Proposal of Bipath Persistent Homology: Interval-Decomposability, Visualization, Algorithm, and Stability

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博　士　論　文

Proposal of Bipath Persistent Homology: Interval-Decomposability, Visualization, Algorithm, and Stability

（バイパスパーシステントホモロジーの導入－区間分解可能性、可視化、アルゴリズム、および安定性－）

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Proposal of Bipath Persistent Homology: Interval-Decomposability, Algorithm, and Stability

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Abstract. Persistent homology is one of the main tools in topological data analysis. It can capture the birth and death of topological features of data across a filtration as a form of intervals, and the persistence diagram that collects these intervals allows us to analyze the hidden structure in the data. As an extension of it, multiparameter persistent homology, which can study topological features in multiparameter data, has been proposed. However, we have no discrete and complete invariants for multiparameter persistent homology like persistence diagrams.

The main contribution of this thesis is the proposal of bipath persistent homology, as an extension of persistent homology, which can also capture topological features as a form of intervals. As related topics, we first study interval covers and interval resolution global dimension. We show that the restriction of interval covers of persistence modules to each direct summand is a monomorphism. This result suggests that we can reduce the computation of the interval covers. Then we show the monotonicity theorem of the interval resolution global dimensions. Using this theorem, we can classify finite connected posets with the interval resolution global dimension of zero. In other words, we show that there only exist two classes of finite connected posets, the so-called A-type posets and the bipath posets, whose persistence modules are always interval-decomposable.

In this direction, we study bipath persistent homology, a persistence module over a bipath poset obtained by applying homology functor to a bipath filtration. Thanks to the interval-decomposability of bipath persistence modules, we can easily visualize topological features in a given bipath filtration. We then propose the bipath persistence diagrams and their visualization. We provide examples of bipath persistence diagrams arising from a bipath filtration and a two-parameter filtration of simplicial complexes. In applying bipath persistent homology for analysis, having algorithms for bipath persistence diagrams is essential. So, we propose an algorithm for computing bipath persistence diagrams and provide its implementation. We have made the implementation publicly available. Finally, we show the stability theorem for bipath persistent homology, which suggests that small changes in the bipath function (except for the ends) imply small changes in the bipath persistence diagram. This result is shown by the isometry theorem for persistence modules over the bipath poset.

Contents

CHAPTER 1

Introduction

This chapter gives a background of the theory discussed in this thesis, an overview of our results, and the organization of this thesis. This chapter is divided into three sections. Section 1.1 gives background on the theory of persistent homology. In this section, we first recall some basic ideas of persistent homology and one of the important theorems in persistence theory the so-called stability theorem. Also, we will see other settings, such as zigzag persistent homology and multiparameter persistent homology. Finally, we will recall a formalization of persistent homology using the language of representation theory. Section 1.2 provides an overview of our results in this thesis. These results are about prime tensor ideals, the homological invariants relative to intervals, and bipath persistent homology. These works are from the author's works [**Tad23**], [**AET23**], [**AET25**], and ongoing work on the stability theorem for bipath persistent homology. In Section 1.3, we give the organization of this thesis.

1.1. Persistent Homology

1.1.1. Persistent Homology. *Persistent homology* [**FL99, LF97, Rob99, CZ05**] is one of the main tools in topological data analysis. It can describe the birth and death of topological features (e.g., connected components, holes, cavities, and so on) of data across a *filtration* via a multiset of intervals called the *persistence diagram* of the filtration. This tool has been applied in various fields to analyze hidden data structures: in the fields of material science [**HNH$^+$16**], evolutionary biology [**CCR13**], and others [**Sou11, SPK11**], [**MVS20, JS20, AAF19, BPC$^+$18**].

We recall a construction of a persistent homology and its persistence diagram from a given data.

(1) We first construct a filtration. For example, let $f\colon X \to \mathbb{R}$ be a real-valued function on a topological space X associated with the given data. We give the diagram of topological spaces $S(f)$ as follows:

$$S(f)\colon X_1 \hookrightarrow X_2 \hookrightarrow \cdots \hookrightarrow X_{n-1} \hookrightarrow X_n, \tag{1.1.1}$$

where each X_i is the topological space given by $X_i := f^{-1}((-\infty, i]) \subseteq X$. By construction, we have $X_i \subseteq X_{i+1}$ for $i = 1, \dots, n-1$. The filtration is called the *sublevelset filtration* of f.

(2) We apply qth homology functor (with coefficient field k) to the sublevelset filtration $S(f)$, and we obtain the following diagram of k-vector spaces $H_q(S(f); k)$ as follows:

$$H_q(S(f); k)\colon H_q(X_1; k) \to H_q(X_2; k) \to \cdots \to H_q(X_{n-1}; k) \to H_q(X_n; k), \tag{1.1.2}$$

where each k-linear morphism between $H_q(X_i; k) \to H_q(X_{i+1}; k)$ is induced by the inclusion map $X_i \hookrightarrow X_{i+1}$. We call the diagram of k-vector spaces the persistent homology of the filtration $S(f)$.

(3) If each $H_q(X_i; k)$ is finite dimensional k-vector space, then the persistent homology of the filtration is uniquely decomposed into the so-called *interval modules* up to isomorphisms by a structure theorem [**Gab72**]. The decomposition gives the multiset of interval modules which we call the persistence diagram. These intervals in the multiset contain information about the persistence of topological features (in this case, qth dimensional holes) across the filtration, and we visualize the persistence diagram by plotting each interval as a point on the plane.

Here, when analyzing data using persistent homology, we construct a filtration of simplicial complexes rather than a filtration of topological spaces, such as the one in (1.1.1). As examples of filtrations of simplicial complexes, we have the Čech complex, the Vietoris-Rips complex, and the alpha complex construction, however, the constructions are not discussed in this thesis.

We note that having algorithms for computing persistence diagrams and their implementations is important for applying persistent homology to data analysis. See, for example, [**ELZ02, CZ05, MMS11**], and [**BKRW17, Bau21, ONH22**] for algorithms and implementations, respectively.

In addition, for its application to data analysis, the *stability theorem* [**CSEH07**], [**CCSG$^+$09**], which guarantees that a small change of a real-valued function implies a small change in the persistence diagram of the sublevelset filtration induced by the function, is considered one of the most important results in the theory of persistent homology because it gives a mathematical justification for analyzing noisy data using persistent homology.

1.1.2. Zigzag Persistent Homology. In [**CDS10**], zigzag persistent homology, which is a generalization of standard persistent homology, is proposed for comparing several subsamplings of data and other purposes [**CDS10**]. In this setting, we consider the deletion and addition of spaces instead of considering a nested family of spaces as the standard setting (1.1.1). For example, we consider the following zigzag diagram of spaces:

$$X_1 \hookrightarrow X_1 \cup X_2 \hookleftarrow X_2 \hookrightarrow X_2 \cup X_3 \hookleftarrow X_3 \hookrightarrow \cdots \hookleftarrow X_m.$$

By applying a homology functor to the zigzag diagram, we obtain a zigzag diagram of vector spaces which we call *zigzag persistent homology.* By the structure theorem [**Gab72**], zigzag persistent homology is decomposed into intervals. In particular, we can easily define a persistence diagram for zigzag persistent homology.

Some applications of zigzag persistent homology are considered [**TC11**], [**MNR$^+$23, TMK20, MMKM23**]. In addition, for the use of zigzag persistent homology, algorithms and implementations are developed, for example, see [**CDS10, MO16**], [**DH22, MBGY14**], respectively.

1.1.3. Multiparameter Persistent Homology. In [**CZ09**], multiparameter persistent homology, a generalization of persistent homology, is proposed for analyzing data with multiple parameters. It can be constructed by applying a homology functor to a multiparameter filtration. We display bifiltration (two-parameter case) below:

$$\begin{tikzcd}
X_{1,n} \arrow[r, hook] & X_{2,n} \arrow[r, hook] & \cdots \arrow[r, hook] & X_{m,n} \\
\vdots \arrow[r, hook] \arrow[u, hook] & \vdots \arrow[r, hook] \arrow[u, hook] & \iddots \arrow[r, hook] \arrow[u, hook] & \vdots \arrow[u, hook] \\
X_{1,2} \arrow[r, hook] \arrow[u, hook] & X_{2,2} \arrow[r, hook] \arrow[u, hook] & \cdots \arrow[r, hook] \arrow[u, hook] & X_{m,2} \arrow[u, hook] \\
X_{1,1} \arrow[r, hook] \arrow[u, hook] & X_{2,1} \arrow[r, hook] \arrow[u, hook] & \cdots \arrow[r, hook] \arrow[u, hook] & X_{m,1} \arrow[u, hook]
\end{tikzcd} . \tag{1.1.3}$$

The multiparameter setting naturally arises in various contexts, for example, when dealing with point cloud data containing outliers. As a well-known problem of persistent homology, it is not robust to outliers [**BL23**]. To address this issue, one approach is to measure the density of each point, identify points with low densities as outliers, and remove them. However, an appropriate density threshold for outlier removal is determined manually. Alternatively, by using density as a second parameter for the filtration, we can construct a bifiltration and the two-parameter persistent homology without fixing a specific threshold, allowing for a more flexible analysis.

Also, the multiparameter setting arises when point cloud data is associated with real-valued functions on points, such as curvature [**CZ09**], atomic mass, partial charge, bond type [**DCG$^+$22**], and others.

Here, in contrast to the standard and zigzag persistent homology, the decomposition of multiparameter persistent homology is not parameterized by the intervals in general. In addition, we have no discrete complete invariant for multiparameter settings [**CZ09**]. Thus, topological features obtained by multiparameter persistent homology are more complicated than those of standard and zigzag persistent homology.

One attempt to tackle the difficulty is to construct discrete invariants to capture the persistence of topological features across a multiparameter filtration. One idea is to use intervals which works well in standard and zigzag persistent homology. For example, the fibered barcode [**LW15**], the homological invariant relative to interval modules [**AENY23a**], the compressed multiplicity [**AENY23b**], the generalized rank invariants [**KM21, DKM24**], and others. There are also various other invariants, for example, the rank invariant [**CZ09**], the dimension vector (Hilbert function), the multigraded Hilbert series [**HOST19**], and others.

See also algorithms or implementations of computing invariants, the generalized rank invariants [**DKM24**], the compressed multiplicity [**AENY23b, Esc21**], the fibered barcode (and other invariants) [**LW15, The20**], and others.

1.1.4. A Formalization: Persistence Modules. In this subsection, we recall a formalization of persistent homology discussed in the last subsections using the language of representation theory, see [**CDS10**], [**CdSGO16**], for example.

For a poset P, we call a diagram of k-vector spaces indexed by P a *persistence module* over P or P-module for short. If every k-vector space in the diagram is finite dimensional, then we call the P-module *pointwise finite dimensional* (pfd for short) persistence module over P. We denote by $\mathsf{Rep}_k(P)$ (resp. $\mathsf{Rep}_k^{\mathrm{pfd}}(P)$ or $\mathsf{rep}_k(P)$) the categories of P-modules (resp. pfd P-modules). By a structure theorem [**CB15**], for any poset P and for any pfd P-module is uniquely decomposed into indecomposable modules.

The persistent homology (1.1.2) discussed in the previous section is a persistence module over an *A-type poset* $A_n(e)$ displayed by the Hasse diagram:

$$A_n(e)\colon 1 \longrightarrow \cdots \longrightarrow n.$$

The zigzag persistent homology discussed in the previous section is a persistence module over A-type poset $A_\ell(z)$ displayed by the Hasse diagram:

$$A_\ell(z)\colon 1 \longrightarrow 2 \longleftarrow 3 \longrightarrow 4 \longleftarrow 5 \longrightarrow \cdots \longleftarrow \ell$$

for an appropriate integer ℓ. Gabriel's theorem [**Gab72**] (for A-type) guarantees that every indecomposable module is isomorphic to an interval module. In particular, standard persistent homology and zigzag persistent homology are decomposed into interval modules.

Multiparameter persistent homology can be considered as a persistence module over a commutative grid. For example, a two-dimensional commutative grid is displayed as follows:

$$\begin{tikzcd}
(1,n) \arrow[r, hook] & (2,n) \arrow[r, hook] & \cdots \arrow[r, hook] & (m,n) \\
\vdots \arrow[u, hook] \arrow[r, hook] & \vdots \arrow[u, hook] \arrow[r, hook] & \iddots \arrow[u, hook] \arrow[r, hook] & \vdots \arrow[u, hook] \\
(1,2) \arrow[u, hook] \arrow[r, hook] & (2,2) \arrow[u, hook] \arrow[r, hook] & \cdots \arrow[u, hook] \arrow[r, hook] & (m,2) \arrow[u, hook] \\
(1,1) \arrow[u, hook] \arrow[r, hook] & (2,1) \arrow[u, hook] \arrow[r, hook] & \cdots \arrow[u, hook] \arrow[r, hook] & (m,1) \arrow[u, hook]
\end{tikzcd}.$$

If the commutative grid is large enough, then the category of persistence modules over the commutative grid becomes *wild representation type* [**Les94**],[**LS00**] (see also [**BBOS20**]). This implies that classifying all isomorphism classes of the category is complicated, making multiparameter settings difficult. In particular, multiparameter persistent homology is not generally decomposed into interval modules in contrast to the standard and zigzag settings.

We note that multiparameter (including single parameter) persistent homology can also be understood as graded modules over graded polynomial rings, see [**CZ05**], [**CZ09**] for example). Also, if the poset P is finite, then we can identify persistence modules over P as modules of the incidence algebra $k[P]$ of the poset P over k through a categorical equivalence between $\mathsf{mod}\, k[P]$ and $\mathsf{rep}_k(P)$, see [**ASS06**, Theorem 1.6] for example. However, this will not be discussed in this thesis.

1.2. Overview of Our Results

This section gives an overview of our results discussed in Chapters 3, 4, and 5 of this thesis. These results are from the author's work [**Tad23**], [**AET23**], [**AET25**], and ongoing work on the stability theorem for bipath persistent homology.

1.2.1. A Classification of Prime Tensor Ideals.

In Chapter 3, we give the author's initial work about prime tensor ideals aiming to relate persistence theory with algebraic geometry (tensor triangulated geometry, see [**Bal05, Pet13**] for example) for giving a new tool in persistence theory.

We consider the following class of posets. Let $A_{\mathbb{Z}}(a) := (\mathbb{Z}, \leq_a)$ be a poset whose underling set is $\mathbb{Z}$, and the order $\leq_\alpha$ satisfies $i \leq_a i+1$ or $i+1 \leq_a i$ for any $i \in \mathbb{Z}$. It has the form:

$$A_{\mathbb{Z}}(a)\colon\ \cdots \longleftrightarrow -1 \longleftrightarrow 0 \longleftrightarrow 1 \longleftrightarrow \cdots ,$$

where $\longleftrightarrow$ is either $\rightarrow$ or $\leftarrow$ depending on the order $\leq_a$. We say that the poset $A_{\mathbb{Z}}(a)$ is *bounded* if there exists a positive number n_a such that the number of elements in $\{z \in \mathbb{Z} \mid x \leq_a z \leq_a y\}$ is less than n_a for any $x, y \in \mathbb{Z}$.

For a poset $A_{\mathbb{Z}}(a)$ and the category $\mathsf{Rep}_k^{\mathrm{pfd}}(A_{\mathbb{Z}}(a))$, we have a full subcategory called *prime tensor ideal* (see Chapter 3). The main result of this chapter is a classification of such full subcategories in the category $\mathsf{Rep}_k^{\mathrm{pfd}}(A_{\mathbb{Z}}(a))$ under a condition.

THEOREM 1.2.1. [**Tad23**, Theorem 1.1] *If the poset $A_{\mathbb{Z}}(a)$ is bounded, then there exist bijections among the following:*

(1) The set of prime tensor ideals of the category $\mathsf{Rep}_k^{\mathrm{pfd}}(A_{\mathbb{Z}}(a))$.
(2) The set of prime ideals of the ring $\prod_{i\in\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z})$.
(3) The set of prime ideals of the boolean algebra $\mathcal{P}(\mathbb{Z})$ (the power set of $\mathbb{Z}$).

Moreover, the above correspondence gives homeomorphisms under their respective Zariski topologies, see [**Tad23**, Proposition 3.4] for details.

This result deviates from the main direction of this thesis. However, we include it in Chapter 3 because it also aims to study persistent homology and provides an insight into the main topic.

1.2.2. Relative Homological Algebra.

Recently, there has been an interest in using relative homological algebra for persistence theory [**BOO21, BBH22, HR22, AENY23a, Asa24, CGR$^+$24, BOOS24**]. The work [**BBH22**] introduced a class of invariants called *homological invariants*, which provide a framework for studying several invariants. For example, it showed that the rank invariant, the dimension vector, and the barcode are homological invariants. The work [**AENY23a**] studied *interval cover* and *interval resolution* and proved the finiteness of the interval resolution global dimension. This result implies that the set of all interval modules gives homological invariants.

In Chapter 4, we study the properties of interval cover and interval resolutions. We give our results below.

1.2.2.1. *Summand Injectivity.* We study the properties of interval cover. The main theorem asserts that the restriction of interval cover to each direct summand is a monomorphism.

THEOREM 1.2.2. [**AET23**, Corollary 3.1] *Let P be a finite poset and $f\colon X = \bigoplus_{i=1}^m X_i \to V$ be an interval cover of $V \in \mathsf{rep}_k(P)$, where each X_i is an interval module for $i = 1, \ldots, m$. Then, we have the following.*

(1) *f is an epimorphism.*
(2) *$f|_{X_i}\colon X_i \to V$ is a monomorphism for every $i \in \{1, \ldots, m\}$.*
(3) $\operatorname{supp} X = \operatorname{supp} V$.

In particular, every X_i can be taken as an interval submodule of V.

The importance of Theorem 1.2.2 is that it can reduce the computation of interval covers. Instead of computing the interval cover of V using all the intervals in P, we only need to see interval submodules in V.

1.2.2.2. *Monotonicity Theorem.* We study the relationship between the interval resolution global dimensions of different posets. The main theorem is as follows.

THEOREM 1.2.3. [**AET23**, Theorem 1.2] *Let P be a finite poset. For any full subposet Q of P, the interval resolution global dimension of Q is smaller than or equal to that of P.*

We note that this monotonicity property does not generally hold in other settings. For example, the (usual) global dimension does not have such a monotonicity property, in general [**IZ90**].

The key to this proof is a functor $\Theta : \mathsf{rep}_k(Q) \to \mathsf{rep}_k(P)$ called the *intermediate extension* [**Kuh94**] (the *prolongement intermédiare* in [**BBD82**]), which is defined by using adjoint triple (see Section 4.3.1). For the pair of the restriction functor $\mathrm{Res}\colon \mathsf{rep}_k(P) \to \mathsf{rep}_k(Q)$ and the functor Θ:

$$\begin{tikzcd} \mathsf{rep}_k(P) \arrow[r, bend right, "\mathrm{Res}"'] & \mathsf{rep}_k(Q), \arrow[l, bend right, "\Theta"'] \end{tikzcd}$$

we will see that

- Res and Θ preserve interval-decomposability of modules,
- $\mathrm{Res} \circ \Theta \cong 1_{\mathsf{rep}_k(Q)}$, and
- Res is an exact functor.

Once we have the above properties, we can directly compare their interval resolution global dimensions via the functors and obtain the monotonicity theorem.

1.2.3. Bipath Persistence. In Chapter 5, we introduce and study bipath persistent homology, a framework that allows us to study two filtrations connected at their ends to compare the two filtrations, as a generalization of standard persistent homology.

A *bipath filtration* S is a diagram of spaces displayed by:

$$S\colon \quad \begin{tikzcd}[row sep=small] & S_1 \arrow[r, hook] & S_2 \arrow[r, hook] & \cdots \arrow[r, hook] & S_n \arrow[dr, hook] & \\ S_{-\infty} \arrow[ur, hook] \arrow[dr, hook] & & & & & S_{+\infty} \\ & S_{1'} \arrow[r, hook] & S_{2'} \arrow[r, hook] & \cdots \arrow[r, hook] & S_{m'} \arrow[ur, hook] & \end{tikzcd} .$$

The *bipath persistent homology* of the bipath filtration S is the diagram of vector spaces obtained by applying a homology functor to the bipath filtration. We can see it as a *bipath persistence module*, or a persistence module over the *bipath poset* $B_{n,m}$, where the poset $B_{n,m}$ is displayed by its Hasse diagram in the following way:

$$B_{n,m}\colon \quad \begin{tikzcd}[row sep=small] & 1 \arrow[r] & 2 \arrow[r] & \cdots \arrow[r] & n \arrow[dr] & \\ -\infty \arrow[ur] \arrow[dr] & & & & & +\infty. \\ & 1' \arrow[r] & 2' \arrow[r] & \cdots \arrow[r] & m' \arrow[ur] & \end{tikzcd} \tag{1.2.1}$$

We give motivations for studying bipath persistence modules.

- Any pfd bipath persistence module is interval-decomposable [**AET23**, Theorem 5.1]. It implies that we can easily visualize the persistence of topological features across a bipath filtration.
- As we explained in the last section, the A-type posets also admit the interval decomposability. In fact, there are no other classes of connected finite posets except for the A-type posets and the bipath posets that admit interval-decomposability for arbitrary pfd persistence modules over the posets [**AET23**, Theorem 5.1]. In this sense, our work filled a gap in the literature concerning interval-decomposability.
- Finally, it can be used to study the persistence of topological features across a pair of filtrations connected at their ends, to compare the two filtrations. Furthermore, a bipath persistence module is naturally obtained by a restriction of a multiparameter persistence module. Thus, the bipath persistence module can describe a part of topological features in the multiparameter persistence module in a simple way.

To give a stability theorem for bipath persistence, we also consider the poset B which we also call *bipath poset*, displayed in the following way:

$$B\colon \quad -\infty \;\bullet \qquad \begin{array}{c} \mathbb{R}\times\{1\} \\ \hline \\ \mathbb{R}\times\{2\} \\ \hline \end{array} \qquad \bullet\; +\infty,$$

where the poset B consists of $\mathbb{R}\times\{1,2\}$ with the global minimum $-\infty$ and the global maximum $+\infty$, and the order on $\mathbb{R}\times\{1,2\}$ is given by $(r,i)\leq(s,j)$ if $i=j$ and $r\leq s$.

1.2.3.1. *Interval-Decomposability of Persistence Modules over B.* In Section 5.1, we prove the following.

THEOREM 1.2.4 (Theorem 5.1.1). *Any pfd B-module is uniquely decomposed into a direct sum of interval modules up to isomorphism.*

A technique for proving the interval-decomposability of $B_{n,m}$-modules is given [**AET25**, Section 4]. In this thesis, we apply essentially the same technique to the case of pfd B-modules and prove the theorem. In addition, our proof works for a more general setting, see Remark 5.1.11.

1.2.3.2. *Complete Classification of Finite Posets: Interval Resolution Global Dimension Zero.* In Section 5.2, we give the complete classification of finite connected posets whose interval resolution global dimension is zero as follows:

THEOREM 1.2.5. [**AET23**, Theorem 5.1] *Let P be a finite connected poset with ℓ vertices and k be a field. Then, the following conditions are equivalent.*

(a) *Every P-module in $\mathsf{rep}_k(P)$ is interval-decomposable.*
(b) *Every indecomposable P-module in $\mathsf{rep}_k(P)$ is an interval module.*
(c) *The interval resolution global dimension of P is zero.*
(d) *The poset P is either an A_ℓ-type poset or a bipath poset $B_{n,m}$ for some positive integers $n,m>0$ with $n+m=\ell-2$.*

In particular, these conditions do not depend on the characteristic of the base field k.

1.2.3.3. *A Visualization of Bipath Persistence Diagrams.* Thanks to the interval-decomposability of bipath persistence modules (in particular, persistence modules over $B_{n,m}$), we can define persistence diagrams for bipath persistence modules which we call *bipath persistence diagram* [**AET25**]. We note that bipath persistence diagrams naturally include standard persistence diagrams. In this point of view, we can regard bipath persistent homology as a generalization of standard persistent homology.

1.2.3.4. *A Computational Method for Bipath Persistence Diagrams.* For applications, giving an algorithm for computing bipath persistence diagrams is important. In this section, we provide Algorithm 2 given by [**AET25**]. By the algorithm, we obtain a bipath persistence diagram by two times the decomposition of standard persistent homology and the permissible operations (Definition 5.1.8) of matrices (which will not be very large). This suggests that bipath persistent homology can be applied to real data. We implemented the algorithm and made it publicly available, see [**Tad24**].

1.2.3.5. *Stability Theorem for Bipath Persistence Diagrams.* It is natural to ask whether we have a stability theorem for bipath persistent homology as analogous to the stability theorem of persistent homology. In Section 5.4, we show that a bipath persistence diagram of *bipath function* (Definition 5.4.28) on a topological space (a pair of B-valued functions on a topological space) is stable under some conditions on the bipath function. This theorem guarantees that the small changes in a bipath filtration (except at the ends) imply the small changes in the corresponding bipath persistence diagram. This result suggests the justification of its application to noisy data.

To state our result more precisely, we set the following notation. Let $f=(f_1,f_2)$ be a bipath function. Then, we obtain a *bipath sublevelset filtration* $(f\leq\cdot)\colon B\to\mathsf{Top}$ and bipath persistent homology $V(f)$ of the bipath sublevelset filtration. If $V(f)$ is pointwise finite dimensional, then we call the bipath function f *tame*. For a tame bipath function f, we denote by $\mathscr{B}(V(f))$ a multisets of intervals in B which we call bipath persistence diagram of $V(f)$. We define a *bottleneck distance* d_{B} (Definition 5.4.7) between bipath persistence diagrams. Then, we have the following:

THEOREM 1.2.6 (Stability Theorem for Bipath Persistence 5.4.29). *Let $f = (f_1, f_2)$ and $g = (g_1, g_2)$ be tame bipath functions on a topological space such that*

$$f_1^{-1}(\{-\infty\}) = f_2^{-1}(\{-\infty\}) = g_1^{-1}(\{-\infty\}) = g_2^{-1}(\{-\infty\}).$$

Then we have the following inequality:

$$d_{\mathrm{B}}(\mathscr{B}(V(f)), \mathscr{B}(V(g))) \leq \|f - g\|_\infty.$$

The key to the proof of the stability theorem is the isometry theorem (algebraic stability theorem) for the bipath persistence module, which says that the interleaving distance d_{I} (see Definition 5.4.2) and bottleneck distance are coincident.

THEOREM 1.2.7 (Isometry Theorem for Bipath Persistence Modules 5.4.30). *Let V and W be pfd B-modules. Then we have the following equality:*

$$d_{\mathrm{I}}(V, W) = d'_{\mathrm{B}}(V, W) = d_{\mathrm{B}}(\mathscr{B}(V), \mathscr{B}(W)).$$

Here, d'_{B} is the distance between two bipath persistence modules, which we call *algebraic bottleneck distance* (Definition 5.4.4). The reason we define two distances d_{B} and d'_{B} is as follows. First, since d_{B} is defined in a more combinatorial way than d'_{B}, the author believes that it is easier to understand. On the other hand, the distance d'_{B} is defined to be compatible with the interleaving distance d_{I}, thus, algebraically easy to handle. For the above reason, we consider two distances d_{B} and d'_{B}.

1.3. Organization

Before closing this chapter, we provide the organization of this thesis.

- In Chapter 2, We provide some basics of posets, persistence modules over posets and their structures, interval modules and their basic properties, and persistent homology of filtrations.
- In Chapter 3, we give the author's initial work for prime tensor ideals.
- In Chapter 4, we first provide the basics of relative homological algebra in Section 4.1. Then, we study interval approximation in Section 4.2 and interval resolution global dimension in Section 4.3.
- In Chapter 5, we introduce and study bipath persistent homology. In Section 5.1, give a structure theorem for persistence modules over bipath posets. In Section 5.2, using a result in Section 5.1 and in Section 4.3, we give the complete classification of finite posets whose interval resolution global dimension is 0. In Section 5.3, we define bipath persistence diagrams and their visualization, and we give a computational method of bipath persistence diagram from a bipath filtration of simplicial complexes. Finally, in Section 5.4, we discuss the stability theorem for bipath persistence diagrams.

CHAPTER 2

Preliminaries

In this chapter, We recall basic knowledge of posets, persistence modules over posets and their structures, interval modules and their properties, and persistent homology of filtrations.

Throughout this chapter, we denote by k a field.

2.1. Partially Ordered Sets

We recall the basics of *posets*. Let P be a set. A *partial order* $\leq$ on P is a relation satisfying the following: for any a, b, and c in P

(1) $a \leq a$,
(2) $a \leq b$ and $b \leq a$ imply $a = b$, and
(3) $a \leq b$ and $b \leq c$ imply $a \leq c$.

We call a set P equipped with a partial order $\leq$ a *partially ordered set* or *poset*, and $(P, \leq)$ denotes the poset. When there is no risk of confusion, the order relation $\leq$ is omitted, and we write P for $(P, \leq)$.

The *Hasse diagram* of a finite poset P is a directed graph. Its vertex set is P, and there exists a unique arrow from a to b if, for any $c \in P$, $a \leq c$ and $c \leq b$ imply $c = a$ or $c = b$. In this way, we identify finite posets with finite directed graphs. For a characterization of finite posets using their Hasse diagrams, we recall the definition of the *underlying graph* of a given directed graph G. The underlying graph of G is the undirected graph that has the same vertices as G, and for each arrow $a \to b$ in G, there exists a corresponding edge $a - b$ in the underlying graph.

Example 2.1.1. (1) The Dynkin A_n*-type posets* are the basic posets in this thesis. The underlying graph of the Hasse diagram of A_n-type posets is of the form:

$$1 \text{ --- } 2 \text{ --- } \cdots \text{ --- } n,$$

We denote by $A_n(e)$ the *equioriented* A_n-type poset, which has the Hasse diagram of the form:

$$A_n(e) \colon 1 \longrightarrow \cdots \longrightarrow n. \tag{2.1.1}$$

This is just a totally ordered set with n elements.

(2) The Dynkin D_n-type posets are also considered in this thesis. The underlying graph of the Hasse diagram of D_n-type posets is the form:

$$\begin{tikzcd} & 1 \arrow[d, no head] & & \\ 2 \arrow[r, no head] & 3 \arrow[r, no head] & \cdots \arrow[r, no head] & n. \end{tikzcd} \tag{2.1.2}$$

(3) The extended Dynkin $\tilde{A}_n$-type posets also considered in this thesis. The underlying graph of the Hasse diagram of $\tilde{A}_n$-type poset is of the form:

$$\begin{tikzcd} & & n+1 \arrow[dll, no head] \arrow[drr, no head] & & \\ 1 \arrow[r, no head] & 2 \arrow[r, no head] & 3 \arrow[r, no head] & \cdots \arrow[r, no head] & n. \end{tikzcd} \tag{2.1.3}$$

We note that the following types of directed graphs cannot arise from $\tilde{A}_n$-type posets as the Hasse diagram of the posets by the definitions of posets and Hasse diagrams

$$\begin{tikzcd} & & n+1 & & \\ 1 \arrow[urr] \arrow[r] & \cdots \arrow[r] & \cdots \arrow[r] & \cdots \arrow[r] & n \arrow[ull] \end{tikzcd} \text{ and } \begin{tikzcd} & & n+1 \arrow[dll] & & \\ 1 \arrow[r] & \cdots \arrow[r] & \cdots \arrow[r] & \cdots \arrow[r] & n \arrow[ull] \end{tikzcd} \cdot$$

A *full subposet* Q of P is a subset of P with the induced partial order on P. That is, we have $q_1 \leq q_2$ in Q if $q_1 \leq q_2$ in P.

The following is an example of a full subposet Q of the D_4-type poset P, displayed by the Hasse diagram.

$$
\begin{tikzcd}
 & 1 \arrow[dl] \arrow[dr] & \\
Q: 2 & & 4
\end{tikzcd}
,\quad
\begin{tikzcd}
 & 1 \arrow[d] & \\
P: 2 & 3 \arrow[l] \arrow[r] & 4.
\end{tikzcd}
$$

2.2. Persistence Modules

We recall the basics of persistence modules over posets. Let P be a poset. We identify P with the category, where its objects are elements of P and its morphisms are given by its order, i.e., a unique morphism exists from x to y if $x \leq y$. For posets P and Q, a map $f\colon P \to Q$ is an *order-preserving map* if $x \leq y$ in P implies $f(x) \leq f(y)$ in Q for any $x \leq y$ in P. An order-preserving map between P and Q is a functor between two posets.

A *persistence module* over P (or P-module for short) is an object in the functor category $\mathsf{Rep}_k(P)$ from the poset P (as a category) to the category of k-vector spaces. Explicitly, a P-module V consists of a family of vector spaces $\{V_p\}_{p\in P}$ and a family of k-linear morphisms $\{V(p,q)\colon V_p \to V_q\}_{p\leq q\in P}$ such that $V(q,r)\circ V(p,q) = V(p,r)$ and $V(p,p) = \mathrm{id}_{V_p}$ for any $p \leq q \leq r$ in P. A morphism between P-modules is a natural transformation. Explicitly, a morphism between V and W is a family of k-linear morphisms $f = \{f_p\colon V_p \to W_p\}_{p\in P}$ such that for any $p \leq q$ in P the following diagram commutes:

$$
\begin{tikzcd}
V_p \arrow[r, "V(p{,}q)"] \arrow[d, "f_p"'] \arrow[dr, phantom, "\circlearrowleft"] & V_q \arrow[d, "f_q"] \\
W_p \arrow[r, "W(p{,}q)"] & W_q.
\end{tikzcd}
$$

We denote by $\mathrm{Hom}_P(V,W)$ the collection of morphisms from V to W. For $(g,f) \in \mathrm{Hom}_P(W,Z) \times \mathrm{Hom}_P(V,W)$, the composition of f and g is given by $g\circ f := \{g_p \circ f_p\}_{p\in P} \in \mathrm{Hom}_P(V,Z)$. For a P-module V, the *identity morphism*, or simply identity is given by the family of k-linear morphisms $\mathrm{id}_V := \{\mathrm{id}_{V_P}\colon V_p \to V_p\}_{p\in P}$, where $\mathrm{id}_{V_P}\colon V_p \to V_p$ is the identity map for each p in P. We have $g \circ \mathrm{id}_V = g$ and $\mathrm{id}_V \circ h = h$ if the compositions are defined respectively.

We say that a morphism $f\colon V \to W$ is an *epimorphism* (resp. *monomorphism*) if $f_p\colon V_p \to W_p$ is surjective (resp. injective) for any $p \in P$. In addition, we call the P-module V (resp. W) a *quotient module* (resp. *submodule*) of the P-module W. We say that a P-module V is isomorphic to a P-module W if there exists a morphism $f\colon V \to W$ and $g\colon W \to V$ such that $f \circ g = \mathrm{id}_W$ and $g \circ f = \mathrm{id}_V$. We call the morphisms f and g *isomorphisms*. If there exists an isomorphism between V and W, we write $V \cong W$. We note that a morphism is an isomorphism if and only if the morphism is an epimorphism and a monomorphism.

We say that a P-module V is a *zero module* if V_p is a zero vector space, and $V(p,q)$ is zero morphisms for any $p \leq q$ in P. We denote by 0 a zero module.

