



Unitarity of S-matrix and its application to dark matter models

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Doctoral Dissertation

Unitarity of S-matrix and
its application to dark matter models
(S行列のユニタリー性と
暗黒物質模型への応用)

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Abstract

Consistency conditions of quantum field theories, such as unitarity and causality, are important tools in particle physics. Recently, 'positivity bounds', that are constraints on the parameter spaces of theories derived from unitarity and analyticity using scattering amplitude techniques, have gained significant attention. When applied to gravitational theories, positivity bounds provide interesting relations between UV and IR parameters, which are called gravitational positivity bounds.

One interesting application of these gravitational positivity bounds is in dark matter physics. In this thesis, we discuss gravitational positivity bounds on models that include a dark $U(1)$ gauge boson as concrete examples of dark sector models. First, we examine gravitational positivity bounds on the massive $U(1)$ gauge boson, derived from the scattering of the gauge bosons themselves. Then, we discuss gravitational positivity bounds on the scattering of the SM photon and the dark $U(1)$ gauge boson.

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Chapter 1

Introduction

Consistency conditions of a theory, such as unitarity and causality, have proven valuable for gaining insight into new physics. One example is the prediction of the Higgs boson based on the perturbative unitarity of W-boson scattering [1, 2].

Recent developments of the study of ‘positivity bounds’ introduce methods to derive an infinite number of constraints on the parameter spaces of effective field theories (EFTs) based on the unitarity of scattering amplitude. In the pioneering work [3], the authors demonstrated that certain coefficients of the low-energy expansion of scattering amplitudes in a given EFT have to be positive, provided the theory has healthy UV completion. A dispersion relation, derived from the analyticity of the scattering amplitude, plays a crucial role in deriving positivity bounds by connecting different energy scales. The original work was limited to scalar EFTs and only utilize data of the forward limit. Subsequently, the method of positivity bounds was extended to spinning EFTs [4, 5], and incorporate data from non-forward scattering [6]. Notably, it was realized that $s-t$ crossing symmetry provides strong two-sided bounds on EFT coefficients [7–9]. Furthermore, positivity bounds have been applied in various contexts, including the Standard Model Effective Field Theory (SMEFT) [10] and massive gravity [11].

It is quite interesting to extend this technique to theories that include gravity. A theory containing gravity can be regarded as an EFT of quantum gravity, and positivity bounds on such a theory can be considered as constraints reflecting the nature of quantum gravity. Recently, it has been realized that gravitational EFTs must satisfy non-trivial constraints by requiring that they have consistent quantum gravity theories as their UV completion. These properties are called swampland bounds [12, 13]. There has been discussion suggesting that positivity bounds may be useful to obtain swampland bounds [14, 15]. However, it is not a straightforward task, because there is a technical obstruction arising from t-channel graviton exchange. Several attempts have been made to derive these bounds. One promising approach was developed in [16], where the authors pointed out that it is possible to subtract the problematic t-channel term. This leads to a dispersion relation that connects EFT quantities with UV parameters, which are tied to the behavior of quantum gravity. As a result, additional assumptions on

the UV parameters are necessary to derive constraints on EFT parameters. Oppositely, this relation can also be interpreted as constraints on UV parameters derived from IR observables. This relation is called gravitational positivity bounds.

Under the assumption about UV parameters, gravitational positivity bounds put lower limits on non-gravitational interactions. Therefore, it is interesting to explore the implications for dark matter physics, where interactions are expected to be tiny. In this thesis, we will investigate gravitational positivity bounds in models that include a dark $U(1)$ gauge boson, providing a concrete setup of dark matter model. The additional $U(1)$ symmetry and its gauge boson is a well-discussed scenario in beyond the Standard Model physics. In particular, a very light $U(1)$ gauge boson, known as a dark photon or hidden photon, can behave as dark matter. First, we examine gravitational positivity bounds on a massive $U(1)$ gauge boson, derived from the scattering of the gauge bosons themselves. Then, we discuss gravitational positivity bounds on the scattering of the Standard Model photon and the dark $U(1)$ gauge boson.

