following local identity.

Proposition 7.1. *Let v be an arbitrary place of F. For a given $f_v \in V_{\pi_v}$ satisfying $\alpha_{\xi,\Lambda,v}(f_v) \neq 0$, there exists $\phi_v \in \mathcal{S}(Z_{D,+}(F_v))$ such that the local integral $\mathcal{K}_v(f_v; \phi_v)$ converges absolutely, $\mathcal{K}_v(f_v; \phi_v) \neq 0$ and the equality*

$$\frac{Z_v(\phi_v, f_v, \pi_v)\, \alpha_{\Lambda_v^{-1}, \psi_{X,v}}(\theta(\phi_v \otimes f_v))}{|\mathcal{K}_v(f_v; \phi_v)|^2} = \frac{\alpha_{\Lambda_v, \psi_{\xi,v}}(f_v)}{(f_v, f_v)_v}$$

holds.

Remark 7.1. *In Corollary 7.1, the existence of f_v with $\alpha_{\Lambda_v, \psi_{\xi,v}}(f_v) \neq 0$ is shown.*

Let us define hermitian inner product on $\mathcal{S}(Z_{D,+}(F_v))$ by

$$\mathcal{B}_{\omega_v, D}(\phi, \phi') = \int_{Z_{D,+}(F_v)} \phi(x) \overline{\phi'}(x)\, dx \quad \text{for} \quad \phi, \phi' \in \mathcal{S}(Z_{D,+}(F_v)).$$

Then we consider the integral

$$Z^\bullet(f, f'; \phi, \phi') = \int_{G^1(F_v)} \langle \pi_v(g) f, f' \rangle_v\, \mathcal{B}_{\omega_v}(\omega_\psi(g)\phi, \phi')\, dg$$

for $f, f' \in \pi_v$ and $\phi, \phi' \in \mathcal{S}(Z_{D,+}(F_v))$. As in Section 6.2, this converges absolutely and gives a $\mathrm{GSU}_{3,D}(F_v)$-invariant hermitian inner product

$$\mathcal{B}_{\sigma_v} : V_{\sigma_v} \times V_{\sigma_v} \to \mathbb{C}$$

by

$$\mathcal{B}_{\sigma_v}(\theta(\phi \otimes f), \theta(\phi' \otimes f')) := Z^\bullet(f, f'; \phi, \phi').$$

By the Rallis inner product formula (5.1.4) and Proposition 5.2, at any place v, there exist f_v, f_v', ϕ, ϕ' such that $Z^\bullet(f, f'; \phi, \phi') \neq 0$ since $\theta_{\psi,D}(\pi) \neq 0$. Thus, $\mathcal{B}_{\sigma_v} \not\equiv 0$.

For $\widetilde{\phi}_i \in \sigma_v$, we define

$$\mathcal{A}(\tilde{\phi}_1, \tilde{\phi}_2) := \int_{N_{3,D}(F_v)}^{st} \int_{M_X(F_v)} \mathcal{B}_{\sigma_v}(\sigma_v(nt)\tilde{\phi}_1, \tilde{\phi}_2) \Lambda_{D,v}(t) \psi_{X,D,v}(n)^{-1}\, dt\, dn.$$

Here, at an archimedean place v, a stable integration means the Fourier transform as in the definition of α_{χ,ψ_N}. Then by an argument similar to the one in [28, 3.2–3.3], we may reduce Proposition 7.1 to the following identity.

Proposition 7.2. *For any $f, f' \in V_{\pi_v}$ and any $\phi, \phi' \in C_c^\infty\left(Z_{D,+}(F_v)\right)$, we have*

$$\mathcal{A}\left(\theta\left(\phi \otimes f\right), \theta\left(\phi' \otimes f'\right)\right) = \int_{N_D(F_v)\backslash G_D^1(F_v)} \int_{N_D(F_v)\backslash G_D^1(F_v)} \alpha_{\Lambda_v, \psi_{\xi,v}}\left(\pi_v\left(h\right) f, \pi_v\left(h'\right) f'\right) \times \left(\omega_{\psi_v}\left(h,1\right)\phi\right)\left(x_0\right) \overline{\left(\omega_{\psi_v}\left(h',1\right)\phi'\right)\left(x_0\right)}\, dh\, dh'. \tag{7.1.2}$$

Before proceeding to a proof of this proposition, we give some corollaries of this identity.

Corollary 7.1. *For an arbitrary place v of F, we have $\alpha_{\Lambda_v, \psi_{\xi,v}} \not\equiv 0$ on π_v.*

Proof. Sine $\mathcal{B}_{\sigma_v} \not\equiv 0$, (7.1.2) implies that $\alpha_{\Lambda_v, \psi_{\xi,v}} \not\equiv 0$ on π_v if and only if $\alpha_{\Lambda_v^{-1}, \psi_{X,v}} \not\equiv 0$ on σ_v. Moreover, by [28, Corollary 5.1], $\alpha_{\Lambda_v^{-1}, \psi_{X,v}} \not\equiv 0$ on σ_v since the theta lift of σ_v to $\mathrm{GU}_{2,2}(F_v)$ is generic. Thus our claim follows. □

As another corollary, a non-vanishing of local theta lifts follows from a non-vanishing of local periods.

Corollary 7.2. *Let k be a local field of characteristic zero and $\mathcal{D}$ be a quaternion algebra over k. Let τ be an irreducible admissible tempered representation of $G_{\mathcal{D}}$ with a trivial central character. Let $S_{\mathcal{D}} \in \mathcal{D}^1$ and χ be a character of $T_{\mathcal{D}, S_{\mathcal{D}}}$. Suppose that $\alpha_{\chi, \psi_{S_{\mathcal{D}}}} \not\equiv 0$ on τ. Then $\mathcal{A} \not\equiv 0$ on $\theta_{\psi,\mathcal{D}}(\tau) \times \theta_{\psi,\mathcal{D}}(\tau)$. In particular $\theta_{\psi,\mathcal{D}}(\tau) \neq 0$ and $Z^\bullet\left(\phi, \phi', f, f'\right) \neq 0$ for some $f, f' \in \tau$ and $\phi, \phi' \in \mathcal{S}(Z_{\mathcal{D},+})$.*

Remark 7.2. *By* [120, Lemma 8.6, Remark 8.4 (1)]*, we know that the existence of such f, f', ϕ, ϕ' is equivalent to the non-vanishing of the theta lift of τ to $\mathrm{GSU}_{3,\mathcal{D}}$ when $k \neq \mathbb{R}$. Though the equivalence is not clear when $k = \mathbb{R}$, we shall use Corollary 7.2 to show that the local non-vanishing of the theta lifts implies the global non-vanishing of the theta lifts in 7.2.*

Proof. By our assumption, the right-hand side of (7.1.2) is not zero for some f, f', ϕ, ϕ' when $F_v \neq \mathbb{R}$. Hence, the left-hand side is not zero, and in particular $Z^\bullet\left(\phi, \phi', f, f'\right) \neq 0$. □

7.1.2. *Local pull-back computation.* Here we shall prove the identity (7.1.2) and thus we complete our proof of Theorem 1.2 when $B_{\xi,\psi,\Lambda} \not\equiv 0$. Here we give a proof of (7.1.2) only in the non-archimedean case since the archimedean case is similarly proved as in the proof of Proposition 6.3. Our proof is a local analogue of the proof of Proposition 3.1 and Proposition 3.2. Moreover we will consider only the case when D is split since the proof is similar and indeed is easier in the non-split case as in the global computation. Since the argument in this subsection is purely local, in order to simplify the notation, we omit subscripts v and we simply write $K(F)$ by K for any algebraic group K defined over $F = F_v$.

From the definition, we may write the left-hand side of (7.1.2) as

$$\int_{N_{4,2}}^{st} \int_{M_X} \int_{G^1} \int_{Z_+} \langle \pi(g) f, f' \rangle (\omega_\psi(g, nt)\phi)(x) \overline{\phi'(x)} \Lambda(t) \psi_X(n)^{-1}\, dx\, dg\, dt\, dn$$

where X is chosen so that $S_X = S$. Further as in (3.1.13), this is equal to

$$\int_F^{st}\int_{F^2}^{st}\int_{F^2}^{st}\int_{M_X}\int_{G^1}\int_{Z_+}(\omega_\psi(g,u_0(s)u_1(s_1,t_1)u_2(s_2,t_2)t)\phi)(x)\overline{\phi'(x)}$$
$$\times\langle\pi(g)f,f'\rangle\Lambda(t)\psi(x_{21}s_1+x_{22}t_1+x_{11}s_2+x_{12}t_2)^{-1}\,dx\,dg\,dt\,ds_2\,dt_2\,ds_1\,dt_1\,ds$$

when we write $X=\begin{pmatrix}x_{11}&x_{12}\\x_{21}&x_{22}\end{pmatrix}$. For each $r\in F$, we may take $A_r=(a_1^r,a_2^r,0,0)\in Z_+$ such that a_1^r,a_2^r are linearly independent and $\langle a_1^r,a_2^r\rangle=r$. Let us denote by Q_r the stabilizer of x_r in G^1. Then as in the proof of Proposition 6.3, for each $r\in F$, there is a Haar measure dq_r of Q_r such that

$$\int_{Z_+}\Phi(x)\,dx=\int_F\int_{Q_r\backslash G^1}\int_{X_+^2}\Phi(h^{-1}\cdot A_r+b)\,db\,dh_r\,dr$$

with $dh_r=dq_r\backslash dh$, provided that the both sides converge. Then applying the Fourier inversion, because of (3.1.15), our integral becomes

$$\int_{F^2}^{st}\int_{F^2}^{st}\int_{M_X}\int_{G^1}\int_{Q_0\backslash G^1}\int_{X_+^2}\langle\pi(g)f,f'\rangle\Lambda(t)\psi(x_{21}s_1+x_{22}t_1+x_{11}s_2+x_{12}t_2)^{-1}$$
$$\times(\omega_\psi(hg,u_1(s_1,t_1)u_2(s_2,t_2)t)\phi)(A_0+b)\overline{(\omega_\psi(h,1)\phi')(A_0+b)}$$
$$db\,dh\,dx\,dg\,dt\,ds_2\,dt_2\,ds_1\,dt_1$$

with $A_0=(x_{-2},x_{-1},0,0)$. This is verified by an argument similar to the one for [76, Lemma 3.20]. We note that $Q_0=N$ from the definition. Moreover, as in [76, Lemma 3.19], the inner integral $\int_{M_X}\int_{G^1}\int_{Q_0\backslash G^1}\int_{X_+^2}$ converges absolutely, and thus this is equal to

$$\int_{F^2}^{st}\int_{F^2}^{st}\int_{Q_0\backslash G^1}\int_{G^1}\int_{M_X}\int_{X_+^2}\langle\pi(g)f,f'\rangle\Lambda(t)\psi(x_{21}s_1+x_{22}t_1+x_{11}s_2+x_{12}t_2)^{-1}$$
$$\times(\omega_\psi(hg,u_1(s_1,t_1)u_2(s_2,t_2)t)\phi)(A_0+b)\overline{(\omega_\psi(h,1)\phi')(A_0+b)}$$
$$db\,dh\,dx\,dg\,dt\,ds_2\,dt_2\,ds_1\,dt_1.$$

From the proof of Lemma 3.2, this integral is equal to

$$\text{(7.1.3)}\quad\int_{F^2}^{st}\int_{F^2}^{st}\int_{Q_0\backslash G^1}\int_{G^1}\int_{M_X}\int_{X_+^2}\langle\pi(g)f,f'\rangle(\omega_\psi(hg,t)\phi)(A_0+b)$$
$$\times\overline{(\omega_\psi(h,1)\phi')(A_0+b)}\,\Lambda(t)\,\psi\left(\mathrm{tr}\begin{pmatrix}s_2&t_2\\s_1&t_1\end{pmatrix}\left(S_0\begin{pmatrix}\langle x_{-2},b_1\rangle&\langle x_{-2},b_2\rangle\\\langle x_{-1},b_1\rangle&\langle x_{-1},b_2\rangle\end{pmatrix}-X\right)\right)$$
$$db\,dh\,dx\,dg\,dt\,ds_2\,dt_2\,ds_1\,dt_1.$$

Now we claim that we may define the stable integral

$$\int_{F^2}^{st}\int_{F^2}^{st}\int_{X_+^2}\langle\pi(g)f,f'\rangle(\omega_\psi(hg,t)\phi)(A_0+b)\overline{(\omega_\psi(h,1)\phi')(A_0+b)}$$
$$\times\Lambda(t)\psi\left(\mathrm{tr}\begin{pmatrix}s_2 & t_2\\ s_1 & t_1\end{pmatrix}\left(S_0\begin{pmatrix}\langle x_{-2},b_1\rangle & \langle x_{-2},b_2\rangle\\ \langle x_{-1},b_1\rangle & \langle x_{-1},b_2\rangle\end{pmatrix}-X\right)\right)db\,ds_2\,dt_2\,ds_1\,dt_1$$

and we may choose a sufficiently large compact open subgroup F_i of F $(1\le i\le 4)$ so that it depends only on ψ and $\int_{F^2}^{st}\int_{F^2}^{st}\cdots=\int_{F_1}\int_{F_2}\int_{F_3}\int_{F_4}\cdots$. This claim easily follows from the following lemma in the one dimensional case.