We define a kernel, cokernel, and image of a morphism using the universal properties. This will simplify the mathematical discussion in Chapter 4. A *kernel* of a morphism $f\colon V \to W$ is $\alpha\colon X \to V$ having the following universal property:

- $f \circ \alpha = 0$.
- For any $h\colon W \to V$ with $f \circ h = 0$, there exist a unique morphism $g\colon W \to X$ such that $h = \alpha \circ g$.

$$
\begin{tikzcd}
X \arrow[r, "\alpha"] & V \arrow[r, "f"] & W \\
W \arrow[u, dotted, "g"] \arrow[ur, "h"'] & &
\end{tikzcd}
$$

We denote by $\ker(f)\colon \mathrm{Ker}(f) \to V$ a kernel of the morphism $f\colon V \to W$. A kernel of a morphism is unique up to isomorphism by construction. For our purpose, we explicitly give a kernel $\mathrm{Ker}(f)$ of the morphism $f\colon V \to W$ as follows. $\mathrm{Ker}(f)$ is given by the family of the vector spaces $\{\mathrm{Ker}(f)_p :=$

$\mathrm{Ker}(f_p)\}_{p\in P}$ and the family of the k-linear morphisms $\{\mathrm{Ker}(f)(p,q)\colon \mathrm{Ker}(f_p) \to \mathrm{Ker}(f_q)\}_{p\leq q\in P}$, where each $\mathrm{Ker}(f)(p,q)\colon \mathrm{Ker}(f_p) \to \mathrm{Ker}(f_q)$ is induced by the universal property of the kernel of f_q for $p \leq q$ in P (see the diagram below). The morphism $\ker(f)\colon \mathrm{Ker}(f) \to V$ is given by the family of inclusion maps $\{\ker(f_p)\colon \mathrm{Ker}(f_p) \to V_p\}_{p\in P}$.

$$\begin{tikzcd}
\mathrm{Ker}(f_p) \arrow[r, tail, "\ker(f_p)"] \arrow[d, dotted, "\mathrm{Ker}(f)(p,q)"'] \arrow[dr, phantom, "\circlearrowleft"] & V_p \arrow[r, "f_p"] \arrow[d, "V(p,q)"] \arrow[dr, phantom, "\circlearrowleft"] & W_p \arrow[d, "W(p,q)"] \\
\mathrm{Ker}\, f_q \arrow[r, tail, "\ker(f_q)"] & V_q \arrow[r, "f_q"] & W_q.
\end{tikzcd}$$

Dually, we have a *cokernel* of the morphism f. An *image* $\mathrm{Im}(f)$ of the morphism f is given by the cokernel of the kernel of f. Explicitly, it is given by the family of vector spaces $\{(\mathrm{Im}\, f)_p := \mathrm{Im}\, f_p\}_{p\in P}$ and the family of k-linear morphisms $\{(\mathrm{Im}\, f)(p,q)\colon \mathrm{Im}\, f_p \to \mathrm{Im}\, f_q\}_{p\leq q\in P}$. Here, each k-linear morphism $(\mathrm{Im}\, f)(p,q)\colon \mathrm{Im}\, f_p \to \mathrm{Im}\, f_q$ is induced by the universal property of the cokernel of the kernel of f_q. The following commutative diagram displays an image of a morphism f:

$$\begin{tikzcd}
\mathrm{Ker}(f_p) \arrow[r, tail, "\ker(f_p)"] \arrow[d, "\mathrm{Ker}(f)(p,q)"'] \arrow[dr, phantom, "\circlearrowleft"] & V_p \arrow[r, two heads, "\pi_p"] \arrow[d, "V(p,q)"] \arrow[dr, phantom, "\circlearrowleft"] & \mathrm{Im}(f_p) \arrow[r, tail, "\mathrm{im}\, f_p"] \arrow[d, dotted, "\mathrm{Im}(f)(p,q)"] \arrow[dr, phantom, "\circlearrowleft"] & W_p \arrow[d, "W(p,q)"] \\
\mathrm{Ker}\, f_q \arrow[r, tail, "\ker(f_q)"] & V_q \arrow[r, two heads, "\pi_q"] & \mathrm{Im}(f_q) \arrow[r, tail, "\mathrm{im}\, f_q"] & W_q,
\end{tikzcd}$$

where $\mathrm{im}\, f_p \circ \pi_p = f_p$ for any $p \in P$. We note that the morphism f factors through its image. We have the following commutative diagram:

$$\begin{tikzcd}
V \arrow[rr, "f"] \arrow[dr, two heads, "\pi"'] & \circlearrowleft & W \\
& \mathrm{Im}(f) \arrow[ur, tail, "\mathrm{im}(f)"'] &
\end{tikzcd}$$

For a family of P-modules $\{V_\lambda\}_{\lambda\in\Lambda}$, the direct sum $\bigoplus_{\lambda\in\Lambda} V_\lambda$ is given by

$$(\bigoplus_{\lambda\in\Lambda} V_\lambda)_p := \bigoplus_{\lambda\in\Lambda}(V_\lambda)_p \text{ and } (\bigoplus_{\lambda\in\Lambda} V_\lambda)(p,q)\colon \bigoplus_{\lambda\in\Lambda}(V_\lambda)_p \to \bigoplus_{\lambda\in\Lambda}(V_\lambda)_q, (v_\lambda)_{\lambda\in\Lambda} \mapsto (V_\lambda(p,q)(v_\lambda))_{\lambda\in\Lambda}.$$

We say that a P-module V is an *indecomposable module* if $V \cong V' \oplus V''$ implies $V' = 0$ or $V'' = 0$.

For a P-module V, we call the subset

$$\mathrm{supp}(V) := \{p \in P \mid V_p \neq 0\}$$

of P the *support* of V.

In this thesis, we consider the full subcategory $\mathsf{Rep}_k^{\mathrm{pfd}}(P)$ of $\mathsf{Rep}_k(P)$ consisting of P-modules V such that V_p is finite dimensional k-vector space for every $p \in P$. We call P-modules in $\mathsf{Rep}_k^{\mathrm{pfd}}(P)$ *pointwise finite dimensional* P-modules (pfd P-modules for short). Following standard notation, we write $\mathsf{rep}_k(P)$ for $\mathsf{Rep}_k^{\mathrm{pfd}}(P)$ if the poset P is finite.

For pfd persistence modules over posets, we have the following characterization of indecomposable modules.

Theorem 2.2.1 (cf. [**BCB20**, Theorem 1.1]). *Let V be a pfd P-module. Then, V is an indecomposable module if and only if the endomorphism ring $\mathrm{End}_P(V)$ is local.*

Moreover, we can say that any pfd P-module is uniquely decomposed into indecomposable pfd P-modules up to isomorphism and permutations of terms.

Theorem 2.2.2 ([**BCB20**, Theorem 1.1], [**Azu50**, Theorem 1 (ii)]). *For any pfd P-module V, there exists a family of indecomposable pfd P-modules $\{V_\lambda\}_{\lambda\in\Lambda}$ such that $V \cong \bigoplus_{\lambda\in\Lambda} V_\lambda$. If $V \cong \bigoplus_{\gamma\in\Gamma} V'_\gamma$ and each V'_γ is a non-zero indecomposable pfd P-module, then there exists a bijection $\sigma\colon \Lambda \to \Gamma$ such that $V_\lambda \cong V'_{\sigma(\lambda)}$.*

For a poset P, we denote by $\mathsf{ind}(P)$ the isomorphism classes of indecomposable modules in $\mathsf{Rep}_k^{\mathrm{pfd}}(P)$. Thanks to Theorem 2.2.2, we have the following map associated with V in $\mathsf{Rep}_k^{\mathrm{pfd}}(P)$:

$$m_V \colon \mathsf{ind}(P) \to \mathbb{Z}_{\geq 0}$$

satisfying $V \cong \bigoplus_{V' \in \mathsf{ind}(P)} (V')^{m_V(V')}$. Using the notation, we denote by $\mathscr{B}(V)$ the multiset given by

$$\mathscr{B}(V) := \{V' \in \mathsf{ind}(P) \text{ with the multiplicity } m_V(V')\}.$$

2.2.1. Interval Modules. In this subsection, we recall interval modules, which are central objects in this thesis, and we see their properties.

DEFINITION 2.2.3. Let I be a non-empty subset of a poset P. The subset I is *interval* if it satisfies the following two conditions:

(1) *Convexity.* For any elements $x \leq y \in I$ and $z \in P$, if we have $x \leq z \leq y$ in P, then $z \in I$ holds.
(2) *Connectivity.* For any elements a and b in I, there exists a sequence of elements $a = c_0, c_1, \ldots, c_\ell = b$ in I such that c_{i-1} and c_i are comparable for every $i \in \{1, \ldots, \ell\}$.

We denote by $\mathbb{I}(P)$ the set of all intervals in P.

We note that the definition of intervals is different from that of *segments*. Any segment, which is of the form $[a, b] := \{p \in P \mid a \leq p \leq b\}$ for some $a \leq b$, is an interval, however, the converse fails in general.

The following class of P-modules are central objects in this thesis.

DEFINITION 2.2.4. Let P be a poset and I be an interval in P. We define the persistence module k_I by

$$(k_I)_a := \begin{cases} k & \text{if } a \in I \\ 0 & \text{otherwise,} \end{cases} \qquad k_I(a, b) := \begin{cases} \mathrm{id}_k & \text{if } a, b \in I \\ 0 & \text{otherwise.} \end{cases} \tag{2.2.1}$$

An *interval module* is a P-module isomorphic to k_I for an interval I of P.

We say that a P-module is *interval-decomposable* if it is isomorphic to $\bigoplus_{I \in \mathbb{I}(P)} k_I^{m_I}$ where m_I is a non-negative integer for each I. By definition, we the zero module 0 is interval-decomposable.

P-modules are not necessarily interval-decomposable in general, however, if P is of a specific form, then any P-module can be decomposed into interval modules. For example, the following results are known.

PROPOSITION 2.2.5 (cf. Gabriel's theorem [**Gab72**]). *Let P be an A_n-type poset for some positive integer n. Any P-module is uniquely decomposed into a direct sum of interval modules, up to isomorphism and permutations.*

We also recall the following theorem.

THEOREM 2.2.6 ([**BCB20**, Theorem 1.2]). *Let P be a totally ordered set. Any pfd P-module is uniquely decomposed into a direct sum of interval modules up to isomorphism and permutations.*

Next, we provide the basic propositions of interval modules given in [**AENY23a**, Lemma 4.4] and [**Asa24**, Lemma 4.8]. The propositions are given for interval modules over finite posets, but they work for the case of infinite posets.

PROPOSITION 2.2.7 (cf. [**AENY23a**, Lemma 4.4], [**Asa24**, Lemma 4.8]). *Let P be a poset. Then we have the following:*

(1) *Any quotient module of an interval module in $\mathsf{Rep}_k^{\mathrm{pfd}}(P)$ is interval-decomposable.*
(2) *Any submodule of an interval module in $\mathsf{Rep}_k^{\mathrm{pfd}}(P)$ is interval-decomposable.*

Next, we consider morphisms between interval modules. Let I and J be intervals in P. We denote by $\Omega(I, J)$ the set of connected components C of $I \cap J$ satisfying the following:

$$I \cap C^{\downarrow} \subseteq C \text{ and } J \cap C^{\uparrow} \subseteq C, \tag{2.2.2}$$

where we define $A^{\uparrow} := \{p \in P \mid \exists\, a \in A,\ a \leq p\}$ and $A^{\downarrow} := \{p \in P \mid \exists\, a \in A,\ p \leq a\}$ for a subset A of P. Then we have the following.

Proposition 2.2.8 ([**BBH22**, Proposition 5.5]). *Let I and J be intervals in P. If $\Omega(I, J) \neq \emptyset$, then for $C \in \Omega(I, J)$, we have the morphism $\phi_C \colon k_I \to k_J$ given by*

$$(\phi_C)_a := \begin{cases} \mathrm{id}_k & \text{if } a \in C \\ 0 & \text{else.} \end{cases}$$

Moreover, $\{\phi_C \mid C \in \Omega(I, J)\}$ is a basis of $\mathrm{Hom}_P(k_I, k_J)$.

Corollary 2.2.9. *Let P be a poset and I be an interval in P. Then, the interval module k_I is indecomposable.*

Proof. By Proposition 2.2.8, we have $\mathrm{End}(k_I) \cong k$. To prove k_I is indecomposable, we assume that there exist non-zero P-modules V' and V'' with $k_I \cong V' \oplus V''$. Then we have the following contradiction:

$$2 \leq \dim_k \mathrm{End}_P(V', V') + \dim_k \mathrm{End}_P(V'', V'') \leq \dim_k \mathrm{End}_P(k_I) = 1.$$

Thus, k_I is an indecomposable module. □

Finally, we construct a functor $\mathsf{Rep}_k^{\mathrm{pfd}}(Q) \to \mathsf{Rep}_k^{\mathrm{pfd}}(P)$ from a given order-preserving map from P to Q, and we show that the functor sends interval modules to interval-decomposable modules.

Let $h \colon P \to Q$ be an order-preserving map. Then, we have the k-linear functor

$$(\cdot\ \circ h) \colon \mathsf{Rep}_k^{\mathrm{pfd}}(Q) \to \mathsf{Rep}_k^{\mathrm{pfd}}(P), V \mapsto V \circ h,$$

where $V \in \mathsf{Rep}_k^{\mathrm{pfd}}(Q)$ is given by $\{(V \circ h)_p := V_{h(p)}\}_{p \in P}$ and $\{(V \circ h)(p, p') \colon (V \circ h)_p \to (V \circ h)_{p'}\}_{p \leq p' \in P}$. Also the functor sends a morphism $f = \{f_p \colon V_p \to W_p\}_{p \in P} \colon V \to W$ to $f \circ h = \{f_{h(p)} \colon (V \circ h)_p \to (W \circ h)_p\}_{p \in P} \colon V \circ h \to W \circ h$.

The above notation is natural since the pfd P-module $V \circ h$ is obtained by the composition of the functor $V \colon Q \to \mathrm{vect}_k$ and the functor $h \colon P \to Q$, where vect_k is the category of finite-dimensional k-vector spaces. We display the composition below.

$$\begin{tikzcd} P \arrow[r, "V \circ h"] \arrow[d, "h"'] & \mathrm{vect}_k \\ Q \arrow[ur, "V"'] & \end{tikzcd}$$

Lemma 2.2.10. *Let $h \colon P \to Q$ be an order-preserving map between posets. Let I be an interval in Q. If $h^{-1}(I)$ is not empty, then $h^{-1}(I)$ is convex in P. In particular, if h is surjective, then $h^{-1}(I)$ is convex since $h^{-1}(I)$ is not empty.*

Proof. We assume $h^{-1}(I)$ is not empty. If we have $a \leq x \leq b$ in P with $a, b \in h^{-1}(I)$, then $h(a) \leq h(x) \leq h(b)$ holds. Since I is convex and both $h(a)$ and $h(b)$ are in I, we have $h(x) \in I$. Thus, $x \in h^{-1}(I)$. This implies that $h^{-1}(I)$ is convex in P. □

Corollary 2.2.11. *Let $h \colon P \to Q$ be an order-preserving map between posets. Then the functor $(\cdot \circ h) \colon \mathsf{Rep}_k^{\mathrm{pfd}}(Q) \to \mathsf{Rep}_k^{\mathrm{pfd}}(P), V \mapsto V \circ h$ sends interval modules to interval-decomposable modules.*

Proof. Let I be an interval in Q. We show that $k_I \circ h$ is interval-decomposable. Let $\{C_a \subseteq P \mid a \in A\}$ be the set of the connected components of $h^{-1}(I)$. If the set $\{C_a \subseteq P \mid a \in A\}$ is empty, then we have $k_I \circ h = 0$. We assume that $\{C_a \subseteq P \mid a \in A\}$ is not empty. Then, each C_a is convex by Lemma 2.2.10. Thus, they are connected and convex, which implies they are intervals in P. We show that the P-module $k_I \circ h$ is isomorphic to $\bigoplus_{a\in A} k_{C_a}$. For any $p \leq p' \in P$, we have

$$\begin{aligned}(k_I \circ h)_p &= \begin{cases} k & \text{if } h(p) \in I \\ 0 & \text{else,} \end{cases} \\ &= \begin{cases} k & \text{if } p \in h^{-1}(I) = \bigsqcup_{a\in A} C_a \\ 0 & \text{else,} \end{cases}, \\ &= (\bigoplus_{a\in A} k_{C_a})_p,\end{aligned}$$

$$\begin{aligned}(k_I \circ h)(p,p') &= (k_I)(h(p), h(p')) \\ &= \begin{cases} \mathrm{id}_k & \text{if } h(p), h(p') \in I \\ 0 & \text{else,} \end{cases} \\ &= \begin{cases} \mathrm{id}_k & \text{if } p, p' \in h^{-1}(I) = \bigsqcup_{a\in A} C_a \\ 0 & \text{else,} \end{cases} \\ &= (\bigoplus_{a\in A} k_{C_a})(p,p'),\end{aligned}$$

where we note that if $p \leq p' \in h^{-1}(I)$, then there exists a unique connected component C_a such that p and p' are in C_a. □

Under certain conditions on $h\colon P \to Q$, the functor $(\cdot \circ h)\colon \mathsf{Rep}_k^{\mathrm{pfd}}(Q) \to \mathsf{Rep}_k^{\mathrm{pfd}}(P)$ sends interval modules to interval modules.

Proposition 2.2.12. *Let $h\colon P \to Q$ be an order-preserving map. If the functor $(\cdot\circ h)\colon \mathsf{Rep}_k^{\mathrm{pfd}}(Q) \to \mathsf{Rep}_k^{\mathrm{pfd}}(P)$ is fully faithful, then, we have the following for any Q-module V.*

(1) *$\mathscr{B}(V) \to \mathscr{B}(V \circ h), V' \mapsto V' \circ h$ is a bijection map.*
(2) *In addition, if we have an order-preserving maps $\iota\colon Q \to P$ with $h \circ \iota = \mathrm{id}_Q$, then for V' in $\mathscr{B}(V)$, the Q-module V' is an interval module if and only if $V' \circ h \in \mathscr{B}(V')$ is an interval module.*

Proof. We first show (1). Let V' be an indecomposable module in $\mathscr{B}(V)$. Then, by Theorem 2.2.1, the endmorphism ring $\mathrm{End}_P(V')$ is local. On the other hand, since $(\cdot\circ h)$ is a fully faithful k-linear functor by our assumption, we have the ring isomorphism $\mathrm{End}_P(V') \cong \mathrm{End}_P(V' \circ h), f \mapsto f \circ h$. By the above discussion, $\mathrm{End}_P(V' \circ h)$ is local. By Theorem 2.2.1, $V' \circ h$ is indecomposable. In particular, the correspondence $\mathscr{B}(V) \to \mathscr{B}(V \circ h), V' \mapsto V' \circ h$ induces a bijection map.

Next, we show (2). If $V' \circ h \in \mathscr{B}(V)circh$ is an interval module, then by Corollary 2.2.11,

$$(V' \circ h) \circ \iota = V' \circ \mathrm{id}_Q = V'$$

is interval-decomposable. Since V' is an indecomposable module, V' is an interval module. Conversely, if $V' \in \mathscr{B}(V)$ is an interval module, then $V \circ h$ is an interval-decomposable module by Corollary 2.2.11. Since it is an indecomposable module by (1), $V \circ h$ is an internal module. This completes the proof. □

2.3. Persistent Homology of Filtrations

Finally, we recall the persistent homology of filtrations, which can describe the persistence of topological features (e.g., connected components, holes, cavities, and so on) across the filtrations. We will relate them with our works in Chapter 5.

For a poset P, a *P-filtration* is a functor $S\colon P \to \mathsf{Top}$ such that $S(p,p')\colon S_p \hookrightarrow S_{p'}$ is an inclusion map for every $p \leq p'$ in P, where Top is the category of topological spaces. Applying the qth homology functor to S (with coefficient field k), we obtain a P-module $H_q(S;k)$ called the *persistent homology* of the filtration S. Explicitly, it consists of the family of k-vector spaces $\{H_q(S;k)_p := H_q(S_p;k)\}_{p\in P}$ and the family of k-linear morphisms $\{H_q(S;k)(p,p')\colon H_q(S;k)_p \to H_q(S;k)_{p'}\}_{p\leq p'\in P}$, where each k-linear morphism is induced by the inclusion map $S_p \hookrightarrow S_{p'}$ for $p \leq p'$ in P.

For example, we can construct a persistent homology of a filtration as follows. Let P be a poset and X be a topological space. For a P-valued function. $f\colon X \to P$, we have the P-filtration $(f \leq \cdot)\colon P \to \mathsf{Top}$ given by

$$\{(f \leq p) := \{x \in X \mid f(x) \leq p\}\}_{p\in P} \text{ and } \{(f \leq p) \hookrightarrow (f \leq p')\}_{p\leq p'\in P}.$$

We denote by $V_q(f)$ the P-module $H_q((f \leq \cdot);k)$.

The persistent homology $H_q(S;k)$ of a given filtration S contains information about the persistence of topological features across the filtration. Indeed, for each $p \in P$, homology generators of the k-vector space $H_q(S;k)_p$ represent the existence of qth dimensional holes in $S(p)$, and the family of k-linear morphisms $\{H_q(S; k(p,p'))\}_{p\leq p'\in P}$ tracks the persistence of these homology generators across the filtration.

In general, persistent homology of a given filtration can be complex to understand the persistence of topological features across the filtration $S\colon P \to \mathsf{Top}$ (e.g., multiparameter filtration (1.1.3)). However, if the poset P is of a specific form such as totally ordered set, then we can describe persistent homology of filtrations compactly, simply, and hence interpretable way using persistence diagrams. Below, we will see that the interval-decomposability of persistence modules over a totally ordered set (Theorem 2.2.6) plays an important role in such a simple description.

2.3.1. Single Parameter Persistent Homology. In this subsection, we recall a setting of the single parameter persistent homology using a simple example (Example 2.3.1). In this setting, we can describe persistent homology in an interpretable way.

Below, we consider the case $P = \mathbb{R}$, or the case of single parameter persistent homology. We first recall that, by Theorem 2.2.6, the set of isomorphism classes of indecomposable modules in $\mathsf{Rep}_k^{\mathrm{pfd}}(\mathbb{R})$ is given by $\mathsf{ind}(\mathbb{R}) = \{k_I \mid I \in \mathbb{I}(\mathbb{R})\}$. Thus, for a pfd $\mathbb{R}$-module V, we have the decomposition

$$V \cong \bigoplus_{V'\in\mathsf{ind}(\mathbb{R})} (V')^{m_V(V')} \cong \bigoplus_{I\in\mathbb{I}(\mathbb{R})} k_I^{m_V(k_I)}.$$

Hence, we identify $\mathscr{B}(V) = \{k_I \text{ with the multiplicity } m_V(k_I)\}$ with the multiset of intervals

$$\{I \in \mathbb{I}(\mathbb{R}) \text{ with the multiplicity } m_V(k_I)\}.$$

We call the multiset $\mathscr{B}(V)$ the *persistence diagram* of V. Thanks to the interval-decomposability, we can visualize the persistence diagram by plotting each interval as a point on the plane, with its given multiplicity. For example, consider the persistence diagram $\{[1,2),[1,2),[4,5)\}$. We plot points at $(1,2)\in\mathbb{R}^2$ with multiplicity two and $(4,5)\in\mathbb{R}^2$ with multiplicity one on the plane.

For an $\mathbb{R}$-filtration S, we have the persistence diagram $\mathscr{B}(H_q(S;k))$. Each interval, say $[b,d)$, in $\mathscr{B}(H_q(S;k))$ corresponds to the persistence of a homology generator across the filtration, which is born at $r = b$ and dies at $r = d$ (or, the homology generator is sent to zero by $H_q(S;k)(b,d)$). In this case, the value $d - b$ represents the lifespan of the homology generator. We see the longer intervals as robust topological features, and we are generally regarded as shorter intervals as noise. The longer intervals are plotted as points on the upper or left side of the plane, and shorter intervals are plotted as points closer to the diagonal in the plane.

Now, we give a simple example of a persistent homology of an $\mathbb{R}$-filtration.

Example 2.3.1. Let $X \subseteq \mathbb{R}^2$ be the set of seven points

$$\{(\cos(\pi i/3),\ \sin(\pi i/3)) \in \mathbb{R}^2 \mid i = 0,1,2,3,4,5\} \cup \{(0,\sqrt{3})\},$$

displayed in Fig. 1. For the real-valued function $f_X\colon \mathbb{R}^2 \to \mathbb{R}, a \mapsto \min\{||x-a|| \mid x \in X\}$, we give the $\mathbb{R}$-filtration $(f_X \leq \cdot)$ as follows:

$$(f_X \leq r) := \bigcup_{x\in X} B(x;r),$$

where $B(x;r) := \{a \in \mathbb{R}^2 \mid ||x-a|| \leq r\}$.

We observe the following by looking at Fig. 1. At $r = 1/2$, a smaller hole is born, which is formed by three balls whose centers are at $(0,\sqrt{3})$, $(\cos(\pi/3),\sin(\pi/3))$, and $(\cos(2\pi/3),\sin(2\pi/3))$. At $r = \sqrt{3}/3$ ($\fallingdotseq 0.5773\cdots$), the hole dies. We record the birth and death of this hole using the interval $[1/2,\sqrt{3}/3)$, which represents its persistence across the filtration. Similarly, at $r = 1/2$, a bigger hole is born, which is formed by the six balls whose centers are given by $\{(\cos(\pi i/3),\sin(\pi i/3)) \in \mathbb{R}^2 \mid i = 0,1,2,3,4,5\}$. At $r = 1$, the holes die. We record its persistence by the interval $[1/2,1)$.

The persistence of the holes can be formalized using $\mathbb{R}$-modules. We take the 1st homology of the $\mathbb{R}$-filtration $(f_X \leq \cdot)$, and we obtain the persistent homology $V_1(f_X) \cong k_{[1/2,\sqrt{3}/3)} \oplus k_{[1/2,1)}$ and its persistence diagram $\{k_{[1/2,\sqrt{3}/3)}, k_{[1/2,1)}\}$. We recall that each interval module is associated with a homology generator. By choosing an appropriate representative, we can interpret the interval modules $k_{[1/2,\sqrt{3}/3)}$ and $k_{[1/2,1)}$ as corresponding to the smaller hole and the bigger hole, respectively.

We visualize the persistence diagram, see Fig. 2.

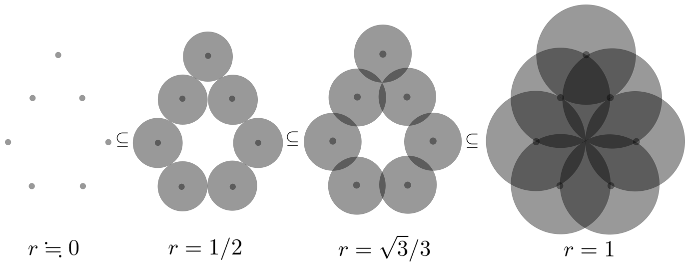

FIGURE 1. A visualization of the filtration $(f_X \leq \cdot)$ at $r \risingdotseq 0$, $r = 1/2$, $r = \sqrt{3}/3$, and $r = 1$:

$$(f_X \leq r \risingdotseq 0) \subseteq (f_X \leq 1/2) \subseteq (f_X \leq \sqrt{3}/3) \subseteq (f_X \leq 1).$$

At $r = 1/2$, a smaller hole and a bigger hole are formed. At $r = \sqrt{3}/3$, the smaller hole dies. At $r = 1$, the bigger hole dies.

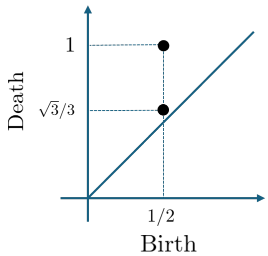

FIGURE 2. A visualization of the persistence diagram of the filtration $(f_X \leq \cdot)$ in Example 2.3.1. The intervals $[1/2, \sqrt{3}/3)$ and $[1/2, 1)$ correspond to the points $(1/2, \sqrt{3}/3)$ and $(1/2, 1)$ respectively. Even for more complex examples, we can compactly summarize the persistence of topological features across the filtration similarly.

CHAPTER 3

A Classification of Prime Tensor Ideals

In this chapter, we give the author's initial work [**Tad23**, Theorem 1.1] aiming to introduce a new research method for multiparameter persistent homology and to relate algebraic geometry (tensor triangulated geometry [**Bal05, Pet13**]) with persistence theory.

We first recall the basics of tensor categories (monoidal categories) and prime tensor ideals. A tensor category $\nu = (\nu, \otimes, I, a, \ell, r)$ is a category ν with a functor $\otimes : \nu \times \nu \to \nu$, an object I in ν, and three natural isomorphisms given by, $a_{X,Y,Z} : (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$, $\ell_X : I \otimes X \to X$, and $r_X : X \otimes I \to X$ such that

$$\begin{tikzcd}
((W \otimes X) \otimes Y) \otimes Z \arrow[rr, "a"] \arrow[dr, "a\otimes 1"'] & & (W \otimes X) \otimes (Y \otimes Z) \arrow[rr, "a"] & & W \otimes (X \otimes (Y \otimes Z)) \\
& (W \otimes (X \otimes Y)) \otimes Z \arrow[rr, "a"] & & W \otimes ((X \otimes Y) \otimes Z) \arrow[ur, "1\otimes a"'] &
\end{tikzcd}$$

and

$$\begin{tikzcd}
(X \otimes I) \otimes Y \arrow[rr, "a"] \arrow[dr, "r\otimes 1"'] & & X \otimes (I \otimes Y) \arrow[dl, "1\otimes \ell"] \\
& X \otimes Y &
\end{tikzcd}$$

commute for any W, X, Y, and Z in ν.

A full subcategory $\mathcal{M}$ of an abelian category $\mathcal{A}$ is said to be *coherent* if it is closed under kernels, cokernels, and extensions. Here, we say that $\mathcal{M}$ is closed under extensions if for any short exact sequence in $\mathcal{A}$

$$0 \to M \to N \to M' \to 0$$

with $M, M' \in \mathcal{M}$, we have $N \in \mathcal{M}$.

Suppose that $\mathcal{A}$ is a tensor abelian category. A *tensor ideal* is a coherent subcategory $\mathcal{M}$ such that $\mathcal{M} \otimes \mathcal{A} \subseteq \mathcal{A}$. A proper tensor ideal $\mathcal{M} \subsetneq \mathcal{A}$ is called *prime ideal* if $A \otimes B \in \mathcal{M}$ implies $A \in \mathcal{M}$ or $B \in \mathcal{M}$.

Let P be a poset. We will see that the category $\mathsf{Rep}_k^{\mathrm{pfd}}(P)$ naturally becomes a tensor abelian category. For any V and W in $\mathsf{Rep}_k^{\mathrm{pfd}}(P)$, we define $V \otimes W$ by $(V \otimes W)_p := V_p \otimes_k W_p$ and $(V \otimes W)(p, q) := V(p, q) \otimes_k W(p, q) \colon (V \otimes W)_p \to (V \otimes W)_q$ for $p \le q$ in $\mathsf{Rep}_k^{\mathrm{pfd}}(P)$. Then, we have natural isomorphisms given by $a_{V,W,Z} : (V \otimes W) \otimes Z \to V \otimes (W \otimes Z)$, $\ell_V : k_P \otimes V \to V$ and $r_V : V \otimes k_P \to V$ for any V, W and Z in P. In particular, $\mathsf{Rep}_k^{\mathrm{pfd}}(P) = (\mathsf{Rep}_k^{\mathrm{pfd}}(P), \otimes, k_P, a, \ell, r)$ becomes a tensor abelian category.

Now, we consider the posets $A_{\mathbb{Z}}(b) := (\mathbb{Z}, \le_b)$, where the underlying set is $\mathbb{Z}$ and the order satisfies either $i \le_b i + 1$ or $i + 1 \le_b i$ for any $i \in \mathbb{Z}$. We can display the poset by:

$$A_{\mathbb{Z}}(b) \colon \ \cdots \longleftrightarrow -1 \longleftrightarrow 0 \longleftrightarrow 1 \longleftrightarrow \cdots ,$$

where $\longleftrightarrow$ is either $\to$ or $\leftarrow$ which depends on the order $\le_b$. We say that the poset $A_{\mathbb{Z}}(b)$ is *bounded* if there exists a positive number n_b such that the cardinality of $\{z \in \mathbb{Z} \mid x \le_b z \le_b y\}$ is at most n_b.

We classify the prime tensor ideals in $\mathsf{Rep}_k^{\mathrm{pfd}}(A_{\mathbb{Z}}(b))$ with the boundedness condition.

THEOREM 3.0.1. [**Tad23**, Theorem 1.1] *If the poset $A_{\mathbb{Z}}(b)$ is bounded, then there exist bijections among the following sets:*

(1) The set of prime tensor ideals of the category $\mathsf{Rep}_k^{\mathrm{pfd}}(A_{\mathbb{Z}}(b))$.
(2) The set of prime ideals of the ring $\prod_{i\in\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z})$.
(3) The set of prime ideals of the boolean algebra $\mathcal{P}(\mathbb{Z})$ (the power set of $\mathbb{Z}$).

Moreover, the above correspondence gives homeomorphisms among the sets with their respective Zariski topologies, see [**Tad23**, Proposition 3.4] for details.

As the study progressed, it remained unclear how this setting was suited for the use of multi-parameter persistent homology for data analysis. Prime tensor ideals are subcategories of $\mathsf{Rep}_k^{\mathrm{pfd}}(A_{\mathbb{Z}}(b))$, while persistent homology analysis mainly focuses on persistence modules derived from data. Thus, the objects of interest lie in different layers.

CHAPTER 4

Interval Approximation

In this chapter, we study persistence modules over finite posets using relative homological algebra. In particular, we aim to study properties of interval approximations and interval resolution global dimensions. We consider these notions for homological invariants relative to intervals [**BBH22**] [**AENY23a**], which can be used to obtain invariants for multiparameter persistent homology.

This chapter consists of three sections. In Section 4.1, we recall the basics of relative homological algebra. In Section 4.2, we show the summand injectivity of interval cover (Theorem 4.2.2). In Section 4.3, we show the monotonicity of interval resolution global dimension of posets (Theorem 4.3.1). These results lead to the theory of bipath persistence in Chapter 5.

Let k be a field throughout this chapter.

4.1. Review on Relative Homological Algebra

In this section, we recall the basics of relative homological algebra. We refer the reader to [**AS93**], [**ARS95**], [**BBH22**], [**AENY23a**], and [**AET23**] for example.

Let P be a finite poset.

DEFINITION 4.1.1. Let $\mathcal{X}$ be a full subcategory of $\mathsf{rep}_k(P)$ and $f\colon X \to V$ be a morphism in $\mathsf{rep}_k(P)$. We say that

(1) the morphism f is *right minimal* if $f \circ g = f$ for $g \in \mathrm{Hom}_P(X, X)$ implies g is an automorphism.

(2) the morphism f is a *right $\mathcal{X}$-approximation* of V if $X \in \mathcal{X}$ and

$$\mathrm{Hom}_P(W, f)\colon \mathrm{Hom}_P(W, X) \to \mathrm{Hom}_P(W, V),\ h \mapsto f \circ h$$

is surjective for any $W \in \mathcal{X}$.

(3) the morphism f is a *right minimal $\mathcal{X}$-approximation* of V if f is right minimal and a right $\mathcal{X}$-approximation.

We give a basic example of a full subcategory $\mathcal{X}$ of $\mathsf{rep}_k(P)$ and approximations by $\mathcal{X}$.

EXAMPLE 4.1.2. We consider an approximation by projective modules. A P-module X is a *projective module* if for any epimorphism $g\colon V \to W$, $\mathrm{Hom}_P(X, g)\colon \mathrm{Hom}_P(X, V) \to \mathrm{Hom}_P(X, W)$ is surjective.

Let $\{k_{\{p\}^{\uparrow}} \mid p \in P\}$ be a set of interval modules, where $\{p\}^{\uparrow}$ is the interval given by $\{p\}^{\uparrow} = \{x \in P \mid p \leq x\}$. In fact, the set of interval modules gives all the indecomposable projective modules in $\mathsf{rep}_k(P)$ (up to isomorphism). In particular, the full subcategory $\mathcal{X} := \mathsf{add}(\{k_{p^{\uparrow}}\}_{p\in P})$ whose objects are isomorphic to finite direct sums of interval modules in $\{k_{\{p\}^{\uparrow}} \mid p \in P\}$ gives the category of projective modules in $\mathsf{rep}_k(P)$. Then, any epimorphism $f\colon X \to V$, where $X \in \mathcal{X}$, is a right $\mathcal{X}$-approximation of V. A right minimal $\mathcal{X}$-approximation of V is called *projective cover* of V.

Let $\mathcal{X}$ be a full subcategory of $\mathsf{rep}_k(P)$ such that it is closed under direct summands. Next, we recall that we can construct a right minimal $\mathcal{X}$-approximation from a given right $\mathcal{X}$-approximation.

For $V \in \mathsf{rep}_k(P)$, we have the *slice category* $\mathsf{rep}_k(P)/V$ whose objects are the morphisms $f\colon X \to V$ in $\mathsf{rep}_k(P)$, and where a morphism $g\colon f \to f'$ from $f\colon X \to V$ to $f'\colon X' \to V$ is a morphism $g\colon X \to X'$ in $\mathsf{rep}_k(P)$ such that $f = f' \circ g$.

$$\begin{tikzcd} X \arrow[d, "g"] \arrow[dr, "f"] & \\ X' \arrow[r, "f'"] & V. \end{tikzcd}$$

If the above $g\colon X \to X'$ is an isomorphism in $\mathsf{rep}_k(P)$, we say that f and f' are isomorphic.

We give an equivalence relation $\sim$ on the objects in the category: for $f\colon X \to V$ and $g\colon X' \to V$, we have $f \sim g$ if and only if there exists $\alpha\colon X \to X'$ and $\beta\colon X' \to X$ such that $g \circ \alpha = f$ and $f \circ \beta = g$. A *right minimal version* of f is a morphism $g\colon Y \to V$ satisfying $f \sim g$ and $\sum_{p\in P} \dim_k(Y_p) \leq \sum_{p\in P} \dim_k(Z_p)$ for any $h\colon Z \to V$ with $h \sim f$, which is unique up to isomorphism ([**ARS95**, Proportion 2.1]). In addition, the following properties are known.

THEOREM 4.1.3. [**ARS95**, Theorem 2.2] *Let $f\colon X \to V$ be a morphism in $\mathsf{rep}_k(P)$. Then, there exists a decomposition $X = X' \oplus X''$ such that $f|_{X'}\colon X' \to V$ is a right minimal version of f and right minimal, and $f|_{X''}\colon X'' \to V$ is zero.*

We obtain a right minimal $\mathcal{X}$-approximation of V from a right $\mathcal{X}$-approximation of V using the above theorem.

COROLLARY 4.1.4. *Let $\mathcal{X}$ be a full subcategory of $\mathsf{rep}_k(P)$ which is closed under direct summands. For a right $\mathcal{X}$-approximation $f\colon X \to V$ of V, a right minimal version of f is a right minimal $\mathcal{X}$-approximation of V.*

PROOF. By Theorem 4.1.3, we have a decomposition of $X = X' \oplus X''$ such that the right minimal version $f|_{X'}$ of f is right minimal. We show that $f|_{X'}$ is a right $\mathcal{X}$-approximation of V and complete the proof.