Organization of this thesis This thesis is organized as follows.

- In Chapter 2, we review the evidence supporting the existence of dark matter. This evidence is derived from various cosmological observations. We discuss rotation curve of galaxy in section 2.1, several evidences from galaxy cluster observations in section 2.2, and we examine the cosmic microwave background in section 2.3.
- In section 3, we review ultra-light dark matter models. While the weakly interacting massive particle (WIMP) scenario has been the most prominent framework for dark matter, another possibility-that dark mater consists of extremely light particles-has been gaining increasing attention. In this chapter, we review the dark photon dark matter model, which is a candidate of ultra-light dark matter. We show that the interaction between the Standard Model electromagnetic current and the dark photon is naturally feeble due to kinetic mixing. Then, we will examine observational constraints on the dark photon model.
- In chapter 4, we review properties and techniques associated with scattering amplitudes. In section 4.1, we introduce the basic properties of scattering amplitudes such as the optical theorem and the partial wave expansion. In section 4.2, we review the Froisart bound, which states that scattering amplitudes $\mathcal{M}(s, t)$ must satisfy the condition $\lim_{s \rightarrow \infty, t: \text{fixed}} \mathcal{M}(s, t) < s^2$. This bound plays a crucial role in deriving the dispersion relation and positivity bounds in chapter 5. In section 4.3, we review the Regge theory. In the pioneering work [17], Regge found that the analytic structure of the partial wave amplitude a_l in the complex l -plane has profound implications. In particular, a pole in the l -plane, called a Regge pole, connects the spectrum of t-channel exchanges with the high-energy

behavior of the scattering amplitude. In this section, we review Regge’s analysis and investigate the scattering amplitude of NS-NS bosons in type-II superstring theory to see an explicit example of a Regge pole.

- In chapter 5, we discuss the derivation of positivity bounds, with particular focus on the role of gravity. In section 5.1, we review the derivation of positivity bounds on non-gravitational theories. By assuming consistency conditions of a quantum field theory, such as unitarity and analyticity of the scattering amplitude, a dispersion relation is derived. This relation connects scattering amplitude evaluated in low-energy a EFT with the imaginary part of the amplitude at high energies, whose behavior is governed by unitarity. From the dispersion relation, it is possible to derive an infinite number of constraints on the parameters of the EFT, which is called positivity bounds. In section 5.2, we discuss how the situation changes for theories with gravity. In the presence of gravity, t-channel graviton exchange contributions prevent us from deriving positivity bounds in the same manner as for non-gravitational theories. Consequently, the positivity bounds for gravitational theories, referred to as gravitational positivity bounds, have dependence on the set of UV parameters (M, σ) . These parameters are closely connected to the nature of UV completion of gravity, namely quantum gravity. By assuming specific properties of (M, σ) , gravitational positivity bounds can be regarded as nontrivial constraints on EFT parameters imposed by quantum gravity. Conversely, they can also be viewed as gravitational positivity bounds as constraints on quantum gravity parameters derived from IR observables. Finally, we present a concrete method for calculating gravitational positivity bounds.
- In chapter 6, we investigate gravitational positivity bounds for the $U(1)$ gauge boson V . We compute these bounds using the $VV \rightarrow VV$ scattering amplitude. In section 6.1, we summarize the setup of the scattering amplitude. Some subtleties which arise in the derivation of gravitational positivity bounds within this setup, along with strategies to resolve these challenges, are discussed. In section 6.2, we analyze the contributions to the gravitational positivity bounds from the Higgs boson, which gives mass to the gauge boson. In section 6.3, we study the contributions to the gravitational positivity bounds from fermions, which are charged under $U(1)$. In section 6.4, we derive constraints on the dark gauge boson and compare them with other theoretical constraints arising from quantum gravity. In section 6.5, we compare our bounds with observational constraints. This chapter is based on our paper [18].
- In chapter 7, we examine gravitational positivity bounds for the dark photon model combined with the Standard Model. We consider the scattering of the ordinary photon A and the dark photon A' mediated by the Standard Model charged particles. In section 7.1, we