Lemma 7.1. *Let f be a locally constant function on F which is in $L^1(F)$. Then there exists a compact open subgroup F_0 of F such that for any compact open subgroups F' and F'' of F containing F_0, we have*

$$\int_{F'}\int_{F}f(x)\psi(xy)\,dx\,dy=\int_{F''}\int_{F}f(x)\psi(xy)\,dx\,dy. \tag{7.1.4}$$

Proof. Suppose that ψ is trivial on $F_0:=\varpi^m\mathcal{O}_F$ and not trivial on $\varpi^{m-1}\mathcal{O}_F$. Put $F'=\varpi^{m'}\mathcal{O}_F$ with $m'\le m$. Then we may write the left-hand side of (7.1.4) as

$$\int_{F'}\int_{F\setminus\mathcal{O}}f(x)\psi(xy)\,dx\,dy+\int_{F'}\int_{\mathcal{O}}f(x)\psi(xy)\,dx\,dy. \tag{7.1.5}$$

The first integral of (7.1.5) converges absolutely. Hence by interchanging the order of integration, it is equal to

$$\int_{F\setminus\mathcal{O}}\int_{F'}f(x)\psi(xy)\,dy\,dx=\int_{F\setminus\mathcal{O}}f(x)\left(\int_{F'}\psi(xy)\,dy\right)dx=0$$

since $y\mapsto\psi(xy)$ is a non-trivial character of F' for each $x\in F\setminus\mathcal{O}$. As for the second integral of (7.1.5), we have

$$\int_{F'}\int_{\mathcal{O}}f(x)\psi(xy)\,dx\,dy$$
$$=\int_{\varpi^m\mathcal{O}}\int_{\mathcal{O}}f(x)\psi(xy)\,dx\,dy+\int_{\varpi^{m'}\mathcal{O}\setminus\varpi^m\mathcal{O}}f(x)\left(\int_{\mathcal{O}}\psi(xy)\,dy\right)dx$$

where the inner integral of the second integral vanishes as above. Thus the left hand side of (7.1.4) is equal to

$$\int_{\varpi^m\mathcal{O}}\int_{\mathcal{O}}f(x)\psi(xy)\,dx\,dy.$$

Similarly the right-hand side of (7.1.4) becomes as above, and our claim follows. □

By Lemma 7.1, we see that (7.1.3) is equal to

$$\int_{N\backslash G^1}\int_{G^1}\int_{M_X}\int_{F^2}^{st}\int_{F^2}^{st}\int_{X_+^2}\langle\pi(g)f,f'\rangle(\omega_\psi(hg,t)\phi)(A_0+b)$$
$$\times\overline{(\omega_\psi(h,1)\phi')(A_0+b)}\,\Lambda(t)\,\psi\left(\mathrm{tr}\begin{pmatrix}s_2 & t_2\\ s_1 & t_1\end{pmatrix}\left(S_0\begin{pmatrix}\langle x_{-2},b_1\rangle & \langle x_{-2},b_2\rangle\\ \langle x_{-1},b_1\rangle & \langle x_{-1},b_2\rangle\end{pmatrix}-X\right)\right)$$
$$db\,dh\,dx\,dg\,dt\,ds_2\,dt_2\,ds_1\,dt_1.$$

Then applying the Fourier inversion, we get

$$\int_{N\backslash G^1}\int_{G^1}\int_{M_X}\langle\pi(g)f,f'\rangle$$
$$\times(\omega_\psi(hg,t)\phi)(A_0+B_0)\overline{(\omega_\psi(h,1)\phi')(A_0+B_0)}\,\Lambda(t)\,db\,dh\,dx\,dg\,dt \tag{7.1.6}$$

where $B_0=(0,0,\frac{x_{21}}{2}x_1+\frac{x_{11}}{2}x_2,-\frac{x_{22}}{2d}x_1-\frac{x_{12}}{2d}x_2)$ and $x_0=A_0+B_0$. By [76, Proposition 3.1], for a sufficiently large compact open subgroup N_0 of N, we have

$$\int_{M_X}\int_N^{st}f(nt)\chi(nt)\,dn\,dt=\int_{N_0}\int_{M_X}f(nt)\chi(nt)\,dn\,dt$$

and thus we may define

$$\int_N^{st}\int_{M_X}f(nt)\chi(nt)\,dn\,dt.$$

Further, we note a simple fact that we have

$$\int_G g(h)\,dh=\int_{N\backslash G}\int_N^{st}g(nh)\,dndh$$

when both sides are defined. Thus (7.1.6) is equal to

$$\int_{N\backslash G^1}\int_{N\backslash G^1}\int_{M_X}\int_N^{st}\langle\pi(g)f,f'\rangle$$
$$\times(\omega_\psi(hg,t)\phi)(A_0+B_0)\overline{(\omega_\psi(h,1)\phi')(A_0+B_0)}\Lambda(t)\,db\,dh\,dx\,dg\,dt.$$

Then the same computation as the one to get (3.1.17) from (3.1.16) may be applied to the above integral, and thus we see that our integral is equal to

$$\int_{N\backslash G^1}\int_{N\backslash G^1}\alpha_{\Lambda,\psi_S}\left(\pi_v\left(h\right)f,\pi_v\left(h'\right)f'\right)$$
$$\times\left(\omega_\psi\left(h,1\right)\phi\right)\left(x_0\right)\overline{\left(\omega_\psi\left(h',1\right)\phi'\right)\left(x_0\right)}\,dh\,dh'.$$

Hence the identity (7.1.2) holds when $B_{\xi,\Lambda,\psi}\not\equiv 0$.

7.2. **Proof of Theorem 1.2 when** $B_{\xi,\Lambda,\psi}\equiv 0$**.** First we note the following proposition concerning the non-vanishing of the L-values.

Proposition 7.3. *Let π be an irreducible cuspidal tempered automorphic representation of $G_D(\mathbb{A})$ with trivial central character. If $G_D\simeq G$ and π is a theta lift from* $\mathrm{GSO}_{3,1}$*, then $L(s,\pi,\mathrm{std}\otimes\chi_E)$ has a simple pole at $s=1$. Otherwise $L(s,\pi,\mathrm{std}\otimes\chi_E)$ is holomorphic and non-zero at $s=1$.*

Proof. Suppose that $G_D \simeq G$, i.e. D is split. Then there exists an irreducible cuspidal globally generic automorphic representation π_0 of $G(\mathbb{A})$ such that π and π_0 are nearly equivalent. Then our claim follows from [120, Lemma 10.2] and [101, Theorem 5.1].

Suppose that D is not split. Let us take a quadratic extension E_0 of F such that π has (E_0, Λ_0)-Bessel period for some character Λ_0 of $\mathbb{A}_{E_0}^\times / E_0^\times$. Then by Theorem 1.1 (1), we see that there exists an irreducible cuspidal tempered automorphic representation π_0 of $G(\mathbb{A})$ such that for a sufficiently large finite set S of places of F containing all archimedean places, $\pi_v, \pi_{0,v}$ are unramified and $\mathrm{BC}_{E_0/F}(\pi_v) \simeq \mathrm{BC}_{E_0/F}(\pi_{0,v})$ for $v \notin S$. This implies that

$$L^S(s, \pi_0, \mathrm{std} \otimes \chi_{E_0}\chi_E) L^S(s, \pi_0, \mathrm{std} \otimes \chi_E) \\ = L^S(s, \pi, \mathrm{std} \otimes \chi_{E_0}\chi_E) L^S(s, \pi, \mathrm{std} \otimes \chi_E).$$

From the case when $G_D \simeq G$, the left-hand side of this identity is not zero at $s = 1$, and thus so is the right-hand side, which possibly has a pole at $s = 1$.

Suppose that $L^S(s, \pi, \mathrm{std} \otimes \chi_{E_0/F}\chi_E)$ has a pole at $s = 1$. We may take a quadratic extension $E_1 \subset D$ of F such that $\chi_{E_1} = \chi_{E_0}\chi_E$. Then by Yamana [120, Lemma 10.2], π is a theta lift from $\mathrm{GSU}_{1,D}$, which is a similitude quaternion unitary group of degree one defined by an element in E_1 as in (2.1.12). In this case, π is not tempered, and thus it contradicts to our assumption on π. Thus, $L^S(s, \pi, \mathrm{std} \otimes \chi_{E_0/F}\chi_E)$ is holomorphic at $s = 1$. Further, by an argument similar to the one for $L^S(s, \pi, \mathrm{std} \otimes \chi_{E_0/F}\chi_E)$, we see that $L^S(s, \pi, \mathrm{std} \otimes \chi_E)$ is holomorphic. Therefore, it is holomorphic and non-zero at $s = 1$. □

Suppose that $B_{\xi,\Lambda,\psi} \equiv 0$ on V_π. We shall show that the right-hand side of (1.6.2) is zero. If $L\left(\frac{1}{2}, \pi \times \mathcal{AI}(\Lambda)\right) = 0$, then there is nothing to prove. Hence, we may suppose that $L\left(\frac{1}{2}, \pi \times \mathcal{AI}(\Lambda)\right) \neq 0$. Then we shall show that for some place v of F, we have $\alpha_{\Lambda_v, \psi_{\xi,v}} \equiv 0$ on π_v.

Assume contrary, i.e. $\alpha_{\Lambda_v, \psi_{\xi,v}} \not\equiv 0$ on π_v for any v. Let us denote by $\pi_+^{B,\mathrm{loc}}$ the unique irreducible constituent of $\pi|_{G_D(\mathbb{A})^+}$ such that $\alpha_{\Lambda_v, \psi_{\xi,v}} \not\equiv 0$ on $\pi_{+,v}^{B,\mathrm{loc}}$ for any v. From our assumption $\alpha_{\Lambda_v, \psi_{\xi,v}} \not\equiv 0$ on π_v and Corollary 7.2, we see that $\alpha_{\Lambda_v^{-1}, \psi_{X,v}} \not\equiv 0$ on the theta lift $\theta_{\psi_v, D}(\pi_v)$ of π_v to $\mathrm{GSU}_{3,D}(F_v)$ and $Z_v(\phi_v, f_v, \pi) \neq 0$ for some $f_v \in \pi_v$ and $\phi_v \in \mathcal{S}(Z_{D,+}(F_v))$. Since π' is nearly equivalent to π, we have $L(1, \pi, \mathrm{std} \otimes \chi_E) \neq 0$. Therefore, the theta lift $\theta_{\psi, D}(\pi_+^{B,\mathrm{loc}})$ of $\pi_+^{B,\mathrm{loc}}$ to $\mathrm{GSU}_{3,D}(\mathbb{A})$ is non-zero by Yamana [120, Theorem 10.3], which states that the non-vanishing of local theta lifts at all places together with the non-vanishing of the L-value implies the non-vanishing of the global theta lift. We note that actually in [120, Theorem 10.3], there is an assumption that D is not split at real places, which was necessary to ensure that the non-vanishing of the local theta lift implies $Z_v(\phi_v, f_v, \pi) \neq 0$ for some $f_v \in \pi_v$ and $\phi_v \in \mathcal{S}(Z_{D,+}(F_v))$. Since the non-vanishing of $Z_v(\phi_v, f_v, \pi)$ for some f_v and ϕ_v is shown in our case by the argument above, we may apply [120, Theorem 10.3] regardless of the assumption.

Recall that from the proof of Theorem 1.1 (1), $\theta_{\psi,D}(\pi_+^{B,\mathrm{loc}})$ is tempered. Let us regard $\theta_{\psi,D}(\pi_+^{B,\mathrm{loc}})$ as automorphic representations of $\mathrm{GU}_{4,\varepsilon}$. By the uniqueness of the Bessel model for $\mathrm{GU}_{4,\varepsilon}$ proved in [29, Proposition A.1], there uniquely exists an irreducible constituent τ of $\theta_{\psi,D}(\pi_+^{B,\mathrm{loc}})|_{\mathrm{U}(4)}$ such that τ has the local $(X, \Lambda_v^{-1}, \psi_v)$-Bessel model at any place v.

On the other hand, we note $L\left(1/2, \tau \times \Lambda^{-1}\right) \neq 0$ since $L\left(\frac{1}{2}, \pi \times \mathcal{AI}\,(\Lambda)\right) \neq 0$. Then by [29, Theorem 1.2], there exists an irreducible cuspidal automorphic representation τ' of $\mathrm{U}(V_0)$ with four dimensional hermitian space V_0 over E such that τ' has (X, Λ_v, ψ_v)-Bessel period. Then we know that τ and τ' have the same L-parameter, in particular, $\tau_v \simeq \tau_v'$ when v is split. At a non-split place v, by the uniqueness of an element of the tempered L-packet which has the same Bessel period due to Beuzart-Plessis [6, 7], we see that $\mathrm{U}(V_0) \simeq \mathrm{U}(J_D)$ and $\tau \simeq \tau'$. Moreover, by Mok [82], we have $\tau = \tau'$. Therefore, $\tau = \tau'$ has (X, Λ^{-1}, ψ)-Bessel period, and this implies that $\theta_{\psi,D}(\pi_+^{B,\mathrm{loc}})$ also has (X, Λ^{-1}, ψ)-Bessel period. Then Proposition 3.1 and 3.2 show that π has (E, Λ)-Bessel period, and this is a contradiction. Thus, (1.6.2) holds when $B_{\xi,\Lambda,\psi} \equiv 0$ on V_π.

8. Generalized Böcherer conjecture

In this section we prove the generalized Böcherer conjecture. In fact, we shall prove Theorem 8.1 below, which is more general than Theorem 1.4 stated in the introduction.

8.1. **Temperedness condition.**

In order to apply Theorem 1.2 to holomorphic Siegel cusp forms of degree two, we need to verify the temperedness for corresponding automorphic representations.

Proposition 8.1. *Suppose that F is totally real. Let τ be an irreducible cuspidal automorphic representation of $G_D(\mathbb{A})$ with a trivial central character such that τ_v is a discrete series representation for every real place v of F. Suppose moreover that τ is not CAP. Then τ is tempered.*

Remark 8.1. *When D is split, i.e. $G_D \simeq G$, Weissauer* [116] *proved that τ_v is tempered at a place v when τ_v is unramified. Moreover, when τ_v is a holomorphic discrete series representation at each archimedean place v, Jorza* [64] *showed the temperedness at finite places not dividing* 2.

Proof. First suppose that $G_D \simeq G$. Let Π denote the functorial lift of τ to $\mathrm{GL}_4(\mathbb{A})$ established by Arthur [3] (see also Cai-Friedberg-Kaplan [14]).

When Π is not cuspidal, since τ is not CAP, Π is of the form $\Pi = \Pi_1 \boxplus \Pi_2$ with irreducible cuspidal automorphic representations Π_i of $\mathrm{GL}_2(\mathbb{A})$. Since τ_v is a discrete series representation for any real place v, $\Pi_{i,v}$ is also a discrete series representation. Then Π_i is tempered by [11] and thus the Langlands parameter of Π_v is tempered at all places v of F. Hence τ is tempered.

Suppose that Π is cuspidal. Then by Raghuram-Sarnobat [93, Theorem 5.6], Π_v is tempered and cohomological at any real place v. Let us take an imaginary quadratic extension E of F such that the base change lift $\mathrm{BC}(\Pi)$ of Π to $\mathrm{GL}_4(\mathbb{A}_E)$

is cuspidal. Note that $\mathrm{BC}(\Pi)$ is cohomological and that $\mathrm{BC}(\Pi)^\vee \simeq \mathrm{BC}(\Pi^\vee) \simeq \mathrm{BC}(\Pi) \simeq \mathrm{BC}(\Pi)^\sigma$. Then Caraiani [15, Theorem 1.2] shows that $\mathrm{BC}(\Pi)$ is tempered at all finite places. This implies that Π_v is also tempered for any finite place v. Thus τ is tempered.

Now let us consider the case when D is not split. Since τ is not CAP, by Proposition 4.1, there exists an irreducible cuspidal automorphic representation τ' of $G(\mathbb{A})$ and a quadratic extension E_0 of F such that τ' is G^{+,E_0}-locally equivalent to τ. Moreover τ is tempered if and only if τ' is tempered. By [75, 80, 85, 86], τ'_v is a discrete series representation at any real place v. Then the temperedness of τ' follows from the split case. Hence τ is also tempered. □

As an application of Proposition 8.1, the following corollary holds.