Since $\mathcal{X}$ is closed under direct summands, the direct summand X' is in $\mathcal{X}$. In addition, for any $g\colon Y \to V$ with $Y \in \mathcal{X}$, we have a morphism $\alpha\colon Y \to X$ such that $f \circ \alpha = g$ since f is a right $\mathcal{X}$-approximation. In addition, we have a natural epimorphism $\beta\colon X \to X'$ such that $f|_{X'} \circ (\beta \circ \alpha) = f \circ \alpha = g$ by Theorem 4.1.3, which makes the following diagram

$$\begin{tikzcd}
Y \arrow[d, "\alpha"] \arrow[dr, "g"] & \\
X \arrow[r, "f"] \arrow[d, "\beta"] & V \\
X' \arrow[ur, "f|_{X'}"'] &
\end{tikzcd}$$

commutes. This implies $f|_{X'}$ is a right $\mathcal{X}$-approximation of V. $\square$

Below, we consider approximations by interval-decomposable modules, which is central notion in this chapter. For the finite set $\mathscr{I}_P = \{k_I \in \mathsf{rep}_k(P) \mid I \in \mathbb{I}(P)\}$, we consider the full subcategory consisting of all the of interval-decomposable modules in $\mathsf{rep}_k(P)$, denoted by $\mathsf{add}(\mathscr{I}_P)$. By definition, the full subcategory $\mathsf{add}(\mathscr{I}_P)$ satisfies the following:

- $\mathsf{add}(\mathscr{I}_P)$ is closed under direct sums and direct summands.
- $\mathsf{add}(\mathscr{I}_P)$ contains all the projective modules in $\mathsf{rep}_k(P)$ (see Example 4.1.2).

PROPOSITION 4.1.5. *We have the following.*

(1) Any P-module $V \in \mathsf{rep}_k(P)$ admits a right minimal $\mathsf{add}(\mathscr{I}_P)$-approximation.
(2) Any right $\mathsf{add}(\mathscr{I}_P)$-approximation is an epimorphism.

PROOF. (1) Let $X_1, \dots, X_n$ be all the interval modules in $\mathscr{I}_P$. For each i in $\{1, \dots, n\}$, let d_i be the k-dimension of $\mathrm{Hom}_P(X_i, V)$ and $\{f_{i,1}, \dots, f_{i,d_i}\}$ be a basis of $\mathrm{Hom}_P(X_i, V)$. Then, we have a morphism $f_i := (f_{i,1}, \dots, f_{i,d_i})\colon X_i^{d_i} \to V$ for each i in $\{1, \dots, n\}$. Then, the morphism $f := (f_1, \dots f_n)\colon \bigoplus_{i=1}^{i=n} X_i^{d_i} \to V$ is a right $\mathsf{add}(\mathscr{I}_P)$-approximation of V. Taking the right minimal version of f, we obtain a right minimal $\mathsf{add}(\mathscr{I}_P)$-approximation of V (Corollary 4.1.4).

(2). Any right $\mathsf{add}(\mathscr{I}_P)$-approximation factors through a projective cover of V, which is an epimorphism. Thus, a right $\mathsf{add}(\mathscr{I}_P)$-approximation of V is an epimorphism. $\square$

Below, we call a right minimal $\mathsf{add}(\mathscr{I}_P)$-approximation of V an *interval cover* of V.

DEFINITION 4.1.6. Let V be in $\mathsf{rep}_k(P)$.

(1) An *interval-resolution* of V is an exact sequence

$$\begin{tikzcd}
\cdots \arrow[rr, "f_2"] \arrow[dr, "\pi_2"'] & & X_1 \arrow[rr, "f_1"] \arrow[dr, "\pi_1"'] & & X_0 \arrow[r, two heads, "f_0"] & V \arrow[r] & 0 \\
& K_2 \arrow[ur, hook, "\operatorname{im} f_2"'] \arrow[u, phantom, "\circlearrowleft"] & & K_1 \arrow[ur, hook, "\operatorname{im} f_1"'] \arrow[u, phantom, "\circlearrowleft"] & & &
\end{tikzcd}$$

such that $f_0\colon X_0 \to V$ is an interval cover of V and $\pi_i\colon X_i \to K_i := \operatorname{Ker}(f_{i-1})(= \operatorname{Im}(f_i))$ is an interval cover of K_i for any $i \in \{1, 2, \ldots\}$.

(2) If V has the interval-resolution of the form

$$0 \to X_m \xrightarrow{f_m} \cdots \xrightarrow{f_2} X_1 \xrightarrow{f_1} X_0 \xrightarrow{f_0} V \to 0$$

with $X_m \neq 0$, then we call the non-negative number m the *interval resolution dimension of* V, and we write int-res-dim$(V) = m$. Otherwise, we say that the interval resolution dimension of V is infinity.

(3) The *interval resolution global dimension* of P is

$$\text{int-res-gldim}(P) := \sup\{\text{int-res-dim}(V) \mid V \in \mathsf{rep}_k(P)\}.$$

We note that the interval resolution global dimension of any finite poset is known to be finite.

THEOREM 4.1.7 ([**AENY23a**, Proposition 4.5]). *Let P be a finite poset. The interval resolution global dimension of P is finite.*

Now we give some computation of interval resolution global dimensions.

EXAMPLE 4.1.8. (1) Let $P := A_n(a)$ be an A_n-type poset. Any P-module V is an interval-decomposable. Thus, an interval resolution of $V \in \mathsf{rep}_k(P)$ is of the form

$$0 \to V \xrightarrow{\mathrm{id}_V} V \to 0.$$

Hence, we have int-res-dim$(V) = 0$ for any V in $\mathsf{rep}_k(P)$. In particular, we have

$$\text{int-res-gldim}(P) = 0.$$

(2) Next, we consider the interval resolution global dimensions of four D_4-type posets $D_4(\alpha_1)$, $D_4(\alpha_2)$, $D_4(\alpha_3)$, and $D_4(\alpha_4)$ respectively, where the posets are displayed by their Hasse diagrams as follows:

$$\begin{tikzcd}[column sep=small]
& 1 \arrow[dl] \arrow[d] \arrow[dr] & \\
2 & 3 & 4,
\end{tikzcd}
\quad
\begin{tikzcd}
& 1 \arrow[d] & \\
3 & 2 \arrow[l] \arrow[r] & 4,
\end{tikzcd}
\quad
\begin{tikzcd}
1 \arrow[r] & 2 \arrow[d] & 3 \arrow[l] \\
& 4 &
\end{tikzcd}
,\quad
\begin{tikzcd}[column sep=small]
1 \arrow[dr] & 2 \arrow[d] & 3 \arrow[dl] \\
& 4 &
\end{tikzcd}
.$$

For each poset $D_4(\alpha_i)$ ($i \in \{1, 2, 3, 4\}$), there exists a unique (up to isomorphism) indecomposable module V_{α_i} that is not an interval module, which is displayed by

$$\begin{tikzcd}
& k^2 \arrow[dl, "{[1\,1]}"'] \arrow[d, "{[1\,0]}"'] \arrow[dr, "{[0\,1]}"] & \\
k & k & k,
\end{tikzcd}
\quad
\begin{tikzcd}
& k \arrow[d, "{{}^t[1\,1]}"] & \\
k & k^2 \arrow[l, "{[1\,0]}"'] \arrow[r, "{[0\,1]}"] & k,
\end{tikzcd}
\quad
\begin{tikzcd}
k \arrow[r, "{{}^t[1\,0]}"] & k^2 \arrow[d, "{[1\,1]}"'] & k \arrow[l, "{{}^t[0\,1]}"'] \\
& k &
\end{tikzcd}
,\quad
\begin{tikzcd}
k \arrow[dr, "{{}^t[1\,0]}"'] & k \arrow[d, "{{}^t[1\,1]}"'] & k \arrow[dl, "{{}^t[0\,1]}"] \\
& k^2 &
\end{tikzcd}
,$$

respectively. By its uniqueness, the interval resolution dimension of V_{α_i} gives the interval resolution global dimension of $D_4(\alpha_i)$ for each i in $\{1, 2, 3, 4\}$. Below, we compute the interval resolution dimension of each V_{α_i}.

We first consider an interval resolution dimension of V_{α_1}. For this purpose, we write the Auslander-Reiten quiver of $\mathsf{rep}_k(D_4(\alpha_1))$, the directed graph whose vertices are the isomorphism classes of indecomposable module in $\mathsf{rep}_k(D_4(\alpha_1))$ and arrows are given by the so-called irreducible morphisms, see [**ASS06**] for more details.

$$\begin{tikzcd}
\substack{0\\1\,0\,0} \arrow[dr] & & \substack{1\\0\,1\,1} \arrow[dr, "a_1"] & & \substack{1\\1\,0\,0} \arrow[dr] & \\
\substack{0\\0\,1\,0} \arrow[r] & \substack{1\\1\,1\,1} \arrow[ur, "b_1"] \arrow[r, "b_2"] \arrow[dr, "b_3"'] & \substack{1\\1\,0\,1} \arrow[r, "a_2"] & V_{\alpha_1} \arrow[ur] \arrow[r] \arrow[dr] & \substack{1\\0\,1\,0} \arrow[r] & \substack{1\\0\,0\,0}, \\
\substack{0\\0\,0\,1} \arrow[ur] & & \substack{1\\1\,1\,0} \arrow[ur, "a_3"'] & & \substack{1\\0\,0\,1} \arrow[ur] &
\end{tikzcd} \tag{4.1.1}$$

In (4.1.1), we represent interval modules by 1 and 0, where 1 is for k and 0 for zero vector spaces, and we omit writing morphisms between the k-vector spaces. For example, we write

$$\begin{matrix} & 1 & \\ 1 & 0 & 1 \end{matrix} \quad \text{for} \quad \begin{tikzcd}[column sep=small] & k \arrow[dl, "\mathrm{id}_k"'] \arrow[d] \arrow[dr, "\mathrm{id}_k"] & \\ k & 0 & k. \end{tikzcd}$$

In this setting, where the number of isomorphism classes is finite, any non-zero nonisomorphism morphism between indecomposable modules is a sum of compositions of irreducible morphisms [**ASS06**, IV. Corollary 5.6]. In particular, by looking at the Auslander-Reiten quiver, we can see that any morphism from an interval module to V_{α_1} factors through the morphism:

$$\substack{1\\0\,1\,1} \oplus \substack{1\\1\,0\,1} \oplus \substack{1\\1\,1\,0} \xrightarrow{[a_1,a_2,a_3]} V_{\alpha_1},$$

giving interval cover of V_{α_1}. Hence, we obtain the following interval resolution of V_{α_1}:

$$0 \longrightarrow \substack{1\\1\,1\,1} \xrightarrow{{}^t[b_1,b_2,b_3]} \substack{1\\0\,1\,1} \oplus \substack{1\\1\,0\,1} \oplus \substack{1\\1\,1\,0} \xrightarrow{[a_1,a_2,a_3]} V_{\alpha_1} \longrightarrow 0.$$

In particular, we have

$$\text{int-res-dim}\, V_{\alpha_1} = 1.$$

Thus, we have

$$\text{int-res-gldim}\, D_4(\alpha_1) = 1.$$

Similarly, we have the Auslander-Reiten quivers of $\mathsf{rep}_k(D_4(\alpha_2))$, $\mathsf{rep}_k(D_4(\alpha_3))$, and $\mathsf{rep}_k(D_4)(\alpha_4)$:

$$\begin{tikzcd}
\substack{0\\1\,0\,0} \arrow[dr] & & \substack{0\\0\,1\,1} \arrow[dr] & & \substack{1\\1\,1\,0} \arrow[dr] & & \\
& \substack{0\\1\,1\,1} \arrow[ur] \arrow[r] \arrow[dr] & \substack{1\\1\,1\,1} \arrow[r] & V_{\alpha_2} \arrow[ur] \arrow[r] \arrow[dr] & \substack{0\\0\,1\,0} \arrow[r] & \substack{1\\0\,1\,0} \arrow[r] & \substack{1\\0\,0\,0}, \\
\substack{0\\0\,0\,1} \arrow[ur] & & \substack{0\\1\,1\,0} \arrow[ur] & & \substack{1\\0\,1\,1} \arrow[ur] & &
\end{tikzcd}$$

$$\begin{tikzcd}
& & \substack{1\,1\,0\\1} \arrow[dr] & & \substack{0\,1\,1\\0} \arrow[dr] & & \substack{1\,0\,0\\0} \\
\substack{0\,0\,0\\1} \arrow[r] & \substack{0\,1\,0\\1} \arrow[ur] \arrow[r] \arrow[dr] & \substack{0\,1\,0\\0} \arrow[r] & V_{\alpha_3} \arrow[ur] \arrow[r] \arrow[dr] & \substack{1\,1\,1\\1} \arrow[r] & \substack{1\,1\,1\\0} \arrow[ur] \arrow[dr] & \\
& & \substack{0\,1\,1\\1} \arrow[ur] & & \substack{1\,1\,0\\0} \arrow[ur] & & \substack{0\,0\,1\\0},
\end{tikzcd}$$

$$\begin{tikzcd}
& \substack{1\,0\,0\\1} \arrow[dr] & & \substack{0\,1\,1\\1} \arrow[dr] & & \substack{1\,0\,0\\0} \\
\substack{0\,0\,0\\1} \arrow[ur] \arrow[r] \arrow[dr] & \substack{0\,1\,0\\1} \arrow[r] & V_{\alpha_4} \arrow[ur] \arrow[r] \arrow[dr] & \substack{1\,0\,1\\1} \arrow[r] & \substack{1\,1\,1\\1} \arrow[ur] \arrow[r] \arrow[dr] & \substack{0\,1\,0\\0} \\
& \substack{0\,0\,1\\1} \arrow[ur] & & \substack{1\,1\,0\\1} \arrow[ur] & & \substack{0\,0\,1\\0},
\end{tikzcd}$$

and we obtain the following interval resolutions of V_{α_2}, V_{α_3}, and V_{α_4}:

$$0 \longrightarrow \begin{smallmatrix} & 0 & \\ 1 & 1 & 1 \end{smallmatrix} \to \begin{smallmatrix} & 0 & \\ 1 & 1 & 0 \end{smallmatrix} \oplus \begin{smallmatrix} & 1 & \\ 1 & 1 & 1 \end{smallmatrix} \oplus \begin{smallmatrix} & 0 & \\ 0 & 1 & 1 \end{smallmatrix} \to V_{\alpha_2} \longrightarrow 0,$$

$$0 \longrightarrow \begin{smallmatrix} 0 & 1 & 0 \\ & 1 & \end{smallmatrix} \to \begin{smallmatrix} 1 & 1 & 0 \\ & 1 & \end{smallmatrix} \oplus \begin{smallmatrix} 0 & 1 & 0 \\ & 0 & \end{smallmatrix} \oplus \begin{smallmatrix} 0 & 1 & 1 \\ & 1 & \end{smallmatrix} \to V_{\alpha_3} \longrightarrow 0,$$

$$0 \longrightarrow \begin{smallmatrix} 0 & 0 & 0 \\ & 1 & \end{smallmatrix} \to \begin{smallmatrix} 1 & 0 & 0 \\ & 1 & \end{smallmatrix} \oplus \begin{smallmatrix} 0 & 1 & 0 \\ & 1 & \end{smallmatrix} \oplus \begin{smallmatrix} 0 & 0 & 1 \\ & 1 & \end{smallmatrix} \to V_{\alpha_4} \longrightarrow 0.$$

Thus, we have

$$\text{int-res-dim}\, V_{\alpha_i} = 1 \text{ and } \text{int-res-gldim}\, D_4(\alpha_i) = 1 \text{ for } i \in \{2, 3, 4\}.$$

Any D_4-type poset is isomorphic to one of the posets $D_4(\alpha_1)$, $D_4(\alpha_2)$, $D_4(\alpha_3)$, or $D_4(\alpha_4)$. Hence, by the above discussion, the interval resolution global dimension of any D_4-type poset is 1.

The fact that the interval resolution global dimension of any D_4-type poset is 1 is used to classify all finite posets whose interval resolution global dimension is 0 (Theorem 5.2.3).

4.2. Summand Injectivity

In this section, we show that the restriction of interval cover to each direct summand is a monomorphism (Theorem 4.2.2). This result is helpful when calculating interval covers.

Let P be a finite poset. We first give the next lemma.

LEMMA 4.2.1. *Let $f_i \colon V_i \to W_i$ ($i = 1, \ldots, m$) be morphisms in $\mathsf{rep}_k(P)$. If the morphism*

$$f := \begin{bmatrix} f_1 & & \\ & \ddots & \\ & & f_m \end{bmatrix} \colon \bigoplus_{i=1}^{m} V_i \to \bigoplus_{i=1}^{m} W_i$$

is a monomorphism, then each f_i is a monomorphism.

PROOF. Let $g_i \colon L_i \to V_i$ be morphisms in $\mathsf{rep}_k(P)$ such that $f_i \circ g_i = 0$ for each i. We show g_i is zero morphism for each i. For the morphism

$$g := \begin{bmatrix} g_1 & & \\ & \ddots & \\ & & g_m \end{bmatrix} \colon \bigoplus_{i=1}^{m} L_i \to \bigoplus_{i=1}^{m} V_i,$$

we have $f \circ g = 0$. Since f is a monomorphism, the morphism g is zero. Hence, each g_i is zero for any i in $\{1, \ldots, m\}$. This implies f_i is a monomorphism for $i \in \{1, \ldots, m\}$. This completes the proof. $\square$

Now, we give our main result in this section.

THEOREM 4.2.2. [**AET23**, Corollary 3.11] *Let P be a finite poset and $\mathscr{I}_P$ be the set of isomorphism classes of interval modules. For $V \in \mathsf{rep}_k(P)$, we take its interval cover $f \colon X = \bigoplus_{i=1}^{m} X_i \to V$, where X_i is an interval module for any $i = 1, \ldots, m$. Then, the following holds.*

(1) *f is an epimorphism.*
(2) *$f|_{X_i} \colon X_i \to V$ is a monomorphism for every $i \in \{1, \ldots, m\}$.*
(3) *$\operatorname{supp} X = \operatorname{supp} V$.*

In particular, every X_i can be taken as an interval submodule of V.

PROOF. (1) is Proposition 4.1.5(2).

(2). Let

$$f = (f_1, \ldots, f_m) \colon X = \bigoplus_{i=1}^{m} X_i \longrightarrow V \tag{4.2.1}$$

be an interval cover of V. We show that f_i is a monomorphism for all i in $\{1, \ldots, m\}$. For each i, the morphism f_i factors through $\operatorname{Im}(f_i)$ as follows

$$X_i \xrightarrow{\pi_i} \operatorname{Im}(f_i) \xrightarrow{\operatorname{im}(f_i)} V$$

where $f_i = \mathrm{im}(f_i) \circ \pi_i$. Since the quotient of interval modules is interval-decomposable by Proposition 2.2.7, $\mathrm{Im}(f_i)$ is interval-decomposable for each i. Hence, since $f\colon X \to V$ is an interval cover of V, there exists $g_i\colon \mathrm{Im}(f_i) \to X$ such that $\mathrm{im}(f_i) = f \circ g_i$. This is displayed by

$$\begin{tikzcd} X \arrow[rr, "f"] & & V \\ X_i \arrow[rr, two heads, "\pi_i"'] & & \mathrm{Im}(f_i). \arrow[u, "\mathrm{im}(f_i)"'] \arrow[llu, dotted, "g_i"] \end{tikzcd}$$

For the morphisms

$$f' := \begin{bmatrix} \pi_1 & & \\ & \ddots & \\ & & \pi_m \end{bmatrix} : \bigoplus_{i=1}^{m} X_i \to \bigoplus_{i=1}^{m} \mathrm{Im}\, f_i \text{ and } g := (g_1, \ldots, g_m) : \bigoplus_{i=1}^{m} \mathrm{Im}\, f_i \to X,$$

we have

$$f \circ g \circ f' = f,$$

which is displayed by the following commutative diagram:

$$\begin{tikzcd} \bigoplus_{i=1}^{m} X_i \arrow[r, "f'"] \arrow[rrd, "f"'] & \bigoplus_{i=1}^{m} \mathrm{Im}(f_i) \arrow[r, "g"] & \bigoplus_{i=1}^{m} X_i \arrow[d, "f"] \\ & & V. \end{tikzcd}$$

Since f is right minimal, $f' \circ g$ is an automorphism. In particular, f' is a monomorphism. By Lemma 4.2.1, each π_i is a monomorphism. Hence $f_i = \mathrm{im}(f_i) \circ \pi_i$ is a monomorphism for each i in $\{1, \ldots, m\}$.

(3). By (1) and (2), we have $\mathrm{supp}(V) \subseteq \mathrm{supp}(X)$ and $\mathrm{supp}(X) \subseteq \mathrm{supp}(V)$. Hence we obtain $\mathrm{supp}(X) = \mathrm{supp}(V)$. This completes the proof. $\square$

The above result reduces the computation of interval covers as follows. We first recall a construction of an interval cover. we first compute an interval approximation as in the proof of Proposition 4.1.5 (1) and then reduce the direct summand to obtain interval cover. To compute an interval approximation, we consider all interval modules as in the proof of Proposition 4.1.5 (1). Here, Theorem 4.2.2 implies that the candidates of the interval modules that form an interval cover of a persistence module are the sub-interval modules. Thus, we need not consider all interval modules. In this way, Theorem 4.2.2 reduces the computation of interval covers.

Remark 4.2.3. We note that Theorem 4.2.2 is essentially the same as [**Asa24**, Proposition 4.9] which states that the following are equivalent for $f\colon Y \to V$ with $Y \in \mathsf{add}(\mathscr{I}_P)$:

1. (1) f is a right $\mathsf{add}(\mathscr{I}_P)$-approximation of V.
2. (2) For any $I \in S_{\mathrm{int}}(V)$ and any monomorphism $g\colon k_I \to V$ in $\mathsf{rep}_k(P)$, $g = f \circ h$ for some $h\colon k_I \to Y$.

Here S_{int} is given by $S_{\mathrm{int}} := \{I \in \mathbb{I}(P) \mid \exists \text{ monomorphism } k_I \to V\}$ [**Asa24**, Definition 4.7].

The direction (1) $\Longrightarrow$ (2) follows from the definition of $\mathsf{add}(\mathscr{I}_P)$-approximation. The direction (2) $\Longrightarrow$ (1) is the essential part. This can be proven by using the fact that interval modules are closed under quotients (similar to the proof of Theorem 4.2.2).

In fact, Theorem 4.2.2 and [**Asa24**, Proposition 4.9] each imply each other by simple insights.

For example, assuming Theorem 4.2.2, we prove (2) $\Longrightarrow$ (1) in [**Asa24**, Proposition 4.9]. We show that any morphism $h\colon J \to V$ with $J \in \mathscr{I}_P$ factors through f, and we obtain (1). Let $g\colon \bigoplus_{i=1}^{n} X_i \to V$ be an interval cover of V. By Theorem 4.2.2, each $g|_{X_i}$ is monomorphism. Thus, there exists $f'\colon X_i \to X$ such that $f \circ f_i' = g|_{X_i}$ by the assumption of (2). On the other hand, since g is an interval cover (and thus an $\mathsf{add}(\mathscr{I}_P)$-approximation) of V, there exists ${}^t(h_1, \ldots, h_n)\colon J \to \bigoplus_{i=1}^{n} X_i$ such that $g \circ {}^t(h_1, \ldots, h_n) = h$. Then, we have $f \circ (f_1', \ldots, f_1') \circ {}^t(h_1, \ldots, h_n) = h$. Thus, h factors through f. This gives (1).

Conversely, [**Asa24**, Proposition 4.9] implies Theorem 4.2.2, see [**Asa24**, Remark 4.12].

In that sense, we claim that they are essentially the same.

4.3. Monotonicity Theorem

This section aims to show one of the main results of this thesis.

THEOREM 4.3.1. [**AET23**, Theorem 4.1] *Let P be a finite poset. For any full subposet Q of P, the following inequality holds.*

$$\text{int-res-gldim}(Q) \leq \text{int-res-gldim}(P).$$

We call this property monotonicity of interval resolution global dimensions.

The interval resolution global dimension of a poset indicates the simplicity of the poset in persistence theory. A-type posets, for example, are one of the simplest posets since their interval resolution global dimensions are zero (Example 4.1.8 (1)), which is equivalent to that any persistence module over the A-type poset is interval-decomposable. This theorem guarantees for the intuitive relation that if Q is contained in P, then the simplicity (interval-resolution global dimension) of Q is less than or equal to that of P.

Such property (monotonicity) fails in general. Indeed, we have the following example given in [**IZ90**].

EXAMPLE 4.3.2. Let Q and P be finite posets given by the following Hasse diagrams

$$Q: \begin{tikzcd} & \bullet \arrow[dl] \arrow[dr] & \\ \bullet \arrow[d] \arrow[drr] & & \bullet \arrow[d] \arrow[dll] \\ \bullet \arrow[dr] & & \bullet \arrow[dl] \\ & \bullet & \end{tikzcd} \quad \text{and} \quad P: \begin{tikzcd} & \bullet \arrow[dl] \arrow[dr] & \\ \bullet \arrow[dr] & & \bullet \arrow[dl] \\ & \bullet \arrow[dl] \arrow[dr] & \\ \bullet \arrow[dr] & & \bullet \arrow[dl] \\ & \bullet & \end{tikzcd}$$

respectively. Then, Q is a full subposet of P, however, the global dimension (which is defined by projective dimensions of modules) of Q is 3 and that of P is 2 (over an arbitrary field), see [**IZ90**, Section 3].

Let P be a poset and Q be a full subposet of P. A key to the proof of Theorem 4.3.1 is a functor $\Theta\colon \mathsf{rep}_k(P) \to \mathsf{rep}_k(Q)$ called the *intermediate extension* [**Kuh94**] (the *prolongement intermédiare* in [**BBD82**]), and the *restriction* functor $\mathrm{Res}\colon \mathsf{rep}_k(P) \to \mathsf{rep}_k(Q), V \mapsto V \circ \iota$, where $\iota\colon Q \hookrightarrow P$. These functors satisfy the following:

1. (1) $\mathrm{Res} \circ \Theta \cong \mathrm{id}_{\mathsf{rep}_k(Q)}$ (Proposition 4.3.6).
2. (2) Θ sends interval modules to interval modules (Proposition 4.3.8).
3. (3) Res sends interval-decomposable modules to interval-decomposable modules (Corollary 2.2.11).
4. (4) Res is an exact functor (Proposition 4.3.4 (2)).

Once, we show the above properties, we can give a proof of Theorem 4.3.1 (Subsection 4.3.4).

In the next three subsections, we recall the construction of the functor Θ and study its properties to prove Theorem 4.3.1.

4.3.1. Adjoint Triple.

Let P be a finite poset and Q be a full subposet of P. In this subsection, we recall the following k-linear functors:

$$\mathrm{Ind}\colon \mathsf{rep}_k(Q) \to \mathsf{rep}_k(P) \text{ and } \mathrm{Coind}\colon \mathsf{rep}_k(Q) \to \mathsf{rep}_k(P)$$

called *induction* and *coinduction* functor respectively. These are used to construct the functor Θ in the next subsection.

We first recall the induction functor. For $V \in \mathsf{rep}_k(Q)$, we give $\mathrm{Ind}(V) \in \mathsf{rep}_k(P)$ by

$$\{ \varinjlim_{q \in Q \cap \{p\}^{\downarrow}} V_q \}_{p \in P} \text{ and } \{ \varinjlim_{q \in Q \cap \{p\}^{\downarrow}} V_q \to \varinjlim_{q \in Q \cap \{p'\}^{\downarrow}} V_q \}_{p \leq p' \in P}$$

where the internal morphisms are induced by the universality of the colimit for each $p \leq p'$ in P. For a morphism $f\colon V \to W \in \mathsf{rep}_k(Q)$, we have an induced morphism $\mathrm{Ind}(V) \to \mathrm{Ind}(W)$, and we write it $\mathrm{Ind}(f)$. Then, these correspondences give rise to the k-linear functor $\mathrm{Ind}\colon \mathsf{rep}_k(Q) \to \mathsf{rep}_k(P)$ called induction functor. By the construction of the induction functor, we have an induced natural transformation

$$\alpha\colon \mathrm{id}_{\mathsf{rep}_k(Q)} \to \mathrm{Res} \circ \mathrm{Ind}. \tag{4.3.1}$$

In addition, by our construction, the natural transformation α is a natural isomorphism.

For each $V \in \mathsf{rep}_k(Q)$, the pair $(\mathrm{Ind}(V), \alpha_V)$ satisfies the following property by the universality of the colimit: for any $W \in \mathsf{rep}_k(P)$ and $g\colon V \to \mathrm{Res}(W)$, there exists a unique morphism $f\colon \mathrm{Ind}V \to W$ such that $g = \mathrm{Res}(f) \circ \alpha_V$, that is, we have the isomorphism as k-vector spaces.

$$\mathrm{Hom}_P(\mathrm{Ind}(V), W) \cong \mathrm{Hom}_Q(V, \mathrm{Res}(W)),\ f \mapsto \mathrm{Res}(f) \circ \alpha_V. \tag{4.3.2}$$

Dually, we have the k-linear functor called the coinduction functor $\mathrm{Coind}\colon \mathsf{rep}_k(Q) \to \mathsf{rep}_k(P)$. For each $V \in \mathsf{rep}_k(Q)$, we give $\mathrm{Coind}(V)$ by

$$\{ \varprojlim_{q \in Q \cap \{p\}^{\uparrow}} V_q \}_{p \in P} \text{ and } \{ \varprojlim_{q \in Q \cap \{p\}^{\uparrow}} V_q \to \varprojlim_{q \in Q \cap \{p'\}^{\uparrow}} V_q \}_{p \le p' \in P}.$$

By the construction, it induces a natural transformation, more strongly a natural isomorphism,

$$\beta\colon \mathrm{Res} \circ \mathrm{Coind} \to \mathrm{id}_{\mathsf{rep}_k(Q)} \tag{4.3.3}$$

giving the isomorphism as k-vector spaces

$$\mathrm{Hom}_P(W, \mathrm{Coind}(V)) \cong \mathrm{Hom}_Q(\mathrm{Res}(W), V),\ f \mapsto \beta_V \circ \mathrm{Res}(f). \tag{4.3.4}$$

Remark 4.3.3. For each $V \in \mathsf{rep}_k(Q)$, the pair $(\mathrm{Ind}(V), \alpha_V)$ (resp. $(\mathrm{Coind}(V), \beta_V)$) with the bijection (4.3.2) (resp. (4.3.4)) is the *left Kan extension* (resp. *right Kan extension*) of V along $\iota\colon Q \to P$, see [**ML78**, X. Theorem 1] for example.

We note that these functors Ind, Coind, and Res are also basic in the representation theory of finite dimensional algebras, see [**ASS06**], [**Ste16**] for example.

In the rest of this subsection, we recall properties of the above three functors (see [**ASS06**, I, Theorem 6.8], [**ML78**] for example) for the construction of the functor Θ in the next subsection.

Proposition 4.3.4. (1) *The natural transformations* $\alpha\colon \mathrm{id}_{\mathsf{rep}_k(Q)} \to \mathrm{Res}\circ\mathrm{Ind}$ (4.3.1) *and* $\beta\colon \mathrm{Res} \circ \mathrm{Coind} \to \mathrm{id}_{\mathsf{rep}_k(Q)}$ (4.3.3) *are natural isomorphisms.*

(2) *The functor* Res *is exact.*

(3) *The functor* Coind *is right adjoint to* Res *and* Ind *is left adjoint to* Res*. That is, we have isomorphisms of k-vector spaces*

$$\begin{aligned} \phi_{V,W}\colon \mathrm{Hom}_P(\mathrm{Ind}(V), W) &\cong \mathrm{Hom}_Q(V, \mathrm{Res}(W)),\ f \mapsto \mathrm{Res}(f) \circ \alpha_V \quad \textit{and} \\ \psi_{V,W}\colon \mathrm{Hom}_P(W, \mathrm{Coind}(V)) &\cong \mathrm{Hom}_Q(\mathrm{Res}(W), V),\ f \mapsto \beta_V \circ \mathrm{Res}(f), \end{aligned}$$

which are natural in $W \in \mathsf{rep}_k(P)$ and $V \in \mathsf{rep}_k(Q)$.

Though it is basic, we give a proof for the readers.

Proof. (1) follows directly from the construction of the functors Ind and Coind, as noted earlier.

We next prove (2). Let

$$V \xrightarrow{f} W \xrightarrow{g} Z$$

be an exact sequence. We show

$$\mathrm{Res}(V) \xrightarrow{\mathrm{Res}(f)} \mathrm{Res}(W) \xrightarrow{\mathrm{Res}(g)} \mathrm{Res}(Z) \tag{4.3.5}$$

is exact. By the definitions of the image and the restriction functor, we have $\mathrm{Res}(\mathrm{Im}(f))_q = \mathrm{Im}(f_q) = \mathrm{Im}(\mathrm{Res}(f))_q$ and $\mathrm{Res}(\mathrm{Im}(f))(q, q') = \mathrm{Im}(\mathrm{Res}(f))(q, q')$ for any $q \le q'$ in Q. Thus, we have $\mathrm{Im}(\mathrm{Res}(f)) = \mathrm{Res}(\mathrm{Im}(f))$. Similarly, we can obtain $\mathrm{Res}(\mathrm{Ker}(f)) = \mathrm{Ker}(\mathrm{Res}(f))$. Hence, we have $\mathrm{Im}(\mathrm{Res}(f)) = \mathrm{Res}(\mathrm{Im}(f)) = \mathrm{Res}(\mathrm{Ker}(g)) = \mathrm{Ker}(\mathrm{Res}(g))$. This implies the sequence (4.3.5) is exact.

(3). We only show that the $\phi_{V,W}\colon \mathrm{Hom}_P(\mathrm{Ind}(V), W) \to \mathrm{Hom}_Q(V, \mathrm{Res}(W)),\ f \mapsto \mathrm{Res}(f) \circ \alpha_V$ gives an isomorphism as k-vector spaces and natural in V and W.

The morphism $\phi_{V,W}$ is a bijection, see Equation (4.3.2). We show the naturality. For any $g\colon V \to V'$ in $\mathsf{rep}_k(Q)$ and for any f in $\mathrm{Hom}_Q(\mathrm{Ind}(V'), W)$, we have

$$\begin{aligned} \phi_{V,W} \circ \mathrm{Hom}_Q(\mathrm{Ind}(g), W)(f) &= \mathrm{Res}(f) \circ \mathrm{Res}(\mathrm{Ind}(g)) \circ \alpha_V \\ &= \mathrm{Res}(f) \circ \alpha_{V'} \circ g \ (\because \alpha \text{ is a natural transformation}) \\ &= \mathrm{Hom}_P(g, \mathrm{Res}(W)) \circ \phi_{V',W}(f). \end{aligned}$$

Thus, $\phi_{V,W}$ is natural in V. The following commutative diagram displays this naturality:

$$\begin{tikzcd}
\mathrm{Hom}_Q(\mathrm{Ind}(V), W) \arrow[r, "\phi_{V,W}"] & \mathrm{Hom}_P(V, \mathrm{Res}(W)) \\
\mathrm{Hom}_Q(\mathrm{Ind}(V'), W) \arrow[r, "\phi_{V',W}"'] \arrow[u, "\mathrm{Hom}_Q(\mathrm{Ind}(g),W)"] \arrow[ur, phantom, "\circlearrowright"] & \mathrm{Hom}_P(V', \mathrm{Res}(W)). \arrow[u, "\mathrm{Hom}_P(g,\mathrm{Res}(W))"']
\end{tikzcd}$$

Next, for any $h\colon W \to W'$ in $\mathsf{rep}_k(P)$ and f in $\mathrm{Hom}_Q(\mathrm{Ind}(V), W)$, we have

$$\begin{aligned}
\phi_{V,W'} \circ \mathrm{Hom}_Q(\mathrm{Ind}(V), h)(f) &= \mathrm{Res}(h) \circ \mathrm{Res}(f) \circ \alpha_V \\
&= \mathrm{Hom}_P(V, \mathrm{Res}(h)) \circ \phi_{V,W'}(f).
\end{aligned}$$

Thus, $\phi_{V,W}$ is natural in W. The following commutative diagram displays this naturality:

$$\begin{tikzcd}
\mathrm{Hom}_Q(\mathrm{Ind}(V), W) \arrow[r, "\phi_{V,W}"] \arrow[d, "\mathrm{Hom}_Q(\mathrm{Ind}(V),h)"'] \arrow[dr, phantom, "\circlearrowright"] & \mathrm{Hom}_P(V, \mathrm{Res}(W)) \arrow[d, "\mathrm{Hom}_P(V,\mathrm{Res}(h))"] \\
\mathrm{Hom}_Q(\mathrm{Ind}(V), W') \arrow[r, "\phi_{V,W'}"'] & \mathrm{Hom}_P(V, \mathrm{Res}(W')).
\end{tikzcd}$$

This completes the proof. □

In the next subsection, we construct the functor Θ using the above properties.

4.3.2. Intermediate Extension. Let P be a poset and Q be a full subposet of P. We construct the intermediate extension functor $\Theta\colon \mathsf{rep}_k(P) \to \mathsf{rep}_k(Q)$, and we see fundamental properties of it.