Corollary 8.1. *Suppose that F is totally real. Let τ be an irreducible cuspidal globally generic automorphic representation of $G(\mathbb{A})$ such that τ_v is a discrete series representation at any real place v. Then τ is tempered and hence the explicit formula (6.2.3) for the Whittaker periods holds for any non-zero decomposable vector in V_τ.*

Proof. Recall that the functorial lift Π of τ to $\mathrm{GL}_4(\mathbb{A})$ is cuspidal or an isobaric sum of irreducible cuspidal automorphic representations of GL_2 by [19]. In particular τ is not CAP by Arthur [3]. Then by Proposition 8.1, τ is tempered and our claim follows from Theorem 6.3. □

8.2. **Vector valued Siegel cusp forms and Bessel periods.** Let $\mathfrak{H}_2$ be the Siegel upper half space of degree two, i.e. the set of two by two symmetric complex matrices whose imaginary parts are positive definite. Then the group $G(\mathbb{R})^+ = \{g \in G(\mathbb{R}) : \nu(g) > 0\}$ acts on $\mathfrak{H}_2$ by

$$g\langle Z\rangle = (AZ+B)(CZ+D)^{-1} \quad \text{for } g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G(\mathbb{R})^+ \text{ and } Z \in \mathfrak{H}_2$$

and the factor of automorphy $J(g, Z)$ is defined by

$$J(g, Z) = CZ + D.$$

For an integer $N \geq 1$, let

$$\Gamma_0(N) = \left\{ \gamma \in G^1(\mathbb{Z}) : \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, C \equiv 0 \pmod{N\mathbb{Z}} \right\}.$$

8.2.1. *Vector valued Siegel cusp forms.* Let (ϱ, V_ϱ) be an algebraic representation of $\mathrm{GL}_2(\mathbb{C})$. Then a holomorphic mapping $\Phi : \mathfrak{H}_2 \to V_\varrho$ is a *Siegel cusp form of weight ϱ with respect to* $\Gamma_0(N)$ when Φ vanishes at the cusps and satisfies

$$\Phi(\gamma\langle Z\rangle) = \varrho(J(\gamma, Z))\,\Phi(Z) \quad \text{for } \gamma \in \Gamma_0(N) \text{ and } Z \in \mathfrak{H}_2. \tag{8.2.1}$$

We denote by $S_\varrho(\Gamma_0(N))$ the complex vector space of Siegel cusp forms of weight ϱ with respect to $\Gamma_0(N)$. Then $\Phi \in S_\varrho(\Gamma_0(N))$ has a Fourier expansion

$$\Phi(Z) = \sum_{T>0} a(T, \Phi) \exp\left[2\pi\sqrt{-1}\,\mathrm{tr}(TZ)\right] \quad \text{where } Z \in \mathfrak{H}_2 \text{ and } a(T, \Phi) \in V_\varrho.$$

Here T runs over positive definite two by two symmetric matrices which are semi-integral, i.e. T is of the form $T = \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}$, $a, b, c \in \mathbb{Z}$. We note that (8.2.1) implies

(8.2.2) $$a\left(\varepsilon T\, {}^t\varepsilon, \Phi\right) = \varrho(\varepsilon)\, a(T, \Phi) \quad \text{for } \varepsilon \in \mathrm{GL}_2(\mathbb{Z}).$$

From now on till the end of this paper, we assume ϱ to be *irreducible*. It is well known that the irreducible algebraic representations of $\mathrm{GL}_2(\mathbb{C})$ are parametrized by

(8.2.3) $$\mathbb{L} = \left\{(n_1, n_2) \in \mathbb{Z}^2 : n_1 \geq n_2\right\}.$$

Namely the parametrization is given by assigning

$$\varrho_\kappa := \mathrm{Sym}^{n_1 - n_2} \otimes \det^{n_2} \quad \text{to } \kappa = (n_1, n_2) \in \mathbb{L}.$$

Suppose that $\varrho = \varrho_\kappa$ with $\kappa = (n + k, k) \in \mathbb{L}$. Then we realize ϱ concretely by taking its space of representation V_ϱ to be $\mathbb{C}[X, Y]_n$, the space of degree n homogeneous polynomials of X and Y, where the action of $\mathrm{GL}_2(\mathbb{C})$ is given by

$$\varrho(g)\, P(X, Y) = (\det g)^k \cdot P((X, Y)\, g) \quad \text{for } g \in \mathrm{GL}_2(\mathbb{C}) \text{ and } P \in \mathbb{C}[X, Y]_n.$$

Let us define a bilinear form

$$\mathbb{C}[X, Y]_n \times \mathbb{C}[X, Y]_n \ni (P, Q) \mapsto (P, Q)_n \in \mathbb{C}$$

by

(8.2.4) $$\left(X^i Y^{n-i}, X^j Y^{n-j}\right)_n = \begin{cases} (-1)^i \dbinom{n}{i} & \text{if } i + j = n; \\ 0 & \text{otherwise.} \end{cases}$$

Then we have

(8.2.5) $$(\varrho(g) P, \varrho(g) Q)_n = (\det g)^{n+2k}\, (P, Q)_n \quad \text{for } g \in \mathrm{GL}_2(\mathbb{C}).$$

We define a positive definite hermitian inner product $\langle\ ,\ \rangle_\varrho$ on V_ϱ by

(8.2.6) $$\langle P, Q \rangle_\varrho := \left(P, \varrho(w_0)\, \overline{Q}\right)_n \quad \text{where } w_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Here $\overline{Q}$ denotes the polynomial obtained from Q by taking the complex conjugates of its coefficients. Then (8.2.5) implies that we have

(8.2.7) $$\langle \varrho(g)\, v, w \rangle_\varrho = \langle v, \varrho\left({}^t\bar{g}\right) w \rangle_\varrho \quad \text{for } g \in \mathrm{GL}_2(\mathbb{C}) \text{ and } v, w \in V_\varrho.$$

In particular the hermitian inner product $\langle\ ,\ \rangle_\varrho$ is $\mathrm{U}_2(\mathbb{R})$-invariant. Then for $\Phi, \Phi' \in S_\varrho(\Gamma_0(N))$, we define the Petersson inner product $\langle \Phi, \Phi' \rangle_\varrho$ by

(8.2.8)
$$\langle \Phi, \Phi' \rangle_\varrho = \frac{1}{\left[\mathrm{Sp}_2(\mathbb{Z}) : \Gamma_0(N)\right]} \int_{\Gamma_0(N) \backslash \mathfrak{H}_2} \langle \Phi(Z), \Phi'(Z) \rangle_\varrho\, (\det Y)^{k-3}\, dX\, dY$$

where $X = \mathrm{Re}(Z)$ and $Y = \mathrm{Im}(Z)$. The space $S_\varrho(\Gamma_0(N))$ has a natural orthogonal decomposition with respect to the Petersson inner product

$$S_\varrho(\Gamma_0(N)) = S_\varrho(\Gamma_0(N))^{\mathrm{old}} \oplus S_\varrho(\Gamma_0(N))^{\mathrm{new}}$$

into the oldspace and the newspace in the sense of Schmidt [100, 3.3]. We note that when n is odd, we have $S_\varrho(\Gamma_0(N)) = \{0\}$ for ϱ with $\kappa = (n+k, k)$ by (8.2.1) since $-1_4 \in \Gamma_0(N)$.

8.2.2. *Adelization.* Given $\Phi \in S_\varrho(\Gamma_0(N))$, its adelization $\varphi_\Phi : G(\mathbb{A}) \to V_\varrho$ is defined as follows (cf. [98, 3.1], [100, 3.2]). For each prime number p, let us define a compact open subgroup $P_{1,p}(N)$ of $G(\mathbb{Q}_p)$ by

$$P_{1,p}(N) := \left\{ g \in G(\mathbb{Z}_p) : g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \, C \equiv 0 \pmod{N\mathbb{Z}_p} \right\}.$$

Then we define a mapping $\varphi_\Phi : G(\mathbb{A}) \to V_\varrho$ by

$$\varphi_\Phi(g) = \nu(g_\infty)^{k+r} \, \varrho\left(J\left(g_\infty, \sqrt{-1}\,1_2\right)\right)^{-1} \Phi\left(g_\infty \langle \sqrt{-1}\,1_2 \rangle\right) \tag{8.2.9}$$

when

$$g = \gamma\, g_\infty\, k_0 \quad \text{with } \gamma \in G(\mathbb{Q}),\, g_\infty \in G(\mathbb{R})^+ \text{ and } k_0 \in \prod_{p<\infty} P_{1,p}(N).$$

Let L be any non-zero linear form on V_ϱ. Then $L(\varphi_\Phi) : G(\mathbb{A}) \to \mathbb{C}$ defined by $L(\varphi_\Phi)(g) = L(\varphi_\Phi(g))$ is a scalar valued automorphic form on $G(\mathbb{A})$. Let $V(\Phi)$ denote the the space generated by right $G(\mathbb{A})$-translates of $L(\varphi_\Phi)$. Then $V(\Phi)$ does not depend on the choice of L and we denote by $\pi(\Phi)$ the right regular representation of $G(\mathbb{A})$ on $V(\Phi)$. Note that the central character of $\pi(\Phi)$ is trivial.

We recall that for scalar valued automorphic forms ϕ, ϕ' on $G(\mathbb{A})$ with a trivial central character, their Petersson inner product $\langle \phi, \phi' \rangle$ is defined by

$$\langle \phi, \phi' \rangle = \int_{Z_G(\mathbb{A})G(\mathbb{Q})\backslash G(\mathbb{A})} \phi(g)\, \overline{\phi'(g)}\, dg$$

where Z_G denotes the center of G and dg is the Tamagawa measure.

Lemma 8.1. *Let L be a non-zero linear form on V_ϱ. Take $v' \in V_\varrho$ such that $L(v) = \langle v, v' \rangle_\varrho$ for any $v \in V_\varrho$.*

Then we have

$$\langle L(\varphi_\Phi), L(\varphi_\Phi) \rangle = C(v') \cdot \langle \Phi, \Phi \rangle_\varrho \quad \textit{for any } \Phi \in S_\varrho(\Gamma_0(N))$$

where

$$C(v') = \frac{\mathrm{Vol}(Z_G(\mathbb{A})\, G(\mathbb{Q}) \backslash G(\mathbb{A}))}{\mathrm{Vol}\left(\mathrm{Sp}_2(\mathbb{Z}) \backslash \mathfrak{H}_2\right)} \cdot \frac{\langle v', v' \rangle_\varrho}{\dim V_\varrho}. \tag{8.2.10}$$

Proof. Let $K_\infty = \mathrm{U}_2(\mathbb{R})$. We identify K_∞ as a subgroup of $\mathrm{Sp}_2(\mathbb{R})$ via

$$K_\infty \ni A + \sqrt{-1}\, B \mapsto \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \in \mathrm{Sp}_2(\mathbb{R}).$$

Let dk be the Haar measure on K_∞ such that $\mathrm{Vol}(K_\infty, dk) = 1$. Then by the Schur orthogonality relations, we have

$$\int_{K_\infty} L\left(\varrho(k)^{-1} v\right) \cdot \overline{L\left(\varrho(k)^{-1} w\right)}\, dk = \frac{\langle v, w \rangle_\varrho \cdot \langle v', v' \rangle_\varrho}{\dim V_\varrho}.$$

On the other hand, it is easily seen that for $\varPhi \in S_{\varrho}\left(\Gamma_0\left(N\right)\right)$, we have

$$\frac{\langle \varPhi, \varPhi\rangle_{\varrho}}{\mathrm{Vol}\left(\mathrm{Sp}_2\left(\mathbb{Z}\right)\backslash\mathfrak{H}_2\right)} = \frac{\langle \varphi_{\varPhi}, \varphi_{\varPhi}\rangle_{\varrho}}{\mathrm{Vol}\left(Z_G\left(\mathbb{A}\right)G\left(\mathbb{Q}\right)\backslash G\left(\mathbb{A}\right)\right)}$$

where

$$\langle \varphi_{\varPhi}, \varphi_{\varPhi}\rangle_{\varrho} := \int_{Z_G(\mathbb{A})G(\mathbb{Q})\backslash G(\mathbb{A})} \langle \varphi_{\varPhi}\left(g\right), \varphi_{\varPhi}\left(g\right)\rangle_{\varrho} dg.$$

Hence

$$\begin{aligned}\langle \varPhi, \varPhi\rangle_{\varrho} =& C\left(v'\right)^{-1}\int_{Z_G(\mathbb{A})G(\mathbb{Q})\backslash G(\mathbb{A})}\int_{K_\infty}\left|L\left(\varrho\left(k\right)^{-1}\varphi_{\varPhi}\left(g\right)\right)\right|^2\, dk\, dg\\ =& C\left(v'\right)^{-1}\int_{K_\infty}\int_{Z_G(\mathbb{A})G(\mathbb{Q})\backslash G(\mathbb{A})}\left|L\left(\varphi_{\varPhi}\left(gk\right)\right)\right|^2\, dg\, dk\\ =& C\left(v'\right)^{-1}\cdot\langle L\left(\varphi_{\varPhi}\right), L\left(\varphi_{\varPhi}\right)\rangle_{\varrho}.\end{aligned}$$

□

8.2.3. *Bessel periods of vector valued Siegel cusp forms.* Let E be an imaginary quadratic field of $\mathbb{Q}$ and $-D_E$ its discriminant. We put

(8.2.11) $$S_E := \begin{cases}\begin{pmatrix}1 & 0\\ 0 & D_E/4\end{pmatrix} & \text{when } D_E \equiv 0 \pmod 4;\\ \begin{pmatrix}1 & 1/2\\ 1/2 & \left(1+D_E\right)/4\end{pmatrix} & \text{when } D_E \equiv -1 \pmod 4.\end{cases}$$

Given $S = S_E$ as above, we define T_S, N and ψ_S as in 2.3.1. Then $T_S\left(\mathbb{Q}\right) \simeq E^\times$.

Let Λ be a character of $T_S\left(\mathbb{A}\right)$ which is trivial on $\mathbb{A}^\times T_S\left(\mathbb{Q}\right)$. Let ψ be the unique character of $\mathbb{A}/\mathbb{Q}$ such that $\psi_\infty\left(x\right) = e^{-2\pi\sqrt{-1}\,x}$ and the conductor of ψ_ℓ is $\mathbb{Z}_\ell$ for any prime number ℓ. Then for a scalar valued automorphic form ϕ on $G\left(\mathbb{A}\right)$ with a trivial central character, we define its (S, Λ, ψ)-Bessel period $B_{S,\Lambda,\psi}\left(\phi\right)$ by (2.3.1) with the Haar measures du on $N\left(\mathbb{A}\right)$ and $dt = dt_\infty\, dt_f$ on $T_S\left(\mathbb{A}\right) = T_S\left(\mathbb{R}\right)\times T_S\left(\mathbb{A}_f\right)$ are taken so that $\mathrm{Vol}\left(N\left(\mathbb{Q}\right)\backslash N\left(\mathbb{A}\right), du\right) = 1$ and

$$\mathrm{Vol}\left(\mathbb{R}^\times\backslash T_S\left(\mathbb{R}\right), dt_\infty\right) = \mathrm{Vol}\left(T_S\left(\mathbb{Z}_p\right), dt_f\right) = 1.$$

Then we note that

$$\mathrm{Vol}(\mathbb{A}^\times T_S\left(\mathbb{Q}\right)\backslash T_S\left(\mathbb{A}\right), dt) = \frac{2h_E}{w(E)} = D_E^{1/2}\cdot L(1, \chi_E).$$

For a V_ϱ-valued automorphic form φ with a trivial central character, it is clear that for a linear form $L : V_\varrho \to \mathbb{C}$ we have

(8.2.12)

$$B_{S,\Lambda,\psi}\left(L\left(\varphi\right)\right) = L\left[\int_{\mathbb{A}^\times T_S(\mathbb{Q})\backslash T_S(\mathbb{A})}\int_{N(\mathbb{Q})\backslash N(\mathbb{A})}\Lambda\left(t\right)^{-1}\psi_S\left(u\right)^{-1}\varphi\left(tu\right)\, dt\, du\right].$$

Recall that we may identify the ideal class group Cl_E of E with the quotient group

$$T_S\left(\mathbb{A}\right)/T_S\left(\mathbb{Q}\right)T_S\left(\mathbb{R}\right)T_S\left(\hat{\mathbb{Z}}\right).$$

Let $\{t_c : c \in \mathrm{Cl}_E\}$ be a set of representatives of Cl_E such that $t_c \in \prod_{p<\infty} T\left(\mathbb{Q}_p\right)$. We write t_c as $t_c = \gamma_c\, m_c\, \kappa_c$ with $\gamma_c \in \mathrm{GL}_2\left(\mathbb{Q}\right)$, $m_c \in \{g \in \mathrm{GL}_2\left(\mathbb{R}\right) : \det g > 0\}$, $\kappa_c \in \prod_{p<\infty} \mathrm{GL}_2\left(\mathbb{Z}_p\right)$. Let $S_c = \left(\det \gamma_c\right)^{-1} \cdot {}^t\gamma_c S \gamma_c$. Then the set $\{S_c : c \in \mathrm{Cl}_E\}$ is a set of representatives for the $\mathrm{SL}_2\left(\mathbb{Z}\right)$-equivalence classes of primitive semi-integral positive definite two by two symmetric matrices of discriminant D_E.

Thus when $\varphi = \varphi_\Phi$ for $\Phi \in S_\varrho\left(\Gamma_0\left(N\right)\right)$ and Λ is a character of Cl_E, we may write (8.2.12) as

$$B_{S,\Lambda,\psi}\left(L\left(\varphi_\Phi\right)\right) = 2 \cdot e^{-2\pi \mathrm{tr}(S)} \cdot L\left(B_\Lambda\left(\Phi; E\right)\right) \tag{8.2.13}$$

where

$$B_\Lambda\left(\Phi; E\right) := w\left(E\right)^{-1} \cdot \pi_\varrho \left(\sum_{c \in \mathrm{Cl}_E} \Lambda(c)^{-1} \cdot a\left(S_c, \Phi\right)\right) \tag{8.2.14}$$

is the vector valued (S, Λ, ψ)-Bessel period where

$$\pi_\varrho = \int_{T_S^1(\mathbb{R})} \varrho\left(t\right)\, dt \quad \text{with } T_S^1 = \mathrm{SL}_2 \cap T_S,\ \mathrm{Vol}\left(T_S^1\left(\mathbb{R}\right), dt\right) = 1 \tag{8.2.15}$$

(e.g. Dickson et al. [21, Proposition 3.5] and Sugano [104, (1-26)]).

Remark 8.2 (An erratum to [27])**.** *The definition of $B\left(\Phi; E\right)$ in the vector valued case in* [27, Theorem 5] *should be replaced by* (8.2.14)*. The statement and the proof of* [27, Theorem 5] *remain valid.*

Suppose that $\varrho = \varrho_\kappa$ where $\kappa = (2r + k, k) \in \mathbb{L}$. We define $Q_{S,\varrho} \in \mathbb{C}\left[X, Y\right]_{2r}$ by

$$Q_{S,\varrho}\left(X, Y\right) := \left(\left(X, Y\right) S \begin{pmatrix} X \\ Y \end{pmatrix} \right)^r \cdot \left(\det S\right)^{-\frac{2r+k}{2}} \quad \text{where } S = S_E \text{ in (8.2.11).} \tag{8.2.16}$$

Then for $\Phi \in S_\varrho\left(\Gamma_0\left(N\right)\right)$, the scalar valued (S, Λ, ψ)-Bessel period $\mathcal{B}_\Lambda\left(\Phi; E\right)$ of Φ is defined by

$$\mathcal{B}_\Lambda\left(\Phi; E\right) := \left(B_\Lambda\left(\Phi; E\right), Q_{S,\varrho}\right)_{2r}. \tag{8.2.17}$$

8.3. **Explicit L-value formula in the vector valued case.** Let us state our explicit formula for holomorphic Siegel modular forms. In what follows, whenever we refer to a type of an admissible representation of G over a non-archimedean local field, we use the standard classification due to Roberts and Schmidt [93].

Let N be a squarefree integer. We say that a non-zero $\Phi \in S_\varrho\left(\Gamma_0\left(N\right)\right)$ is a *newform* if

1. (1) $\Phi \in S_\varrho\left(\Gamma_0\left(N\right)\right)^{\mathrm{new}}$.
2. (2) Φ is an eigenform for the local Hecke algebras for all primes p not dividing N and an eigenfunction of the local $U\left(p\right)$ operator (see Saha and Schmidt [99, 2.3]) for all primes dividing N.
3. (3) The representation $\pi\left(\Phi\right)$ of $G\left(\mathbb{A}\right)$ is irreducible.

Then the following theorem is derived from Theorem 1.2 exactly as Dickson, Pitale, Saha and Schmidt [21, Theorem 1.13] except that we need to compute local Bessel periods at the real place adapting to the vector valued case. We perform the computation of them in Appendix B.

Theorem 8.1. *Let $N \geq 1$ be an odd squarefree integer. Let $\varrho = \varrho_\kappa$ where $\kappa = (2r+k, k)$ with $k \geq 2$. Let Φ be a non-CAP newform in $S_\varrho(\Gamma_0(N))$. Suppose that $\left(\frac{D_E}{p}\right) = -1$ for all primes p dividing N. When $k = 2$, suppose moreover that $\pi(\Phi)$ is tempered.*

Then we have

$$\frac{|\mathcal{B}_\Lambda(\Phi; E)|^2}{\langle \Phi, \Phi \rangle_\varrho} = \frac{2^{4k+6r-c}}{D_E} \cdot \frac{L(1/2, \pi(\Phi) \times \mathcal{AI}(\Lambda))}{L(1, \pi(\Phi), \mathrm{Ad})} \cdot \prod_{p \mid N} J_p \tag{8.3.1}$$

where $c = 5$ if Φ is a Yoshida lift in the sense of Saha [98, Section 4] *and $c = 4$ otherwise. The quantities J_p for p dividing N are given by*

$$J_p = \left(1 + p^{-2}\right)\left(1 + p^{-1}\right) \times \begin{cases} 1 & \text{if } \pi(\Phi)_p \text{ is of type IIIa;} \\ 2 & \text{if } \pi(\Phi)_p \text{ is of type VIb;} \\ 0 & \text{otherwise.} \end{cases}$$

Remark 8.3. *When $k \geq 3$, $\pi(\Phi)$ is tempered by Proposition 8.1.*

Remark 8.4. *Since $\mathcal{B}(\Phi; E) = 2^k D_E^{-\frac{k}{2}} \cdot B(\Phi; E)$ when $r = 0$,* (1.8.2) *follows from* (8.3.1) *by putting $N = 1$ and $r = 0$.*

Remark 8.5. *In the statement of the theorem, we used the notion of Yoshida lifts in the sense of Saha* [98]. *Though it is necessary to extend the arguments concerning Yoshida lifts in* [98, Section 4] *in the scalar valued case to the vector valued case to be rigorous, we omit it here since it is straightforward. We also mention that the arguments in* [98, 4.4] *now work unconditionally since the classification theory in Arthur* [3] *is complete for $\mathbb{G} = \mathrm{PGSp}_2 \simeq \mathrm{SO}(3,2)$.*

Remark 8.6. *Recall that the L-functions in* (8.3.1) *are complete L-functions. We may rewrite the explicit formula in terms of the finite parts of the L-functions by observing that the relevant archimedean L-factors are given by*

$$L(1/2, \pi(\Phi)_\infty \times \mathcal{AI}(\Lambda)_\infty) = 2^4 (2\pi)^{-2(k+r)} \Gamma(k+r-1)^2 \Gamma(r+1)^2$$

and

$$L(1, \pi(\Phi)_\infty, \mathrm{Ad}) = 2^6 (2\pi)^{-(4k+6r+1)} \times \Gamma(k+2r)\Gamma(k-1)\Gamma(2r+2)\Gamma(2k+2r-2)$$

respectively.

Remark 8.7. *Let us consider the case when D is a quaternion algebra over $\mathbb{Q}$ which is split at the real place, i.e. $D(\mathbb{R}) \simeq \mathrm{Mat}_{2\times 2}(\mathbb{R})$. Assuming that the endoscopic classification holds for $\mathbb{G}_D = G_D/Z_D$, we may apply Theorem 1.2 to holomorphic modular forms on $\mathbb{G}_D(\mathbb{A})$. In this case, Hsieh-Yamana* [55] *compute local Bessel* *periods and show an explicit formula for Bessel periods such as* (8.3.1) *for scalar valued holomorphic modular forms, including the case when* $G_D = G$ *and* N *is an even squarefree integer. Meanwhile we shall maintain* N *to be odd in Theorem 8.1, since our computation of the local Bessel period at the real place in the vector valued case in Appendix B is performed under the assumption that* N *is odd.*

As we noted in Remark 1.5, after the submission of this paper, Ishimoto [59] *showed the endoscopic classification of* $\mathrm{SO}(4,1)$ *for generic Arthur parameters. Therefore, we may apply our theorem to the case of* $\mathbb{G}_D \simeq \mathrm{SO}(4,1)$.

Remark 8.8. *A global explicit formula such as* (8.3.1) *is obtained in a certain non-squarefree level case by Pitale, Saha and Schmidt* [90, Theorem 4.8].

Appendix A. Explicit formula for the Whittaker periods on $G = \mathrm{GSp}_2$

Here we shall prove Theorem 6.3.

Let (π, V_π) be an irreducible cuspidal globally generic automorphic representation of $G(\mathbb{A})$. Then Soudry [103] has shown that the theta lift of π to $\mathrm{GSO}_{3,3}$ is non-zero and globally generic. We may divide into two cases according to whether the theta lift of π to $\mathrm{GSO}_{3,3}$ is cuspidal or not.

Suppose that the theta lift of π to $\mathrm{GSO}_{3,3}$ is cuspidal. Since $\mathrm{PGSO}_{3,3} \simeq \mathrm{PGL}_4$ and the explicit formula for the Whittaker periods on GL_n is known by Lapid and Mao [71], the arguments in 6.2 and 6.2.3, which are used to obtain (6.1.6) in Theorem 6.1 from (6.2.3), work mutatis mutandis to obtain (6.2.3) from the Lapid–Mao formula in the case of GL_4.

Suppose that the theta lift of π to $\mathrm{GSO}_{3,3}$ is not cuspidal. Then the theta lift of π to $\mathrm{GSO}_{2,2}$ is non-zero and cuspidal.

Thus here we give a proof of Theorem 6.3 only in the case when π is a theta lift from $\mathrm{GSO}_{2,2}$. Recall that $\mathrm{PGSO}_{2,2} \simeq \mathrm{PGL}_2 \times \mathrm{PGL}_2$. Our argument is similar to the one for [76, Theorem 4.3]. Indeed we shall prove (6.2.3) by pushing forward the Lapid–Mao formula for $\mathrm{GSO}_{2,2}$ to G.

A.1. **Global pull-back computation.** Let $(X, \langle\ ,\ \rangle)$ be the 4 dimensional symplectic space as in 3.1.2 and let $\{x_1, x_2, x_{-1}, x_{-2}\}$ be the standard basis of X given by (3.1.5).

Let $Y = F^4$ be an orthogonal space with a non-degenerate symmetric bilinear form defined by

$$(v_1, v_2) = {}^t v_1 J_4 v_2 \quad \text{for } v_1, v_2 \in Y$$

where J_4 is given by (2.1.6). We take a standard basis $\{y_{-2}, y_{-1}, y_1, y_2\}$ of $Y = F^4$ given by

$$y_{-2} = {}^t(1,0,0,0), \quad y_{-1} = {}^t(0,1,0,0), \quad y_1 = {}^t(0,0,1,0), \quad y_2 = {}^t(0,0,0,1).$$

We note that $\left(y_i, y_{-j}\right) = \delta_{ij}$ for $1 \le i, j \le 2$.

Put $Z = X \otimes Y$. Then Z is naturally a symplectic space over F. We take a polarization $Z = Z_+ \oplus Z_-$ where

$$Z_\pm = X_\pm \otimes Y$$

and $X_\pm = F \cdot x_{\pm 1} + F \cdot x_{\pm 2}$. Here all the double signs correspond. When $z_+ = x_1 \otimes a_1 + x_2 \otimes a_2 \in Z_+(\mathbb{A})$ where $a_1, a_2 \in Y$, we write $z_+ = (a_1, a_2)$ and $\phi(z_+) = \phi(a_1, a_2)$ for $\phi \in \mathcal{S}(Z_+(\mathbb{A}))$.

Let $N_{2,2}$ denote the group of upper triangular unipotent matrices of $\mathrm{GO}_{2,2}$, i.e.

$$N_{2,2}(F) = \left\{ \begin{pmatrix} 1 & x & y & -xy \\ 0 & 1 & 0 & -y \\ 0 & 0 & 1 & -x \\ 0 & 0 & 0 & 1 \end{pmatrix} \,|\, x, y \in F \right\}.$$

We define a non-degenerate character $\psi_{2,2}$ of $N_{2,2}(\mathbb{A})$ by

$$\psi_{2,2} \begin{pmatrix} 1 & x & y & -xy \\ 0 & 1 & 0 & -y \\ 0 & 0 & 1 & -x \\ 0 & 0 & 0 & 1 \end{pmatrix} = \psi(x+y).$$

Then for a cusp form f on $\mathrm{GSO}_{2,2}(\mathbb{A})$, we define its Whittaker period $W_{2,2}(f)$ by

$$W_{2,2}(f) = \int_{N_{2,2}(F)\backslash N_{2,2}(\mathbb{A})} f(n)\, \psi_{2,2}(n)^{-1}\, dn.$$

The following identity is stated in [42, p.113] but without a proof. Though it is shown by an argument similar to the one for [42, Proposition 2.6], here we give a proof for the convenience of the reader.

Proposition A.1. *Let φ be a cusp form on $\mathrm{GO}_{2,2}(\mathbb{A})$. For $\phi \in \mathcal{S}(Z(\mathbb{A})_+)$, let $\Theta_\psi(\varphi, \phi)$ (resp. $\theta_\psi(\varphi, \phi)$) be the theta lift of σ (resp. the restriction of φ to $\mathrm{GSO}_{2,2}(\mathbb{A})$) to $G(\mathbb{A})$.*

Then we have

(A.1.1)

$$W_{\psi_{U_G}}(\Theta_\psi(\varphi, \phi)) = \int_{N_0(\mathbb{A})\backslash \mathrm{O}_{2,2}(\mathbb{A})} \phi(g^{-1}(y_{-2}, y_{-1} + y_1)) W_{\psi_{2,2}}(\sigma(g)\varphi)\, dg$$

where N_0 denotes the unipotent subgroup

$$N_0 = \left\{ \begin{pmatrix} 1 & x & -x & x^2 \\ 0 & 1 & 0 & -x \\ 0 & 0 & 1 & x \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\},$$

which is the stabilizer of y_{-2} and $y_{-1} + y_1$.