For each $V \in \mathsf{rep}_k(Q)$ we have the bijection between $\mathrm{Hom}_P(\mathrm{Ind}(V), \mathrm{Coind}(V))$ and $\mathrm{Hom}_Q(V, V)$ by Proposition 4.3.4, which is given by

$$\mathrm{Hom}_P(\mathrm{Ind}(V), \mathrm{Coind}(V)) \cong \mathrm{Hom}_Q(\mathrm{Res}(\mathrm{Ind}V), V) \cong \mathrm{Hom}_Q(V, V), f \mapsto (\beta_V \circ \mathrm{Res}(f)) \circ \alpha_V,$$

or

$$\mathrm{Hom}_P(\mathrm{Ind}(V), \mathrm{Coind}(V)) \cong \mathrm{Hom}_Q(V, \mathrm{Res}(\mathrm{Coind}(V))) \cong \mathrm{Hom}_Q(V, V), f \mapsto \beta_V \circ (\mathrm{Res}(f) \circ \alpha_V).$$

Then, we have the morphism $\theta_V \in \mathrm{Hom}_P(\mathrm{Ind}(V), \mathrm{Coind}(V))$ corresponding to $\mathrm{id}_V \in \mathrm{Hom}_Q(V, V)$ via the bijection:

$$\begin{tikzcd}
\mathrm{Hom}_Q(V, V) \arrow[r, "\sim"'] & \mathrm{Hom}_P(\mathrm{Ind}(V), \mathrm{Coind}(V)) \\
\mathrm{id}_V \arrow[u, phantom, "\Cup" description] \arrow[r, mapsto] & \theta_V, \arrow[u, phantom, "\Cup" description]
\end{tikzcd}$$

and, we have $\Theta(V) \in \mathsf{rep}_k(P)$ given by

$$\Theta(V) := \mathrm{Im}(\theta_V) \subseteq \mathrm{Coind}(V).$$

This correspondence $V \mapsto \Theta(V)$ induces the functor $\Theta\colon \mathsf{rep}_k(Q) \to \mathsf{rep}_k(P)$. We prove that the functor is k-linear functor for the readers.

PROPOSITION 4.3.5. *The family of morphisms* $(\theta_V\colon \mathrm{Ind}(V) \to \mathrm{Coind}(V))_{V \in \mathsf{rep}_k(Q)}$ *gives a natural transformation* $\theta\colon \mathrm{Ind} \to \mathrm{Coind}$. *In particular, it gives rise to the* k*-linear functor*

$$\Theta\colon \mathsf{rep}_k(Q) \to \mathsf{rep}_k(P),\ V \mapsto \mathrm{Im}(\theta_V). \tag{4.3.6}$$

Proof. We first show that θ is a natural transformation. For any morphism $f\colon V \to W$ in $\mathsf{rep}_k(Q)$, we have the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Hom}_Q(V,V) \arrow[r, "\sim"] \arrow[d, "\mathrm{Hom}_Q(V,f)"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathrm{Hom}_P(\mathrm{Ind}(V),\mathrm{Coind}(V)) \arrow[d, "\mathrm{Hom}_P(\mathrm{Ind}(V),\mathrm{Coind}(f))"] \\
\mathrm{Hom}_Q(V,W) \arrow[r, "\sim"] \arrow[dr, phantom, "\circlearrowleft"] & \mathrm{Hom}_P(\mathrm{Ind}(V),\mathrm{Coind}(W)) \\
\mathrm{Hom}_Q(W,W) \arrow[r, "\sim"'] \arrow[u, "\mathrm{Hom}_Q(f,W)"] & \mathrm{Hom}_P(\mathrm{Ind}(W),\mathrm{Coind}(W)). \arrow[u, "\mathrm{Hom}_P(\mathrm{Ind}(f),\mathrm{Coind}(W))"']
\end{tikzcd}$$

In particular, we have $\theta_W \circ \mathrm{Ind}(f) = \mathrm{Coind}(f) \circ \theta_V$, which is displayed by the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Ind}(V) \arrow[r, "\theta_V"] \arrow[d, "\mathrm{Ind}(f)"] \arrow[dr, phantom, "\circlearrowleft"] & \mathrm{Coind}(V) \arrow[d, "\mathrm{Coind}(f)"] \\
\mathrm{Ind}(W) \arrow[r, "\theta_W"'] & \mathrm{Coind}(W).
\end{tikzcd} \tag{4.3.7}$$

Thus, θ is a natural transformation.

Next, we show that Θ is a k-linear functor. The natural transformation θ induces the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Ind}(V) \arrow[r, two heads, "\pi_V"] \arrow[d, "\mathrm{Ind}(f)"] \arrow[dr, phantom, "\circlearrowleft"] & \Theta(V) \arrow[r, tail, "\mathrm{im}\,\theta_V"] \arrow[d, "\Theta(f)"] \arrow[dr, phantom, "\circlearrowleft"] & \mathrm{Coind}(V) \arrow[d, "\mathrm{Coind}(f)"] \\
\mathrm{Ind}(W) \arrow[r, two heads, "\pi_W"] & \Theta(W) \arrow[r, tail, "\mathrm{im}\,\theta_W"] & \mathrm{Coind}(W),
\end{tikzcd} \tag{4.3.8}$$

where $\mathrm{im}\,\theta_V$ (resp. $\mathrm{im}\,\theta_W$) is a monomorphism such that $\mathrm{im}\,\theta_V \circ \pi_V = \theta_V$ (resp. $\mathrm{im}\,\theta_W \circ \pi_W = \theta_W$), and $\Theta(f)$ is induced by the universal property of the images. We can check that $\Theta(\mathrm{id}_V) = \mathrm{id}_{\Theta(V)}$ and $\Theta(g \circ h) = \Theta(g) \circ \Theta(h)$ by the universal property of the images. Hence Θ is a functor. In addition, since Ind (and Coind) is a k-linear functor the functor Θ is a k-linear functor. This completes the proof. $\square$

The following property is needed for the monotonicity theorem (Theorem 4.3.1).

Proposition 4.3.6. *We have a natural isomorphism $\gamma\colon 1_{\mathsf{rep}_k(Q)} \to \mathrm{Res} \circ \Theta$.*

Proof. Let $\alpha\colon 1_{\mathsf{rep}_k(Q)} \to \mathrm{Res} \circ \mathrm{Ind}$ and $\beta\colon \mathrm{Res} \circ \mathrm{Coind} \to 1_{\mathsf{rep}_k(Q)}$ be the natural isomorphisms given in (4.3.1), and (4.3.3). For $V \in \mathsf{rep}_k(Q)$, we have $\theta_V = \mathrm{im}\,\theta_V \circ \pi_V$ as in (4.3.8). Then we have the commutative diagram:

$$\begin{tikzcd}
V \arrow[r, "\mathrm{id}_V", "\sim"'] \arrow[d, "\alpha_V"', "\wr"] \arrow[dr, phantom, "\circlearrowleft"] & V \arrow[r, "\mathrm{id}_V", "\sim"'] \arrow[d, "\gamma_V"'] \arrow[dr, phantom, "\circlearrowleft"] & V \arrow[d, "{\beta_V}^{-1}"', "\wr"] \\
\mathrm{Res}(\mathrm{Ind}(V)) \arrow[r, two heads, "\mathrm{Res}(\pi_V)"] & \mathrm{Res}(\Theta(V)) \arrow[r, tail, "\mathrm{Res}(\mathrm{im}\,\theta_V)"] & \mathrm{Res}(\mathrm{Coind}(V)),
\end{tikzcd} \tag{4.3.9}$$

where $\gamma_V := \mathrm{Res}(\pi_V) \circ \alpha_V$, which satisfies ${\beta_V}^{-1} = \mathrm{Res}(\mathrm{im}\,\theta_V) \circ \gamma_V$. Since Res is an exact functor by Proposition 4.3.4(2), the morphisms $\mathrm{Res}(\pi_V)$ and $\mathrm{Res}(\mathrm{im}\,\theta_V)$ is an epimorphism and a monomorphism respectively. In particular, γ_V is an epimorphism and monomorphism. Hence, γ_V is an isomorphism.

Next, we show the naturality of γ. For any $f\colon V \to W$ in $\mathsf{rep}_k(Q)$, we have

$$\begin{aligned}
\mathrm{Res}(\Theta(f)) \circ \gamma_V &= \mathrm{Res}(\Theta(f)) \circ \mathrm{Res}(\pi_V) \circ \alpha_V \\
&= \mathrm{Res}(\pi_W) \circ \mathrm{Res}(\mathrm{Ind}(f)) \circ \alpha_V \ (\because \text{the left of the commutative diagram (4.3.8)}) \\
&= \mathrm{Res}(\pi_W) \circ \alpha_W \circ f \ (\because \text{the naturality of } \alpha) \\
&= \gamma_W \circ f.
\end{aligned}$$

Thus γ is a natural isomorphism. This completes the proof. □

We will use the following characterization of intermediate extensions to show that the functor Θ sends interval modules in $\mathsf{rep}_k(Q)$ to interval modules in $\mathsf{rep}_k(P)$ (Proposition 2.2.10).

PROPOSITION 4.3.7 ([**Kuh94**, Proposition 4.6 (3)]). *Let V be a Q-module. For a P-module X, we have $\Theta(V) \cong X$ if and only if X satisfies the conditions* (i)–(iii) *below:*

(i) $\mathrm{Res}(X) \cong V$.
(ii) *For any proper submodule Y of X, we have* $\mathrm{Res}(Y) \not\cong V$.
(iii) *For any proper quotient Z of X, we have* $\mathrm{Res}(Z) \not\cong V$.

4.3.3. Intermediate Extensions Sends Interval Modules to Interval Modules. Let P be a finite poset, Q be a full subposet of P, and $\iota\colon Q \hookrightarrow P$ be an inclusion map.

In this subsection, we prove the following proposition.

PROPOSITION 4.3.8. [**AET23**, Proposition 4.15] *The intermediate extension functor Θ sends interval modules to interval modules. More explicitly, for a given interval $I \in \mathbb{I}(Q)$, we have $\Theta(k_I) \cong k_{\mathrm{conv}(I)}$, where $\mathrm{conv}(I)$ is the smallest interval in $\mathbb{I}(P)$ containing $\iota(I)$.*

We prove the above proposition using the characterization of the intermediate extensions (Proposition 4.3.7).

For our convenience, we set the following map

$$\mathrm{conv}\colon \mathbb{I}(Q) \longrightarrow \mathbb{I}(P), \tag{4.3.10}$$

which sends $I \in \mathbb{I}(Q)$ to the smallest interval in $\mathbb{I}(P)$ containing $\iota(I)$. We use the symbol conv as a map since $\mathrm{conv}(I)$ is the convex hull of $\iota(I)$ (i.e., $\mathrm{conv}(I)$ is the smallest convex full subposet containing $\iota(I)$) in P. We note that we have

$$\mathrm{conv}(I) \cap Q = I \text{ and } \mathrm{conv}(J \cap Q) \subseteq J \tag{4.3.11}$$

for intervals $I \in \mathbb{I}(Q)$ and $J \in \mathbb{I}(P)$.

LEMMA 4.3.9. *Let k_I be an interval module in $\mathsf{rep}_k(P)$. Then, we have the following:*

(1) For any proper submodule Y of k_I, we have $\mathrm{supp}(Y) \subsetneq I$.
(2) For any proper quotient Z of k_I, we have $\mathrm{supp}(Z) \subsetneq I$.

PROOF. We prove (1), the proof of (2) is dual. For any proper submodule Y of k_I, we have a monomorphism $f\colon Y \to k_I$. To prove $\mathrm{supp}(Y) \subsetneq I$, we assume $\mathrm{supp}(Y) = I$ and then show that this leads to a contradiction. For any $a \in P$, the k-linear morphism $f_a\colon Y_a \to (k_I)_a$ is injective. In addition, it is surjective since the k-dimension of $(k_I)_a$ is at most 1 and the assumption $\mathrm{supp}(Y) = I$. This implies that f_a is isomorphism as k-vector spaces for any $a \in P$. In particular, $f\colon Y \to k_I$ is an isomorphism in $\mathsf{rep}_k(P)$. Hence, Y is not a proper submodule of k_I. This is a contradiction. Hence we have $\mathrm{supp}(Y) \subsetneq I$. □

Now, we are ready to prove Proposition 4.3.8.

PROOF OF PROPOSITION 4.3.8. Let I be an interval of Q. To prove the assertion, it suffices to check that $k_{\mathrm{conv}(I)}$ satisfies all properties (i)–(iii) in Proposition 4.3.7.

(i) We have $\mathrm{Res}(k_{\mathrm{conv}(I)}) = k_{\mathrm{conv}(I)} \circ \iota = k_{\mathrm{conv}(I) \cap Q} = k_I$, where $\iota\colon Q \hookrightarrow P$.
(ii) Let Y be a proper submodule of $k_{\mathrm{conv}(I)}$. By Lemma 4.3.9, we have $\mathrm{supp}(Y) \subsetneq \mathrm{conv}(I)$. Then,

$$\mathrm{supp}(Y) \cap Q \subsetneq \mathrm{conv}(I) \cap Q (= I)$$

holds. Indeed, if we assume that $\operatorname{supp}(Y) \cap Q = \operatorname{conv}(I) \cap Q$ holds, then we have $\operatorname{supp}(Y) \subsetneq \operatorname{conv}(I) = \operatorname{conv}(\operatorname{supp}(Y) \cap Q) \subseteq \operatorname{supp}(Y)$ by (4.3.11). This is a contradiction. Thus, we have $\operatorname{supp}(Y) \cap Q \subsetneq I$. In particular, we have

$$\operatorname{supp}(\operatorname{Res}(Y)) = \operatorname{supp}(Y) \cap Q \subsetneq I = \operatorname{supp}(k_I).$$

This implies $\operatorname{Res}(Y)$ is not isomorphic to k_I.

(iii) This is the dual of (ii).

This completes the proof. □

4.3.4. Proof of Monotonicity Theorem Using Intermediate Extensions. In this subsection, we prove the monotonicity theorem (Theorem 4.3.1).

PROOF. For any $V \in \mathsf{rep}_k(Q)$, we consider the interval resolutions of V and $\Theta(V)$

$$\begin{tikzcd}
\cdots \arrow[rr, "f_2"] \arrow[dr, "\pi_2"'] & & X_1 \arrow[rr, "f_1"] \arrow[dr, "\pi_1"'] & & X_0 \arrow[r, two heads, "f_0"] & V \arrow[r] & 0 \\
& K_2 \arrow[ur, "\mathrm{im}(f_2)"'] \arrow[u, phantom, "\circlearrowleft"] & & K_1 \arrow[ur, "\mathrm{im}(f_1)"'] \arrow[u, phantom, "\circlearrowleft"] & & &
\end{tikzcd}$$

$$\begin{tikzcd}
\cdots \arrow[rr, "g_2"] \arrow[dr, "\pi'_2"'] & & Y_1 \arrow[rr, "g_1"] \arrow[dr, "\pi'_1"'] & & Y_0 \arrow[r, two heads, "g_0"] & \Theta(V) \arrow[r] & 0 \\
& L_2 \arrow[ur, "\mathrm{im}(g_2)"'] \arrow[u, phantom, "\circlearrowleft"] & & L_1 \arrow[ur, "\mathrm{im}(g_1)"'] \arrow[u, phantom, "\circlearrowleft"] & & &
\end{tikzcd}$$

respectively, where $K_j := \operatorname{Ker}(f_{i-1}) = \operatorname{Im}(f_i)$ and $L_j := \operatorname{Ker}(g_{i-1}) = \operatorname{Im}(g_i)$ for $j \in \{1, 2, \ldots\}$. We write K_0 and L_0 for V and $\Theta(V)$ respectively.

We show that there exist

$$h_j \colon \Theta(K_j) \to L_j \text{ and } h'_j \colon \operatorname{Res}(L_j) \to K_j \tag{4.3.12}$$

such that the composition of morphisms $K_j \xrightarrow{\gamma_{K_j}} \operatorname{Res} \circ \Theta(K_j) \xrightarrow{\operatorname{Res}(h_j)} \operatorname{Res}(L_j) \xrightarrow{h'_j} K_j$ satisfies

$$h'_j \circ \operatorname{Res}(h_j) \circ \gamma_{K_j} = \mathrm{id}_{K_j} \tag{4.3.13}$$

for any $j \in \{0, 1, \ldots\}$, where γ is the natural isomorphism $\gamma \colon \mathrm{id}_{\mathsf{rep}_k(Q)} \to \operatorname{Res} \circ \Theta$ given in Proposition 4.3.6.

We note that, if we have (4.3.13) for any $j \in \{0, 1, \ldots\}$, then K_j is a direct summand of $\operatorname{Res}(L_j)$ for each j. In particular, $L_j = 0$ implies $K_j = 0$ for each j. Hence, we have

$$\text{int-res-dim}(V) \leq \text{int-res-dim}(\Theta(V))$$

for any $V \in \mathsf{rep}_k(Q)$. Thus, we obtain the desired inequality

$$\text{int-res-gldim}(Q) \leq \text{int-res-gldim}(P).$$

Thus, showing the existence of h_j and h'_j satisfying (4.3.13) for $j \in \{0, 1, \ldots\}$ completes the proof. We prove it by mathematical induction on j.

For $j = 0$, we let $h_0 := \mathrm{id}_{\Theta(V)} \colon \Theta(V) \to \Theta(V)$ and $h'_0 := \gamma_V^{-1} \colon \operatorname{Res} \circ \Theta(V) \to V$. Then, h_0 and h'_0 satisfy (4.3.13). Indeed, we have

$$h'_0 \circ \operatorname{Res}(h_{-1}) \circ \gamma_{K_0} = \gamma_V^{-1} \circ \mathrm{id}_{\operatorname{Res}(\Theta(V))} \circ \gamma_{K_0} = \mathrm{id}_{K_0}.$$

Next, we assume that we have morphisms h_i and h'_i satisfying (4.3.13) for some $i \in \{0, 1, \ldots\}$, and we show the existence of h_{i+1} and h'_{i+1} satisfying (4.3.13). For the exact sequence

$$0 \to K_{i+1} \xrightarrow{\mathrm{im}(f_{i+1})} X_{i+1} \xrightarrow{\pi_{i+1}} K_i \to 0,$$

we apply the functor Θ to the exact sequence and obtain the following commutative diagram:

$$\begin{tikzcd}
& \Theta(K_{i+1}) \arrow[r, "\Theta(\mathrm{im}(f_{i+1}))"] \arrow[d, dotted, "h_{i+1}"] \arrow[dr, phantom, "\circlearrowleft"] & \Theta(X_{i+1}) \arrow[r, "\Theta(\pi_{i+1})"] \arrow[d, dotted, "\delta"] \arrow[dr, phantom, "\circlearrowleft"] & \Theta(K_i) \arrow[d, "h_i"] & \\
0 \arrow[r] & L_{i+1} \arrow[r, "\mathrm{im}(g_{i+1})"] & Y_{i+1} \arrow[r, "\pi'_{i+1}"] & L_i \arrow[r] & 0,
\end{tikzcd} \tag{4.3.14}$$

where we have a morphism $\delta\colon \Theta(X_{i+1}) \to Y_{i+1}$ satisfying $\pi'_{i+1} \circ \delta = h_i \circ \Theta(\pi_{i+1})$ since $\Theta(X_{i+1})$ is interval-decomposable module by Proposition 4.3.8 and π'_{i+1} is an interval cover of L_i, and we have a morphism $h_{i+1}\colon \Theta(K_{i+1}) \to L_{i+1}$ satisfying $\delta \circ \Theta(\mathrm{im}(f_{i+1})) = \mathrm{im}(g_{i+1}) \circ h_{i+1}$ by the universality of kernel.

By applying the restriction functor Res, which is exact by Proposition 4.3.4 (2), to the above commutative diagram, we obtain the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & K_{i+1} \arrow[r, "\mathrm{im}(f_{i+1})"] \arrow[d, "\gamma_{K_{i+1}}"] \arrow[dr, phantom, "\circlearrowleft"] & X_{i+1} \arrow[r, "\pi_{i+1}"] \arrow[d, "\gamma_{X_{i+1}}"] \arrow[dr, phantom, "\circlearrowleft"] & K_i \arrow[r] \arrow[d, "\gamma_{K_i}"] & 0 \\
0 \arrow[r] & \mathrm{Res}\Theta(K_{i+1}) \arrow[r, "\mathrm{Res}\Theta(\mathrm{im}(f_{i+1}))"] \arrow[d, "\mathrm{Res}(h_{i+1})"] \arrow[dr, phantom, "\circlearrowleft"] & \mathrm{Res}\Theta(X_{i+1}) \arrow[r, "\mathrm{Res}\Theta(\pi_{i+1})"] \arrow[d, "\mathrm{Res}(\delta)"] \arrow[dr, phantom, "\circlearrowleft"] & \mathrm{Res}\Theta(K_i) \arrow[r] \arrow[d, "\mathrm{Res}(h_i)"] & 0 \\
0 \arrow[r] & \mathrm{Res}(L_{i+1}) \arrow[r, "\mathrm{Res}(\mathrm{im}(g_{i+1}))"] \arrow[d, dotted, "\iota'"] \arrow[dr, phantom, "\circlearrowleft"] & \mathrm{Res}(Y_{i+1}) \arrow[r, "\mathrm{Res}(\pi'_{i+1})"] \arrow[d, dotted, "\iota"] \arrow[dr, phantom, "\circlearrowleft"] & \mathrm{Res}(L_i) \arrow[r] \arrow[d, "h'_i"] & 0 \\
0 \arrow[r] & K_{i+1} \arrow[r, "\mathrm{im}(f_{i+1})"] & X_{i+1} \arrow[r, "\pi_{i+1}"] & K_i \arrow[r] & 0,
\end{tikzcd} \tag{4.3.15}$$

where the morphism $\iota\colon \mathrm{Res}(Y_{i+1}) \to X_{i+1}$ satisfying $\pi_{i+1} \circ \iota = h'_i \circ \mathrm{Res}(\pi'_{i+1})$ is induced since $\mathrm{Res}(Y_{i+1})$ is interval-decomposable module by Corollary 2.2.11 and π_{i+1} is an interval cover of K_i, and the morphism $\iota'\colon \mathrm{Res}(L_{i+1}) \to K_{i+1}$ satisfying $\iota \circ \mathrm{Res}(\mathrm{im}(g_{i+1})) = \mathrm{im}(f_{i+1}) \circ \iota$ is induced by the universality of kernel.

Using the assumption of mathematical induction, we have

$$\pi_{i+1} = \mathrm{id}_{K_i} \circ \pi_{i+1} = (h'_i \circ \mathrm{Res}(h_i) \circ \gamma_{K_i}) \circ \pi_{i+1} = \pi_{i+1} \circ (\iota \circ \mathrm{Res}(\delta) \circ \gamma_{X_{i+1}}).$$

Since π_{i+1} is right minimal, the morphism $\iota \circ \mathrm{Res}(\delta) \circ \gamma_{X_{i+1}}\colon X_{i+1} \to X_{i+1}$ is an automorphism. We write ϕ for the automorphism $\iota \circ \mathrm{Res}(\delta) \circ \gamma_{X_{i+1}}$ for simplicity. Then, we have

$$\begin{tikzcd}
0 \arrow[r] & K_{i+1} \arrow[r, "\mathrm{im}(f_{i+1})"] \arrow[d, "\iota' \circ \mathrm{Res}(h_{i+1}) \circ \gamma_{K_{i+1}}"'] \arrow[dr, phantom, "\circlearrowleft"] & X_{i+1} \arrow[r, "\pi_{i+1}"] \arrow[d, "\phi", "\wr"'] \arrow[dr, phantom, "\circlearrowleft"] & K_i \arrow[r] \arrow[d, "\mathrm{id}_{K_i}", "\wr"'] & 0 \\
0 \arrow[r] & K_{i+1} \arrow[r, "\mathrm{im}(f_{i+1})"] & X_{i+1} \arrow[r, "\pi_{i+1}"] & K_i \arrow[r] & 0.
\end{tikzcd} \tag{4.3.16}$$

By the universality of kernel, the morphism $\iota' \circ \mathrm{Res}(h_{i+1}) \circ \gamma_{K_{i+1}}\colon K_{i+1} \to K_{i+1}$ is an automorphism. We write it μ, and we let h'_{i+1} be the morphism $\mu^{-1} \circ \iota'\colon \mathrm{Res}(L_{i+1}) \to K_{i+1}$. Then, we obtain the equation

$$h'_{i+1} \circ \mathrm{Res}(h_{i+1}) \circ \gamma_{K_{i+1}} = \mu^{-1} \circ (\iota' \circ \mathrm{Res}(h_{i+1}) \circ \gamma_{K_{i+1}}) = \mu \circ \mu^{-1} = \mathrm{id}_{K_{i+1}},$$

which satisfies (4.3.13) for $j = i + 1$.

By mathematical induction, we have morphisms h_j and h'_j satisfying (4.3.13) for any $j \in \{0, 1, \ldots\}$. This completes the proof. $\square$

Using the monotonicity theorem, we will classify all finite connected posets whose interval resolution global dimension is zero (Theorem 5.2.3) in Subsection 5.2 in the next chapter. By the classification, we naturally introduced a class of posets which we call bipath posets (Definition 5.0.1) in [**AET25**]. The next chapter studies persistence modules over bipath posets, and more specifically, bipath persistent homology.

CHAPTER 5

Bipath Persistence

This chapter aims to develop theoretical aspects of *bipath persistence modules*, proposed by our work [**AET25**], as a new tool for topological data analysis. It can be used

- to study the persistence of topological features across a pair of filtrations connected at their ends, to compare the two filtrations (Example 5.3.3), and
- to obtain a part of the information from a multiparameter persistent homology (Example 5.3.4),

for example.

Here, bipath persistence modules are persistence modules over *bipath posets*.

Definition 5.0.1. (1) For the set $B := \mathbb{R} \times \{1, 2\} \sqcup \{\pm\infty\}$, we give a partial order on B as follows. For $a, b \in B$, if
- $a = -\infty$ or $b = +\infty$, or
- $a = (s, i)$ and $b = (t, i)$ with $s \leq t$ and $i \in \{1, 2\}$,

then we set $a \leq b$.

(2) For non-negative integers n and m, we define the full subposet $B_{n,m}$ of the poset B by $B := (\{1, \ldots, n\} \times \{1\}) \sqcup (\{1, \ldots, m\} \times \{2\}) \sqcup \{\pm\infty\}$.

We call the posets B and $B_{n,m}$ *bipath posets*.

The bipath poset B is displayed as follows:

$$B\colon \quad -\infty \;\bullet \quad \begin{array}{c} \mathbb{R} \times \{1\} \\ \hline \\ \mathbb{R} \times \{2\} \\ \hline \end{array} \quad \bullet\; +\infty,$$

and $B_{n,m}$ is displayed by the following Hasse diagram

$$\begin{tikzcd}[row sep=small] & 1 \arrow[r] & 2 \arrow[r] & \cdots \arrow[r] & n \arrow[dr] & \\ B_{n,m}\colon \; -\infty \arrow[ur] \arrow[dr] & & & & & +\infty \\ & 1' \arrow[r] & 2' \arrow[r] & \cdots \arrow[r] & m' \arrow[ur] & \end{tikzcd} \tag{5.0.1}$$

where we write $1, 2, \ldots, n$ (resp. $1', 2', \ldots, m'$) for $(1, 1), (2, 1), \ldots, (n, 1)$ (resp. $(1, 2), (2, 2), \ldots, (m, 2)$) in $\mathbb{R} \times \{1\}$ (resp. $\mathbb{R} \times \{2\}$) for simplicity. We may write ∞ for $+\infty$ to save space if there is no risk of confusion.

This chapter consists of four sections. In Section 5.1, we show that any pfd B-module (and pfd $B_{n,m}$-module) is interval-decomposable. In Section 5.2, we go back to the theory of interval resolution global dimension and give the complete classification of finite connected posets whose interval resolution global dimension is zero using the observation given in Section 5.1 and using the monotonicity theorem (Theorem 4.3.1). In Section 5.3, we define *bipath persistence diagrams* and propose a visualization of the bipath persistence diagrams (as analogous to a standard persistence diagram). In addition, we propose an algorithm for computing bipath persistence diagrams. In Section 5.4, we prove a stability theorem for bipath persistence diagrams.

Throughout this chapter, let k be a field.

5.1. Interval Decomposability of Bipath Persistence Modules

In this section, we show the next theorem.

Theorem 5.1.1. *Any pfd B-module is uniquely decomposed into a direct sum of interval modules up to isomorphism.*

Our argument follows [**AET25**, Section 4], which gives the interval-decomposition of $B_{n,m}$-modules. We apply their method to the interval decomposition of arbitrary pfd B-modules.

5.1.1. A Key Proposition. This subsection aims to show Proposition 5.1.4 for Theorem 5.1.1. We will see that the proposition immediately proves Theorem 5.1.1.

For our purpose, we first give a poset B', and then we relate B'-modules and B-modules via an order-preserving map between them. The poset B', whose underlying set is $\overline{\mathbb{R}} \times \{1,2\}$, is displayed as follows, where $\overline{\mathbb{R}} := \mathbb{R} \sqcup \{\pm\infty\}$:

$$
B'\colon \quad \begin{array}{ccccc}
(-\infty,1) \ \bullet & & \overset{\mathbb{R}\times\{1\}}{\rule{6cm}{0.4pt}} & & \bullet \ (+\infty,1) \\
\rotatebox{90}{$\leq$} & & & & \rotatebox{90}{$\leq$} \\
(-\infty,2) \ \bullet & & \overset{\mathbb{R}\times\{2\}}{\rule{6cm}{0.4pt}} & & \bullet \ (+\infty,2)
\end{array}.
$$

For relating pfd B'-modules and pfd B-modules, we give the order-preserving map $h\colon B' \to B$ given by

$$
h((r,i)) := \begin{cases} (r,i) & \text{if } r \in \mathbb{R} \\ \pm\infty & \text{if } r = \pm\infty \end{cases} \tag{5.1.1}
$$

for any $(r,i) \in B'$. Then we have the functor $(-\circ h)\colon \mathsf{Rep}_k^{\mathrm{pfd}}(B) \to \mathsf{Rep}_k^{\mathrm{pfd}}(B'), V \mapsto V \circ h$. We note that $(V\circ h)((\pm\infty,1),(\pm\infty,2))$ is the identity $\mathrm{id}_{V_{\pm\infty}}\colon V_{\pm\infty} \to V_{\pm\infty}$ by construction. This functor has the following properties:

Lemma 5.1.2. *Let V be a pfd B-module and $h\colon B' \to B$ be the order-preserving map defined by Equation* (5.1.1)*. We have the following.*

(1) The functor $(-\circ h)\colon \mathsf{Rep}_k^{\mathrm{pfd}}(B) \to \mathsf{Rep}_k^{\mathrm{pfd}}(B')$ is a fully faithful k-linear functor. In particular, we have the bijection of multisets

$$
\mathscr{B}(V) \to \mathscr{B}(V\circ h), V' \mapsto V'\circ h.
$$

(2) $V' \in \mathscr{B}(V)$ is an interval module if and only if $V'\circ h \in \mathscr{B}(V\circ h)$ is an interval module.

Proof. First, we prove (1). We define the injective map $\iota\colon B \to B'$ by

$$
\iota(b) := \begin{cases} (-\infty,1) & \text{if } b = -\infty \\ (+\infty,2) & \text{if } b = +\infty \\ b & \text{else.} \end{cases}
$$

Then, we have $h\circ\iota = \mathrm{id}_B$. In particular, for the functors $(\cdot\circ\iota)\colon \mathsf{Rep}_k^{\mathrm{pfd}}(B') \to \mathsf{Rep}_k^{\mathrm{pfd}}(B)$ and $(\cdot\circ h)\colon \mathsf{Rep}_k^{\mathrm{pfd}}(B) \to \mathsf{Rep}_k^{\mathrm{pfd}}(B')$, we obtain $(\cdot\circ\iota)(\cdot\circ h) = 1_{\mathsf{Rep}_k^{\mathrm{pfd}}(B)}$. This implies that $(\cdot\circ h)$ is faithful. To show that it is full, we recall that we have $(V\circ h)((\pm\infty,1),(\pm\infty,2))$ is the identity $\mathrm{id}_{V_{\pm\infty}}\colon V_{\pm\infty} \to V_{\pm\infty}$. It is easy to see that for any morphisms $f\colon W\circ h \to W'\circ h$ with $W, W' \in \mathsf{Rep}_k^{\mathrm{pfd}}(B)$, we have $(f\circ\iota)\circ h = f$. It implies that $(\cdot\circ h)$ is full. By the above discussion, the functor $(\cdot\circ h)$ is fully faithful.

Next, we prove (2). Now, we have $h\circ\iota = \mathrm{id}_B$. By (1) and Proposition 2.2.12, we obtain the desired result. This completes the proof. □

By the above lemma, we have the following.

Corollary 5.1.3. *Let V be a pfd B-module. Then, V is interval-decomposable if and only if $V\circ h$ is interval-decomposable.*

Proof. It follows from Lemma 5.1.2. □

Next, for pfd B-module V, we construct a pfd B'-module $Z \cong V\circ h$, and we give Proposition 5.1.4.

We denote by V_U the pfd $\overline{\mathbb{R}}\times\{1\}$-module such that $(V_U)_{(s,1)} := (V\circ h)_{(s,1)}$ and $V_U((s,1),(t,1)) := (V\circ h)((s,1),(t,1))$ for $s \leq t$ in $\overline{\mathbb{R}}$. Similarly, we denote by V_D the pfd $\overline{\mathbb{R}}\times\{2\}$-module such that $(V_D)_{(s,2)} := (V\circ h)_{(s,2)}$ and $(V_U)((s,2),(t,2)) := (V\circ h)((s,2),(t,2))$ for $s \leq t$ in $\overline{\mathbb{R}}$. By the structure theorem (Theorem 2.2.6), there exist isomorphisms $\phi\colon X \to V_U$ in $\mathsf{Rep}_k^{\mathrm{pfd}}(\overline{\mathbb{R}} \times \{1\})$ and $\phi\colon V_D \to Y$ in $\mathsf{Rep}_k^{\mathrm{pfd}}(\overline{\mathbb{R}} \times \{2\})$ as follows:

$$\phi\colon X := \bigoplus_{I \in \mathscr{B}(V_U)} k_I \to V_U \text{ and } \psi\colon V_D \to \bigoplus_{J \in \mathscr{B}(V_D)} k_J =: Y,$$

where we identify the interval modules in $\mathscr{B}(V_U)$ and $\mathscr{B}(V_D)$ as the corresponding intervals.

For the isomorphisms ϕ and ψ, we give the k-linear morphisms $\Lambda(\phi, \psi)$ and $\Gamma(\phi, \psi)$, which depend on ϕ and ψ, as follows:

$$\Lambda(\phi,\psi) \;:=\; \psi_{(-\infty,2)} \circ \phi_{(-\infty,1)}\colon X_{(-\infty,1)} \to Y_{(-\infty,2)} \text{ and} \tag{5.1.2}$$

$$\Gamma(\phi,\psi) \;:=\; \psi_{(+\infty,2)} \circ \phi_{(+\infty,1)}\colon X_{(+\infty,1)} \to Y_{(+\infty,2)}. \tag{5.1.3}$$

See the left and right sides of Diagarm (5.1.5) for the morphisms $\Lambda(\phi, \psi)$ and $\Gamma(\phi, \psi)$, respectively. We note that we have $(V_U)_{(-\infty,1)} = (V_D)_{(-\infty,2)}$ and $(V_U)_{(+\infty,1)} = (V_D)_{(+\infty,2)}$, hence $\Lambda(\phi, \psi)$ and $\Gamma(\phi, \psi)$ are defined.

Then, we have the pfd B'-module $Z := Z(\phi, \psi)$, which depends on the isomorphisms ϕ and ψ, defined by

$$Z_{(s,i)} := \begin{cases} X_{(s,1)} & \text{if } i = 1 \\ Y_{(s,2)} & \text{if } i = 2 \end{cases}, \; Z(a,b) := \begin{cases} X(a,b) & \text{if } a = (s,1),\ b = (t,1) \text{ with } s \le t \\ Y(a,b) & \text{if } a = (s,2),\ b = (t,2) \text{ with } s \le t \\ \Lambda(\phi,\psi) & \text{if } a = (-\infty,1),\ b = (-\infty,2) \\ \Gamma(\phi,\psi) & \text{if } a = (+\infty,1),\ b = (+\infty,2). \end{cases} \tag{5.1.4}$$

The pfd B'-module $Z(\phi, \psi)$ is displayed as the outside of the following commutative diagram:

$$\begin{tikzcd}
X_{(-\infty,1)} \arrow[r] \arrow[d, "\phi_{(-\infty,1)}"] \arrow[ddd, bend right=60, "\Lambda(\phi,\psi)"'] & X_{(s,1)} \arrow[r] \arrow[d, "\phi_{(s,1)}"] & X_{(t,1)} \arrow[r] \arrow[d, "\phi_{(t,1)}"] & X_{(\infty,1)} \arrow[d, "\phi_{(\infty,1)}"] \arrow[ddd, bend left=60, "\Gamma(\phi,\psi)"] \\
(V_U)_{(-\infty,1)} \arrow[r] \arrow[d, "\mathrm{id}"] & (V_U)_{(s,1)} \arrow[r] & (V_U)_{(t,1)} \arrow[r] & (V_U)_{(\infty,1)} \arrow[d, "\mathrm{id}"] \\
(V_D)_{(-\infty,2)} \arrow[r] \arrow[d, "\psi_{(-\infty,2)}"] & (V_D)_{(s,2)} \arrow[r] \arrow[d, "\psi_{(s,2)}"] & (V_D)_{(t,2)} \arrow[r] \arrow[d, "\psi_{(t,2)}"] & (V_D)_{(\infty,2)} \arrow[d, "\psi_{(\infty,2)}"] \\
Y_{(-\infty,2)} \arrow[r] & Y_{(s,2)} \arrow[r] & Y_{(t,2)} \arrow[r] & Y_{(\infty,2)}
\end{tikzcd} \tag{5.1.5}$$

where $s \le t \in \mathbb{R}$. In particular, we have $V \circ h \cong Z(\phi, \psi)$ as pfd B'-modules.