Similarly we have

(A.1.2)

$$W_{\psi_{U_G}}(\theta_\psi(\varphi, \phi)) = \int_{N_0(\mathbb{A})\backslash \mathrm{SO}_{2,2}(\mathbb{A})} \phi(g^{-1}(y_{-2}, y_{-1} + y_1)) W_{\psi_{2,2}}(\sigma(g)\varphi)\, dg.$$

Proof. Since the proofs are similar, we prove only (A.1.1). From the definition of the theta lift, we may write

$$\int_{N(F)\backslash N(\mathbb{A})} \Theta_\psi(\varphi,\phi)(ug)\,\psi_{U_G}(u)^{-1}\,du$$
$$= \int_{\mathrm{O}_{2,2}(F)\backslash \mathrm{O}_{2,2}(\mathbb{A})} \sum_{(a_1,a_2)\in \mathcal{X}} \omega_\psi(g,h)\,\phi(a_1,a_2)\varphi(h)\,dh$$

where

$$\mathcal{X} = \left\{(a_1,a_2)\in Y(F)^2 : \begin{pmatrix} (a_1,a_1) & (a_1,a_2)\\ (a_2,a_1) & (a_2,a_2)\end{pmatrix} = \begin{pmatrix} 0 & 0\\ 0 & 1\end{pmatrix}\right\}.$$

Then as in [23, Lemma 1], only $(a_1,a_2)\in\mathcal{X}$ such that a_1 and a_2 are linearly independent contributes in the above sum. Thus, by Witt's theorem, we may rewrite the above integral as

$$\int_{\mathrm{O}_{2,2}(F)\backslash \mathrm{O}_{2,2}(\mathbb{A})} \sum_{\gamma\in N_0(F)\backslash \mathrm{O}_{2,2}(F)} \omega_\psi(g,h)\phi(\gamma^{-1}y_{-2},\gamma^{-1}(y_{-1}+y_1))\,\varphi(h)\,dh$$
$$= \int_{\mathrm{O}_{2,2}(F)\backslash \mathrm{O}_{2,2}(\mathbb{A})} \sum_{\gamma\in N_0(F)\backslash \mathrm{O}_{2,2}(F)} \omega_\psi(g,\gamma h)\phi(y_{-2},y_{-1}+y_1)\,\varphi(h)\,dh$$
$$= \int_{N_0(F)\backslash \mathrm{O}_{2,2}(\mathbb{A})} \omega_\psi(g,h)\phi(y_{-2},y_{-1}+y_1)\,\varphi(h)\,dh$$
$$= \int_{N_0(\mathbb{A})\backslash \mathrm{O}_{2,2}(\mathbb{A})} \int_{N_0(F)\backslash N_0(\mathbb{A})} \omega_\psi(g,h)\phi(y_{-2},y_{-1}+y_1)\,\varphi(nh)\,dn\,dh.$$

Thus by (6.2.1) we have

$$W_{\psi_{U_G}}(\Theta_\psi(\varphi,\phi)) = \int_{N_0(\mathbb{A})\backslash \mathrm{O}_{2,2}(\mathbb{A})} \int_{N_2(F)\backslash N_2(\mathbb{A})} \int_{N_0(F)\backslash N_0(\mathbb{A})} \omega_\psi(m(u)g,h)\phi(y_{-2},y_{-1}+y_1)\varphi(nh)\psi_{U_G}(m(u))^{-1}\,dh\,du. \tag{A.1.3}$$

Here we have

$$\omega_\psi(m(u)g,h)\phi(y_{-2},y_{-1}+y_1) = \omega_\psi(g,m_0(u)h)\phi(y_{-2},y_{-1}+y_1)$$

where $m_0(u) = \begin{pmatrix} 1 & \frac{a}{2} & \frac{a}{2} & \frac{a^2}{4}\\ 0 & 1 & 0 & -\frac{a}{2}\\ 0 & 0 & 1 & -\frac{a}{2}\\ 0 & 0 & 0 & 1\end{pmatrix}$ for $u = \begin{pmatrix} 1 & a\\ 0 & 1\end{pmatrix}$, since $\psi_{U_G}(m(u))^{-1} = \psi(-a)$.

By noting the decomposition

$$\begin{pmatrix} 1 & x & y & -xy\\ 0 & 1 & 0 & -y\\ 0 & 0 & 1 & -x\\ 0 & 0 & 0 & 1\end{pmatrix} = \begin{pmatrix} 1 & \frac{x+y}{2} & \frac{x+y}{2} & \frac{(x+y)^2}{4}\\ 0 & 1 & 0 & -\frac{x+y}{2}\\ 0 & 0 & 1 & -\frac{x+y}{2}\\ 0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix} 1 & \frac{x-y}{2} & -\frac{x-y}{2} & \frac{(x-y)^2}{4}\\ 0 & 1 & 0 & -\frac{x-y}{2}\\ 0 & 0 & 1 & -\frac{x-y}{2}\\ 0 & 0 & 0 & 1\end{pmatrix},$$

the required identity (A.1.1) follows from (A.1.3). □

Recall the exact sequence

$$1 \to \mathrm{GSO}_{2,2} \to \mathrm{GO}_{2,2} \to \mu_2 \to 1.$$

Hence we have

$$\Theta_\psi(\varphi,\phi)(g) = \int_{\mu_2(F)\backslash\mu_2(\mathbb{A})} \theta_\psi(\varphi^\varepsilon : \phi^\varepsilon)(g)\, d\varepsilon$$

where $\varphi^\varepsilon = \sigma(\varepsilon)\varphi$ and $\phi^\varepsilon = \omega_\psi(\varepsilon)\phi$. Thus we have

$$\left|W_{\psi_{U_G}}(\Theta_\psi(\varphi,\phi))\right|^2 = \int_{\mu_2(F)\backslash\mu_2(\mathbb{A})} \mathbb{W}_{\psi_{U_G}}(\theta_\psi(\varphi^\varepsilon,\phi^\varepsilon))\, d\varepsilon$$

where

$$\mathbb{W}_{\psi_{U_G}}(\theta_\psi(\varphi^\varepsilon,\phi^\varepsilon)) = \int_{\mu_2(F)\backslash\mu_2(\mathbb{A})} W_{\psi_{U_G}}(\theta_\psi(\varphi^\varepsilon,\phi^\varepsilon))\, \overline{W_{\psi_{U_G}}(\theta_\psi(\varphi,\phi))}\, d\varepsilon.$$

A.2. **Lapid-Mao formula.** Let us recall the Lapid-Mao formula in the GL_2 case. Let (τ, V_τ) denote an irreducible cuspidal unitary automorphic representation of $\mathrm{GL}_2(\mathbb{A})$. Then for $f \in V_\tau$, its Whittaker period is defined by

$$W_2(f) = \int_{F\backslash\mathbb{A}} f\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \psi(-x)\, dx$$

with the Tamagawa measure $dx = \prod dx_v$. Let v be a place of F. For $f_v \in \tau_v$ and $\widetilde{f_v} \in \overline{\tau}_v$, by [76] (see also [71, Section 2]), we may define

$$\mathcal{W}_2(f_v, \widetilde{f_v}) = \int_F^{st} \mathcal{B}_{\tau_v}(\tau_v(x_v) f_v, \widetilde{f_v}) \psi_v(-x_v)\, dx_v.$$

Put

$$\mathcal{W}_2^\natural(f_v, \widetilde{f_v}) = \frac{L(1,\tau_v,\mathrm{Ad})}{\zeta_{F_v}(2)} \mathcal{W}_2(f_v, \widetilde{f_v})$$

which is equal to 1 at almost all places v by [71, Proposition 2.14]. Let us define

$$\langle f, f\rangle = \int_{\mathbb{A}^\times \mathrm{GL}_2(F)\backslash\mathrm{GL}_2(\mathbb{A})} |f(g)|^2\, dg$$

where dg is the Tamagawa measure. We note that $\mathrm{Vol}\,(\mathbb{A}^\times\mathrm{GL}_2(F)\backslash\mathrm{GL}_2(\mathbb{A}), dg) = 2$. Further, let us take a local $\mathrm{GL}_2(F_v)$-invariant pairing $\langle\,,\,\rangle_v$ on $\tau_v \times \tau_v$ such that $\langle f, f\rangle = \prod \langle f_v, f_v\rangle_v$. Then by [71, Theorem 4.1], we have

$$|W_2(f)|^2 = \frac{1}{2}\cdot\frac{\zeta_F(2)}{L(1,\tau,\mathrm{Ad})} \prod \mathcal{W}_2^\natural(f_v, \overline{f}_v). \tag{A.2.1}$$

for a factorizable vector $f = \otimes f_v \in V_\tau$.

A.3. **Local pull-back computation.** We fix a place v of F which will be suppressed from the notation in this section. Further, we simply write $X(F)$ by X for any object X defined over F. Let σ be an irreducible tempered representation of $\mathrm{GO}_{2,2}$ such that its big theta lift $\Theta(\sigma)$ to H is non-zero. Because of the Howe duality proved by Howe [52], Waldspurger [113] and Gan-Takeda [37], combined with Roberts [95], $\Theta(\sigma)$ has a unique irreducible quotient, which we denote by π. Put $R = \{(g,h) \in G \times \mathrm{GO}_{2,2} : \lambda(g) = \nu(h)\}$. Then we have a unique R-equivariant map

$$\theta : \omega_\psi \otimes \sigma \to \pi.$$

Let $\mathcal{B}_\omega : \omega_\psi \otimes \overline{\omega_\psi} \to \mathbb{C}$ be the canonical bilinear pairing defined by

$$\mathcal{B}_\omega(\phi, \tilde{\phi}) = \int_{V^2} \phi(x)\widetilde{\phi}(x)\, dx.$$

By [39, Lemma 5.6], the pairing $\mathcal{Z} : (\sigma \otimes \overline{\sigma}) \otimes (\omega_\psi \otimes \overline{\omega_\psi}) \to \mathbb{C}$, defined as

$$\mathcal{Z}(\varphi, \widetilde{\varphi}, \phi, \widetilde{\phi}) = \frac{\zeta_F(2)\zeta_F(4)}{L(1,\sigma,\mathrm{std})} \int_{\mathrm{O}_{2,2}} \mathcal{B}_\omega(\omega_\psi(h)\phi, \widetilde{\phi}) \langle \sigma(h)\varphi, \widetilde{\varphi}\rangle\, dh,$$

which converges absolutely by [76, Lemma 3.19], gives a pairing $\mathcal{B}_\pi : \pi \otimes \overline{\pi} \to \mathbb{C}$ by

$$\mathcal{B}_\pi(\theta(\varphi,\phi), \theta(\widetilde{\varphi}, \widetilde{\phi})) = \mathcal{Z}(\varphi, \widetilde{\varphi}, \phi, \widetilde{\phi}).$$

Proposition A.2. *We write $y_0 = (y_{-2}, y_{-1} + y_1)$. For any $u \in N_2$,*

$$\left(\frac{\zeta_F(2)\zeta_F(4)}{L(1,\sigma,\mathrm{std})}\right)^{-1} \int_{N_H}^{st} \mathcal{B}_\pi(\pi(nm(u))\theta(\varphi,\phi), \theta(\widetilde{\varphi}, \widetilde{\phi}))\psi_{U_H}(n)^{-1}\, dn$$
$$= \int_{\mathrm{O}_{2,2}} \int_{N_0\backslash \mathrm{SO}_{2,2}} \left(\omega_\psi(g, m(u))\phi\right)(y_0)\overline{\widetilde{\phi}(h^{-1}\cdot y_0)} \langle \sigma(g)\varphi, \sigma(h)\widetilde{\varphi}\rangle\, dg\, dh.$$

Let us define

$$\mathcal{W}_{\psi_{U_H}}(f_1, f_2)) = \int_{U_H}^{st} \mathcal{B}_\pi(\pi(u)f_1, f_2)\psi_{U_H}^{-1}(u)\, du.$$

Take the measure $dh_0 = 2dh|_{\mathrm{SO}_{2,2}}$. Then

$$\left(\frac{\zeta_F(2)\zeta_F(4)}{L(1,\sigma,\mathrm{std})}\right)^{-1} \mathcal{W}_{\psi_{U_H}}(\theta(\varphi,\phi), \theta(\widetilde{\varphi}, \widetilde{\phi}))$$
$$= \int_{N_2}^{st} \int_{\mathrm{O}_{2,2}} \int_{N_0\backslash \mathrm{SO}_{2,2}} \left(\omega_\psi(g, m(u))\phi\right)(y_0)\overline{\widetilde{\phi}(h^{-1}\cdot y_0)}$$
$$\times \langle \sigma(g)\varphi, \sigma(h)\widetilde{\varphi}\rangle dg\, dh\, du.$$

By an argument similar to the one for [28, Section 3.4.2] and [29, Section 5.4], we see that this is equal to

$$\int_{N_0\backslash \mathrm{O}_{2,2}} \int_{N_0\backslash \mathrm{SO}_{2,2}} \int_{N_{2,2}}^{st} \left(\omega_\psi(g, m(u))\phi\right)(y_0)\overline{\widetilde{\phi}(h^{-1}\cdot y_0)} \langle \sigma(g)\varphi, \sigma(h)\widetilde{\varphi}\rangle dg\, dh\, du.$$

Further, it is equal to

$$\text{(A.3.1)} \quad \sum_{\varepsilon=\pm 1} \int_{N_0\backslash \mathrm{SO}_{2,2}} \int_{N_0\backslash \mathrm{SO}_{2,2}} \int_{N_{2,2}}^{st} \left(\omega_\psi(g, m(u))\phi^\varepsilon\right)(y_0)\overline{\widetilde{\phi}(h^{-1}\cdot y_0)} \times \langle \sigma(g)\varphi^\varepsilon, \sigma(h)\widetilde{\varphi}\rangle \, dg\, dh\, du$$

$$= \sum_{\varepsilon=\pm 1} \int_{N_0\backslash \mathrm{SO}_{2,2}} \int_{N_0\backslash \mathrm{SO}_{2,2}} \phi^\varepsilon(g^{-1}\cdot y_0)\widetilde{\phi}(h^{-1}\cdot y_0)\mathcal{W}_{2,2}(\sigma(g)\varphi^\varepsilon, \sigma(h)\widetilde{\varphi})\, dg\, dh$$

where we define

$$\mathcal{W}_{2,2}(\varphi_1, \varphi_2) := \int_{N_{2,2}}^{st} \langle \sigma(u)\varphi_1, \varphi_2\rangle \psi_{2,2}^{-1}(u)\, du \quad \text{for} \quad \varphi_i \in V_\sigma.$$

Let us introduce a measure $d'h = \zeta_F(2)^2 dh$. Then we get

$$\mathcal{W}^\natural_{\psi_{U_H}}(\theta(\varphi, \phi), \theta(\widetilde{\varphi}, \widetilde{\phi})) = \sum_{\varepsilon=\pm 1} \int_{N_0\backslash \mathrm{SO}_{2,2}} \int_{N_0\backslash \mathrm{SO}_{2,2}} \phi^\varepsilon(g^{-1}\cdot y_0)\widetilde{\phi}(h^{-1}\cdot y_0) \times \mathcal{W}^\natural_{2,2}(\sigma(g)\varphi^\varepsilon, \sigma(h)\widetilde{\varphi})\, dg\, d'h.$$

Here

$$\mathcal{W}^\natural_{2,2}(\sigma(g)\varphi^\varepsilon, \sigma(h)\widetilde{\varphi}) = \frac{L(1, \sigma_1, \mathrm{Ad})L(1, \sigma_2, \mathrm{Ad})}{\zeta_F(2)^2}\mathcal{W}_{2,2}(\sigma(g)\varphi^\varepsilon, \sigma(h)\widetilde{\varphi}).$$