The following is a key proposition to Theorem 5.1.1.

Proposition 5.1.4. *Using the above notation, there exist bases of $X_{(\pm\infty,1)}$ and $Y_{(\pm\infty,2)}$, and automorphisms $\alpha\colon X \to X$ and $\beta\colon Y \to Y$ such that the representation matrices*

$$\Lambda(\phi\alpha, \beta\psi)\colon X_{(-\infty,1)} \to Y_{(-\infty,2)} \text{ and } \Gamma(\phi\alpha, \beta\psi)\colon X_{(+\infty,1)} \to Y_{(+\infty,2)}$$

relative to the bases are permutation matrices.

Using Proposition 5.1.4, we show Theorem 5.1.1.

Proof of Theorem 5.1.1. By Proposition 5.1.4, we have bases of $X_{(\pm\infty,1)}$ and $Y_{(\pm\infty,2)}$, and automorphisms α and β of X and Y respectively such that $\Lambda(\phi\alpha, \beta\psi)\colon X_{(-\infty,1)} \to Y_{(-\infty,2)}$ and $\Gamma(\phi\alpha, \beta\psi)\colon X_{(+\infty,1)} \to Y_{(+\infty,2)}$ are permutation matrices. In particular, $Z(\phi\alpha, \beta\psi)$ is interval-decomposable. Since we have $V \circ h \cong Z(\phi, \psi) \cong Z(\phi\alpha, \beta\psi)$, the B'-module $V \circ h$ is interval-decomposable. By Corollary 5.1.3, V is interval-decomposable. This completes the proof. ☐

When showing Proposition 5.1.4, the following observation makes the construction of automorphisms α and β of X and Y simple.

Observation 5.1.5. Intervals in $\mathscr{B}(X)$ (resp. $\mathscr{B}(Y)$) that do not contain both $(-\infty, 1)$ and $(+\infty, 1) \in \overline{\mathbb{R}} \times \{1\}$ (resp. $(-\infty, 2)$ and $(+\infty, 2) \in \overline{\mathbb{R}} \times \{2\}$) can be regarded as intervals in $\mathscr{B}(Z(\phi, \psi))$. Thus, when showing interval-decomposability of the pfd B'-module $Z(\phi, \psi)$, by removing such interval modules in $\mathscr{B}(Z(\phi, \psi))$ in advance, we can assume that the support of every indecomposable module in $\mathscr{B}(Z(\phi, \psi))$ contains $(\pm\infty, 1)$ and $(\pm\infty, 2)$.

Under the assumption, $\mathscr{B}(X)$ (resp. $\mathscr{B}(Y)$) consists of intervals containing $(-\infty, 1)$ or $(+\infty, 1)$ (resp. $(-\infty, 2)$ or $(+\infty, 2)$). Then the multiset $\mathscr{B}(X)$ (resp. $\mathscr{B}(Y)$) is finite since $\dim_k(X_{(\pm\infty,1)})$ and $\dim_k(Y_{(\pm\infty,2)})$ are finite.

By the above observation, we can assume that $\mathscr{B}(X)$ (resp. $\mathscr{B}(Y)$) is a finite multiset of intervals containing $(-\infty, 1)$ or $(+\infty, 1)$ (resp. $(-\infty, 2)$ or $(+\infty, 2)$) when showing Proposition 5.1.4.

Below, we will show Proposition 5.1.4 and complete the proof of Theorem 5.1.1. By Observation 5.1.5, we assume that every interval in $\mathscr{B}(X)$ contains $(-\infty, 1)$ or $(+\infty, 1)$, and every interval in $\mathscr{B}(Y)$ contains $(-\infty, 2)$ or $(+\infty, 2)$. In particular, $\mathscr{B}(X)$ and $\mathscr{B}(Y)$ are finite multisets of intervals.

5.1.2. Elementary Automorphisms. We keep using the notation in the last subsection. In this subsection, we define *elementary automorphisms* of X and Y. These automorphisms are used to construct automorphisms for the proof of Proposition 5.1.4.

For simple notation, we identify $\overline{\mathbb{R}}$ with $\overline{\mathbb{R}} \times \{i\} (i = 1, 2)$ as poset. We define an order $\trianglelefteq_\ell$ (resp. $\trianglelefteq_r$) on the set of intervals in $\overline{\mathbb{R}}$ containing $-\infty$ (resp. $+\infty$) by

$$J \trianglelefteq_\ell I (\text{resp. } J \trianglelefteq_r I) \text{ if and only if } \dim_k \operatorname{Hom}_{\overline{\mathbb{R}}}(k_I, k_J) \neq 0.$$

We have the following.

PROPOSITION 5.1.6. *The orders $\trianglelefteq_\ell$ and $\trianglelefteq_r$ are total.*

PROOF. We only show that the order $\trianglelefteq_\ell$ is total since we can show that the order $\trianglelefteq_r$ is total in a similar discussion.

Let $\mathbb{I}_\ell(\overline{\mathbb{R}})$ be the subset of $\mathbb{I}(\overline{\mathbb{R}})$ whose intervals contains $-\infty$. By definition, for any I and J in $\mathbb{I}_\ell(\overline{\mathbb{R}})$, we have $(I \cap J)^{\downarrow} = I \cap J$ and $(I \cap J)^{\uparrow} = \overline{\mathbb{R}}$. In particular, using Proposition 2.2.8, we have

$$\begin{aligned}\dim_k \operatorname{Hom}_{\overline{\mathbb{R}}}(k_I, k_J) &= |\Omega(I, J)| \\ &= |\{I \cap J : (I \cap J)^{\downarrow} \cap I \subseteq I \cap J \text{ and } (I \cap J)^{\uparrow} \cap J \subseteq I \cap J\}| \\ &= |\{I \cap J : J \subseteq I\}|.\end{aligned}$$

Hence, we have

$$J \trianglelefteq_\ell I :\Leftrightarrow \dim_k \operatorname{Hom}_{\overline{\mathbb{R}}}(k_I, k_J) \neq 0 \Leftrightarrow \{I \cap J : J \subseteq I\} \neq \emptyset \Leftrightarrow J \subseteq I.$$

Thus, the poset $(\mathbb{I}_\ell(\overline{\mathbb{R}}), \trianglelefteq_\ell)$ and the poset $(\mathbb{I}_\ell(\overline{\mathbb{R}}), \subseteq)$ (whose order is defined by the inclusion relation) are isomorphic as posets. Since $(\mathbb{I}_\ell(\overline{\mathbb{R}}), \subseteq)$ is total, $(\mathbb{I}_\ell(\overline{\mathbb{R}}), \trianglelefteq_\ell)$ is total. This completes the proof. □

Using the orders $\trianglelefteq_\ell$ and $\trianglelefteq_r$, we define the total order $\trianglelefteq$ on the set of intervals containing $-\infty$ or $+\infty$ as follows: for I and J containing $-\infty$ or $+\infty$, we have $J \trianglelefteq I$ if and only if

- $-\infty \in I \cap J$ with $J \trianglelefteq_\ell I$, or
- $\infty \in I \cap J$ with $J \trianglelefteq_r I$, or
- $-\infty \in J$ and $+\infty \in I$.

Then, for $J \trianglelefteq I$, by Proposition 2.2.8, we have the following morphism $f_{J,I} \colon k_I \to k_J$ given by

$$(f_{J,I})_r := \begin{cases} \mathrm{id}_k & \text{if } r \in I \cap J \\ 0 & \text{else.} \end{cases} \tag{5.1.6}$$

By definition, if $K \trianglelefteq J$ and $J \trianglelefteq I$ then we have $f_{K,I} = f_{K,J} \circ f_{J,I}$.

Now, we define elementary automorphisms for a pfd $\overline{\mathbb{R}}$-module. Let W be a pfd $\overline{\mathbb{R}}$-module such that any $I \in \mathscr{B}(W)$ contains $-\infty$ or $+\infty$. Then, $\mathscr{B}(W)$ is a finite multiset of intervals, and we can order the intervals in $\mathscr{B}(W) = \{I_1 \trianglelefteq \cdots \trianglelefteq I_m\}$ so that $i \leq j$ implies $I_i \trianglelefteq I_j$. Then, for intervals I_i and I_j with $i < j$ in $\mathscr{B}(W)$ and a non-zero scalar $\lambda \in k$, we have the automorphisms

$E^W_{I_i,\lambda}$ and $E^W_{I_i,I_j,\lambda}$ of $W = k_{I_1} \oplus \cdots \oplus k_{I_m}$ given by

$$E^W_{I_i,\lambda} = \begin{array}{c|c} & \begin{matrix} I_1 & \cdots & I_i & \cdots & I_m \end{matrix} \\ \begin{matrix} I_1 \\ \vdots \\ I_i \\ \vdots \\ I_m \end{matrix} & \begin{bmatrix} f_{I_1,I_1} & & & & \\ & \ddots & & & \\ & & \lambda f_{I_i,I_i} & & \\ & & & \ddots & \\ & & & & f_{I_m,I_m} \end{bmatrix} \end{array}, \quad E^W_{I_j,I_i,\lambda} = \begin{array}{c|c} & \begin{matrix} I_1 & \cdots & I_i & \cdots & I_j & \cdots & I_m \end{matrix} \\ \begin{matrix} I_1 \\ \vdots \\ I_i \\ \vdots \\ I_j \\ \vdots \\ I_m \end{matrix} & \begin{bmatrix} f_{I_1,I_1} & & & & & & \\ & \ddots & & & & & \\ & & f_{I_i,I_i} & & \lambda f_{I_i,I_j} & & \\ & & & \ddots & & & \\ & & & & f_{I_j,I_j} & & \\ & & & & & \ddots & \\ & & & & & & f_{I_m,I_m} \end{bmatrix} \end{array}, \tag{5.1.7}$$

where their inverse morphisms are $E^W_{I_i,\lambda^{-1}}$ and $E^W_{I_j,I_i,-\lambda}$, respectively. We call the above automorphisms of W (whose columns and rows are labeled by intervals in $\mathscr{B}(W) = \{I_1 \trianglelefteq \cdots \trianglelefteq I_\ell\}$ with a given order) *elementary automorphisms* of W.

Now, we recall that $\mathscr{B}(X)$ (resp. $\mathscr{B}(Y)$) is finite multisets of intervals containing $(-\infty, 1)$ or $(\infty, 1)$ (resp. $(-\infty, 2)$ or $(\infty, 2)$) by our assumption (see Observation 5.1.5). Thus, we order $\mathscr{B}(X)$ and $\mathscr{B}(Y)$ by

$$\mathscr{B}(X) = \{I_1 \trianglelefteq \cdots \trianglelefteq I_m\} \text{ and } \mathscr{B}(Y) = \{J_1 \trianglelefteq \cdots \trianglelefteq J_n\} \tag{5.1.8}$$

respectively, and we fix their orders respectively. By the above discussion, we have the elementary automorphisms of X and Y respectively.

5.1.3. Permissible Operations. In this subsection, we give a block matrix $M(\phi, \psi)$ (associated with $\phi\colon X \to V_U$ and $\psi\colon V_D \to Y$) with *permissible operations* (Definition 5.1.8). We relate the permissible operations of the block problem $M(\phi, \psi)$ with the elementary automorphisms for the proof of Proposition 5.1.4.

We first set the following notation. We divide the direct summands of X into

$$X = X_\ell \oplus X_c \oplus X_r$$

where X_c consists of all terms that are non-zero in $(\pm\infty, 1)$, X_ℓ consists of all terms that are zero in $(+\infty, 1)$ and non-zero in $(-\infty, 1)$, X_r consists all terms that are zero at $(-\infty, 1)$ and non-zero at $(+\infty, 1)$. We note that we have $I \trianglelefteq J \trianglelefteq K$ for $I \in \mathscr{B}(X_\ell)$, $J \in \mathscr{B}(X_c)$, and $K \in \mathscr{B}(X_r)$. Thus, our notation $X_\ell \oplus X_c \oplus X_r$ is compatible with the order $\mathscr{B}(X)$ given in Equation (5.1.8). In addition, we take the standard bases of the vector spaces

$$\begin{aligned} X_{(-\infty,1)} &= (k_{I_1} \oplus \cdots \oplus k_{I_n})_{(-\infty,1)} (= (X_\ell \oplus X_c)_{(-\infty,1)}) \text{ and} \\ X_{(-\infty,2)} &= (k_{I_1} \oplus \cdots \oplus k_{I_n})_{(-\infty,2)} (= (X_c \oplus X_r)_{(-\infty,2)}) \end{aligned}$$

using the order given in (5.1.8).

We also give the decomposition $Y = Y_\ell \oplus Y_c \oplus Y_r$, and we give the standard bases of $Y_{(-\infty,2)}$ and $Y_{(+\infty,2)}$ using the order given in (5.1.8).

Below, we construct a block matrix $M(\phi, \psi)$. We write Λ and Γ for $\Lambda(\phi, \psi)$ and $\Gamma(\phi, \psi)$ given in Equation (5.1.2) for simplicity, and we regard both k-linear morphisms as the representation matrices with respect to the given bases of $X_{(\pm\infty,1)}$ and $Y_{(\pm\infty,2)}$.

Using the above notation, Diagram (5.1.5) induces the following commutative diagram:

$$\begin{tikzcd} (X_\ell \oplus X_c)_{(-\infty,1)} \arrow[r, "{\left[\begin{smallmatrix} 0 & E \\ 0 & 0 \end{smallmatrix}\right]}"] \arrow[d, "\Lambda"] & (X_c \oplus X_r)_{(+\infty,1)} \arrow[d, "\Gamma"] \\ (Y_\ell \oplus Y_c)_{(-\infty,2)} \arrow[r, "{\left[\begin{smallmatrix} 0 & E \\ 0 & 0 \end{smallmatrix}\right]}"] & (Y_c \oplus Y_r)_{(+\infty,2)} \end{tikzcd} \tag{5.1.9}$$

where 0 and E are block matrices of appropriate sizes. On the other hand, we can write Λ and Γ by the following block matrix forms:

$$\Lambda = \begin{bmatrix} \Lambda_{\ell,\ell} & \Lambda_{\ell,c} \\ \Lambda_{c,\ell} & \Lambda_{c,c} \end{bmatrix} : (X_\ell \oplus X_c)_{(-\infty,1)} \to (Y_\ell \oplus Y_c)_{(-\infty,2)}$$

$$\Gamma = \begin{bmatrix} \Gamma_{c,c} & \Gamma_{c,r} \\ \Gamma_{r,c} & \Gamma_{r,r} \end{bmatrix} : (X_c \oplus X_r)_{(+\infty,1)} \to (Y_c \oplus Y_r)_{(+\infty,2)}.$$

By the commutativity of Diagram (5.1.9), we have

$$\Lambda_{c,\ell} = 0, \Gamma_{r,c} = 0, \text{ and } \Lambda_{c,c} = \Gamma_{c,c}. \tag{5.1.10}$$

Thus, Λ and Γ have the following block diagonal forms:

$$\Lambda = \begin{bmatrix} \Lambda_{\ell,\ell} & \Lambda_{\ell,c} \\ 0 & \Lambda_{c,c} \end{bmatrix} : (X_\ell \oplus X_c)_{(-\infty,1)} \to (Y_\ell \oplus Y_c)_{(-\infty,2)} \tag{5.1.11}$$

$$\Gamma = \begin{bmatrix} \Gamma_{c,c} & \Gamma_{c,r} \\ 0 & \Gamma_{r,r} \end{bmatrix} : (X_c \oplus X_r)_{(+\infty,1)} \to (Y_c \oplus Y_r)_{(+\infty,2)}. \tag{5.1.12}$$

Then, we have the following.

Lemma 5.1.7. *The blocks $\Lambda_{\ell,\ell}$, $\Lambda_{c,c} = \Gamma_{c,c}$, and $\Gamma_{r,r}$ in Equations* (5.1.11) *and* (5.1.12) *are invertible.*

Proof. We have

$$\begin{aligned} (\dim_k (X_c)_{(-\infty,1)} = \dim_k (X_c)_{(+\infty,1)}) &= \dim_k \operatorname{Im} X_c((-\infty,1),(+\infty,1)) \\ &= \dim_k \operatorname{Im}(Y_c((-\infty,2),(+\infty,2)) \\ &= (\dim_k (Y_c)_{(-\infty,2)} = \dim_k (Y_c)_{(+\infty,2)}). \end{aligned}$$

Thus, both $\Lambda_{c,c} \colon (X_c)_{(-\infty,1)} \to (Y_c)_{(-\infty,2)}$ and $\Gamma_{c,c} \colon (X_c)_{(+\infty,1)} \to (Y_c)_{(+\infty,2)}$ are square matrices. In particular, since Λ and Γ are square matrices, so are $\Lambda_{\ell,\ell}$ and $\Gamma_{r,r}$. On the other hand, since Λ and Γ are invertible matrices, we have $0 \neq \det(\Lambda) = \det(\Lambda_{\ell,\ell})\det(\Lambda_{c,c})$, and $0 \neq \det(\Gamma) = \det(\Gamma_{c,c})\det(\Gamma_{r,r})$. Thus, we have $\Lambda_{\ell,\ell} \neq 0$, $\det(\Lambda_{c,c}) = \det(\Gamma_{c,c}) \neq 0$, and $\det(\Gamma_{r,r}) \neq 0$. This implies that $\Lambda_{\ell,\ell}$, $\Lambda_{c,c} = \Gamma_{c,c}$, and $\Gamma_{r,r}$ are invertible matrices. □

Here we remark that the columns of the blocks

$$\begin{bmatrix} \Lambda_{\ell,c} \\ \Lambda_{c,c} \end{bmatrix} \text{ and } \begin{bmatrix} \Gamma_{c,c} \\ 0 \end{bmatrix}$$

of Λ and Γ are labeled by the same intervals in the same order. Similarly, the rows of the blocks

$$\begin{bmatrix} 0 & \Lambda_{c,c} \end{bmatrix}, \text{ and } \begin{bmatrix} \Gamma_{c,c} & \Gamma_{c,r} \end{bmatrix}$$

of Λ and Γ are labeled by the same intervals in the same order. Since the labeling of the blocks of Λ and Γ shares the same intervals, we let Λ and Γ be combined in a single matrix as follows

$$M(\phi,\psi) := \begin{bmatrix} \Lambda_{\ell,\ell} & \Lambda_{\ell,c} & \emptyset \\ 0 & \Lambda_{c,c} = \Gamma_{c,c} & \Gamma_{c,r} \\ \emptyset & 0 & \Gamma_{r,r} \end{bmatrix}, \tag{5.1.13}$$

where the top right and the bottom left blocks $\emptyset$ are just symbols. We label the columns and rows of $M(\phi,\psi)$ by the intervals $\mathscr{B}(X)$ and $\mathscr{B}(Y)$ with their total order respectively.

Now, we define the permissible operations below.

Definition 5.1.8. Let $M(\phi,\psi)$ be the block matrix defined by Equation (5.1.13). We call the following operations for the block matrix $M(\phi,\psi)$ *permissible operations.*

- Multiplying a row (or column) by a non-zero scalar,
- Adding a row (or column) multiplied by a non-zero scalar to a row (or column) above (or right) it.

Here, we let the blocks $\emptyset$ remain $\emptyset$ themselves by the column and row operations. We also call the above matrix operations permissible operations for any matrix.

Now, we relate the permissible operation on $M(\phi,\psi)$ with the elementary automorphisms of X and Y. The matrix obtained by the permissible operation which multiplies the ith row (resp. column) of $M(\phi,\psi)$ by $0 \neq \lambda \in k$ is equal to the matrix $M(\phi, E^Y_{J_i,\lambda}\psi)$ (resp. $M(\phi E^X_{I_i,\lambda},\psi)$). The matrix obtained by the permissible operation which adds the jth row (or column) multiplied by $0 \neq \lambda \in k$ to ith row (or column) with $i < j$ is equal to the matrix $M(\phi, E^Y_{J_i,J_j,\lambda}\psi)$ (resp. $M(\phi E^X_{I_i,I_j,\lambda},\psi)$).

5.1.4. Proof of Proposition 5.1.4. We keep using the notation in the last subsections. In this subsection, we give a proof of Proposition 5.1.4. We first give the following lemma.

LEMMA 5.1.9. *Let A be an invertible matrix. Then, A can be transformed into a permutation matrix by permissible operations.*

PROOF. It follows from elementary linear algebra. □

Now, we can give a proof of Proposition 5.1.4.

PROOF OF PROPOSITION 5.1.4. Since $\Lambda_{c,c} = \Gamma_{c,c}$ is an invertible matrix by Lemma 5.1.7, we can transform them to a permutation matrix P_c by permissible operations by Lemma 5.1.9. Thus, there exists permissible operations of $M(\phi,\psi)$ that transforms to

$$\begin{bmatrix} \Lambda_{\ell,\ell} & * & \emptyset \\ 0 & P_c & * \\ \emptyset & 0 & \Gamma_{r,r} \end{bmatrix}. \tag{5.1.14}$$

Here, we note that the transformation does not change the blocks $\Lambda_{\ell,\ell}$ and $\Gamma_{r,r}$.

Since $\Lambda_{\ell,\ell}$ is invertible by Lemma 5.1.7, we can transform it to a permutation matrix P_ℓ by permissible operations by Lemma 5.1.9. Thus, there exist permissible operations that transform the matrix (5.1.14) to

$$\begin{bmatrix} P_\ell & * & \emptyset \\ 0 & P_c & * \\ \emptyset & 0 & \Gamma_{r,r} \end{bmatrix}. \tag{5.1.15}$$

Here, we note that the transformation does not change the blocks P_c and $\Gamma_{r,r}$.

Similarly, since $\Lambda_{r,r}$ is invertible by Lemma 5.1.7, we can transform it to a permutation matrix P_r by permissible operations by Lemma 5.1.9. Thus, there exist permissible operations that transform the matrix (5.1.15) to

$$\begin{bmatrix} P_\ell & * & \emptyset \\ 0 & P_c & * \\ \emptyset & 0 & P_r \end{bmatrix}. \tag{5.1.16}$$

Here, we note that the transformation does not change the blocks P_ℓ and P_c.

We apply permissible operations to matrices (5.1.16), and obtain

$$\begin{bmatrix} P_\ell & 0 & \emptyset \\ 0 & P_c & 0 \\ \emptyset & 0 & P_r \end{bmatrix}. \tag{5.1.17}$$

Since the above permissible operations are induced by elementary automorphisms of X and Y, we have an automorphism α of X and an automorphism β of Y such that

$$M(\phi\alpha, \beta\psi) = \begin{bmatrix} P_\ell & 0 & \emptyset \\ 0 & P_c & 0 \\ \emptyset & 0 & P_r \end{bmatrix}.$$

Thus, we have $\Lambda(\phi\alpha,\beta\psi) = \begin{bmatrix} P_\ell & 0 \\ 0 & P_c \end{bmatrix}$ and $\Gamma(\phi\alpha,\beta\psi) = \begin{bmatrix} P_c & 0 \\ 0 & P_r \end{bmatrix}$. This completes the proof. □

REMARK 5.1.10. We give two observations for the computational aspect of the interval decomposition (for Section 5.3) following the above proof. First, once we obtain the representation matrices Λ and Γ with appropriate bases of the form:

$$\Lambda = \begin{bmatrix} \Lambda_{\ell,\ell} & \Lambda_{\ell,c} \\ 0 & \Lambda_{c,c} \end{bmatrix} \text{ and } \Gamma = \begin{bmatrix} \Gamma_{c,c} & \Gamma_{c,r} \\ 0 & \Gamma_{r,r} \end{bmatrix}, \tag{5.1.18}$$

as (5.1.11) and (5.1.12), then we can see that $\mathscr{B}(V)$ contains k_B with the multiplicity $\operatorname{rank}(\Lambda_{c,c}) = \operatorname{rank}(\Gamma_{c,c})$(= the size of $\Lambda_{c,c}$). Secondly, the permutation matrices P_ℓ and P_r can be obtained without constructing $M(\phi, \psi)$ as in (5.1.13), by applying permissible operations to $\Lambda_{\ell,\ell}$ and $\Gamma_{r,r}$, respectively.

The observations are used in Subsection 5.3.2 for giving an algorithm for interval-decomposition of persistence modules over (finite version of) bipath posets.

REMARK 5.1.11. Our proof works in a more general setting. Let P and Q be totally ordered sets. We denote by $B(P, Q)$ a poset whose underlying set is $P \sqcup Q \sqcup \{\pm\infty\}$ with the order $\leq$ given by $a \leq b$ in $B(P, Q)$ if and only if the following holds:

- $a = -\infty$ or $b = +\infty$, or
- $a \leq b$ in P, or
- $a \leq b$ in Q.

We can show that any pfd $B(P, Q)$-module is interval-decomposable.

Remark 5.1.11 will be applied in the next section to classify all finite connected posets whose interval resolution global dimension is zero (or equivalently, whose persistence modules are always interval-decomposable).

5.2. Complete Classification of Finite Posets: Interval Resolution Global Dimension Zero

This section aims to give the complete classification of finite connected posets whose interval resolution global dimension is zero [**AET23**, Theorem 5.1]. The theorem asserts that such finite posets are either the A_ℓ-type poset or the finite bipath posets $B_{n,m}$ for some non-negative integers ℓ, n and m.

For our purpose, we first show the interval-decomposability of pfd $B_{n,m}$-modules, which is immediate from Remark 5.1.11.

PROPOSITION 5.2.1. *Let n and m be positive integers. Any pfd $B_{n,m}$-module is uniquely decomposed into a direct sum of interval modules up to isomorphism.*

PROOF. For the two totally ordered sets $A_n(e)$ and $A_m(e)$, the bipath poset $B_{n,m}$ is isomorphic to the poset $B(A_n(e), A_m(e))$ (see Remark 5.1.11). By Remark 5.1.11, any pfd $B(A_n(e), A_m(e))$-module is interval-decomposable. Thus, any pfd $B_{n,m}$-module is interval-decomposable. □

We give the following lemma for the main result in this section. While the lemma can be proven, for example, using Gabriel's theorem, we give an elementary proof for the readers unfamiliar with the quiver representation theory.

LEMMA 5.2.2. *Let $Q_e := \{s_1, \ldots, s_{2e+1}\} \cup \{t_2, \ldots, t_{2e+2}\}$ be the $\tilde{A}_{2e+1}$-type poset for some positive integer e, where its order is given by $s_i \leq t_{i+1}, t_{i-1}$ for $i = 1, 3 \ldots, 2e+1$, where t_0 denotes t_{2e+2}. Then, we have the following.*

(1) $1 \leq \text{int-res-gldim}(Q_e)$.

(2) Let P be an $\tilde{A}_\ell$-type poset for an positive integer ℓ. If P is not isomorphic to the bipath poset $B_{n,m}$ as posets for any n and m with $n + m = \ell - 1$, then we have $1 \leq \text{int-res-gldim}(P)$.

PROOF. We give $V \in \mathsf{rep}_k(Q_e)$ as follows:

$$V_a := \begin{cases} k & \text{if } a \in Q_e \setminus \{s_1\} \\ k^2 & \text{if } a = s_1, \end{cases} \quad \text{and } V(a,b) := \begin{cases} \mathrm{id}_k & \text{if } a \neq s_1 \\ (1,0) & \text{if } a = s_1, b = s_2 \\ (0,1) & \text{if } a = s_1, b = s_{2n+2}. \end{cases}$$

Then we can easily check $\operatorname{End}_{Q_e}(V) \cong k$, and therefore V is indecomposable. Since it is not an interval module, we have $1 \leq \text{int-res-dim}(V)$, in particular, we have $1 \leq \text{int-res-gldim}(Q_e)$.

Next, let P be an $\tilde{A}_\ell$-type poset that is not isomorphic to the bipath poset $B_{n,m}$ for any n and m with $n + m = \ell - 1$. Then, P contains Q_e for some positive integer e as a full subposet. By the monotonicity theorem (Theorem 4.3.1), we have $1 \leq \text{int-res-gldim}(Q_e) \leq \text{int-res-gldim}(P)$. This completes the proof. □

The following is the main result in this section.

THEOREM 5.2.3. [**AET23**, Theorem 5.1] *Let P be a finite connected poset with ℓ vertices and k be a field. Then, the following conditions are equivalent.*

(a) *Every P-module in $\mathsf{rep}_k(P)$ is interval-decomposable.*
(b) *Every indecomposable P-module in $\mathsf{rep}_k(P)$ is interval.*
(c) int-res-gldim$(P) = 0$.
(d) *The poset P is either an A_ℓ-type poset or a bipath poset $B_{n,m}$ for some positive integers $n, m > 0$ with $n + m = \ell - 2$.*

In particular, these conditions do not depend on the characteristic of the base field k.

PROOF. The equivalence between (a), (b), and (c) is trivial. We show the equivalence of (b) and (d).

First, we show that (d) implies (b). We recall that any persistence module in $\mathsf{rep}_k(P)$ for an A_ℓ-type poset P is interval-decomposable by Gabriel's theorem [**Gab72**] (see also Proposition 2.2.5). Also by Proposition 5.2.1, any $B_{n,m}$-module in $\mathsf{rep}_k(B_{n,m})$ is interval-decomposable. We obtain (b) from (d).

We next show that (b) implies (d). We first see that the Hasse diagram of the poset P has no vertices such that their degrees are more than or equal to three. To prove this, we assume that the Hasse diagram of P has a vertex with a degree more than or equal to three and then show that this leads to a contradiction. Under the assumption, we have a full subposet Q of P which is a Dynkin D_4-type poset. Then, by Example 4.1.8 (2), we have $1 = \text{int-res-gldim}(Q)$. On the other hand, by the monotonicity theorem (Theorem 4.3.1), we have $\text{int-res-gldim}(Q) \leq \text{int-res-gldim}(P)(= 0)$. Combining the above we have:

$$1 = \text{int-res-gldim}(Q) \leq \text{int-res-gldim}(P) = 0.$$

This is a contradiction. Thus, the Hasse diagram of P has no vertices such that their degree are more than or equal to three. In particular, we can say that P is either A_ℓ-type poset or $\tilde{A}_{\ell-1}$-type poset.

If P is an A_ℓ-type poset, then we have nothing to show. Suppose that P is an $\tilde{A}_{\ell-1}$-type poset. We assume that P is not the bipath poset $B_{n,m}$ for any positive integers $n, m > 0$ with $n+m = \ell-2$ and leads to a contradiction. Under the assumption, by Lemma 5.2.2, we have $1 \leq \text{int-res-gldim}(P)$. On the other hand, we have $0 = \text{int-res-gldim}(P)$ by (c) $\iff$ (b), a contradiction. Hence P is a bipath poset $B_{n,m}$ for some positive integers $n, m > 0$ with $n + m = \ell - 2$. This completes the proof.

□

5.3. Bipath Persistence Diagrams and Its Computation

In this section, we define bipath persistence diagrams and propose their visualization analogous to (standard) persistence diagrams. We also propose a computational method for the bipath persistence diagrams.

5.3.1. Bipath Persistence Diagrams. This subsection aims to define bipath persistence diagrams, which can be considered a generalization of standard persistence diagrams, and propose their visualization following [**AET25**, Section 2]. We will give simple examples of bipath persistence diagrams using bipath filtrations of simplicial complexes.

Let $B_{n,m}$ be a bipath poset for some positive integer n and m. We give U and D by the upper and the lower path of $B_{n,m}$ respectively, displayed by

$$U\colon\ -\infty \to 1 \to 2 \to \cdots \to n \to +\infty, \text{ and } \quad D\colon\ -\infty \to 1' \to 2' \to \cdots \to m' \to +\infty. \quad (5.3.1)$$

Then, using the notation, we divide the set of all intervals in $B_{n,m}$ into the following five sets satisfying

$$\mathbb{I}(B_{n,m}) = \mathcal{B}(B_{n,m}) \sqcup \mathcal{L}(B_{n,m}) \sqcup \mathcal{R}(B_{n,m}) \sqcup \mathcal{U}(B_{n,m}) \sqcup \mathcal{D}(B_{n,m}),$$

where

- the set $\mathcal{B}(B_{n,m})$ is the one point set $\{B_{n,m}\}$,
- the set $\mathcal{L}(B_{n,m})$ consists of $[-\infty, t] \cup [-\infty, s]$ for some $t \in U$ and $s \in D$ with $s, t \neq \infty$,
- the set $\mathcal{R}(B_{n,m})$ consists of $[s, \infty] \cup [t, \infty]$ for some $s \in U$ and $t \in D$ with $s, t \neq -\infty$,

- the set $\mathcal{U}(B_{n,m})$ consists of $[s, t]$ for some $s, t \in U$ with $s, t \notin \{-\infty, +\infty\}$, and
- the set $\mathcal{D}(B_{n,m})$ consists of $[t, s]$ for some $s, t \in D$ with $s, t \notin \{-\infty, +\infty\}$.

Here, $[a, b]$ is an interval given by $\{x \in B_{n,m} \mid a \leq x \leq b\}$. If an interval $J(\neq B_{n,m})$ of $B_{n,m}$ has one of the above forms, we write J by the pair of elements $\langle s, t\rangle$. Table 1 displays each interval.

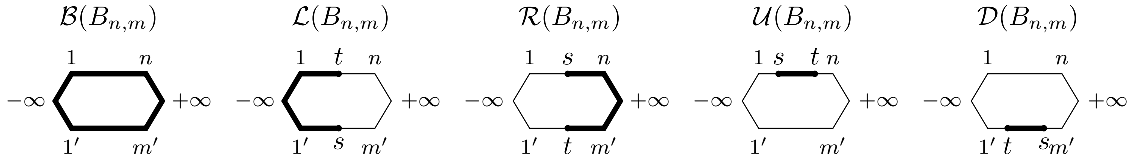

Table 1. All intervals of $B_{n,m}$. The leftmost one is $B_{n,m}$. For each symbol $\mathcal{X} \in \{\mathcal{L}, \mathcal{R}, \mathcal{U}, \mathcal{D}\}$, we represent intervals $J = \langle s, t\rangle \in \mathcal{X}(B_{n,m})$ by thick lines.

Under the above notation, we have the following.

Proposition 5.3.1. *Let $B_{n,m}$ be a bipath poset. For any $V \in \mathsf{rep}_k(B_{n,m})$, we have the decomposition $V \cong V_{\mathcal{B}} \oplus V_{\mathcal{L}} \oplus V_{\mathcal{R}} \oplus V_{\mathcal{U}} \oplus V_{\mathcal{D}}$, where*

$$V_{\mathcal{X}} \cong \bigoplus_{J \in \mathcal{X}(B_{n,m})} k_J^{m_V(J)} \quad \text{for } \mathcal{X} \in \{\mathcal{B}, \mathcal{L}, \mathcal{R}, \mathcal{U}, \mathcal{D}\}.$$

Proof. It follows from Proposition 5.2.1. □

We define a persistence diagram for every bipath persistence module as follows.

Definition 5.3.2. Let $V \in \mathsf{rep}_k(B_{n,m})$ be decomposed as in Proposition 5.3.1. The *bipath persistence diagram* of V is the multiset given by

$$\begin{aligned} \mathscr{B}(V) &:= \{k_J \in \mathsf{rep}_k(B_{n,m}) \text{ with multiplicity } m_V(J)\} \\ &= \mathscr{B}(V_{\mathcal{B}}) \sqcup \mathscr{B}(V_{\mathcal{L}}) \sqcup \mathscr{B}(V_{\mathcal{R}}) \sqcup \mathscr{B}(V_{\mathcal{U}}) \sqcup \mathscr{B}(V_{\mathcal{D}}). \end{aligned}$$

If there is no risk of confusion, we identify $\mathscr{B}(V)$ with the multisets of intervals

$$\{J \in \mathbb{I}(B_{n,m}) \text{ with multiplicity } m_V(J)\}.$$

When applying bipath persistent homology to data analysis, visualizing topological features across a bipath filtration can help analysts find hidden structures in the data. To this end, we propose a visualization of bipath persistence diagrams, analogous to standard persistence diagrams.

Let $B_{n,m}$ be the bipath poset. For $V \in \mathsf{rep}_k(P)$, we visualize the bipath persistence diagram $\mathscr{B}(V)$ of V as follows:

- We set elements $1, 2, \ldots, n, \infty, m', \ldots, 1', -\infty$ of $B_{n,m}$ on the horizontal axis in this order and set elements $-\infty, 1, \ldots, n, \infty, m', \ldots, 1'$ of $B_{n,m}$ on the vertical axis in this order as shown in Fig. 1. We obtain a plane.
- We set the region $\mathcal{B}$ (consisting of one point set) at the upper left of the plane. We plot points with multiplicity $m_V(B_{n,m})$. This point corresponds to the interval $B_{n,m}$.
- For an interval $J \neq B_{n,m}$, we can write $J = \langle s, t\rangle$. We plot the point at (s, t) with the multiplicity $m_V(J)$. The point corresponds to the interval $\langle s, t\rangle$.