A.4. **Proof of Theorem 6.3.** Let (σ, V_σ) be an irreducible cuspidal automorphic representation of the group $\mathrm{GO}_{2,2}(\mathbb{A})$. Suppose that σ is induced by the representation $\sigma_1 \boxtimes \sigma_2$ of $\mathrm{GL}_2(\mathbb{A}) \times \mathrm{GL}_2(\mathbb{A})$. For $f = f_1 \otimes f_2 \in V_{\sigma_1} \otimes V_{\sigma_2}$, we have

$$W_{U_H}(f) = \int_{F\backslash \mathbb{A}} f_1\left(\begin{pmatrix}1 & x\\ & 1\end{pmatrix} h_1\right)\psi(-x)\, dx \int_{F\backslash \mathbb{A}} f_2\left(\begin{pmatrix}1 & x\\ & 1\end{pmatrix} h_2\right)\psi(-x)\, dx$$

for $h = (h_1, h_2) \in \mathrm{SO}_{2,2}(\mathbb{A})$. Moreover, for any place v of F, we have

$$\mathcal{W}^\natural_{2,2}(\varphi_v, \widetilde{\varphi}_v) = \mathcal{W}^\natural_2(\varphi_{1,v}, \widetilde{\varphi}_{1,v})\mathcal{W}^\natural_2(\varphi_{2,v}, \widetilde{\varphi}_{2,v})$$

with $\varphi_v = (\varphi_{1,v}, \varphi_{2,v})$ and $\widetilde{\varphi}_v = (\widetilde{\varphi}_{1,v}, \widetilde{\varphi}_{2,v})$. Then by (A.1.2) and the Lapid-Mao formula (A.2.1), we obtain

$$\mathbb{W}_{\psi_{U_H}}(\theta_\psi(\varphi^\varepsilon, \phi^\varepsilon)) = \frac{1}{4}\frac{\zeta_F(2)^2}{L(1, \sigma_1, \mathrm{Ad})L(1, \sigma_2, \mathrm{Ad})}$$

$$\times \int_{\mu_2(F)\backslash \mu_2(\mathbb{A})} \prod_v \int \int_{(N_0(F_v)\backslash \mathrm{SO}_{2,2})^2} \left(\prod_{\alpha=1,2} \mathcal{W}^\natural_2((\sigma(g_v)\varphi_v^\varepsilon)_\alpha, (\overline{\sigma}(h_v)\overline{\varphi}_v)_\alpha)\right) \times \phi_v^\varepsilon(g_v^{-1}\cdot y_0)\overline{\phi}_v(h_v^{-1}\cdot y_0)\, dg\, dh$$

$$= \frac{1}{4}\frac{\zeta_F(2)^2}{L(1, \sigma_1, \mathrm{Ad})L(1, \sigma_2, \mathrm{Ad})} \int_{\mu_2(F)\backslash \mu_2(\mathbb{A})} \prod_v \int \int_{(N_0(F_v)\backslash \mathrm{SO}_{2,2})^2} \mathcal{W}^\natural_{2,2}(\sigma(g_v)\varphi_v, \overline{\sigma}_v(h_v)\overline{\varphi}_v)\phi_v^\varepsilon(g_v^{-1}\cdot y_0)\overline{\phi}_v(h_v^{-1}\cdot y_0)\, dg\, dh.$$

By (A.3.1), this is equal to

$$\frac{1}{4}\frac{\zeta_F(2)\zeta_F(4)}{L(1,\sigma_1,\mathrm{Ad})L(1,\sigma_2,\mathrm{Ad})}\prod \mathcal{W}^{\natural}_{\psi_{U_H}}(\theta(\varphi_v,\phi_v),\theta(\overline{\varphi}_v,\overline{\phi}_v)),$$

and thus this completes our proof of Theorem 6.3.

Appendix B. Explicit computation of local Bessel periods at the real place

The goal of this appendix is to compute explicitly the local Bessel periods at the real place and to complete our proof of Theorem 8.1. In this section, we use the same notation as in Section 8.

For a newform $\Phi \in S_\varrho(\Gamma_0(N))$ in Theorem 8.1, we define a scalar valued automorphic form $\phi_{\Phi,S}$ on $G(\mathbb{A})$ by

$$\phi_{\Phi,S}(g) = \left(\varphi_\Phi(g), Q_{S,\varrho}\right)_{2r} \quad \text{for } g \in G(\mathbb{A}), \tag{B.0.1}$$

where φ_Φ is the adelization of Φ given by (8.2.9) and $Q_{S,\varrho}$ by (8.2.16). We note that by the argument in [21, 3.2], $\phi_{\Phi,S}$ is a factorizable vector $\phi_{\Phi,S} = \otimes_v \phi_{\Phi,S,v}$. For a place v of $\mathbb{Q}$, we define J_v by

$$J_v = \frac{\alpha_v^{\natural}\left(\phi_{\Phi,S,v},\phi_{\Phi,S,v}\right)}{\langle \phi_{\Phi,S,v},\phi_{\Phi,S,v}\rangle_v}. \tag{B.0.2}$$

It is clear that J_v remains invariant under replacing $\phi_{\Phi,S,v}$ by its non-zero scalar multiple. Further, we put

$$C = C_\xi \cdot \frac{\zeta_{\mathbb{Q}}(2)\ \zeta_{\mathbb{Q}}(4)}{L(1,\chi_E)} \tag{B.0.3}$$

with the Haar measure constant C_ξ defined by (1.6.1). Then the following identity holds.

Theorem B.1.

$$C\left(Q_{S,\varrho}\right)CJ_\infty = \frac{2^{4k+6r-1}e^{-4\pi\,\mathrm{tr}(S)}}{D_E}. \tag{B.0.4}$$

Recall that $C\left(Q_{S,\varrho}\right)$ is defined by (8.2.10) for $v' = Q_{S,\varrho}$.

Remark B.1. *In the scalar valued case, i.e. $r = 0$, the explicit computation of J_∞ is done in Dickson et al.* [21, 3.5] *using the explicit formula for matrix coefficients when $k \geq 3$. Meanwhile Hsieh and Yamana* [55, Proposition 5.7] *compute J_∞ in a different way when $k \geq 2$, based on Shimura's work on confluent hypergeometric functions.*

We note that the left hand side of (B.0.4) depends only on the archimedean representation $\pi(\Phi)_\infty$ and the vector $\phi_{\Phi,S,\infty}$. Thus our strategy is to first obtain an explicit formula (B.1.12) for the Bessel periods of vector valued Yoshida lifts by combining the results in Hsieh and Namikawa [53, 54], Chida and Hsieh [18], Martin and Whitehouse [78], and, then to evaluate $C\left(Q_{S,\varrho}\right)CJ_\infty$ by singling out the real place contribution, comparing (B.1.12) with (1.6.2).

B.1. **Explicit formula for Bessel periods of Yoshida lifts.** For a prime number p, let

$$\Gamma_0^{(1)}(p) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : c \equiv 0 \pmod{p} \right\}$$

and $S_k\left(\Gamma_0^{(1)}(p)\right)$ the space of cusp forms of weight k with respect to $\Gamma_0^{(1)}(p)$.

In order to insure what follows to be non-vacuous, first we shall prove the following technical lemma.

Lemma B.1. *Let k_1 and k_2 be integers with $k_1 \geq k_2 \geq 0$. Then there is a constant $N = N(k_1, k_2, E) \in \mathbb{R}$ such that for any prime $p > N$, there exist distinct normalized newforms $f_i \in S_{2k_i+2}\left(\Gamma_0^{(1)}(p)\right)$ for $i = 1, 2$ satisfying the condition:*

(B.1.1) *the Atkin-Lehner eigenvalues of f_i at p for $i = 1, 2$ coincide.*

Proof. We divide into the following two cases:

(B.1.2a) $$k_1 \equiv k_2 \pmod{2};$$

(B.1.2b) $$k_1 + 1 \equiv k_2 \equiv 0 \pmod{2}.$$

Suppose that (B.1.2a) holds. Then by Iwaniec, Luo and Sarnak [60, Corollary 2.14], there is a constant $N(k_1, k_2)$ such that, for any prime $p > N(k_1, k_2)$, there exist distinct normalized newforms $f_i \in S_{2k_i+2}\left(\Gamma_0^{(1)}(p)\right)$ for $i = 1, 2$ such that

$$\varepsilon(1/2, \pi_1) = \varepsilon(1/2, \pi_2)$$

where π_i denotes the automorphic representation of $\mathrm{GL}_2(\mathbb{A})$ corresponding to f_i for $i = 1, 2$. Since π_i is unramified at all prime numbers different from p, we have

$$(-1)^{k_1+1} \cdot \varepsilon_p(1/2, \pi_1) = (-1)^{k_2+1} \cdot \varepsilon_p(1/2, \pi_2).$$

Hence $\varepsilon_p(1/2, \pi_1) = \varepsilon_p(1/2, \pi_2)$ by (B.1.2a). Then by the relationship between the local ε-factor at p and the Atkin-Lehner eigenvalue at p (e.g. [54, 4.4]), we see that (B.1.1) holds.

Suppose that (B.1.2b) holds. Then by Michel and Ramakrishnan [79, Theorem 3] or Ramakrishnan and Rogawski [94, Corollary B], there exists a constant $N_1 = N_1(k_1, E)$ such that for any prime $p > N_1$, there exists a normalized newform $f_1 \in S_{2k_1+2}\left(\Gamma_0^{(1)}(p)\right)$ such that

$$L(1/2, \pi_1)\, L(1/2, \pi_1 \times \chi_E) \neq 0.$$

In particular, $\varepsilon(1/2, \pi_1) = 1$, and thus as in the previous case, we have

$$(-1)^{k_1+1} \cdot \varepsilon_p(1/2, \pi_1) = 1.$$

Moreover, by [60, Corollary 2.14], there exists a constant $N_2 = N_2(k_2)$ such that for any prime $p > N_2$, there exists a normalized newform $f_2 \in S_{2k_2+2}\left(\Gamma_0^{(1)}(p)\right)$ such that

$$\varepsilon(1/2, \pi_2) = -1.$$

Then by taking the constant N to be $\max(N_1, N_2)$, the condition (B.1.1) holds by the same argument as above. $\square$

B.1.1. *Vector valued Yoshida lift.* As for the Yoshida lifting, we refer the details to our main references Hsieh and Namikawa [53, 54].

Let k_1 and k_2 be integers with $k_1 \geq k_2 \geq 0$. Then by Lemma B.1, we may take a prime number p satisfying the condition:

(B.1.3) $$p \text{ is odd, and inert and unramified in } E$$

and may take distinct normalized newforms $f_i \in S_{2k_i+2}\left(\Gamma_0^{(1)}(p)\right)$ $(i = 1, 2)$ satisfying the condition (B.1.1).

For a non-negative integer r, we denote by $(\tau_r, \mathcal{W}_r)$ the representation (ϱ, V_ϱ) of $\mathrm{GL}_2(\mathbb{C})$ where $\varrho = \varrho_{(r,-r)}$, i.e. $\tau_r = \mathrm{Sym}^{2r} \otimes \det^{-r}$. We note that the action of the center of $\mathrm{GL}_2(\mathbb{C})$ on $\mathcal{W}_r$ by τ_r is trivial and the pairing $(\ ,\ )_{2r}$ is $\mathrm{GL}_2(\mathbb{C})$-invariant by (8.2.5). Let p be a prime number and $D = D_{p,\infty}$ the unique division quaternion algebra over $\mathbb{Q}$ which ramifies precisely at p and ∞. Let $\mathcal{O}_D$ be the maximal order of D specified as in [53, 3.2] and we put $\hat{\mathcal{O}}_D = \mathcal{O}_D \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}$.

Definition B.1. $\mathcal{A}_r\left(D^\times(\mathbb{A}), \hat{\mathcal{O}}_D\right)$*, the space of automorphic forms of weight r and level $\hat{\mathcal{O}}_D$ on $D^\times(\mathbb{A})$ is a space of functions $\mathbf{g} : D^\times(\mathbb{A}) \to \mathcal{W}_r$ satisfying*

$$\mathbf{g}(z\gamma h u) = \tau_r(h_\infty)^{-1}\,\mathbf{g}\left(h_f\right)$$

for $z \in \mathbb{A}^\times$, $\gamma \in D^\times(\mathbb{Q})$, $u \in \hat{\mathcal{O}}_D^\times$ and $h = \left(h_\infty, h_f\right) \in D^\times(\mathbb{R}) \times D^\times\left(\mathbb{A}_f\right)$.

For $i = 1, 2$, let π_i be the irreducible cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A})$ corresponding to f_i. Let π_i^D be the Jacquet-Langlands transfer of π_i to $D^\times(\mathbb{A})$. We denote by $\mathcal{A}_{k_i}\left(D^\times(\mathbb{A}), \hat{\mathcal{O}}_D\right)\left[\pi_i^D\right]$ the π_i^D-isotypic subspace of $\mathcal{A}_{k_i}\left(D^\times(\mathbb{A}), \hat{\mathcal{O}}_D\right)$. Then $\mathcal{A}_{k_i}\left(D^\times(\mathbb{A}), \hat{\mathcal{O}}_D\right)\left[\pi_i^D\right]$ has a subspace of newforms, which is one dimensional. Let us take newforms $\mathbf{f}_i \in \mathcal{A}_{k_i}\left(D^\times(\mathbb{A}), \hat{\mathcal{O}}_D\right)\left[\pi_i^D\right]$ for $i = 1, 2$ and fix. Then to the pair $\mathbf{f} = (\mathbf{f}_1, \mathbf{f}_2)$, Hsieh and Namikawa [53, 3.7] associate the *Yoshida lift* $\theta_{\mathbf{f}}$, a V_ϱ-valued cuspidal automorphic form on $G(\mathbb{A})$ where $\varrho = \varrho_\kappa$ with

$$\kappa = (k_1 + k_2 + 2, k_1 - k_2 + 2) \in \mathbb{L}.$$

The *classical Yoshida lift* $\theta_{\mathbf{f}}^* \in S_\varrho(\Gamma_0(p))$ is also attached to $\mathbf{f}$ in [53, 3.7] so that $\theta_{\mathbf{f}}$ is obtained from $\theta_{\mathbf{f}}^*$ by the adelization procedure in (8.2.9).

B.1.2. *Bessel periods of Yoshida lifts.* Let $\phi_{\mathbf{f},S}$ denote a scalar valued automorphic form attached to $\theta_{\mathbf{f}}^*$ as in (B.0.1). Hsieh and Namikawa evaluated the Bessel periods of $\phi_{\mathbf{f},S}$ in [53].

First we remark that by [53, Theorem 5.3], for any sufficiently large prime number q which is different from p, we may take a character Λ_0 of $\mathbb{A}_E^\times$ satisfying:

(B.1.4a) $$L(1/2, \pi_1 \otimes \mathcal{AI}(\Lambda_0))\, L\left(1/2, \pi_2 \otimes \mathcal{AI}\left(\Lambda_0^{-1}\right)\right) \neq 0;$$

(B.1.4b) $$\text{the conductor of } \Lambda_0 \text{ is } q^m \mathcal{O}_E \text{ where } m > 0;$$

(B.1.4c) $$\Lambda_0\,|_{\mathbb{A}^\times} \text{ is trivial;}$$

(B.1.4d) $\Lambda_{0,\infty}$ is trivial.

Then [53, Proposition 4.7] yields the following formula.