Since we divided $\mathscr{B}(V)$ into the five components $\mathscr{B}(V) = \mathscr{B}(V_{\mathcal{B}}) \sqcup \mathscr{B}(V_{\mathcal{L}}) \sqcup \mathscr{B}(V_{\mathcal{R}}) \sqcup \mathscr{B}(V_{\mathcal{U}}) \sqcup \mathscr{B}(V_{\mathcal{D}})$, the plane which visualizes the bipath persistence diagram also has five components.

Now, we give two examples of bipath persistence diagrams of bipath filtrations of simplicial complexes. In the first example, we explain how to relate points in a bipath persistence diagram to topological features in a bipath filtration. We also recall our motivation for bipath persistent homology. In the second example, while we do not give the interpretation of the bipath persistence diagram, we relate bipath persistent homology to multiparameter persistent homology, as a potential application of bipath persistent homology.

Example 5.3.3. We give an example of bipath filtration of simplicial complexes, the visualization of the bipath persistence diagram, and its interpretations below.

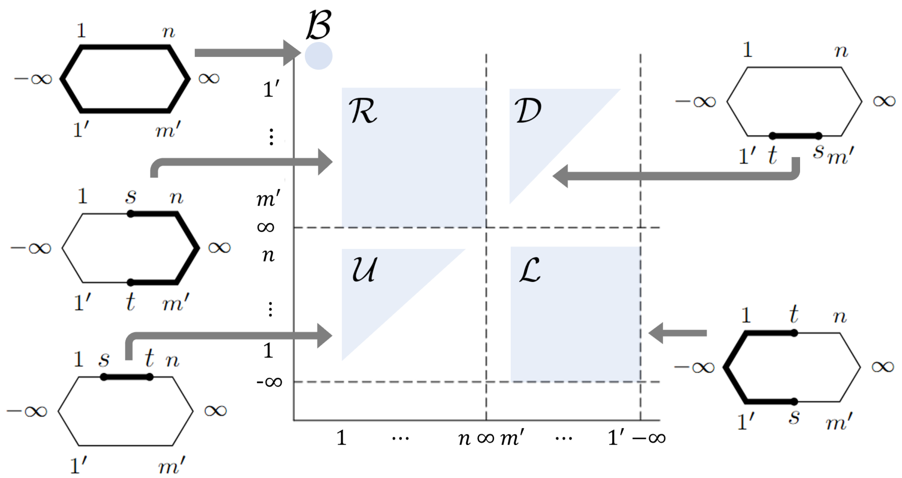

FIGURE 1. This is a visualization of a bipath persistence diagram. Points in the region $\mathcal{X} \in \{\mathcal{B}, \mathcal{L}, \mathcal{R}, \mathcal{U}, \mathcal{D}\}$ correspond to intervals in $\mathcal{X}(B_{n,m})$. Here, the region $\mathcal{U}$ can be seen as a standard persistence diagram, that is, a point (s,t) corresponds to the interval $[s,t]$ in the upper path $[1,n]$. On the other hand, a point (s',t') in the region $\mathcal{D}$ corresponds to the interval $[t',s']$ in the lower path $[1',m']$. We note that the notation in the region $\mathcal{D}$ is the reverse of that in the region $\mathcal{U}$. In each region, points toward the top or left indicate longer intervals.

We consider an abstract simplicial complex Δ whose j-faces are given by

$$
\begin{aligned}
\Delta^0 &:= \{a,b,c,d,e\},\\
\Delta^1 &:= \{\{a,b\},\{a,c\},\{a,d\},\{a,e\},\{b,c\},\{c,d\},\{c,e\},\{d,e\}\},\\
\Delta^2 &:= \{\{a,c,e\},\{c,d,e\}\},
\end{aligned}
$$

and the $B_{3,2}$-filtration

$$
S\colon\quad
\begin{tikzcd}[column sep=small]
 & S_1 \arrow[r, hook] & S_2 \arrow[r, hook] & S_3 \arrow[dr, hook] & \\
S_{-\infty} \arrow[ur, hook] \arrow[dr, hook] & & & & S_{+\infty}\\
 & S_{1'} \arrow[rr, hook] & & S_{2'} \arrow[ur, hook] &
\end{tikzcd}
$$

for Δ defined as follows. Let $S_{-\infty} := \Delta^0 \cup \{\{a,b\},\{a,c\},\ \{b,c\},\{c,d\},\ \{c,e\},\{d,e\}\}$ and $S_\infty := \Delta$. In addition, let

$$
\begin{aligned}
S_1 &:= S_{-\infty} \cup \{\{a,e\}\},\\
S_2 &:= S_1 \cup \{\{a,d\}\},\\
S_3 &:= S_2 \cup \{\{c,d,e\}\},\\
S_{1'} &:= S_{-\infty} \cup \{\{a,d\}\},\\
S_{2'} &:= S_{1'} \cup \{\{a,e\},\{a,c,e\}\}.
\end{aligned}
$$

We apply the 1st homology functor with coefficient field $k = \mathbb{F}_2$ (a field with two elements) to S and obtain the bipath persistent homology $V := H_1(S;k) \in \mathsf{rep}_k(B_{3,2})$.

$$
\begin{tikzcd}[column sep=small]
 & H_1(S_1;k) \arrow[r] & H_1(S_2;k) \arrow[r] & H_1(S_3;k) \arrow[dr] & \\
H_1(S_{-\infty};k) \arrow[ur] \arrow[dr] & & & & H_1(S_{+\infty};k).\\
 & H_1(S_{1'};k) \arrow[rr] & & H_1(S_{2'};k) \arrow[ur] &
\end{tikzcd}
$$

We decompose V into intervals as follows:

$$
V \cong k_{B_{3,2}} \oplus k_{\langle 2',2\rangle} \oplus k_{\langle 2,1'\rangle} \oplus k_{\langle 1,3\rangle},
$$

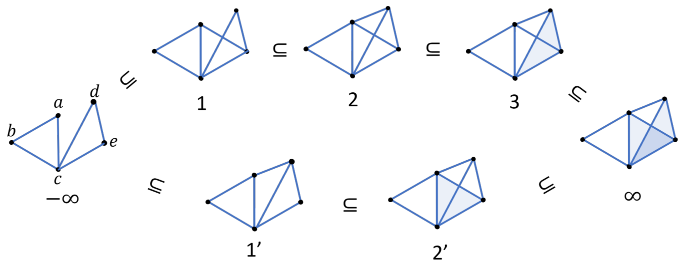

FIGURE 2. A geometric realization of the bipath filtration S.

and we obtain the bipath persistence diagram of V:

$$\mathscr{B}(V) = \{B_{3,2}, \langle 2', 2\rangle, \langle 2, 1'\rangle, \langle 1, 3\rangle\}.$$

We give a visualization $\mathscr{B}(V)$ in Fig. 3

We provide an intuitive explanation of how the persistence of topological features, holes, across the bipath filtration S corresponds to intervals in the bipath persistence diagram $\mathscr{B}(V)$. Here, by "holes" we mean representatives of the homology classes in each homology module. We note that holes, representatives of the homology classes, are generally not unique. In the following explanation, we have chosen specific representatives.

The interval $B_{3,2}$ corresponding to the point (A) in Fig. 3 says that a hole, formed by the simplices $\{a, b\}$, $\{b, c\}$, and $\{a, c\}$ in S, exists throughout the filtration S. The interval $\langle 2, 1'\rangle$ corresponding to the point (B) in Fig. 3 says that there exists a hole that is born at 2 in the upper filtration and $1'$ in the lower filtration which does not die. This interval $\langle 2, 1'\rangle$ corresponds to a hole formed by the simplices $\{a, d\}, \{a, c\}, \{c, e\}$, and $\{d, e\}$. The interval $\langle 1, 3\rangle$ corresponding to the point (C) in Fig. 3 says that there is a hole that is born at 1 in the upper filtration and dies at 3. This interval $\langle 1, 3\rangle$ corresponds to a hole formed by the the simplices $\{a, c\}, \{a, e\}$, and $\{c, e\}$. Finally, the interval $\langle 2', 2\rangle$ corresponding to the point (D) in Fig. 3 says that there is a hole that is born at $-\infty$ and dies at 2 in the upper filtration and at $2'$ in the lower filtration. This interval $\langle 2', 2\rangle$ corresponds to a hole formed by simplices $\{c, d\}, \{c, e\}$, and $\{d, e\}$.

Before closing the example, we go back to our motivation for bipath persistent homology. As noted earlier, bipath persistent homology can be used to study the differences in the persistence of topological features in the upper and lower filtrations. For example, the interval $\langle 1, 3\rangle$ corresponds to the point (C) in Fig. 3 can not be seen in the lower path. This may be understood as the feature that is robust in the upper path, however it is not robust in the lower path. On the other hand, the intervals $B_{3,2}$, $\langle 2, 1'\rangle$, and $\langle 2', 2\rangle$, which correspond to the point (A), (B), and (D) respectively, are robust features in the upper and lower filtrations. In this way, we can compare the topological features in the upper and lower filtrations.

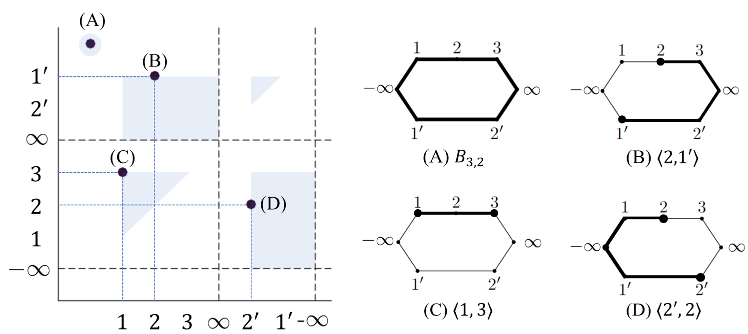

FIGURE 3. This is a visualization of the bipath persistence diagram $\mathscr{B}(V)$ of $V := H_1(S; k)$. Each point has a multiplicity one in this case.

Using the next example of the bipath persistence diagram, we relate bipath persistent homology with multiparameter persistent homology.

EXAMPLE 5.3.4. We first note that bipath persistent homology is naturally obtained from multiparameter persistent homology. Hence we can visualize a part of multiparameter persistent homology using bipath persistence diagrams.

The left picture in Fig. 4, for example, is a bifiltration of simplicial complexes, and the middle picture is a bipath filtration of simplicial complexes obtained by restricting to a bipath. We computed 1st bipath persistent homology of the bipath filtration (using a field $k = \mathbb{F}_2$), and we display it in Fig. 4.

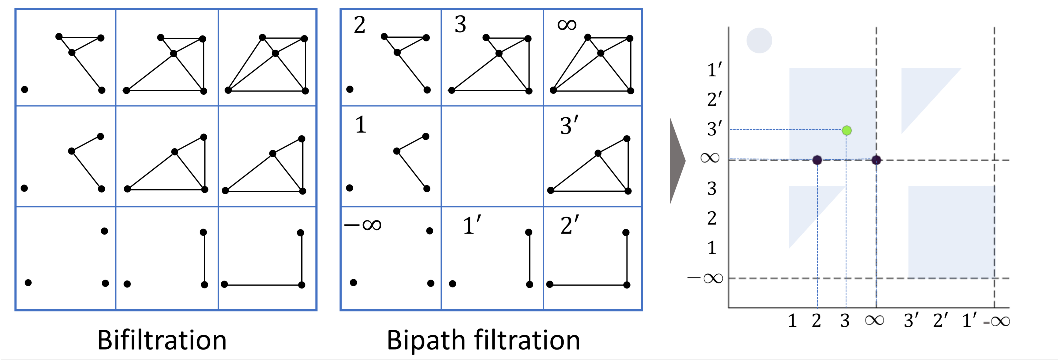

FIGURE 4. This is a visualization of the bipath persistence diagram obtained from a restriction of bifiltration to a bipath filtration. The black points $(2, \infty)$ and (∞, ∞) in the bipath persistence diagram have a multiplicity of one, and the green point $(3, 3')$ has a multiplicity of two.

In this way, bipath persistent homology can capture and visualize a part of topological features within multiparameter persistent homology.

For applying bipath persistent homology to data analysis, having algorithms for computing bipath persistence diagrams is crucial. In the next subsection, we propose an algorithm for bipath persistence diagrams.

5.3.2. A Computation of Bipath Persistence Diagrams. This subsection aims to provide a computational method for bipath persistence diagrams of (1) bipath persistence modules and (2) bipath persistent homology of a bipath filtration of simplicial complexes. These methods are given in [**AET25**, Section 4].

5.3.2.1. *A Computation of Bipath Persistence Diagrams From Bipath Persistence Modules.* Here, we aim to give a computational method for bipath persistence diagrams of bipath persistence modules.

For $V \in \mathsf{rep}_k(B_{n,m})$, we compute the bipath persistence diagram

$$\mathscr{B}(V) = \mathscr{B}(V_{\mathcal{B}}) \sqcup \mathscr{B}(V_{\mathcal{L}}) \sqcup \mathscr{B}(V_{\mathcal{R}}) \sqcup \mathscr{B}(V_{\mathcal{U}}) \sqcup \mathscr{B}(V_{\mathcal{D}})$$

of V following Section 5.1 (and Remark 5.1.10). Similar to Section 5.1, we identify intervals and interval modules for simplicity if there is no risk of confusion.

We first outline our computation of $\mathscr{B}(V)$.

(1) For the persistence modules V_U and V_D

$$V_U\colon\ V_{-\infty} \xrightarrow{V(-\infty,1)} V_1 \xrightarrow{V(1,2)} \cdots \xrightarrow{V(n-1,n)} V_n \xrightarrow{V(n,+\infty)} V_{+\infty},$$
$$V_D\colon\ V_{-\infty} \xrightarrow{V(-\infty,1')} V_{1'} \xrightarrow{V(1',2')} \cdots \xrightarrow{V(m'-1,m')} V_{m'} \xrightarrow{V(m',+\infty)} V_{+\infty}.$$

over the totally ordered sets U and D (5.3.1), we decompose them into intervals (their interval-decomposability is guaranteed by structure theorem (e.g., [**Gab72, BCB20**])).

We separate the multiset $\mathscr{B}(V_X)(X \in \{U, D\})$ by

$$\begin{aligned}
\mathscr{B}_\ell(V_X) &:= \{I \in \mathscr{B}(V_X) \mid -\infty \in I \text{ and } +\infty \notin I\},\\
\mathscr{B}_c(V_X) &:= \{I \in \mathscr{B}(V_X) \mid \pm\infty \in I\},\\
\mathscr{B}_r(V_X) &:= \{I \in \mathscr{B}(V_X) \mid -\infty \notin I \text{ and } +\infty \in I\}, \text{ and}\\
\mathscr{B}_o(V_X) &:= \mathscr{B}(V_X) \setminus (\mathscr{B}_\ell(V_X) \sqcup \mathscr{B}_c(V_X) \sqcup \mathscr{B}_r(V_X)).
\end{aligned}$$

Then, we obtain the intervals

$$\mathscr{B}(V_{\mathcal{U}}) = \mathscr{B}_o(V_U) \text{ and } \mathscr{B}(V_{\mathcal{D}}) = \mathscr{B}_o(V_D).$$

(2) Compute representation matrices

$$\Lambda = \begin{bmatrix} \Lambda_{\ell,\ell} & \Lambda_{\ell,c} \\ 0 & \Lambda_{c,c} \end{bmatrix} \text{ and } \Gamma = \begin{bmatrix} \Gamma_{c,c} & \Gamma_{c,r} \\ 0 & \Gamma_{r,r} \end{bmatrix}$$

with appropriate bases as in (5.1.5) to connect intervals in $\mathscr{B}_\ell(V_U)$ and $\mathscr{B}_\ell(V_D)$, $\mathscr{B}_c(V_U)$ and $\mathscr{B}_c(V_D)$, and $\mathscr{B}_r(V_U)$ and $\mathscr{B}_r(V_D)$ respectively. Here, we obtain

$$\mathscr{B}(V_{\mathcal{B}}) = \{B_{n,m} \text{ with the multiplicity } |\mathscr{B}_c(V_U)| = |\mathscr{B}_c(V_D)|\}.$$

(3) Reduce $\Lambda_{\ell,\ell}$ and $\Gamma_{r,r}$ to permutation matrices P_ℓ and P_r respectively by the permissible operations (Definition 5.1.8).

(4) Read off intervals from P_ℓ and P_r, and obtain

$$\mathscr{B}(V_{\mathcal{L}}) \text{ and } \mathscr{B}(V_{\mathcal{R}}).$$

Then we obtain the desired decomposition:

$$\mathscr{B}(V) = \mathscr{B}(V_{\mathcal{B}}) \sqcup \mathscr{B}(V_{\mathcal{L}}) \sqcup \mathscr{B}(V_{\mathcal{R}}) \sqcup \mathscr{B}(V_{\mathcal{U}}) \sqcup (V_{\mathcal{D}}).$$

The following is a detailed explanation of each of these steps.

Step 1. We decompose V_U and V_D into intervals (for example, see [**CDS10**, Section 4]) and denote by ϕ and ψ isomorphisms

$$\phi\colon X := \bigoplus_{I \in \mathscr{B}(V_U)} k_I \to V_U, \text{ and } \psi\colon V_D \to \bigoplus_{J \in \mathscr{B}(V_D)} k_J =: Y.$$

We regard intervals $\mathscr{B}_o(V_U)$ and $\mathscr{B}_o(V_D)$ as the intervals $\mathscr{B}(V_{\mathcal{U}})$ and $\mathscr{B}(V_{\mathcal{D}})$ respectively.

Step 2. We first order the intervals $\mathscr{B}_\ell(V_U) \sqcup \mathscr{B}_c(V_U) \sqcup \mathscr{B}_r(V_U)$ and $\mathscr{B}_\ell(V_D) \sqcup \mathscr{B}_c(V_D) \sqcup \mathscr{B}_r(V_D)$ by the order $\trianglelefteq$ given in Section 5.1. More explicitly, we first order all intervals in $\mathscr{B}_\ell(V_U)$ (resp. $\mathscr{B}_\ell(V_D)$) by

$$I_{\ell 1} \trianglelefteq I_{\ell 2} \trianglelefteq \cdots \trianglelefteq I_{\ell m_\ell} \text{ (resp. } J_{\ell 1} \trianglelefteq J_{\ell 2} \trianglelefteq \cdots \trianglelefteq J_{\ell m_\ell})$$

so that $I_{\ell i} \subseteq I_{\ell j}$ (resp. $J_{\ell i} \subseteq J_{\ell j}$) whenever $i \le j \in \{1, \dots, m_\ell\}$. Here we have $m_\ell = |\mathscr{B}_\ell(V_U)| = |\mathscr{B}_\ell(V_D)|$. Next, we order all intervals in $\mathscr{B}_c(V_U)$ (resp. $\mathscr{B}_c(V_D)$) by

$$I_{c1} \trianglelefteq I_{c2} \trianglelefteq \cdots \trianglelefteq I_{cm_c} \text{ (resp. } J_{c1} \trianglelefteq J_{c2} \trianglelefteq \cdots \trianglelefteq J_{cm_c})$$

without any condition, where we have $m_c = |\mathscr{B}_c(V_U)| = |\mathscr{B}_c(V_D)|$, and $U = I_{c1} = \cdots = I_{cm_c}$ and $D = J_{c1} = \cdots = J_{cm_c}$. Finally, we order all intervals in $\mathscr{B}_r(V_U)$ (resp. $\mathscr{B}_r(V_D)$) by

$$I_{r1} \trianglelefteq I_{r2} \trianglelefteq \cdots \trianglelefteq I_{rm_r} \text{ (resp. } J_{r1} \trianglelefteq J_{r2} \trianglelefteq \cdots \trianglelefteq J_{rm_r})$$

so that $I_{rj} \subseteq I_{ri}$ (resp $J_{rj} \subseteq J_{ri}$) whenever $i \le j \in \{1, \dots, m_r\}$, where $m_r = |\mathscr{B}_r(V_U)| = |\mathscr{B}_r(V_D)|$. Using the above, We order all intervals in $\mathscr{B}_\ell(V_U) \sqcup \mathscr{B}_c(V_U) \sqcup \mathscr{B}_r(V_U)$ and $\mathscr{B}_\ell(V_D) \sqcup \mathscr{B}_c(V_D) \sqcup \mathscr{B}_r(V_D)$ by

$$\begin{aligned}
&I_{\ell 1} \trianglelefteq \cdots \trianglelefteq I_{\ell m_\ell} \trianglelefteq I_{c1} \trianglelefteq \cdots \trianglelefteq I_{cm_c} \trianglelefteq I_{r1} \trianglelefteq \cdots \trianglelefteq I_{rm_r}, \text{ and}\\
&J_{\ell 1} \trianglelefteq \cdots \trianglelefteq J_{\ell m_\ell} \trianglelefteq J_{c1} \trianglelefteq \cdots \trianglelefteq J_{cm_c} \trianglelefteq J_{r1} \trianglelefteq \cdots \trianglelefteq J_{rm_r}
\end{aligned}$$

respectively.

Then, we give standard bases for $X_{\pm\infty}$ and $Y_{\pm\infty}$ associated with the order of intervals given above. For the bases, we denote Λ and Γ by the representation matrices of the k-linear morphisms $\psi_{-\infty} \circ \phi_{-\infty} \colon X_{-\infty} \to Y_{-\infty}$ and $\psi_{+\infty} \circ \phi_{+\infty} \colon X_{+\infty} \to Y_{+\infty}$. Both matrices have the form:

$$\begin{array}{cc} & \begin{array}{cccccc} I_{\ell 1} & \cdots & I_{\ell m_\ell} & I_{c1} & \cdots & I_{cm_c}\end{array} \\ \begin{array}{c} J_{\ell 1}\\ \vdots \\ J_{\ell m_\ell} \\ J_{c1} \\ \vdots \\ J_{cm_c}\end{array} & \left[\begin{array}{cc} \boxed{\Lambda_{\ell,\ell}} & \begin{array}{ccc} * & \cdots & * \\ \vdots & \ddots & \vdots \\ * & \cdots & *\end{array} \\ \begin{array}{ccc} 0 & \cdots & 0\\ \vdots&\ddots&\vdots\\0&\cdots&0\end{array} & \boxed{\Lambda_{c,c}} \end{array}\right]\end{array} \quad \text{and} \quad \begin{array}{cc} & \begin{array}{cccccc} I_{c1} & \cdots & I_{cm_c} & I_{r1} & \cdots & I_{rm_r}\end{array} \\ \begin{array}{c} J_{c1}\\ \vdots \\ J_{cm_c} \\ J_{r1} \\ \vdots \\ J_{rm_r}\end{array} & \left[\begin{array}{cc} \boxed{\Gamma_{c,c}} & \begin{array}{ccc} * & \cdots & * \\ \vdots & \ddots & \vdots \\ * & \cdots & *\end{array} \\ \begin{array}{ccc} 0 & \cdots & 0\\ \vdots&\ddots&\vdots\\0&\cdots&0\end{array} & \boxed{\Gamma_{r,r}} \end{array}\right]\end{array}. \tag{5.3.2}$$

Here, we have

$$\mathscr{B}(V_{\mathcal{B}}) = \{B_{n,m} \text{ with the multiplicity } m_c\}.$$

Step 3. We apply the permissible operations (Definition 5.1.8) to $\Lambda_{\ell,\ell}$ and $\Gamma_{r,r}$ and obtain permutation matrices P_ℓ and P_r. Here, we can apply the algorithm OneWayReduce [**AET25**, Algorithm 1] to the invertible matrices $\Lambda_{\ell,\ell}$ and $\Gamma_{r,r}$, which transforms an input invertible matrix into a permutation matrix by permissible operations and outputs it.

Algorithm 1 Reduction of an invertible $d \times d$ matrix A

```
procedure OneWayReduce(A)
    for col ← 1, 2, ..., d do
        row ← max{i | A_{i,col} ≠ 0}
        Multiply the row-th row by (A_{row,col})^{-1}.
        Use entry at (row, col) to zero out all entries to its right using column operations.
        Use entry at (row, col) to zero out all entries to its above using row operations.
    end for
    return A
end procedure
```

Step 4. Using the permutation matrices P_ℓ and P_r, we get

$$\mathscr{B}(V_{\mathcal{L}}) = \{I \cup J \subseteq B_{n,m} \mid (I, J) \in \mathscr{B}_\ell(V_U) \times \mathscr{B}_\ell(V_D) \text{ with } (P_\ell)_{J,I} = 1_k\},$$
$$\mathscr{B}(V_{\mathcal{R}}) = \{I \cup J \subseteq B_{n,m} \mid (I, J) \in \mathscr{B}_r(V_U) \times \mathscr{B}_r(V_D) \text{ with } (P_r)_{J,I} = 1_k\}.$$

Then, we obtain the desired decomposition:

$$\mathscr{B}(V) = \mathscr{B}(V_{\mathcal{B}}) \sqcup \mathscr{B}(V_{\mathcal{L}}) \sqcup \mathscr{B}(V_{\mathcal{R}}) \sqcup \mathscr{B}(V_{\mathcal{U}}) \sqcup \mathscr{B}(V_{\mathcal{D}}).$$

Next, we give an algorithm for computing bipath persistence diagrams of bipath persistent homology from bipath filtration of simplicial complexes.

5.3.2.2. *A Computation of Bipath Persistence Diagrams From Bipath Filtrations.* Here, we aim to provide an algorithm for computing bipath persistence diagrams of bipath persistent homology from bipath filtrations of simplicial complexes, rather than from the level of persistence modules discussed in the last subsubsection. This allows us to give an efficient computational method for Step 1 given in the last subsubsection. The computational method given here will be useful when applying bipath persistent homology.

Consider a $B_{n,m}$-filtration S :

$$\begin{tikzcd} & S_1 \arrow[r, hook] & S_2 \arrow[r, hook] & \cdots \arrow[r, hook] & S_n \arrow[dr, hook] & \\ S_{-\infty} \arrow[ur, hook] \arrow[dr, hook] & & & & & S_{+\infty} \\ & S_{1'} \arrow[r, hook] & S_{2'} \arrow[r, hook] & \cdots \arrow[r, hook] & S_{m'} \arrow[ur, hook] & \end{tikzcd}.$$

By applying qth homology functor $H_q(-;k)$, we obtain the $B_{n,m}$-module $V = H_q(S;k)$ as follows:

$$\begin{tikzcd}[row sep=small, column sep=small]
 & H_q(S_1;k) \arrow[r] & H_q(S_2;k) \arrow[r] & \cdots \arrow[r] & H_q(S_n;k) \arrow[dr] & \\
H_q(S_{-\infty};k) \arrow[ur] \arrow[dr] & & & & & H_q(S_{+\infty};k). \\
 & H_q(S_{1'};k) \arrow[r] & H_q(S_{2'};k) \arrow[r] & \cdots \arrow[r] & H_q(S_{m'};k) \arrow[ur] &
\end{tikzcd}$$

To obtain $\mathscr{B}(V)$, we compute the interval decomposition of

$$V_U\colon\ H_q(S_{-\infty};k) \longrightarrow H_q(S_1;k) \longrightarrow \cdots \longrightarrow H_q(S_n;k) \longrightarrow H_q(S_{+\infty};k),\ \text{and}$$

$$V_D\colon\ H_q(S_{-\infty};k) \longrightarrow H_q(S_{1'};k) \longrightarrow \cdots \longrightarrow H_q(S_{m'};k) \longrightarrow H_q(S_{+\infty};k)$$

respectively following the last subsection (Step 1). Here, instead of computing the interval-decomposition of V_U and V_D from the level of persistence modules, we can apply the standard persistent homology algorithm (see [**ELZ02**], [**OPT$^+$17**] for example) to the two filtrations

$$S_U\colon\ S_{-\infty} \subseteq S_1 \subseteq S_2 \subseteq \cdots \subseteq S_n \subseteq S_{+\infty},$$

and

$$S_D\colon\ S_{-\infty} \subseteq S_{1'} \subseteq S_{2'} \subseteq \cdots \subseteq S_{m'} \subseteq S_{+\infty}$$

respectively to obtain its interval decomposition.

REMARK 5.3.5. After computing interval decomposition of V_U and V_D by standard persistent homology algorithm respectively, we construct Λ and Γ as in Step 2. Here, the standard persistent homology algorithm records the representatives of homology classes with bases for the images of the boundary operation at $-\infty$ and $+\infty$. Thus, we can compute Λ and Γ as change-of-basis matrices for the bases at $-\infty$ and $+\infty$ of the upper and lower path.

Using the above observation, we give Algorithm 2, which is given in [**AET25**, Algorithm 2] for computing bipath persistence diagrams of bipath persistent homology.

Algorithm 2 Main Algorithm – for bipath filtrations

Require: Bipath filtration S

1: **procedure** INTERVALDECOMPOSE(S, q)
2: Compute the standard qth persistence diagram of S_U and S_D (Step 1) (with additional information noted in Remark 5.3.5).
3: Construct the matrices Λ and Γ as in Remark 5.3.5 (Step 2).
4: Reduce blocks of Λ and Γ to permutation matrices (Step 3).
5: **return** The bipath persistence diagram (Step 4)
6: **end procedure**

The Algorithm 2 can compute the bipath persistence diagram of a given bipath filtration of a simplicial complex using the standard persistent homology algorithm twice and permissible operations of two invertible matrices Λ and Γ whose size depends on the number of intervals which is usually smaller than the number of simplices. This observation suggests that bipath persistence diagrams can be computed without much more effort than the standard persistent homology algorithm.

REMARK 5.3.6. We provide an implementation of Algorithm 2 in `https://github.com/ShunsukeTada1357/Bipathposets`.

This section defined bipath persistence diagrams and proposed their visualization and computational methods. The next section gives the stability theorem for the bipath persistence diagrams.

5.4. Stability Theorem for Bipath Persistence

This section shows the stability theorem for bipath persistence diagrams (Theorem 5.4.29) under a condition on *bipath functions* (Definition 5.4.28). This theorem will provide a mathematical justification for applying bipath persistent homology to noisy data, analogous to the stability theorem for standard persistent homology [**CSEH07**], [**CCSG$^+$09**].

This section is divided into six subsections. The first subsection (Subsection 5.4.1) reviews interleaving and bottleneck distances. We recall the relation between both distances through the language of graph theory. The observations are important in proving the isometry theorem for the bipath persistence modules (Theorem 5.4.30). Subsection 5.4.2 revisits the standard stability theorem. We recall that standard isometry theory easily implies standard stability theorem. The third to sixth subsections are our main work in this section. Subsection 5.4.3 studies types of interval modules over the bipath poset. We divide intervals in the bipath poset B into five types, and we study morphisms between interval modules of different types. Subsection 5.4.4 gives the main result — the stability theorem for bipath persistent homology — assuming the isometry theorem for bipath persistence modules. Finally, Subsections 5.4.5 and 5.4.6 give a proof of the isometry theorem for bipath persistence modules (Theorem 5.4.30).

5.4.1. Review on Distances Between Persistence Modules. In this subsection, we recall the interleaving distance and bottleneck distances, and we relate them using the language of graph theory.

We fix a poset P. An *action* on P is a family of endofunctors $\Lambda := \{\Lambda_\epsilon\colon P \to P\}_{\epsilon\in\mathbb{R}_{\geq 0}}$ satisfying the following:

- $\Lambda_0 = \mathrm{id}_P$.
- For all $p \in P$, we have $p \leq \Lambda_\epsilon(p)$.
- $\Lambda_\epsilon \circ \Lambda_\zeta = \Lambda_{\epsilon+\zeta}$.

This is just a special case of the superlinear family of translations, see [**BDSS15, dSMS18**] for example.

5.4.1.1. *Interleaving Distance.* Here, we define an interleaving distance between pfd P-modules. Let Λ be an action on P and ϵ be a non-negative real number. A *shift functor* $(\cdot)(\epsilon)\colon \mathsf{Rep}_k^{\mathrm{pfd}}(P) \to \mathsf{Rep}_k^{\mathrm{pfd}}(P)$ is the functor $(\cdot \circ \Lambda_\epsilon)\colon \mathsf{Rep}_k^{\mathrm{pfd}}(P) \to \mathsf{Rep}_k^{\mathrm{pfd}}(P)$. We denote by $V_{0\to\epsilon}\colon V \to V(\epsilon)$ the morphism given by $(V_{0\to\epsilon})_p := V(p, \Lambda_\epsilon(p))$ for every $p \in P$.

Definition 5.4.1. An *ϵ-interleaving* between two pfd P-modules V and W is a pair of morphisms $\alpha\colon V \to W(\epsilon)$ and $\beta\colon W \to V(\epsilon)$ such that $\beta(\epsilon)\circ\alpha = V_{0\to 2\epsilon}$ and $\alpha(\epsilon)\circ\beta = W_{0\to 2\epsilon}$.

If there exists an ϵ-interleaving between V and W, we say that V and W are *ϵ-interleaved.* Then, we define the interleaving distance between pfd P-modules.

Definition 5.4.2. The *interleaving distance* d_{I} between pfd P-modules is an extended pseudometric defined by

$$d_{\mathrm{I}}(V, W) := \inf\{\epsilon \in \mathbb{R}_{\geq 0} \mid V \text{ and } W \text{ are } \epsilon\text{-interleaved}\} \tag{5.4.1}$$

for pfd P-modules V and W.

5.4.1.2. *Algebraic Bottleneck Distance.* Here, we define a bottleneck distance between pfd P-modules. For multisets X and Y, a *partial bijection* between X and Y is a bijection $\sigma\colon X' \to Y'$ for some subsets $X' \subseteq X$ and $Y' \subseteq Y$, and we write it by $\sigma\colon X \nrightarrow Y$. For the bijection, we write X' and Y' by $\mathrm{Coim}\,\sigma$ and $\mathrm{Im}\,\sigma$ respectively.

We say that a pfd P-module V is *ϵ-trivial* if $V_{0\to\epsilon} = 0$. If V is not ϵ-trivial, we say that V is *ϵ-significant.* Notice that $V_{0\to 2\epsilon} = 0$ if and only if V and 0 are ϵ-interleaved.

Definition 5.4.3 (Algebraic ϵ-matching). Let V and W be pfd P-modules. We define an *algebraic ϵ-matching* between $\mathscr{B}(V)$ and $\mathscr{B}(W)$ by a partial bijection $\sigma\colon \mathscr{B}(V) \nrightarrow \mathscr{B}(W)$ satisfying the following properties:

(1) For all $M \in \mathscr{B}(V) \setminus \mathrm{Coim}\,\sigma$, the pfd P-module M is 2ϵ-trivial,
(2) For all $N \in \mathscr{B}(W) \setminus \mathrm{Im}\,\sigma$, the pfd P-module N is 2ϵ-trivial,
(3) For all $M \in \mathrm{Coim}\,\sigma$, the pfd P-modules M and $\sigma(M)$ are ϵ-interleaved.

We say that V and W (or $\mathscr{B}(V)$ and $\mathscr{B}(W)$) are *algebraic ϵ-matched* if there exists an algebraic ϵ-matching between $\mathscr{B}(V)$ and $\mathscr{B}(W)$.

Definition 5.4.4 (Bottleneck distance for modules). The *bottleneck distance* d_{B} between pfd P-modules is an extended pseudometric defined by

$$d_{\mathrm{B}}(V,W) := \inf\{\epsilon \in \mathbb{R}_{\geq 0} \mid V \text{ and } W \text{ are algebraic } \epsilon\text{-matched}\} \tag{5.4.2}$$

for pfd P-modules V and W.

By definition, the following holds.

Proposition 5.4.5. *Let V and W be pfd P-modules. If there exists an algebraic ϵ-matching between V and W, then there exists an ϵ-interleaving between V and W. In particular, we have the following inequality.*

$$d_{\mathrm{I}}(V,W) \leq d_{\mathrm{B}}(V,W). \tag{5.4.3}$$

5.4.1.3. *Bottleneck Distance.* Here, we define a bottleneck distance between multisets of intervals in $\mathbb{I}(P)$. Let I be an interval in $\mathbb{I}(P)$ and let ϵ be a non-negative real number. We give the subset $\mathrm{Ex}^{\epsilon}_{\Lambda}(I)$ of P by

$$\mathrm{Ex}^{\epsilon}_{\Lambda}(I) := \Lambda_{\epsilon}^{-1}(I)^{\uparrow} \cap \Lambda_{\epsilon}(I)^{\downarrow}, \tag{5.4.4}$$

where we define $\emptyset^{\uparrow} = P$. We note that $\mathrm{Ex}^{\epsilon}_{\Lambda}(I)$ is convex. If there is no risk of confusion, we write $\mathrm{Ex}^{\epsilon}(I)$ for $\mathrm{Ex}^{\epsilon}_{\Lambda}(I)$ for simplicity.

Definition 5.4.6 (ϵ-matching). Let A and B be multisets of intervals. We define an *ϵ-matching* between A and B by a partial bijection $\sigma\colon A \nrightarrow B$ satisfying the following properties:

(1) For all $I \in A \setminus \mathrm{Coim}\,\sigma$, the interval module k_I is 2ϵ-trivial.
(2) For all $J \in B \setminus \mathrm{Im}\,\sigma$, the interval module k_J is 2ϵ-trivial.
(3) If $\sigma(I) = J$ then $I \subseteq \mathrm{Ex}^{\epsilon}(J)$ and $J \subseteq \mathrm{Ex}^{\epsilon}(I)$.

We say that A and B are *ϵ-matched* if there exists an ϵ-matching between A and B.

We define the bottleneck distance between multisets of intervals as follows.

Definition 5.4.7. The *bottleneck distance* d_{B} between the multisets A and B of intervals in $\mathbb{I}(P)$ is an extended pseudometric given by

$$d_{\mathrm{B}}(A,B) := \inf\{\epsilon \in \mathbb{R}_{\geq 0} \mid A \text{ and } B \text{ are } \epsilon\text{-matched}\}. \tag{5.4.5}$$

5.4.1.4. *Two Bottleneck Distances.* Let Λ be an action on P. This subsubsection aims to prove Proposition 5.4.13, which relates the bottleneck distances d'_{B} and d_{B}, under the following condition on the action.

$$\text{For any interval } I \in \mathbb{I}(P) \text{ and } \epsilon \in \mathbb{R}_{\geq 0},\ \text{if } p \in I^{\uparrow} \text{ and } \Lambda_{\epsilon}(p) \in \Lambda_{\epsilon}(I)^{\downarrow}, \text{then } p \in I. \tag{5.4.6}$$

Example 5.4.8.
(1) Let P be a poset with an action Λ such that $\Lambda_{\epsilon}\colon P \to P$ is a bijection for any $\epsilon \in \mathbb{R}_{\geq 0}$. Then the action Λ satisfies Condition (5.4.6).
(2) Let $(\mathbb{R}, \leq)$ be the poset of all real numbers with the action Λ given by $\Lambda_{\epsilon}(r) := r + \epsilon$ for any $r \in \mathbb{R}$ and $\epsilon \in \mathbb{R}_{\geq 0}$. The map Λ_{ϵ} is a bijection, therefore, the action satisfies Condition (5.4.6) by the above example (1).
(3) Let $(B, \leq)$ be the bipath poset (Definition 5.0.1 (1)). We define the action Λ on B by

$$\Lambda_{\epsilon}(b) := \begin{cases} (r+\epsilon, i) & \text{if } b = (r,i) \in \mathbb{R} \times \{i\} \text{ for } i = 1,2 \\ \pm\infty & \text{if } b = \pm\infty, \end{cases}$$

for each $\epsilon \in \mathbb{R}_{\geq 0}$. Since it is a bijection for each $\epsilon \in \mathbb{R}_{\geq 0}$, the action satisfies Condition (5.4.6) by the above example (1).

This thesis considers actions Λ on P such that Λ_{ϵ} is a bijection for each $\epsilon \in \mathbb{R}_{\geq 0}$. By Example 5.4.8 (1), such actions satisfy Condition (5.4.6). Therefore, we may assume from the beginning that Λ_{ϵ} is a bijection for each $\epsilon \in \mathbb{R}_{\geq 0}$ rather than assuming Condition (5.4.6). However, we set Condition (5.4.6) to clarify the conditions we will use in this subsubsection.

Let us return to the main discussion. For our purpose in this subsubsection, we first show that intervals I and J with $I \subseteq \mathrm{Ex}^{\epsilon}(J)$ and $J \subseteq \mathrm{Ex}^{\epsilon}(I)$ induce an ϵ-interleaving between k_I and k_J (Proposition 5.4.12), and then prove Proposition 5.4.13. We begin with the following lemma.

LEMMA 5.4.9. *Let C be a convex full subposet of P. We have $C^{\uparrow} \cap C^{\downarrow} = C$.*

PROOF. For any $a \in C^{\uparrow} \cap C^{\downarrow}$, there exist x, y in C such that $x \leq a \leq y$. Since C is convex, we have $a \in C$. This completes the proof. $\square$

Then we have the following.

LEMMA 5.4.10. *Let I and J are intervals in P. Let Λ be an action on P satisfying Condition* (5.4.6). *If $I \subseteq \mathrm{Ex}^{\epsilon}(J)$ and $J \subseteq \mathrm{Ex}^{\epsilon}(I)$, then the set of connected components in $I \cap \Lambda_{\epsilon}^{-1}(J)$ (resp. $J \cap \Lambda_{\epsilon}^{-1}(I)$) coincides with $\Omega(I, \Lambda_{\epsilon}^{-1}(J))$ (resp. $\Omega(J, \Lambda_{\epsilon}^{-1}(I))$).*

PROOF. It suffices to show that any connected component of $I \cap \Lambda_{\epsilon}^{-1}(J)$ satisfies (2.2.2). Let C be a connected component of $I \cap \Lambda_{\epsilon}^{-1}(J)$. We first show $I \cap C^{\downarrow} \subseteq C$. For any $a \in I \cap C^{\downarrow} \subseteq C$, there exists $c \in C$ such that $a \leq c$. In particular, by $c \in I \cap \Lambda_{\epsilon}^{-1}(J)$, we have $a \in \Lambda_{\epsilon}^{-1}(J)^{\downarrow}$. On the other hand, by $a \in I \subseteq \mathrm{Ex}^{\epsilon}(J) \subseteq \Lambda_{\epsilon}^{-1}(J)^{\uparrow}$, we have $a \in \Lambda_{\epsilon}^{-1}(J)^{\downarrow} \cap \Lambda_{\epsilon}^{-1}(J)^{\uparrow} = \Lambda_{\epsilon}^{-1}(J)$. Hence $a \in I \cap \Lambda_{\epsilon}^{-1}(J)$. By $a \leq c$, the element a is in the same component as c. We obtain $a \in C$.

Next, we show $\Lambda_{\epsilon}^{-1}(J) \cap C^{\uparrow} \subseteq C$. For any $b \in \Lambda_{\epsilon}^{-1}(J) \cap C^{\uparrow} \subseteq C$, there exists $c \in C$ such that $c \leq b$. In particular, by $c \in I \cap \Lambda_{\epsilon}^{-1}(J) \subseteq I$, we have $b \in I^{\uparrow}$. On the other hand, we have $\Lambda_{\epsilon}(b) \in J \subseteq \mathrm{Ex}^{\epsilon}(I) \subseteq \Lambda_{\epsilon}(I)^{\downarrow}$. By Condition (5.4.6) on Λ, we have $b \in I$. Hence we obtain $b \in I \cap \Lambda_{\epsilon}^{-1}(J)$. By $c \leq b$, the element b is in the same component as c. We obtain $b \in C$.

We can show that any connected component of $J \cap \Lambda_{\epsilon}^{-1}(I)$ satisfies (2.2.2) in a similar way. This completes the proof. $\square$

LEMMA 5.4.11. *Let I and J be intervals in P. Let Λ be an action on P satisfying Condition* (5.4.6). *If $I \subseteq \mathrm{Ex}^{\epsilon}(J)$ and $J \subseteq \mathrm{Ex}^{\epsilon}(I)$, then we have $I \cap \Lambda_{\epsilon}^{-1}(J) \cap \Lambda_{2\epsilon}^{-1}(I) = I \cap \Lambda_{2\epsilon}^{-1}(I)$ and $J \cap \Lambda_{\epsilon}^{-1}(I) \cap \Lambda_{2\epsilon}^{-1}(J) = J \cap \Lambda_{2\epsilon}^{-1}(J)$.*

PROOF. We first show $I \cap \Lambda_{\epsilon}^{-1}(J) \cap \Lambda_{2\epsilon}^{-1}(I) = I \cap \Lambda_{2\epsilon}^{-1}(I)$. For any $a \in I \cap \Lambda_{2\epsilon}^{-1}(I)$, we show $a \in \Lambda_{\epsilon}^{-1}(J)$. By assumption, we have $a \in I \subseteq \mathrm{Ex}^{\epsilon}(J) \subseteq \Lambda_{\epsilon}^{-1}(J)^{\uparrow}$. Thus, there exists $x \in \Lambda_{\epsilon}^{-1}(J)$ such that $x \leq a$. In particular, we have $J \ni \Lambda_{\epsilon}(x) \leq \Lambda_{\epsilon}(a)$, which implies $\Lambda_{\epsilon}(a) \in J^{\uparrow}$. On the other hand, since $a \in \Lambda_{2\epsilon}^{-1}(I)$, we have $\Lambda_{\epsilon}(\Lambda_{\epsilon}(a)) = \Lambda_{2\epsilon}(a) \in I \subseteq \mathrm{Ex}^{\epsilon}(J) \subseteq \Lambda_{\epsilon}(J)^{\downarrow}$. By Condition (5.4.6), we have $\Lambda_{\epsilon}(a) \in J$. Thus, we obtain $a \in \Lambda_{\epsilon}^{-1}(J)$. Hence, we have the desired inclusion relation.

Similarly, we can show $J \cap \Lambda_{\epsilon}^{-1}(I) \cap \Lambda_{2\epsilon}^{-1}(J) = J \cap \Lambda_{2\epsilon}^{-1}(J)$. This completes the proof. $\square$

PROPOSITION 5.4.12. *Let I and J be intervals in P. Let Λ be an action on P satisfying Condition* (5.4.6). *If $I \subseteq \mathrm{Ex}^{\epsilon}(J)$ and $J \subseteq \mathrm{Ex}^{\epsilon}(I)$, then k_I and k_J are ϵ-interleaved.*

PROOF. By Lemma 2.2.8 and Lemma 5.4.10, we have the morphisms $f\colon k_I \to k_J(\epsilon)$ and $g\colon k_J \to k_I(\epsilon)$ defined by

$$f_a := \begin{cases} \mathrm{id}_k & \text{if } a \in I \cap \Lambda_{\epsilon}^{-1}(J) \\ 0 & \text{else,} \end{cases} \quad \text{and } g_b := \begin{cases} \mathrm{id}_k & \text{if } b \in J \cap \Lambda_{\epsilon}^{-1}(I) \\ 0 & \text{else,} \end{cases}$$

for any $a \in I$ and $b \in J$. The pair f and g is ϵ-interleaving between k_I and k_J. Indeed, we have $(g(\epsilon) \circ f)_a = \mathrm{id}_k$ if $a \in I \cap \Lambda_{\epsilon}^{-1}(J) \cap \Lambda_{2\epsilon}^{-1}(I)$ and 0 if $a \neq I \cap \Lambda_{\epsilon}^{-1}(J) \cap \Lambda_{2\epsilon}^{-1}(I)$. By Lemma 5.4.11, we have $I \cap \Lambda_{\epsilon}^{-1}(J) \cap \Lambda_{2\epsilon}^{-1}(I) = I \cap \Lambda_{2\epsilon}^{-1}(I)$. Hence $g(\epsilon) \circ f = (k_I)_{0 \to 2\epsilon}$. Similarly, we have $f(\epsilon) \circ g = (k_J)_{0 \to 2\epsilon}$. Hence the pair f and g is an ϵ-interleaving between k_I and k_J. This completes the proof. $\square$

The following is the main proposition in this subsubsection.

PROPOSITION 5.4.13. *Let V and W be interval-decomposable pfd P-modules. Let Λ be an action on P satisfying Condition* (5.4.6). *If there exists an ϵ-matching between $\mathscr{B}(V)$ and $\mathscr{B}(W)$, then it gives an algebraic ϵ-matching between V and W. In particular, we have*

$$d'_{\mathrm{B}}(V, W) \leq d_{\mathrm{B}}(\mathscr{B}(V), \mathscr{B}(W)). \tag{5.4.7}$$

PROOF. If there exists an ϵ-matching between multisets $\mathscr{B}(V)$ and $\mathscr{B}(W)$ of $\mathbb{I}(P)$, then by Proposition 5.4.12, it gives an algebraic ϵ-matching between V and W. This completes the proof. $\square$

We immediately have the following.

COROLLARY 5.4.14. *Let V and W be interval-decomposable pfd P-modules. Let Λ be an action on P satisfying Condition* (5.4.6) *and ϵ be a non-negative real number. we have the following:*

- *If there exists an ϵ-matching between $\mathscr{B}(V)$ and $\mathscr{B}(W)$, then there exists an algebraic ϵ-matching between V and W.*
- *If there exists an algebraic ϵ-matching between V and W, then there exists an ϵ-interleaving between V and W.*

Therefore, we have

$$d_{\mathrm{I}}(V,W) \leq d'_{\mathrm{B}}(V,W) \leq d_{\mathrm{B}}(\mathscr{B}(V),\mathscr{B}(W)).$$

PROOF. It follows from Proposition 5.4.5 and Proposition 5.4.13. □

5.4.1.5. *Relation to Graph Matching.* Let V and W be pfd P-modules, and ϵ be a non-negative real number. In this subsubsection, we give a sufficient condition to have an algebraic ϵ-matching between V and W (Theorem 5.4.20) using the language of graph theory. This condition is used in proving the isometry theorem for bipath persistence modules (Theorem 5.4.30).

We first recall the basic terminology of graph theory. Let $G = (V, E)$ be an undirected graph. A *matching* in G is a subset of edges that do not share common vertices. Let V' be a subset of vertices and E' be a subset of edges in a graph. We say that E' *covers* V' if any vertex in V' is an endpoint of an edge in E'. A *full subgraph* induced by a subset of vertices V' in G is a subgraph formed from a V' and from all of the edges that have both endpoints in the subset V'. A *neighborhood* of a vertex v in G is the set of vertices that are connected to v by an edge. We denote by $N_G(v)$ the neighborhood of v in G. For a subset V' of vertices in G, we write $N_G(V')$ for $\bigcup_{v\in V'} N_G(v)$. A *bipartite graph* is a graph $G = (X \sqcup Y, E)$ such that every edge in E connects a vertex in X to a vertex in Y. For subsets of vertices $X' \subseteq X$ and $Y' \subseteq Y$, we denote by $G(X' \sqcup Y')$ a full subgraph of G.

Next, we recall the graph-theoretical propositions.

CONDITION 5.4.15. Let $G = (X \sqcup Y, E)$ be a bipartite graph. We say that G satisfies Conditions (H) and (H′) if it satisfies the following respectively.

(H) For any finite subset X' of X, we have $|X'| \leq |N_G(X')|$.
(H′) For any finite subset Y' of Y, we have $|Y'| \leq |N_G(Y')|$.

The following theorem is called Hall's marriage theorem.

THEOREM 5.4.16 (cf. [**Hal35**, Theorem 1]). *For a bipartite graph $G = (X \sqcup Y, E)$ such that $N_G(x)$ is finite for all $x \in X$, the following are equivalent:*

(a) *The graph G satisfies Condition* (H).
(b) *There exists a matching in G that covers X.*

PROPOSITION 5.4.17. [**Bje21**, P.111] *Let $G = (X \sqcup Y, E)$ be a bipartite graph. Let X' and Y' be subsets of X and Y such that the neighborhood of an arbitrary x in X' and an arbitrary y in Y' are finite respectively. If the full subgraphs $G(X' \sqcup Y)$ and $G(X \sqcup Y')$ satisfy Conditions* (H) *and* (H′) *respectively, then there exists a matching in G that covers $X' \sqcup Y'$.*

Now, we relate a graph matching and an ϵ-interleaving between pfd P-modules. To this end, we give a construction of a bipartite graph from two P-modules V and W whose vertices are given by $\mathscr{B}(V) \sqcup \mathscr{B}(W)$ and undirected edges are induced by ϵ-interleavings between elements in $\mathscr{B}(V)$ and $\mathscr{B}(W)$.

Let V and W be pfd P-modules, and let ϵ be a non-negative real number. For an indecomposable direct summand $M \in \mathscr{B}(V)$ and $A \subseteq \mathscr{B}(V)$, we give $\mu = \mu_\epsilon$ by

$$\mu(M) := \{N \in \mathscr{B}(W) \mid M \text{ and } N \text{ are } \epsilon\text{-interleaved}\} \text{ and } \mu(A) := \bigcup_{I\in A} \mu(I) \tag{5.4.8}$$

We immediately have the following.

LEMMA 5.4.18. *Let V and W be pfd P-modules and ϵ be a non-negative real number. For any 2ϵ-significant pfd P-module $M \in \mathscr{B}(V)$, the multiset $\mu(M)$ given by* (5.4.8) *is finite.*

PROOF. Since M is 2ϵ-significant, there exists $a \in P$ such that $(M_{0\to 2\epsilon})_a \neq 0$. For this a, we have $N_{\Lambda_\epsilon(a)} \neq 0$ for any $N \in \mu(M) \subseteq \mathscr{B}(W)$. Since we have

$$|\mu(M)| \leq \sum_{N\in\mu(M)} \dim_k(N_{\Lambda_\epsilon(a)}) \leq \dim_k(W_{\Lambda_\epsilon(a)}) < \infty,$$

we obtain that $\mu(M)$ is finite. This completes the proof. □

Let

$$G(V, W, \mu) := (\mathscr{B}(V) \sqcup \mathscr{B}(W), E_\mu) \tag{5.4.9}$$

be the bipartite graph where E_μ is the set of undirected edges given by $\bigcup_{M\in\mathscr{B}(V)}\{(M, N) \mid N \in \mu(M)\}$. We relate a notion of algebraic ϵ-matching and matching in a graph.

PROPOSITION 5.4.19. *Let V and W be pfd P-modules, and let ϵ be a non-negative real number. The following are equivalent.*

(a) *There exists an algebraic ϵ-matching between $\mathscr{B}(V)$ and $\mathscr{B}(W)$.*
(b) *There exists a matching in the bipartite graph $G(V, W, \mu)$ that covers 2ϵ-significant elements in $\mathscr{B}(V) \sqcup \mathscr{B}(W)$.*

PROOF. If there exists an algebraic ϵ-matching σ between V and W, then we have a matching of graph $\{(M, \sigma(M)) \mid M \in \operatorname{Coim}\sigma\}$ which covers $\mathscr{B}_{2\epsilon}(V) \sqcup \mathscr{B}_{2\epsilon}(W)$.

Conversely, if there exists a matching $E = \{(M_\gamma, N_\gamma) \mid \gamma \in \Gamma, M_\gamma \in \mathscr{B}(V), N_\gamma \in \mathscr{B}(W)\} \subseteq E_\mu$ of the graph that covers $\mathscr{B}_{2\epsilon}(V) \sqcup \mathscr{B}_{2\epsilon}(W)$, then we have a bijection $\sigma'\colon \{M_\gamma \mid \gamma \in \Gamma\} \to \{N_\gamma \mid \gamma \in \Gamma\}, M_\gamma \mapsto N_\gamma$, which induces an algebraic ϵ-matching. This completes the proof. □

For a multiset of pfd P-modules A, we denote by $A_{2\epsilon}$ the multiset of 2ϵ-significant elements in A. The following observation is given in [**Bje21**, P.111].

THEOREM 5.4.20. *Let V and W be pfd P-modules and let ϵ be a non-negative real number. For the bipartite graph $G = G(V, W, \mu)$, if the full subgraphs $G(\mathscr{B}_{2\epsilon}(V)\sqcup\mathscr{B}(W))$ and $G(\mathscr{B}(V)\sqcup\mathscr{B}_{2\epsilon}(W))$ satisfy Conditions* (H) *and* (H′) *respectively, then there exists an algebraic ϵ-matching between $\mathscr{B}(V)$ and $\mathscr{B}(W)$.*

PROOF. By Lemma 5.4.18, the neighborhoods of any $M \in \mathscr{B}_{2\epsilon}(V)$ and any $N \in \mathscr{B}_{2\epsilon}(W)$ are finite, respectively. Thus, by our assumption and Proposition 5.4.17, there exists a matching in $G(V, W, \mu)$ that covers $\mathscr{B}_{2\epsilon}(V) \sqcup \mathscr{B}_{2\epsilon}(W)$. By Proposition 5.4.19, we have an algebraic ϵ-matching between $\mathscr{B}(V)$ and $\mathscr{B}(W)$. This completes the proof. □

5.4.2. Review on Stability Theorem. In this subsection, we recall the stability theorem for standard persistence diagrams [**CSEH07**], [**CCSG$^+$09**], and we outline the proof of the theorem. This outline will give insight into the stability theorem for bipath persistence diagrams (Theorem 5.4.29).

Now, we consider the case where $P = \mathbb{R}$. Let Λ be the action on $\mathbb{R}$ given by Example 5.4.8 (2) as follows: $\Lambda_\epsilon(r) := r + \epsilon$ for any $r \in \mathbb{R}$ and $\epsilon \in \mathbb{R}_{\geq 0}$. For any interval I of $\mathbb{R}$, we can explicitly denote by $\mathrm{Ex}^\epsilon(I)$ as follows:

$$\mathrm{Ex}^\epsilon(I) = \{r \in \mathbb{R} \mid \exists i \in I \text{ with } |r - i| \leq \epsilon\}.$$

For example, if an interval is of the from $[a, b)$ with $a < b$, then we have $\mathrm{Ex}^\epsilon(I) = [a - \epsilon, b + \epsilon)$.

Now we can state the stability theorem.

THEOREM 5.4.21 ([**CCSG$^+$09**], [**CSEH07**]). *Let f and g be tame real-valued functions over a topological space X. Then we have*

$$d_{\mathrm{B}}(\mathscr{B}(V(f)), \mathscr{B}(V(g))) \leq \|f - g\|_\infty \tag{5.4.10}$$

where $\|f - g\|_\infty := \sup\{|f(x) - g(x)| \mid x \in X\}$.

The algebraic stability theorem and isometry theorem below will give us an observation to show the stability theorem.

THEOREM 5.4.22 ([**CCSG$^+$09**], [**Les15**]). *Let V and W be pfd $\mathbb{R}$-modules.*

(1) If V and W are ϵ-interleaved, then V and W are algebraic ϵ-matched.
(2) If V and W are algebraic ϵ-matched, then $\mathscr{B}(V)$ and $\mathscr{B}(W)$ are ϵ-matched.

In particular, we have the following equality:

$$d_{\mathrm{I}}(V,W) = d'_{\mathrm{B}}(V,W) = d_{\mathrm{B}}(\mathscr{B}(V),\mathscr{B}(W)). \tag{5.4.11}$$

We give an outline of the proof of the stability theorem (Theorem 5.4.21). We take an arbitrary $\epsilon \geq 0$ with $\|f-g\|_\infty \leq \epsilon$, then we obtain the following commutative diagrams of topological spaces and vector spaces for every $r \in \mathbb{R}$:

$$\begin{tikzcd}
(f \leq r) \arrow[r] \arrow[dr] & (f \leq r+\epsilon) \arrow[r] \arrow[dr] & (f \leq r+2\epsilon) \\
(g \leq r) \arrow[r] \arrow[ur] & (g \leq r+\epsilon) \arrow[r] \arrow[ur] & (g \leq r+2\epsilon),
\end{tikzcd}
\quad
\begin{tikzcd}
V(f)_r \arrow[r] \arrow[dr] & V(f)_{r+\epsilon} \arrow[r] \arrow[dr] & V(f)_{r+2\epsilon} \\
V(g)_r \arrow[r] \arrow[ur] & V(g)_{r+\epsilon} \arrow[r] \arrow[ur] & V(g)_{r+2\epsilon}.
\end{tikzcd} \tag{5.4.12}$$

The right diagram in (5.4.12) says that we have an ϵ-interleaving between $V(f)$ and $V(g)$, which implies $d_{\mathrm{I}}(V(f),V(g)) \leq \|f-g\|_\infty$. By the isometry theorem 5.4.22, we obtain $d_{\mathrm{B}}(\mathscr{B}(V(f)),\mathscr{B}(V(g))) \leq \|f-g\|_\infty$.

To complete the proof of the stability theorem, we need to see $d_{\mathrm{I}}(V,W) = d'_{\mathrm{B}}(V,W)$ and $d'_{\mathrm{B}}(V,W) = d_{\mathrm{B}}(\mathscr{B}(V),\mathscr{B}(W))$ hold for pfd $\mathbb{R}$-modules V and W. The former equation follows from a special case of a theorem given in [**Bje21**, Theorem 4.3].

For the latter equation, we give the following lemma.

Lemma 5.4.23. *Let I and J be intervals in $\mathbb{R}$. If k_I and k_J are ϵ-interleaved, then both k_I and k_J are 2ϵ-trivial, or we have $J \subseteq \mathrm{Ex}^\epsilon(I)$ and $I \subseteq \mathrm{Ex}^\epsilon(J)$.*

Proof. We can easily check this lemma. □

Proposition 5.4.24. *Let V and W be pfd $\mathbb{R}$-modules. If V and W are algebraic ϵ-matched, then V and W are ϵ-matched. In particular, we have $d_{\mathrm{B}}(V,W) \leq d'_{\mathrm{B}}(V,W)$.*

Proof. Let V and W be pfd $\mathbb{R}$-modules. If there exists an algebraic ϵ-matching $\sigma\colon \mathscr{B}(V) \nrightarrow \mathscr{B}(W)$, then for the pair $k_I \in \operatorname{Coim}\sigma$ and $k_J = \sigma(k_I)$, they are 2ϵ-trivial, or $J \subseteq \mathrm{Ex}^\epsilon(I)$ and $I \subseteq \mathrm{Ex}^\epsilon(J)$ by Lemma 5.4.23. By removing all the pair $(k_I,k_J) \in \operatorname{Coim}\sigma \times \operatorname{Im}\sigma$ such that they are 2ϵ-trivial from $\operatorname{Coim}\sigma \times \operatorname{Im}\sigma$, we obtain a new algebraic ϵ-matching σ' which satisfies $J \subseteq \mathrm{Ex}^\epsilon(I)$ and $I \subseteq \mathrm{Ex}^\epsilon(J)$ for $I \in \operatorname{Coim}\sigma'$ and $k_J = \sigma'(I)$. Thus the new algebraic ϵ-matching is an ϵ-matching between $\mathscr{B}(V)$ and $\mathscr{B}(W)$. This imp $d_{\mathrm{B}}(V,W) \leq d'_{\mathrm{B}}(V,W)$. □

5.4.3. Types of Interval Modules Over the Bipath Poset. Let B be the bipath poset (5.0.1). In this subsection, we give a notion of types of interval modules over the bipath poset. We first study the structure of $\mathsf{Rep}_k^{\mathrm{pfd}}(B)$ by morphisms between the same or different types of interval modules. The observations are used to prove the isometry theorem for bipath persistence modules (Theorem 5.4.30). Then, we study the relation between shift functors and the types of interval modules.

We first divide $\mathbb{I}(B)$ into the following five sets of intervals.

- $\mathcal{U}(B) := \{I \in \mathbb{I}(B) \mid I \subseteq \mathbb{R} \times \{1\}\}$
- $\mathcal{D}(B) := \{I \in \mathbb{I}(B) \mid I \subseteq \mathbb{R} \times \{2\}\}$
- $\mathcal{B}(B) := \{B\}$
- $\mathcal{L}(B) := \{I \in \mathbb{I}(B) \setminus \mathcal{B}(B) \mid -\infty \in I\}$
- $\mathcal{R}(B) := \{I \in \mathbb{I}(B) \setminus \mathcal{B}(B) \mid +\infty \in I\}$

We say that intervals I and J are the *same type* of intervals if I and J are in the same set $\mathcal{X}(B)$ where the symbol $\mathcal{X}$ is either $\mathcal{U}$, $\mathcal{D}$, $\mathcal{B}$, $\mathcal{L}$, or $\mathcal{R}$. Otherwise, we say that I and J are not the same type of intervals. We also say that interval modules k_I and k_J are (resp. not) the same type of interval modules if I and J are (resp. not) the same type of interval modules.

Let T be the set of symbols $\{\mathcal{U},\mathcal{D},\mathcal{B},\mathcal{L},\mathcal{R}\}$. We define an order $\leq_T$ on T such that $\mathcal{U}$, $\mathcal{D}$, $\mathcal{B}$ are incomparable and $\mathcal{R}$ (resp. $\mathcal{L}$) is the global maximal (minimal) element on T. The poset T is displayed by the following Hasse diagram.

$$\begin{tikzcd} & \mathcal{U} \arrow[dr] & \\ \mathcal{R} \arrow[ur] \arrow[r] \arrow[dr] & \mathcal{B} \arrow[r] & \mathcal{L} \\ & \mathcal{D} \arrow[ur] & \end{tikzcd}$$

LEMMA 5.4.25. *Let I and J be intervals such that $I \in \mathcal{X}(B)$ and $J \in \mathcal{Y}(B)$ for $\mathcal{X}, \mathcal{Y} \in \{\mathcal{U}, \mathcal{D}, \mathcal{B}, \mathcal{L}, \mathcal{R}\}$ respectively. If $\mathcal{X} \not\leq_T \mathcal{Y}$ then* $\mathrm{Hom}_B(k_I, k_J) = 0$.

PROOF. We have $\Omega(I, J) = \emptyset$. Hence, by Proposition 2.2.8, we have $\mathrm{Hom}_B(k_I, k_J) = 0$. This completes the proof. $\square$

REMARK 5.4.26. If I_1 and I_2 are the same types of intervals and J is not the same type of I_i $(i = 1, 2)$, then any morphism $k_{I_1} \to k_{I_2}$ that factors through k_J must be zero morphisms by Lemma 5.4.25.

PROPOSITION 5.4.27. *Let I and J be intervals such that $I, J \in \mathcal{L}(B)$ (resp. I and $J \in \mathcal{R}(B)$). Then* $\mathrm{Hom}_B(k_I, k_J) \neq 0$ *if and only if $J \subseteq I$ (resp. $I \subseteq J$).*

PROOF. Suppose that $I, J \in \mathcal{L}(B)$. Then, both I and J contain the minimal element in B. By Proposition 2.2.8, we have

$$\begin{aligned} \mathrm{Hom}_B(k_I, k_J) \neq 0 :&\iff \Omega(I, J) \neq \emptyset \\ &\iff \{I \cap J \mid (I \cap J)^{\downarrow} \cap I \subseteq I \cap J \text{ and } (I \cap J)^{\uparrow} \cap J \subseteq I \cap J\} \neq \emptyset \\ &\iff \{I \cap J \mid J \subseteq I\} \neq \emptyset \\ &\iff J \subseteq I. \end{aligned}$$

For $I, J \in \mathcal{R}(B)$, we obtain the $\mathrm{Hom}_B(k_I, k_J) \neq \emptyset \iff I \subseteq J$ similarly. This completes the proof. $\square$

For the next subsection, we consider the relation between the type of intervals and the action Λ on B, which is given in Example 5.4.8 (3) as follows:

$$\Lambda_\epsilon(b) := \begin{cases} (r + \epsilon, i) & \text{if } b = (r, i) \in \mathbb{R} \times \{i\} \text{ for } i = 1, 2 \\ \pm\infty & \text{if } b = \pm\infty, \end{cases} \tag{5.4.13}$$

for $b \in B$ and $\epsilon \in \mathbb{R}_{\geq 0}$. For an interval $I \in \mathbb{I}(B)$, we explicitly write $\mathrm{Ex}^\epsilon(I)$, which is defined by Equation (5.4.4), for the readers.

$$\mathrm{Ex}^\epsilon(I) = \begin{cases} \{(r, i) \in \mathbb{R} \times \{i\} \mid \exists s \in \mathbb{R} \text{ with } |r - s| \leq \epsilon\} & \text{if } I \subseteq \mathbb{R} \times \{i\} \\ \Lambda_\epsilon(I) & \text{if } -\infty \in I \\ \Lambda_\epsilon^{-1}(I) & \text{if } +\infty \in I \\ I & \text{if } I = B. \end{cases} \tag{5.4.14}$$

Here, for an interval I in B and a non-negative real number ϵ, we have the following.

- I and $\Lambda_\epsilon^{-1}(I)$ are the same type of intervals.
- I and $\mathrm{Ex}^\epsilon(I)$ are the same type of intervals.

The first statement implies that k_I and $k_I(\epsilon)$ are the same type of interval modules for any $\epsilon \in \mathbb{R}_{\geq 0}$. We can say that the shift functors do not change the type of the intervals.

5.4.4. Stability Theorem for Bipath Persistence Diagrams. In this subsection, we give a stability theorem for bipath persistence diagrams (Theorem 5.4.29). This theorem guarantees that the bottleneck distances between two bipath persistence diagrams obtained from two *bipath functions* (Definition 5.4.28) are less than or equal to the difference between the two bipath functions.

To state the stability theorem for bipath persistent homology, we first introduce bipath functions on a topological space and a distance between them.

DEFINITION 5.4.28. Let $f_i\colon X \to (\mathbb{R}\times\{i\})\sqcup\{\pm\infty\} \subseteq B$ $(i = 1, 2)$ be two maps on a topological space X. We call the pair $f = (f_1, f_2)$ *bipath function* on X if it satisfies the hollowing:

$$f_1^{-1}(\{-\infty\}) = f_2^{-1}(\{-\infty\}).$$

For a bipath function $f = (f_1, f_2)$ on a topological space X, the condition $f_1^{-1}(\{-\infty\}) = f_2^{-1}(\{-\infty\})$ is needed to define the following B-filtration $(f \le \cdot)$ (a functor from B to Top):

$$(f \le b) := \begin{cases} \{x \in X \mid f_i(x) \le b\} & \text{if } b \in \mathbb{R}\times\{i\} \text{ for } i = 1, 2, \\ f_1^{-1}(\{-\infty\}) & \text{if } b = -\infty, \\ X & \text{if } b = +\infty. \end{cases}$$

To give a distance between bipath functions, we define

$$|b_1 - b_2|_B := \begin{cases} 0 & \text{if } b_1 = b_2 \\ |r_1 - r_2| & \text{if } b_1 = (r_1, i), b_2 = (r_2, i) \in \mathbb{R}\times\{i\} \\ +\infty & \text{else.} \end{cases}$$

Then, for two bipath functions $f = (f_1, f_2)$ and $g = (g_1, g_2)$, we define

$$||f - g||_\infty := \max\{||f_1 - g_1||_\infty,\ ||f_2 - g_2||_\infty\},$$

where $||f_i - g_i||_\infty := \sup\{|f_i(x) - g_i(x)|_B \mid x \in X\}$.

Before stating the stability theorem, we recall that, by applying the qth homology functor $H_q\colon \mathsf{Top} \to \mathrm{Vect}_k$, we obtain $V(f) := H_q \circ (f \le \cdot) \in \mathsf{Rep}_k(B)$ for a bipath function f. We say that a bipath function f is *tame* if $V(f)_b$ is a finite-dimensional vector space for every $b \in B$.

The main statement in this section is the following:

THEOREM 5.4.29 (Stability Theorem for Bipath Persistence). *Let $f = (f_1, f_2)$ and $g = (g_1, g_2)$ be tame bipath functions on a topological space such that*

$$f_1^{-1}(\{-\infty\}) = f_2^{-1}(\{-\infty\}) = g_1^{-1}(\{-\infty\}) = g_2^{-1}(\{-\infty\}).$$

Then we have the following inequality:

$$d_{\mathrm{B}}(\mathscr{B}(V(f)), \mathscr{B}(V(g))) \le \|f - g\|_\infty. \tag{5.4.15}$$

Isometry theorem (Theorem 5.4.30), which relates the interleaving distance d_{I} (Definition 5.4.2) and the bottleneck distance d_{B} (Definition 5.4.7), plays an important role in proving the stability theorem (Theorem 5.4.29).

THEOREM 5.4.30 (Isometry Theorem for Bipath Persistence Modules). *Let V and W be pfd B-modules and let ϵ be a non-negative real number. The following are equivalent.*

(a) *There exists an ϵ-interleaving between V and W.*
(b) *There exists an algebraic ϵ-matching between V and W.*
(c) *There exists an ϵ-matching between $\mathscr{B}(V)$ and $\mathscr{B}(W)$.*

Therefore, we have the following equality:

$$d_{\mathrm{I}}(V, W) = d'_{\mathrm{B}}(V, W) = d_{\mathrm{B}}(\mathscr{B}(V), \mathscr{B}(W)). \tag{5.4.16}$$

Once we show the isometry theorem, we can prove the stability theorem (Theorem 5.4.29) in the following way.

PROOF OF THEOREM 5.4.29. Let f and g be bipath functions. If $\|f - g\|_\infty = +\infty$, then we have $d_{\mathrm{B}}(V(f), V(g)) \le +\infty = \|f - g\|_\infty$. Otherwise, we assume $\epsilon := \|f - g\|_\infty < +\infty$. For ϵ and any $b \in B$, we have

$$(f \le b) \subseteq (g \le \Lambda_\epsilon(b)) \text{ and } (g \le b) \subseteq (f \le \Lambda_\epsilon(b)).$$

Indeed, if $b = -\infty$, then by the assumption on the bipath functions, we have $(f \le b) = (g \le b)$. In addition, since $b = -\infty$ is a fixed point of Λ_ϵ, we have $(f \le b) \subseteq (g \le b) = (g \le \Lambda_\epsilon(b))$ and $(g \le b) \subseteq (f \le b) = (f \le \Lambda_\epsilon(b))$. If $b \ne -\infty$, then we can easily check $(f \le b) \subseteq (g \le \Lambda_\epsilon(b))$ and $(g \le b) \subseteq (f \le \Lambda_\epsilon(b))$ for any $b \in B$.