Lemma B.2. *We have*

$$B_{S,\Lambda_0,\psi}\left(\phi_{\mathbf{f},S}\right) = q^{2m}\cdot\left(-2\sqrt{-1}\right)^{k_1+k_2}\cdot e^{-2\pi\,\mathrm{Tr}(S)}\cdot\prod_{i=1}^{2}P\left(\mathbf{f}_i,\Lambda_0^{\alpha_i},1_2\right) \tag{B.1.5}$$

where $\alpha_i = (-1)^{i+1}$ *and*

$$P\left(\mathbf{f}_i,\Lambda_0^{\alpha_i},1_2\right) = \int_{E^\times\mathbb{A}^\times\backslash\mathbb{A}_E^\times}\left((XY)^{k_i},\mathbf{f}_i(t)\right)_{2k_i}\cdot\Lambda_0^{\alpha_i}(t)\,dt.$$

From (B.1.5), we have

$$\left|B_{S,\Lambda_0,\psi}\left(\phi_{\mathbf{f},S}\right)\right|^2 = q^{4m}\cdot 2^{2(k_1+k_2)}\cdot e^{-4\pi\,\mathrm{tr}(S)}\cdot\prod_{i=1}^{2}\left|P\left(\mathbf{f}_i,\Lambda_0^{\alpha_i},1_2\right)\right|^2. \tag{B.1.6}$$

Since p is odd and inert in E, we may evaluate the right hand side of (B.1.6) by Martin and Whitehouse [78]. Namely the following formula holds by [78, Theorem 4.1].

Lemma B.3. *We have*

$$\frac{\left|P(\mathbf{f}_i,\Lambda_0^{\alpha_i},1_2)\right|^2}{\left\|\phi_{\mathbf{f}_i}\right\|^2} = \frac{1}{4}\cdot\frac{\xi(2)}{\zeta_{\mathbb{Q}_p}(2)}\cdot\frac{L\left(1/2,\pi_i\otimes\mathcal{AI}\left(\Lambda_0^{\alpha_i}\right)\right)}{L\left(1,\pi_i,\mathrm{Ad}\right)}\cdot\left(1+p^{-1}\right)^{-1} \times\frac{\Gamma\left(2k_i+2\right)}{2\,q^m\,\pi\,D_E^{1/2}\,\Gamma\left(k_i+1\right)^2} \tag{B.1.7}$$

where $\xi(s)$ *denotes the complete Riemann zeta function,* $\phi_{\mathbf{f}_i}$ *the scalar valued automorphic form on* $D^\times(\mathbb{A})$ *defined by*

$$\phi_{\mathbf{f}_i}(h) = \left((XY)^{k_i},\mathbf{f}_i(h)\right)_{2k_i}\quad\text{for } h\in D^\times(\mathbb{A})$$

and

$$\left\|\phi_{\mathbf{f}_i}\right\|^2 = \int_{\mathbb{A}^\times D^\times(\mathbb{Q})\backslash D^\times(\mathbb{A})}\left|\phi_{\mathbf{f}_i}(h)\right|^2\,dh.$$

Here dh *is the Tamagawa measure on* $\mathbb{A}^\times\backslash D^\times(\mathbb{A})$*, and thus*

$$\mathrm{Vol}(\mathbb{A}^\times D^\times(\mathbb{Q},)\backslash D^\times(\mathbb{A}),dh) = 2.$$

Remark B.2. *The factor* $\frac{1}{4}$ *in* (B.1.7) *originates from the difference of measures between the one used here and the one in* [78].

In order to utilize the explicit inner product formula for vector valued Yoshida lifts in Hsieh and Namikawa [54], we need the following lemma.

Lemma B.4. *Let us define an inner product* $\langle\mathbf{f}_i,\mathbf{f}_i\rangle$ *for* $i = 1,2$ *by*

$$\langle\mathbf{f}_i,\mathbf{f}_i\rangle = \sum_a\langle\mathbf{f}_i(a),\mathbf{f}_i(a)\rangle_{\tau_{k_i}}\cdot\frac{1}{\#\Gamma_a} \tag{B.1.8}$$

where $\langle\,,\,\rangle_{\tau_{k_i}}$ *is defined by* (8.2.6)*,* a *runs over double coset representatives of* $D^\times(\mathbb{Q})\backslash D^\times(\mathbb{A}_f)/\hat{O}_D^\times$ *and* $\Gamma_a = \left(a\,\hat{O}_D^\times\,a^{-1}\cap D^\times(\mathbb{Q})\right)/\{\pm 1\}$.

Then for $i = 1, 2$*, we have*

$$\|\phi_{\mathbf{f}_i}\|^2 = 2^3\cdot 3\cdot p^{-1}\left(1-p^{-1}\right)^{-1}\cdot\frac{\Gamma(k_i+1)^2}{\Gamma(2k_i+1)}\cdot\frac{1}{(2k_i+1)^2}\cdot\langle\mathbf{f}_i,\mathbf{f}_i\rangle. \tag{B.1.9}$$

Proof. Since $\|\phi_{\mathbf{f}_i}\|^2 = \|\pi_i^D(h_\infty)\phi_{\mathbf{f}_i}\|^2$ for $h_\infty\in D^\times(\mathbb{R})$, we have

$$\|\phi_{\mathbf{f}_i}\|^2 = \frac{1}{\mathrm{Vol}(\mathbb{R}^\times\backslash D^\times(\mathbb{R}), dh_\infty)} \times\int_{\mathbb{R}^\times\backslash D^\times(\mathbb{R})}\int_{\mathbb{A}^\times D^\times(\mathbb{Q})\backslash D^\times(\mathbb{A})}\left|\phi_{\mathbf{f}_i}(hh_\infty)\right|^2 dh\,dh_\infty.$$

By interchanging the order of integration, we have

$$\|\phi_{\mathbf{f}_i}\|^2 = \frac{1}{\mathrm{Vol}(\mathbb{R}^\times\backslash D^\times(\mathbb{R}), dh_\infty)} \times\int_{\mathbb{A}^\times D^\times(\mathbb{Q})\backslash D^\times(\mathbb{A})}\int_{\mathbb{R}^\times\backslash D^\times(\mathbb{R})}\left|\phi_{\mathbf{f}_i}(hh_\infty)\right|^2 dh_\infty\,dh.$$

Here the Schur orthogonality implies

$$\frac{1}{\mathrm{Vol}(\mathbb{R}^\times\backslash D^\times(\mathbb{R}), dh_\infty)}\int_{\mathbb{R}^\times\backslash D^\times(\mathbb{R})}\left|\left((XY)^{k_i},\mathbf{f}_i(hh_\infty)\right)_{2k_i}\right|^2 dh_\infty = d_i^{-1}\cdot\left((XY)^{k_i},(XY)^{k_i}\right)_{2k_i}\cdot\left(\mathbf{f}_i(h),\overline{\mathbf{f}_i(h)}\right)_{2k_i}$$

where $d_i = \dim\mathrm{Sym}^{2k_i} = 2k_i+1$ and $\left((XY)^{k_i},(XY)^{k_i}\right)_{2k_i} = (-1)^{k_i}\binom{2k_i}{k_i}^{-1}$. Hence

$$\|\phi_{\mathbf{f}_i}\|^2 = \binom{2k_i}{k_i}^{-1}(2k_i+1)^{-1}\int_{\mathbb{A}^\times D^\times(\mathbb{Q})\backslash D^\times(\mathbb{A})}\left(\mathbf{f}_i(h),\overline{\mathbf{f}_i(h)}\right)_{2k_i}dh.$$

By [53, Lemma 6], we have

$$\int_{\mathbb{A}^\times D^\times(\mathbb{Q})\backslash D^\times(\mathbb{A})}\left(\mathbf{f}_i(h),\overline{\mathbf{f}_i(h)}\right)_{2k_i}dh = \frac{(-1)^{k_i}}{2k_i+1}\int_{\mathbb{A}^\times D^\times(\mathbb{Q})\backslash D^\times(\mathbb{A})}\langle\mathbf{f}_i(h),\mathbf{f}_i(h)\rangle_{\tau_{k_i}}dh. \tag{B.1.10}$$

Finally by Chida and Hsieh [18, (3.10)] with the following Remark B.3, we obtain (B.1.9). ◻

Remark B.3. *In* [18]*, the Eichler mass formula is used to express the right hand side of* (B.1.10) *in terms of the inner product defined by* (B.1.8)*. There is a typo in the Eichler mass formula in* [18, p.103]*. The right hand side of the formula quoted there should be multiplied by* 2*.*

Let us recall the inner product formula for $\theta_{\mathbf{f}}^*$ by Hsieh and Namikawa [54, Theorem A].

Proposition B.1. *We have*

$$\text{(B.1.11)}\quad \frac{\langle \theta_{\mathbf{f}}^*, \theta_{\mathbf{f}}^* \rangle_{\varrho}}{\langle \mathbf{f}_1, \mathbf{f}_1 \rangle \langle \mathbf{f}_2, \mathbf{f}_2 \rangle} = L(1, \pi_1 \times \pi_2) \cdot \frac{2^{-(2k_1+6)}}{(2k_1+1)(2k_2+1)} \cdot \frac{1}{p^2 \left(1+p^{-1}\right)\left(1+p^{-2}\right)}.$$

Here $\langle \theta_{\mathbf{f}}^*, \theta_{\mathbf{f}}^* \rangle_{\varrho}$ *is given by*

$$\langle \theta_{\mathbf{f}}^*, \theta_{\mathbf{f}}^* \rangle_{\varrho} = \frac{1}{[\mathrm{Sp}_2(\mathbb{Z}) : \Gamma_0(p)]} \int_{\Gamma_0(p) \backslash \mathfrak{H}_2} \langle \theta_{\mathbf{f}}^*(Z), \theta_{\mathbf{f}}^*(Z) \rangle_{\varrho} (\det Y)^{k_1-k_2-1} \, dX \, dY$$

with $\varrho = \varrho_\kappa$ *where* $\kappa = (k_1+k_2+2, k_1-k_2+2)$.

Thus by combining (B.1.6), (B.1.7), (B.1.9) and (B.1.11), we have

$$\text{(B.1.12)}\quad \frac{\left|B_{S,\Lambda_0,\psi}\left(\phi_{\mathbf{f},S}\right)\right|^2}{\langle \theta_{\mathbf{f}}^*, \theta_{\mathbf{f}}^* \rangle_{\varrho}} = \frac{2^{4k_1+2k_2+5} e^{-4\pi \operatorname{tr}(S)}}{D_E} \cdot 2\left(1+p^{-1}\right)\left(1+p^{-2}\right) \cdot q^{2m} \times \frac{L\left(1/2, \pi_1 \otimes \mathcal{AI}(\Lambda_0)\right) L\left(1/2, \pi_2 \otimes \mathcal{AI}\left(\Lambda_0^{-1}\right)\right)}{L(1, \pi_1, \mathrm{Ad}) L(1, \pi_2, \mathrm{Ad}) L(1, \pi_1 \times \pi_2)}.$$

Here we note that the both sides of (B.1.12) are non-zero due to the conditions (B.1.1) and (B.1.4).

B.2. **Proof of Theorem B.1.** Since the Ichino-Ikeda type formula has been proved for Yoshida lifts by Liu [76, Theorem 4.3], the computations in Dickson et al. [21] implies

$$\text{(B.2.1)}\quad \frac{\left|B_{S,\Lambda_0,\psi}\left(\phi_{\mathbf{f},S}\right)\right|^2}{\langle \phi_{\mathbf{f},S}, \phi_{\mathbf{f},S} \rangle} = \frac{C J_\infty}{2^2} \cdot 2\left(1+p^{-1}\right)\left(1+p^{-2}\right) \cdot J_q \times \frac{L\left(1/2, \pi_1 \otimes \mathcal{AI}(\Lambda_0)\right) L\left(1/2, \pi_2 \otimes \mathcal{AI}\left(\Lambda_0^{-1}\right)\right)}{L(1, \pi_1, \mathrm{Ad}) L(1, \pi_2, \mathrm{Ad}) L(1, \pi_1 \times \pi_2)}.$$

Thus in order to evaluate J_∞, we need to determine J_q.

Here we use a scalar valued Yoshida lift to evaluate J_q. First we recall that (B.0.4) holds in the scalar valued case, i.e. when $k_2 = 0$, as we noted in Remark B.1. By Lemma B.1, when q is large enough, there also exist distinct normalized newforms $f_1' \in S_{2k_1+2}\left(\Gamma_0^{(1)}(p)\right)$ and $f_2' \in S_2\left(\Gamma_0^{(1)}(p)\right)$ satisfying the condition (B.1.1), and, a character λ_0' of $\mathbb{A}_E^\times$ satisfying the conditions (B.1.4) for π_i' $(i = 1, 2)$ where π_i' is the automorphic representation of $\mathrm{GL}_2(\mathbb{A})$. Define $\mathbf{f}'$ similarly for π_1' and π_2'.

Since (B.0.4) is valid in the scalar valued case, we have

$$\frac{\left|B_{S,\Lambda_0,\psi}\left(\phi_{\mathbf{f}',S}\right)\right|^2}{\langle\phi_{\mathbf{f}',S},\phi_{\mathbf{f}',S}\rangle}=\frac{2^{4k_1+5}e^{-4\pi\,\mathrm{tr}(S)}}{D_E}\cdot C\left(Q_{S,\varrho_{(k_1,k_1)}}\right)^{-1}$$
$$\cdot 2\left(1+p^{-1}\right)\left(1+p^{-2}\right)\cdot J_q\cdot\frac{L\left(1/2,\pi_1'\otimes\mathcal{AI}\left(\Lambda_0'\right)\right)L\left(1/2,\pi_2'\otimes\mathcal{AI}\left(\Lambda_0'^{-1}\right)\right)}{L\left(1,\pi_1',\mathrm{Ad}\right)L\left(1,\pi_2',\mathrm{Ad}\right)L\left(1,\pi_1'\times\pi_2'\right)}.$$

We note that J_q here is the same as the one in (B.2.1). Then by comparing the formula above with (B.1.12) for $\mathbf{f}'$ and Λ_0', we have $J_q=q^{2m}$.

Finally by comparing (B.1.12) with (B.2.1) substituting $J_q=q^{2m}$, we have

$$C\left(Q_{S,\varrho}\right)CJ_\infty=\frac{2^{4k_1+2k_2+7}e^{-4\pi\,\mathrm{tr}(S)}}{D_E} \tag{B.2.2}$$

in the general case.

For Φ in Theorem 8.1, a scalar valued automorphic form $\phi_{\Phi,S}$ defined by

$$\phi_{\Phi,S}\left(g\right)=\left(\varphi_\Phi\left(g\right),Q_{S,\varrho}\right)_{2r}\quad\text{for } g\in G\left(\mathbb{A}\right)$$

is factorizable, i.e. $\phi_{\Phi,S}=\otimes_v\,\phi_{\Phi,S,v}$. Let us choose k_1 and k_2 so that

$$(2r+k,k)=(k_1+k_2+2,k_1-k_2+2)\,,\quad\text{i.e.}\quad k_1=r+k-2,\ k_2=r.$$

Then for $\phi_{\mathbf{f},S}=\otimes_v\,\phi_{\mathbf{f},S,v}$ in (B.2.1), the archimedean factor $\phi_{\mathbf{f},S,\infty}$ is a non-zero scalar multiple of $\phi_{\Phi,S,\infty}$. Thus (B.0.4) follows from (B.2.2). □

B.3. **Proof of Theorem 8.1.** Let us complete our proof of Theorem 8.1. By Theorem 1.2, we have

$$\frac{\left|B_{S,\Lambda,\psi}\left(\phi_{\Phi,S}\right)\right|^2}{\langle\phi_{\Phi,S},\phi_{\Phi,S}\rangle}=\frac{CJ_\infty}{2^{c-3}}\cdot\frac{L\left(1/2,\pi\left(\Phi\right)\times\mathcal{AI}\left(\Lambda\right)\right)}{L\left(1,\pi\left(\Phi\right),\mathrm{Ad}\right)}\cdot\prod_{p\mid N}J_p \tag{B.3.1}$$

where c is as stated in Theorem 8.1. By (8.2.13) and (8.2.17), we have

$$B_{S,\psi,\Lambda}\left(\phi_{\Phi,S}\right)=2\cdot e^{-2\pi\,\mathrm{tr}(S)}\cdot\mathcal{B}_\Lambda\left(\Phi;E\right).$$

Since $\langle\phi_{\Phi,S},\phi_{\Phi,S}\rangle=C\left(Q_{S,\varrho}\right)\cdot\langle\Phi,\Phi\rangle_\varrho$ by Lemma 8.1, we have

$$\frac{\left|\mathcal{B}_\Lambda\left(\Phi;E\right)\right|^2}{\langle\Phi,\Phi\rangle_\varrho}=\frac{\left|B_{S,\Lambda,\psi}\left(\phi_{\Phi,S}\right)\right|^2}{\langle\phi_{\Phi,S},\phi_{\Phi,S}\rangle}\cdot 2^{-2}e^{4\pi\,\mathrm{tr}(S)}C\left(Q_{S,\varrho}\right). \tag{B.3.2}$$

Thus by combining (B.3.1), (B.3.2) and (B.0.4), the identity (8.3.1) holds.

Appendix C. Meromorphic continuation of L-functions for $\mathrm{SO}(5)\times\mathrm{SO}(2)$

As we remarked in Remark 1.3, here we show the meromorphic continuation of $L^S(s,\pi\times\mathcal{AI}(\Lambda))$ in Theorem 1.1, when $\mathcal{AI}\ (\Lambda)$ is cuspidal and S is a sufficiently large finite set of places of F containing all archimedean places. The following theorem clearly suffices.

Theorem C.1. *Let π (resp. τ) be an irreducible unitary cuspidal automorphic representation π of $G_D(\mathbb{A})$ (resp. $\mathrm{GL}_2(\mathbb{A})$) with a trivial central character. Then $L^S(s, \pi \times \tau)$ has a meromorphic continuation to $\mathbb{C}$ and it is holomorphic at $s = \frac{1}{2}$ for a sufficiently large finite set S of places of F containing all archimedean places.*

When D is split, then $G_D \simeq G$ and the theorem follows from Arthur [3]. Hence from now on we assume that D is non-split.

By [74], for some ξ and Λ, π has the (ξ, Λ, ψ)-Bessel period. Thus we may use the the integral representation of the L-function for $G_D \times \mathrm{GL}_2$ introduced in [84]. Then the meromorphic continuation of the Siegel Eisenstein series on $\mathrm{GU}_{3,3}$, which is used in the integral representation is known by the main theorem of Tan [107] (see also [89, Proposition 3.6.2]). Hence by the standard argument, our theorem is reduced to the analysis of the local zeta integrals. Meanwhile the non-archimedean local integrals are already studied in [84, Lemma 5.1]. Hence it suffices for us to investigate the archimedean ones. Since the case when E_v is a quadratic extension field of F_v is similar to, and indeed simpler than, the split case, here we only consider the split case.

Let us briefly recall our local zeta integral (see [84, (28)]). Let v be an archimedean place of F. Since we consider the split case, D_v is split and we may assume that $G_D(F_v) = G(F_v) = \mathrm{GSp}_2(F_v)$ and $\xi = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Then we have

$$T_\xi(F_v) = \left\{ g \in \mathrm{GL}_2(F) \mid {}^t g \xi g = \det(g)\, \xi \right\} = \left\{ \begin{pmatrix} x & y \\ y & x \end{pmatrix} \in \mathrm{GL}_2(F) \right\}.$$

In what follows, we omit the subscript v from any object in order to simplify the notation. Let Λ be a unitary character of $F^\times$. Then we regard Λ as a character of $T_\xi(F)$ by

$$\Lambda \begin{pmatrix} x & y \\ y & x \end{pmatrix} = \Lambda\left(\frac{x+y}{x-y} \right) \quad \text{for } \begin{pmatrix} x & y \\ y & x \end{pmatrix} \in T_S(F).$$

For a non-trivial character ψ of F, let $\mathcal{B}_{\xi,\Lambda,\psi}(\pi)$ denote the (ξ, Λ, ψ)-Bessel model of π, i.e. the space of functions $B : G(F) \to \mathbb{C}$ such that

$$B(tug) = \Lambda(t)\psi_\xi(u)B(g) \quad \text{for } t \in T_\xi(F), u \in N(F) \text{ and } g \in G(F),$$

which affords π by the right regular representation. Let $\mathcal{W}(\tau)$ denote the Whittaker model of π, i.e. the space of functions $W : \mathrm{GL}_2(F) \to \mathbb{C}$ such that

$$W\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g \right) = \psi(-x)W(g) \quad \text{for } x \in F \text{ and } g \in \mathrm{GL}_2(F),$$

which affords τ by the right translation. Let $G_0(F) = \mathrm{GL}_2(F) \times G(F)$ and we regard G as a subgroup of $\mathrm{GL}_6(F)$ by the embedding

$$\iota : G_0 \ni \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} A & B \\ C & C \end{pmatrix} \right) \hookrightarrow \begin{pmatrix} a & 0 & b & 0 \\ 0 & A & 0 & B \\ c & 0 & d & 0 \\ 0 & C & 0 & D \end{pmatrix} \in \mathrm{GL}_6(F).$$

Let us define a subgroup H_0 of G_0 by

$$H_0(F) = \left\{ \nu(h) \left(\begin{pmatrix} 1 & \mathrm{tr}(\xi X) \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} h & 0 \\ 0 & \det h \cdot {}^t h^{-1} \end{pmatrix} \begin{pmatrix} 1_2 & X \\ 0 & 1_2 \end{pmatrix} \right) \mid X = {}^t X, h \in T_\xi\,(F) \right\}$$

where

$$\nu(h) = x - y \quad \text{for } h = \begin{pmatrix} x & y \\ y & x \end{pmatrix} \in T_\xi\,(F).$$

Let P_3 be the maximal parabolic subgroup of GL_6 defined by

$$P_3 = \left\{ \begin{pmatrix} h_1 & X \\ 0 & h_2 \end{pmatrix} : h_1, h_2 \in \mathrm{GL}_3 \right\}.$$

Then we consider a principal series representation

$$I(\Lambda, s) = \left\{ f_s : \mathrm{GL}_6(F) \to \mathbb{C} \mid f_s \left(\left(\begin{smallmatrix} h_1 & X \\ 0 & h_2 \end{smallmatrix} \right) h \right) = \Lambda \left(\frac{\det h_1}{\det h_2} \right) \left| \frac{\det h_1}{\det h_2} \right|^{3s+\frac{3}{2}} f_s(h) \right\}.$$

For $f_s \in I(\Lambda, s)$, $B \in \mathcal{B}_{\xi,\Lambda,\psi}(\pi)$ and $W \in \mathcal{W}(\tau)$, our local zeta integral $Z(f_s, B, W)$ is given by

$$Z(f_s, B, W) = \int_{Z_0(F)H_0(F)\backslash G_0(F)} f_s\,(\theta_0\,\iota\,(g_1, g_2))\, B(g_2) W(g_1)\, dg_1\, dg_2$$

where Z_0 denote the center of G_0 and

$$\theta_0 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & 0 & -1 \end{pmatrix}.$$

As explained above, Theorem C.1 follows by the standard argument if we prove the following lemma.

Lemma C.1. *Let s_0 be an arbitrary point in $\mathbb{C}$. Then we may choose f_s, B and W so that $Z(f_s, B, W)$ has a meromorphic continuation to $\mathbb{C}$ and is holomorphic and non-zero at $s = s_0$.*

Proof. For $\varphi \in C_c^\infty(\mathrm{GL}_6(F))$, we may define $P_s[\varphi] \in I(\Lambda, s)$ by

$$P_s[\varphi](h) = \int_{\mathrm{GL}_3(F)} \int_{\mathrm{GL}_3(F)} \int_{\mathrm{Mat}_{3\times 3}(F)} \varphi \left(\begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix} \begin{pmatrix} 1_3 & X \\ & 1_3 \end{pmatrix} h \right) \times \left| \frac{\det h_1}{\det h_2} \right|^{-3s+\frac{3}{2}} \Lambda \left(\frac{\det h_1}{\det h_2} \right)^{-1} dh_1\, dh_2\, dX.$$

In what follows we construct φ of a special form, whose support is contained in the open double coset $P_3\,(F)\,\theta_0 G_0\,(F)$ in $\mathrm{GL}_6\,(F)$.

Let B_0 be the group of upper triangular matrices in GL_2, and, P_0 the mirabolic subgroup of GL_2, i.e.

$$P_0(F) = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \mid a \in F^\times,\, b \in F \right\}.$$

We define a subgroup M_0 of G by

$$M_0(F) = \left\{ \begin{pmatrix} h & 0 \\ 0 & \lambda \cdot {}^t h^{-1} \end{pmatrix} \mid \lambda \in F^\times, h \in B_0(F) \right\}$$

and $M = \iota(P_0, M_0)$. Then by the Iwasawa decomposition for $G_0(F)$ and the inclusion

$$H_0(F) \subset G_0(F) \cap \theta_0^{-1} P_3(F)\, \theta_0, \tag{C.0.1}$$

we have

$$P_3(F)\, \theta_0 G_0(F) = P_3(F)\, \theta_0 M(F)\, K_0$$

where K_0 is a maximal compact subgroup of $G_0(F)$. We take $K_0 = \iota(K_1, K_2)$ where K_1 (resp. K_2) is a maximal compact subgroup of $\mathrm{GL}_2(F)$ (resp. $G(F)$). By direct computations, we see that

$$\begin{cases} \theta_0\, N(F)\ \theta_0^{-1} \cap P_3(F) = \{1_6\}; \\ \theta_0\, M(F)\ \theta_0^{-1} \cap P_3(F) = \theta_0\, A(F)\ \theta_0^{-1}; \\ \theta_0\, K_0\, \theta_0^{-1} \cap P_3(F) = \{1_6\}, \end{cases}$$

where

$$A(F) = \left\{ \begin{pmatrix} a \cdot 1_3 & \\ & 1_3 \end{pmatrix} : a \in F^\times \right\}.$$

Let us define subgroups T_0, N_0 of G_0 by

$$T_0(F) = \left\{ \iota\left(\begin{pmatrix} a & \\ & 1 \end{pmatrix}, \begin{pmatrix} x & & & \\ & y & & \\ & & \lambda x^{-1} & \\ & & & \lambda y^{-1} \end{pmatrix} \right) : x, y, \lambda \in F^\times \right\};$$

$$N_0(F) = \left\{ \iota\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}, \begin{pmatrix} 1 & y & & \\ & 1 & & \\ & & 1 & \\ & & -y & 1 \end{pmatrix} \right) : x, y \in F \right\}.$$

Then for $\varphi_1 \in C_c^\infty(N_0(F))$, $\varphi_2 \in C_c^\infty(T_0(F))$, $\varphi_3, \varphi_4 \in C_c^\infty(\mathrm{GL}_3(F))$, $\varphi_5 \in C_c^\infty(\mathrm{Mat}_{3\times 3}(F))$ and $\varphi_6 \in C_c^\infty(K_0)$, we may construct $\varphi' \in C_c^\infty(\mathrm{GL}_6(F))$, whose support is contained in $P_3(F)\, \theta_0 G_0(F)$, by

$$\varphi'\left(\begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix} \begin{pmatrix} 1_3 & X \\ 0 & 1_3 \end{pmatrix} \theta_0\, n_0\, t_0\, k \right) = \varphi_6(k)\varphi_3(h_1)\varphi_4(h_2)\varphi_5(X)\varphi_1(n_0) \int_{A(F)} \varphi_2(t_0\, a)\, d^\times a$$

where $n_0 \in N_0(F)$, $t_0 \in T_0(F)$ and $k \in K_0$.

Then the local zeta integral $Z(P_s[\varphi'], B, W)$ is written as

$$Z(P_s[\varphi'], B, W) = \int \varphi' \left(\begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix} \begin{pmatrix} 1_3 & X \\ & 1_3 \end{pmatrix} \iota(n_{0,1}, n_{0,2}) \iota(t_{0,1}, t_{0,2}) \iota(k_1, k_2) \right)$$

$$\times \left| \frac{\det h_1}{\det h_2} \right|^{-3s+\frac{3}{2}} \Lambda \left(\frac{\det h_1}{\det h_2} \right)^{-1} W(n_{0,1} t_{0,1} k_1) B(n_{0,2} t_{0,2} k_2)\, dh_1\, dh_2\, dX\, dn_0\, dt_0\, dk$$

$$= \int \varphi_6(\iota(k_1, k_2)) \varphi_3(h_1) \varphi_4(h_2) \varphi_5(X) \varphi_1(n_0) \varphi_2(t_0 a) \left| \frac{\det h_1}{\det h_2} \right|^{-3s+\frac{3}{2}} \Lambda \left(\frac{\det h_1}{\det h_2} \right)^{-1}$$
$$\times W(n_{0,1} t_{0,1} k_1) B(n_{0,2} t_{0,2} k_2)\, d^\times a\, dh_1\, dh_2\, dX\, dn_0\, dt_0\, dk$$

$$= \int \varphi_6(\iota(k_1, k_2)) \varphi_3(h_1) \varphi_4(h_2) \varphi_5(X) \varphi_1(n_0) \varphi_2(t_0) \left| \frac{\det h_1}{\det h_2} \right|^{-3s+\frac{3}{2}} \Lambda \left(\frac{\det h_1}{\det h_2} \right)^{-1}$$
$$\times \Lambda(\lambda) |\lambda|^{3s-\frac{9}{2}} W \left(n_{0,1} \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix} t_{0,1} k_1 \right) B \left(n_{0,2} \begin{pmatrix} \lambda \cdot 1_2 & 0 \\ 0 & 1_2 \end{pmatrix} t_{0,2} k_2 \right)$$
$$d^\times \lambda\, dh_1\, dh_2\, dX\, dn_0\, dt_0\, dk$$

where we write $n_0 = \iota(n_{0,1}, n_{0,2}) \in N_0(F)$, $t_0 = \iota(t_{0,1}, t_{0,2}) \in T_0(F)$ and $k = \iota(k_1, k_2) \in K_0$. Since we may vary φ_i $(1 \le i \le 6)$, our assertion in Lemma C.1 follows from the same assertion for the integral

$$\int_{F^\times} \Lambda(\lambda) |\lambda|^{3s-\frac{9}{2}} B \begin{pmatrix} \lambda \cdot 1_2 & 0 \\ 0 & 1_2 \end{pmatrix} W \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix} d^\times \lambda. \tag{C.0.2}$$

For any $\phi \in C_c^\infty(F^\times)$, there exists $W_\phi \in W(\tau)$ such that $W_\phi \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} = \phi(a)$ by the theory of Kirillov model for $\mathrm{GL}_2(\mathbb{R})$ by Jacquet [61, Proposition 5] and for $\mathrm{GL}_2(\mathbb{C})$ by Kemarsky [66, Theorem 1]. Thus our assertion clearly holds for the integral (C.0.2). □

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