Thus, we have the families of inclusion maps $\{(f \leq b) \to (g \leq \Lambda_\epsilon(b))\}_{b\in B}$ and $\{(g \leq b) \to (f \leq \Lambda_\epsilon(b))\}_{b\in B}$, which is displayed by the following diagram of topological spaces:

$$\begin{tikzcd}
(f \leq b) \arrow[r] \arrow[dr] & (f \leq \Lambda_\epsilon(b)) \arrow[r] \arrow[dr] & (f \leq \Lambda_{2\epsilon}(b)) \\
(g \leq b) \arrow[r] \arrow[ur] & (g \leq \Lambda_\epsilon(b)) \arrow[r] \arrow[ur] & (g \leq \Lambda_{2\epsilon}(b)).
\end{tikzcd}$$

By applying the qth homology functor to the families of inclusion maps, we obtain the morphisms

$$\begin{aligned}
\alpha &:= \{\alpha_b \colon V(f)_b \to V(g)_{\Lambda_\epsilon(b)}\}_{b\in B} \colon V(f) \to V(g)(\epsilon) \text{ and} \\
\beta &:= \{\beta_b \colon V(g)_b \to V(f)_{\Lambda_\epsilon(b)}\}_{b\in B} \colon V(g) \to V(f)(\epsilon)
\end{aligned}$$

that satisfy $\beta_{\Lambda_\epsilon(b)} \circ \alpha_b = V(f)(b, \Lambda_{2\epsilon}(b))$ and $\alpha_{\Lambda_\epsilon(b)} \circ \beta_b = V(g)(b, \Lambda_{2\epsilon}(b))$ for all $b \in B$. This means that the pair of morphisms α and β is an ϵ-interleaving between $V(f)$ and $V(g)$. Thus we obtain $d_{\mathrm{I}}(V(f), V(g)) \leq \epsilon = \|f - g\|_\infty$. By Theorem 5.4.30, we obtain the desired inequality:

$$d_{\mathrm{B}}(\mathscr{B}(V(f)), \mathscr{B}(V(g))) = d_{\mathrm{I}}(V(f), V(g)) \leq \epsilon = \|f - g\|_\infty.$$

□

5.4.5. Proof of Isometry Theorem for Bipath Persistence Modules. In this subsection, we prove the isometry theorem (Theorem 5.4.30).

5.4.5.1. *The Equality of Two Bottleneck Distances.* In this subsubsection, we aim to show Proposition 5.4.33, which induces the equality of the two bottleneck distances d_{B} and d'_{B}.

Lemma 5.4.31. *Let I and J be intervals of the same type. Then, we have the following.*

(1) If I and J are subsets of $\mathbb{R} \times \{i\}$ for some $i \in \{1, 2\}$, then k_I and k_J are ϵ-interleaved if and only if I and J are 2ϵ-trivial, or $J \subseteq \mathrm{Ex}^\epsilon(I)$ and $I \subseteq \mathrm{Ex}^\epsilon(J)$.

(2) If I and J contains $+\infty$ or $-\infty$, then k_I and k_J are ϵ-interleaved if and only if $J \subseteq \mathrm{Ex}^\epsilon(I)$ and $I \subseteq \mathrm{Ex}^\epsilon(J)$.

Proof. (1) follows from Lemma 5.4.23 and Proposition 5.4.12.

Let I and J be the same type of intervals containing $+\infty$ or $-\infty$. Then the interval modules k_I and k_J are not 2ϵ-trivial. Let $\alpha \colon k_I \to k_J(\epsilon)$ and $\beta \colon k_J \to k_I(\epsilon)$ be an ϵ-interleaving between k_I and k_J. Then, we have $0 \neq (k_I)_{0\to 2\epsilon} = \beta(\epsilon) \circ \alpha$, which implies $\alpha \neq 0$ and $\beta \neq 0$. By Proposition 5.4.27, we have ${\Lambda_\epsilon}^{-1}(J) \subseteq I$ (resp. $I \subseteq {\Lambda_\epsilon}^{-1}(J)$) and ${\Lambda_\epsilon}^{-1}(I) \subseteq J$ (resp. $J \subseteq {\Lambda_\epsilon}^{-1}(I)$). Thus we obtain $I \subseteq \mathrm{Ex}^\epsilon(I)$ and $J \subseteq \mathrm{Ex}^\epsilon(I)$. The converse follows from Proposition 5.4.12. This completes the proof of (2). □

Lemma 5.4.32. *If k_I and k_J are ϵ-interleaved, then I and J are the same type of intervals, or I and J are 2ϵ-trivial.*

Proof. Let the pair $\alpha \colon k_I \to k_J(\epsilon)$ and $\beta \colon k_J \to k_I(\epsilon)$ be an ϵ-interleaving between k_I and k_J. If I and J are not the same type, then we have $0 = \beta(\epsilon) \circ \alpha = (k_I)_{0\to 2\epsilon}$ and $0 = \alpha(\epsilon) \circ \beta = (k_J)_{0\to 2\epsilon}$ by Remark 5.4.26. Thus k_I and k_J are 2ϵ-trivial. □

Now, we can show Proposition 5.4.33.

Proposition 5.4.33. *Let V and W be pfd B-modules. If there exists an algebraic ϵ-matching between V and W, then there exists an ϵ-matching between V and W. In particular, we have the following:*

$$d_{\mathrm{B}}(\mathscr{B}(V), \mathscr{B}(W)) \leq d'_{\mathrm{B}}(V, W).$$

Proof. Let V and W be pfd B-modules and $\sigma \colon A \twoheadrightarrow B$ be the given algebraic ϵ-matching between $\mathscr{B}(V)$ and $\mathscr{B}(W)$. We will construct an ϵ-matching from the algebraic ϵ-matching and complete our proof.

By Lemma 5.4.32, if $I \in \operatorname{Coim} \sigma$ and $\sigma(I) \in \operatorname{Im} \sigma$ are not the same type of intervals, then they are ϵ-trivial. By removing these intervals from $\operatorname{Coim} \sigma$ and $\operatorname{Im} \sigma$ respectively, we obtain a new algebraic ϵ-matching $\sigma' \colon A \twoheadrightarrow B$ so that $I \in \operatorname{Coim} \sigma'$ and $\sigma'(I)$ are the same type.

By Lemma 5.4.31, if $I \in \operatorname{Coim} \sigma'$ and $\sigma'(I)$ are ϵ-interleaved, then both I and $\sigma'(I)$ are 2ϵ-trivial, or $I \subseteq \mathrm{Ex}^\epsilon(\sigma'(I))$ and $\sigma'(I) \subseteq \mathrm{Ex}^\epsilon(I)$ for all $I \in \operatorname{Coim} \sigma'$. By removing these 2ϵ-trivial intervals I and $\sigma'(I)$ from $\operatorname{Coim}\sigma'$ and $\operatorname{Im}\sigma'$ respectively, we can construct a new algebraic ϵ-matching σ'' such that $I \subseteq \operatorname{Ex}^{\epsilon}(\sigma''(I))$ and $\sigma''(I) \subseteq \operatorname{Ex}^{\epsilon}(I)$ for all $I \in \operatorname{Coim}\sigma''$. This is an ϵ-matching between $\mathscr{B}(V)$ and $\mathscr{B}(W)$. This completes the proof. $\square$

5.4.5.2. *The Equality of Interleaving Distance and Algebraic Bottleneck Distance.* In this sub-subsection, we aim to show Proposition 5.4.36, and we give a proof of the isometry theorem (Theorem 5.4.30).

We write T for the set of symbols $\{\mathcal{U}, \mathcal{D}, \mathcal{B}, \mathcal{L}, \mathcal{R}\}$, see Subsection 5.4.3. Any pfd B-module V is decomposed into $V \cong \bigoplus_{\mathcal{X}\in T} V_{\mathcal{X}}$ where $V_{\mathcal{X}}$ is a direct sum of interval modules of type $\mathcal{X}$. We also recall that shift functors do not change the type of intervals.

PROPOSITION 5.4.34. *Let $V = \bigoplus_{\mathcal{X}\in T} V_{\mathcal{X}}$ and $W = \bigoplus_{\mathcal{X}\in T} W_{\mathcal{X}}$ be bipath modules. If V and W are ϵ-interleaved, then $V_{\mathcal{X}}$ and $W_{\mathcal{X}}$ are ϵ-interleaved for each $\mathcal{X} \in T$.*

PROOF. Let the pair $\phi\colon V \to W(\epsilon)$ and $\psi\colon W \to V(\epsilon)$ be an ϵ-interleaving between V and W. Then, they are of the form $\phi = (\phi_{\mathcal{Y},\mathcal{X}}\colon V_{\mathcal{X}} \to W_{\mathcal{Y}}(\epsilon))_{\mathcal{X},\mathcal{Y}\in T}$ and $\psi = (\phi_{\mathcal{Y},\mathcal{X}}\colon W_{\mathcal{X}} \to V_{\mathcal{Y}}(\epsilon))_{\mathcal{X},\mathcal{Y}\in T}$. By Remark 5.4.26, we have

$$\begin{aligned}(V_{\mathcal{X}})_{0\to 2\epsilon} &= \sum_{\mathcal{Y}\in T} \psi_{\mathcal{X},\mathcal{Y}}(\epsilon)\circ\phi_{\mathcal{Y},\mathcal{X}} = \psi_{\mathcal{X},\mathcal{X}}(\epsilon)\circ\phi_{\mathcal{X},\mathcal{X}} \text{ and}\\ (W_{\mathcal{X}})_{0\to 2\epsilon} &= \sum_{\mathcal{Y}\in T} \phi_{\mathcal{X},\mathcal{Y}}(\epsilon)\circ\psi_{\mathcal{Y},\mathcal{X}} = \phi_{\mathcal{X},\mathcal{X}}(\epsilon)\circ\psi_{\mathcal{X},\mathcal{X}}\end{aligned}$$

for each $\mathcal{X} \in T$. This shows $V_{\mathcal{X}}$ and $W_{\mathcal{X}}$ are ϵ-interleaved for each $\mathcal{X} \in T$. This completes the proof. $\square$

PROPOSITION 5.4.35. *Let $V_{\mathcal{X}}$ and $W_{\mathcal{X}}$ be pfd B-modules for $\mathcal{X} \in \{\mathcal{U}, \mathcal{D}, \mathcal{B}, \mathcal{L}, \mathcal{R}\}$. If $V_{\mathcal{X}}$ and $W_{\mathcal{X}}$ are ϵ-interleaved, then $V_{\mathcal{X}}$ and $W_{\mathcal{X}}$ are algebraic ϵ-matched.*

We give a proof of Proposition 5.4.35 in the next subsection. If we admit Proposition 5.4.35, we can give a proof of Proposition 5.4.36.

PROPOSITION 5.4.36. *Let V and W be pfd B-modules. If V and W are ϵ-interleaved, then they are algebraic ϵ-matched. In particular, we have the following:*

$$d'_{\mathrm{B}}(V, W) \leq d_{\mathrm{I}}(V, W).$$

PROOF. By our assumption, V and W are ϵ-interleaved. Thus, by Proposition 5.4.34, we have an ϵ-interleaving between $V_{\mathcal{X}}$ and $W_{\mathcal{X}}$ for each $\mathcal{X} \in \{\mathcal{U}, \mathcal{D}, \mathcal{B}, \mathcal{L}, \mathcal{R}\}$. Then, by Proposition 5.4.35, there exists an algebraic ϵ-matching $\sigma_{\mathcal{X}}$ between $\mathscr{B}(V_{\mathcal{X}})$ and $\mathscr{B}(W_{\mathcal{X}})$ for each $\mathcal{X} \in T$. Using the algebraic ϵ-matchings, we can construct an algebraic ϵ-matching σ between $\mathscr{B}(V)$ and $\mathscr{B}(W)$ such that $\operatorname{Coim}\sigma = \bigsqcup_{\mathcal{X}\in T}\operatorname{Coim}\sigma_{\mathcal{X}}$ and $\operatorname{Im}\sigma = \bigsqcup_{\mathcal{X}\in T}\operatorname{Im}\sigma_{\mathcal{X}}$ which sends $I \in \operatorname{Coim}\sigma$ to $\sigma_{\mathcal{X}}(I) \in \operatorname{Im}\sigma_{\mathcal{X}}$ if $I \in \operatorname{Coim}\sigma_{\mathcal{X}}$. This completes the proof. $\square$

Using Proposition 5.4.36, we have a proof of the isometry theorem (Theorem 5.4.30). We first recall the statement of the theorem and then provide a proof.

THEOREM (Theorem 5.4.30). *Let V and W be pfd B-modules and let ϵ be a non-negative real number. The following are equivalent.*

(a) *There exists an ϵ-interleaving between V and W.*
(b) *There exists an algebraic ϵ-matching between V and W.*
(c) *There exists an ϵ-matching between $\mathscr{B}(V)$ and $\mathscr{B}(W)$.*

Therefore, we have the following equality:

$$d_{\mathrm{I}}(V, W) = d'_{\mathrm{B}}(V, W) = d_{\mathrm{B}}(\mathscr{B}(V), \mathscr{B}(W)).$$

PROOF OF THEOREM 5.4.30. We first note that the action on B given in (5.4.13) satisfies (5.4.6). Thus, by Corollary 5.4.14, (c) implies (b), and (b) implies (a). By Proposition 5.4.33, (b) implies (c). By Proposition 5.4.36, (a) implies (b). This completes the proof. $\square$

We prove Proposition 5.4.35 in the next subsection.

5.4.6. Proof of Proposition 5.4.35: Existence of an Algebraic ϵ-Matching. Let V and W be pfd B-modules. We have their decomposition of the forms $V \cong V_{\mathcal{U}} \oplus V_{\mathcal{D}} \oplus V_{\mathcal{B}} \oplus V_{\mathcal{L}} \oplus V_{\mathcal{R}}$ and $W \cong W_{\mathcal{U}} \oplus W_{\mathcal{D}} \oplus W_{\mathcal{B}} \oplus W_{\mathcal{L}} \oplus W_{\mathcal{R}}$. Let $V_{\mathcal{X}}$ and $W_{\mathcal{X}}$ be pfd B-modules for $\mathcal{X} \in \{\mathcal{U}, \mathcal{D}, \mathcal{B}, \mathcal{L}, \mathcal{R}\}$ and ϵ be a non-negative real number. This subsection aims to show Proposition 5.4.35, or the existence of an algebraic ϵ-matching between $V_{\mathcal{X}}$ and $W_{\mathcal{X}}$ if they are ϵ-interleaved.

Remark 5.4.37. As for $\mathcal{X} = \mathcal{U}, \mathcal{D}$, or $\mathcal{B}$, we can construct an algebraic ϵ-matching between $V_{\mathcal{X}}$ and $W_{\mathcal{X}}$ if they are ϵ-interleaved. Indeed, we obtain them as follows:

- If $\mathcal{X}$ is type $\mathcal{U}$ or $\mathcal{D}$, then there exists an algebraic ϵ-matching between $\mathscr{B}(V_{\mathcal{X}})$ and $\mathscr{B}(W_{\mathcal{X}})$ by [**Bje21**, Theorem 4.3]. (This setting is a special case considered in [**Bje21**].)
- If $\mathcal{X} = \mathcal{B}$, we can check an ϵ-interleaving between $V_{\mathcal{X}}$ and $W_{\mathcal{X}}$ induces $V_{\mathcal{X}} \cong W_{\mathcal{X}}$. It implies that they are algebraic ϵ-matched.

For the case $\mathcal{X} = \mathcal{L}$ and $\mathcal{X} = \mathcal{R}$, we need to construct an algebraic ϵ-matching between $V_{\mathcal{X}}$ and $W_{\mathcal{X}}$ under the assumption that they are ϵ-interleaved. We only consider the case $\mathcal{X} = \mathcal{L}$ since they are dual.

Following the above remark, in the rest of this subsection, we concentrate on the proof of the existence of an algebraic ϵ-matching between $V_{\mathcal{L}}$ and $W_{\mathcal{L}}$ if they are ϵ-interleaved. For this purpose, we set some notation to give a preorder on the set $\mathcal{L}(B)(= \{I \in \mathbb{I}(B) \setminus \{B\} \mid -\infty \in I\})$ and then go back to the proof of the existence of an algebraic ϵ-matching between $V_{\mathcal{L}}$ and $W_{\mathcal{L}}$, and we complete the proof of Proposition 5.4.35.

Decorated numbers are elements of the following set:

$$\overline{\mathbb{R}}^* := \{r^+ \mid r \in \overline{\mathbb{R}}\} \sqcup \{r^- \mid r \in \overline{\mathbb{R}}\}. \tag{5.4.17}$$

We denote by $\mathbb{R}^*$ the subset $\{r^+ \mid r \in \mathbb{R}\} \sqcup \{r^- \mid r \in \mathbb{R}\}$ of $\overline{\mathbb{R}}^*$. When we write r^*, it is either r^+ or r^-. Using this notation, we define intervals (s^σ, t^τ) on $\overline{\mathbb{R}}$ by

$$(s^\sigma, t^\tau) := \begin{cases} [s,t] & \text{if } (\sigma,\tau) = (-,+), \\ [s,t) & \text{if } (\sigma,\tau) = (-,-), \\ (s,t] & \text{if } (\sigma,\tau) = (+,+), \\ (s,t) & \text{if } (\sigma,\tau) = (+,-). \end{cases} \tag{5.4.18}$$

Any intervals in $\overline{\mathbb{R}}$ can be written of the form (s^σ, t^τ). We note that there is a total order $\leq_*$ on the set $\overline{\mathbb{R}}$ given by $s^\sigma \leq_* t^\tau$ for $s < t$, or $s = t$ with $(\sigma, \tau) = (-, \pm)$ for all s and t in $\overline{\mathbb{R}}$. We write $s^\sigma = t^\tau$ if $s = t$ and $\sigma = \tau$. We write $s^\sigma <_* t^\tau$ if we have $s < t$ or $(\sigma, \tau) = (-, +)$. We can also add decorated numbers and real numbers by letting $a^+ + x := (a + x)^+$ and $a^- + x := (a + x)^-$ for $a \in \overline{\mathbb{R}}$ and $x \in \mathbb{R}$. With the above notation, we can write any interval in $\mathcal{L}(B)$ by $[-\infty, t^\tau) \times \{1\} \cup [-\infty, s^\sigma) \times \{2\}$ for $s^\sigma, t^\tau \in \overline{\mathbb{R}}^* \setminus \{+\infty^+\}$. We denote by the interval the pair $\langle s^\sigma, t^\tau \rangle$.

We define a preorder $\leq_\alpha$ on $\mathcal{L}(B)$. Firstly, we define pairwise disjoint subsets of $\mathcal{L}(B)$ by

1. $\mathcal{L}_1 := \{\langle -\infty^+, \infty^+ \rangle\}(= \{\{-\infty\}\})$
2. $\mathcal{L}_2 := \{\langle -\infty^+, t^\tau \rangle \in \mathbb{I}(B) \mid t^\tau \in \mathbb{R}^* \cup \{+\infty^-\}\} \cup \{\langle s^\sigma, -\infty^+ \rangle \in \mathbb{I}(B) \mid s^\sigma \in \mathbb{R}^* \cup \{+\infty^-\}\}$
3. $\mathcal{L}_3 := \{\langle s^\sigma, t^\tau \rangle \in \mathbb{I}(B) \mid s^\sigma, t^\tau \in \mathbb{R}^*\}$
4. $\mathcal{L}_4 := \{\langle +\infty^+, t^\tau \rangle \in \mathbb{I}(B) \mid t^\tau \in \mathbb{R}^*\} \cup \{\langle s^\sigma, +\infty^+ \rangle \in \mathbb{I}(B) \mid s^\sigma \in \mathbb{R}^*\}$
5. $\mathcal{L}_5 := \{\langle +\infty^-, +\infty^- \rangle\}(= \{B \setminus \{+\infty\}\})$

so that $\mathcal{L}(B) = \bigsqcup_{i=1}^{i=5} \mathcal{L}_i$, see Table 2.

We give the map $\alpha \colon \mathcal{L}(B) \to \overline{\mathbb{R}}^*$:

$$\alpha(\langle s^\sigma, t^\tau \rangle) := \begin{cases} -\infty^+ & \text{if } \langle s^\sigma, t^\tau \rangle \in \mathcal{L}_1, \\ \max\{s^\sigma, t^\tau\} & \text{if } \langle s^\sigma, t^\tau \rangle \in \mathcal{L}_2, \\ s^\sigma + t^\tau & \text{if } \langle s^\sigma, t^\tau \rangle \in \mathcal{L}_3, \\ \min\{s^\sigma, t^\tau\} & \text{if } \langle s^\sigma, t^\tau \rangle \in \mathcal{L}_4, \\ +\infty^- & \text{if } \langle s^\sigma, t^\tau \rangle \in \mathcal{L}_5, \end{cases} \tag{5.4.19}$$

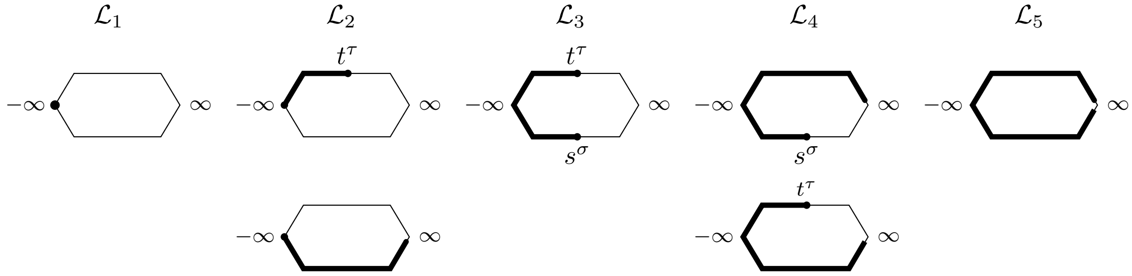

Table 2. Some pictures of $\mathcal{L}_i$ for $i \in \{1,2,3,4,5\}$.

where we define an addition on $\mathbb{R}^*$ by

$$s^\sigma + t^\tau := \begin{cases} (s+t)^+ & \text{if } (\sigma,\tau) = (+,+), \\ (s+t)^- & \text{else.} \end{cases}$$

For intervals $I, J \in \mathcal{L}(B)$, we define $I \leq_\alpha J$ by

- $I \in \mathcal{L}_i$ and $J \in \mathcal{L}_j$ for $i < j \in \{1,2,3,4,5\}$, or
- $I, J \in \mathcal{L}_i$ for some $i \in \{1,2,3,4,5\}$ and $\alpha(I) \leq_* \alpha(J)$.

This order is equipped with reflexivity ($I \leq_\alpha I$) and transitivity ($I \leq_\alpha I'$ and $I' \leq_\alpha I''$ implies $I \leq_\alpha I''$), hence it is a preorder. We note that any intervals I and J are comparable.

Now, we return to the proof of the existence of an algebraic ϵ-matching between $V_{\mathcal{L}}$ and $W_{\mathcal{L}}$. By the definition of $\leq_\alpha$, we have the following:

Lemma 5.4.38. *Let R, S, and T be intervals in $\mathcal{L}(B)$, and ϵ be a non-negative real number. We have the following:*

(1) *$R \subseteq S$ implies $R \leq_\alpha S$.*
(2) *If R in $\mathcal{L}_t$ for some $t \in \{1,2,3,4,5\}$, then $\Lambda_\epsilon^{-1}(R)$ is in $\mathcal{L}_t$.*
(3) *If we have $R \leq_\alpha S$ and $S \leq_\alpha T$ with $R \in \mathcal{L}_t$ and $T \in \mathcal{L}_t$ for some $t \in \{1,2,3,4,5\}$, then $S \in \mathcal{L}_t$.*

Proof. We can easily check these from the definitions of $\leq_\alpha$ and $\mathcal{L}_t$ ($t = 1,2,3,4,5$). □

Proposition 5.4.39. *Let R, S, and T be intervals in $\mathcal{L}(B)$ such that $R \leq_\alpha T$. If there exists non-zero morphisms $k_R \to k_S(\epsilon)$ and $k_S \to k_T(\epsilon)$, then S is ϵ-interleaved with either R or T.*

Proof. We note that we have

$$\Lambda_{2\epsilon}{}^{-1}(T) \subseteq \Lambda_\epsilon^{-1}(S) \subseteq R \tag{5.4.20}$$

by the existence of non-zero morphisms $k_R \to k_S(\epsilon)$ and $k_S \to k_T(\epsilon)$, see Proposition 5.4.27. By Lemma 5.4.38 (1), we have $\Lambda_{2\epsilon}^{-1}(T) \leq_\alpha \Lambda_\epsilon^{-1}(S)$ and $\Lambda_\epsilon^{-1}(S) \leq_\alpha R$. In addition, by our assumption, we have $R \leq_\alpha T$. Thus, if T is in $\mathcal{L}_i$ for some $i \in \{1,2,3,4,5\}$, then S and R are also in the sane set $\mathcal{L}_i$ by Lemma 5.4.38 (3) and (2).

If T is in $\mathcal{L}_1$ or $\mathcal{L}_5$, then we have $R = S = T$ by Lemma 5.4.38. Hence k_S is ϵ-interleaved with k_R and k_T.

If $T \in \mathcal{L}_2$ (resp. $T \in \mathcal{L}_4$), then $R \in \mathcal{L}_2$ (resp. $R \in \mathcal{L}_4$). By $R \leq_\alpha T$ with $\Lambda_{2\epsilon}^{-1}(T) \subseteq R$ (Equation (5.4.20)), we have $R \subseteq T$. Thus, we have $\Lambda_\epsilon^{-1}(T) \subseteq S \subseteq \Lambda_\epsilon(T)$ and $\Lambda_\epsilon^{-1}(R) \subseteq S \subseteq \Lambda_\epsilon(R)$ by (5.4.20). Hence, by Lemma 5.4.31 (2), both k_T and k_R are ϵ-interleaved with k_S.

If $T \in \mathcal{L}_3$, we write R, S, and T for $\langle r_1^{\rho_1}, r_2^{\rho_2} \rangle$, $\langle s_1^{\sigma_1}, s_2^{\sigma_2} \rangle$, and $\langle t_1^{\tau_1}, t_2^{\tau_2} \rangle$, respectively. We show k_S is ϵ-interleaved with either k_R and k_T by contradiction. If k_S and k_R are not ϵ-interleaved, and k_S and k_T are not ϵ-interleaved, then we have $\Lambda_\epsilon^{-1}(S) \nsubseteq R$ or $\Lambda_\epsilon^{-1}(R) \nsubseteq S$, and we have $\Lambda_\epsilon^{-1}(S) \nsubseteq T$ or $\Lambda_\epsilon^{-1}(T) \nsubseteq S$. By (5.4.20), we have $\Lambda_\epsilon^{-1}(R) \nsubseteq S$ and $\Lambda_\epsilon^{-1}(S) \nsubseteq T$. This says that there exists i and p in $\{1,2\}$ such that

$$s_i^{\sigma_i} <_* (r_i - \epsilon)^{\rho_i} \text{ and } t_p^{\tau_p} <_* (s_p - \epsilon)^{\sigma_p}. \tag{5.4.21}$$

We take j and q from $\{1,2\}$ so that $i \neq j$ and $p \neq q$. For the j and q, by (5.4.20), we have

$$(s_j - \epsilon)^{\sigma_j} \leq_* r_j^{\rho_j} \text{ and } (t_q - \epsilon)^{\tau_q} \leq_* s_q^{\sigma_q}. \tag{5.4.22}$$

In particular, we have $s_j - \epsilon \le r_j$ and $t_q - \epsilon \le s_q$.

By (5.4.21) and (5.4.22), we have

$$\begin{aligned} s_1^{\sigma_1} + s_2^{\sigma_2} &\le_* (r_i - \epsilon)^{\rho_i} + (r_j + \epsilon)^{\rho_j} \\ &= \begin{cases} (r_1 + r_2)^+ & \text{if } (\rho_1, \rho_2) = (+,+) \\ (r_1 + r_2)^- & \text{else,} \end{cases} \\ &= r_1^{\rho_1} + r_2^{\rho_2}. \end{aligned}$$

Similarly, we obtain $t_1^{\tau_2} + t_2^{\tau_2} \le_* s_1^{\sigma_1} + s_2^{\sigma_2}$. Since we have $R \le_\alpha T$ ($\iff r_1^{\rho_1} + r_2^{\rho_2} \le_* t_1^{\tau_1} + t_2^{\tau_2}$) by assumption, we obtain

$$s_1^{\sigma_1} + s_2^{\sigma_2} = r_1^{\rho_1} + r_2^{\rho_2} = t_1^{\tau_2} + t_2^{\tau_2}. \tag{5.4.23}$$

In particular, we have $s_1 + s_2 = r_1 + r_2 = t_1 + t_2$.

By $s_i^{\sigma_i} <_* (r_i - \epsilon)^{\rho_i}$ from (5.4.21), we have either $s_i < r_i - \epsilon$, or $s_i = r_i - \epsilon$ with $(\sigma_i, \rho_i) = (-,+)$. If $s_i < r_i - \epsilon$, then $s_j - \epsilon \le r_j$ from (5.4.22) gives $s_1 + s_2 < r_1 + r_2$, which contradicts to (5.4.23). Hence we must have $s_i = r_i - \epsilon$ with $(\sigma_i, \rho_i) = (-,+)$. In particular, we have $s_j - \epsilon = r_1 + r_2 - (s_i + \epsilon) = r_j$ and $\rho_j = -$ by (5.4.23). By substituting each for $(s_j - \epsilon)^{\sigma_j} \le_* r_j^{\rho_j}$, we obtain $r_j^{\sigma_j} \le_* r_j^-$, which implies $\sigma_j = -$. We note that we have $(\sigma_1, \sigma_2) = (-,-)$.

By $t_p^{\tau_p} <_* (s_p - \epsilon)^{\sigma_p}$ from (5.4.21), we have either $t_p < s_p - \epsilon$, or $t_p = s_p - \epsilon$ with $(\tau_p, \sigma_p) = (-,+)$. Since we have $(\sigma_1, \sigma_2) = (-,-)$, the latter is false, hence we must have $t_p < s_p - \epsilon$. In this case, by $t_q - \epsilon \le s_q$ from (5.4.22), we obtain $t_1 + t_2 < s_1 + s_2$. This contradicts to (5.4.23). Thus the former case is false. This is a contradiction. Therefore k_S is ϵ-interleaved with either k_R or k_T. □

Using Proposition 5.4.39, we show the key lemma.

LEMMA 5.4.40. *Let $G(V_\mathcal{L}, W_\mathcal{L}, E_\mu)$ be the bipartite graph defined by* (5.4.9). *We have the following.*

(1) *Let X be a subset of $\mathscr{B}(V_\mathcal{L})$, then $|X| \le |\mu(X)|$.*
(2) *Let Y be a subset of $\mathscr{B}(W_\mathcal{L})$, then $|Y| \le |\mu(Y)|$.*

PROOF. We first show (1). Let n be the number of elements in X. We order $X = \{I_1, \ldots, I_n\}$ so that $I_i \le_\alpha I_{i'}$ for $1 \le i \le i' \le n$. We write an ϵ-interleaving between $V_\mathcal{L} = \bigoplus_{I \in \mathscr{B}(V_\mathcal{L})} k_I$ and $W_\mathcal{L} = \bigoplus_{J \in \mathscr{B}(W_\mathcal{L})} k_J$ by the pair

$$\alpha = (\alpha_{J,I})_{I \in \mathscr{B}(V_\mathcal{L}), J \in \mathscr{B}(W_\mathcal{L})} \colon V_\mathcal{L} \to W_\mathcal{L}(\epsilon) \text{ and } \beta = (\beta_{I,J})_{I \in \mathscr{B}(V_\mathcal{L}), J \in \mathscr{B}(W_\mathcal{L})} \colon W_\mathcal{L} \to V_\mathcal{L}(\epsilon).$$

For any I in X, we have

$$(k_I)_{0 \to 2\epsilon} = \sum_{J \in \mathscr{B}(W_\mathcal{L})} \beta_{I,J}(\epsilon) \circ \alpha_{J,I} = \sum_{J \in \mu(I)} \beta_{I,J}(\epsilon) \circ \alpha_{J,I} + \sum_{J \in \mathscr{B}(W_\mathcal{L}) \setminus \mu(I)} \beta_{I,J}(\epsilon) \circ \alpha_{J,I}.$$

By Proposition 5.4.39, $\sum_{J \in \mathscr{B}(W_\mathcal{L}) \setminus \mu(I)} \beta_{I,J}(\epsilon) \circ \alpha_{J,I}$ must be 0. Hence, the above equation is rewritten as follows:

$$(k_I)_{0 \to 2\epsilon} = \sum_{J \in \mu(X)} \beta_{I,J}(\epsilon) \circ \alpha_{J,I} \tag{5.4.24}$$

for any I in X.

On the other hand, for $I_i, I_{i'} \in X$ with $i < i'$, we have

$$0 = \sum_{J \in \mathscr{B}(W)} \beta_{I_{i'},J}(\epsilon) \circ \alpha_{J,I_i} = \sum_{J \in \mu(I)} \beta_{I_{i'},J}(\epsilon) \circ \alpha_{J,I_i} + \sum_{J \in \mathscr{B}(W) \setminus \mu(I)} \beta_{I_{i'},J}(\epsilon) \circ \alpha_{J,I_i}.$$

By Proposition 5.4.39, $\sum_{J \in \mathscr{B}(W) \setminus \mu(I)} \beta_{I_{i'},J}(\epsilon) \circ \alpha_{J,I_i}$ must be zero. Hence, the above equation is rewritten as follows:

$$0 = \sum_{J \in \mu(I)} \beta_{I_{i'},J}(\epsilon) \circ \alpha_{J,I_i} \tag{5.4.25}$$

for $I_i, I_{i'} \in X$ with $i < i'$.

We denote by f the composition of $(\alpha_{J,I_i})_{J\in\mu(X),I_i\in X}$ and $(\beta_{I_i,J}(\epsilon))_{J\in\mu(X),I_i\in X}$, which is displayed by

$$\begin{tikzcd}
(\bigoplus_{I_i\in X} k_{I_i}) \arrow[rr, "f"] \arrow[dr, "(\alpha_{J,I_i})_{J\in\mu(X),I_i\in X}"'] & & (\bigoplus_{I_i\in X} k_{I_i}(2\epsilon)) \\
& (\bigoplus_{J\in\mu(X)} k_J). \arrow[ur, "(\beta_{I_i,J}(\epsilon))_{J\in\mu(X),I_i\in X}"'] &
\end{tikzcd}$$

By the above equations (5.4.24) and (5.4.25), the morphism f is of the form:

$$f = \begin{bmatrix} (k_{I_1})_{0\to 2\epsilon} & * & \cdots & * \\ 0 & (k_{I_2})_{0\to 2\epsilon} & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & (k_{I_n})_{0\to 2\epsilon} \end{bmatrix}.$$

Since every interval in $\mathcal{L}(B)$ contains the minimal element $-\infty \in B$, the k-linear morphism $f_{-\infty}\colon (\bigoplus_{I_i\in X} k_{I_i})_{-\infty} \to (\bigoplus_{I_i\in X} k_{I_i}(2\epsilon))_{-\infty}$ is an isomorphism. Thus, the k-linear morphism

$$((\alpha_{J,I_i})_{J\in\mu(X),I_i\in X})_{-\infty}\colon (\bigoplus_{I_i\in X} k_{I_i})_{-\infty}(\cong k^{|X|}) \to (\bigoplus_{J\in\mu(X)} k_J)_{-\infty}(\cong k^{|\mu(X)|})$$

is injective. Thus, we have

$$|X| = \dim_k(\bigoplus_{I_i\in X} k_{I_i})_{-\infty} \le \dim_k(\bigoplus_{J\in\mu(X)} k_J)_{-\infty} = |\mu(X)|.$$

We get the desired inequality.

By changing the role of X and Y, we can show (2) similarly. This completes the proof. $\square$

Finally, we give a proof of Proposition 5.4.35 as follows.

Proof of Proposition 5.4.35. We only consider the case of $\mathcal{X} = \mathcal{L}$ by Remark 5.4.37.

We note that any interval module in $\mathscr{B}(V_{\mathcal{L}}) \sqcup \mathscr{B}(W_{\mathcal{L}})$ is 2ϵ-significant, that is, we have $\mathscr{B}_{2\epsilon}(V_{\mathcal{L}}) = \mathscr{B}(V_{\mathcal{L}})$ and $\mathscr{B}_{2\epsilon}(W_{\mathcal{L}}) = \mathscr{B}(W_{\mathcal{L}})$. In addition, both $\mathscr{B}(V_{\mathcal{L}})$ and $\mathscr{B}(W_{\mathcal{L}})$ are finite. By Lemma 5.4.40, the bipartite graph $G(V_{\mathcal{L}}, W_{\mathcal{L}}, \mu_\epsilon = \mu)$ satisfies both Conditions (H) and (H$'$) given in Condition 5.4.15. Thus, by Theorem 5.4.20, we have an algebraic ϵ-matching between $V_{\mathcal{L}}$ and $W_{\mathcal{L}}$. This completes the proof. $\square$

Discussion

In Chapter 4, we study the properties of interval covers, and our results simplify their computation. However, we currently lack a method for computing interval covers similar to the way projective covers are computed. As a future work, we could study an explicit description of interval covers, analogous to projective covers. Additionally, in a different direction, we could ask whether each interval submodule can capture topological features, or if there exists a notion of approximation at the level of filtrations of the simplicial complexes that are compatible with the interval cover of the persistent homology of the filtration.

We study Θ to prove the monotonicity theorem. The functor Θ maps interval modules to interval modules. This property is useful for studying intervals. Thus, we can study more about the properties of the functor.

In Chapter 5, as future work, we need to find examples of data where bipath persistent homology is essential. We also need to implement constructions of bipath filtrations from a given input data. It is also essential in applying bipath persistent homology to analysis.

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