introduce the dark photon model. In section 7.2, we present the explicit Lagrangian relevant to photon-dark photon scattering. In section 7.3, the setup for the scattering process is summarized. In section 7.4, we summarize the contribution of each charged particle to gravitational positivity bounds. In section 7.5, the implication of gravitational positivity bounds for the dark photon model is discussed. The contribution of higher-dimensional operator are also examined. This chapter is based on our paper [19].

- Chapter 8 is devoted to the conclusion.

Chapter 2

Evidence for the existence of dark matter

There are a number of evidences of the existence of dark matter from astrophysical and cosmological observation. In this chapter, we review these evidences with reference to [20, 21].

2.1 Rotation curve of galaxy

The motion of stars which compose a galaxy is one of the oldest and strongest evidences of dark matter. Stars move in circular orbits under the gravitational fields of a galaxy, and the relationship between their orbital velocity and radius is referred to as the galaxy's rotation curve. The velocity of an object in circular motion under the gravitational force due to a spherically symmetric mass distribution with density $\rho(r)$ is given by

$$v_c(r) = \sqrt{\frac{GM(r)}{r}}, \quad (2.1.1)$$

where r is the orbital radius, G is the gravitational constant, and $M(r)$ is the total mass enclosed within radius r .

The enclosed mass $M(r)$ is determined as

$$M(r) = 4\pi \int_0^r dr' r'^2 \rho(r'). \quad (2.1.2)$$

Firstly, let us assume that dark matter does not exist in a galaxy and consider how the rotation curve would behave. The mass distribution of the visible parts of the galaxy, inferred from light, is not spherically symmetric but rather disk-shaped. However, for circular orbits with radii much larger than the radius of the galaxy's disk R_{disk} , equation (2.1.1) applies, and the rotation velocity is given by

$$v_{c,\text{noDM}}(r) = \sqrt{\frac{GM_{\text{visible}}}{r}}, \quad (2.1.3)$$

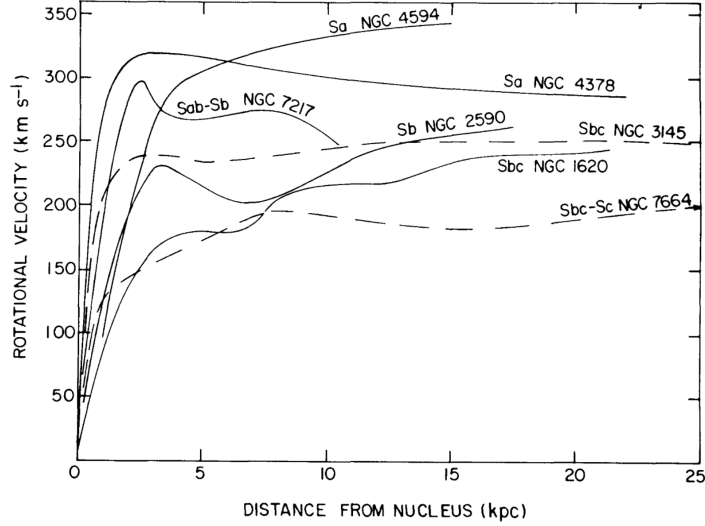


Figure 2.1: The rotation curve of the galaxy, cited from [22].

where M_{visible} is the total mass of the visible portion of the galaxy. Thus, far outside the galaxy's disk, the rotation curve is predicted to behave as $v \propto r^{-1/2}$, decreasing with increasing radius.

Fig.2.1 shows the observed rotation curve. Contrary to the prediction of (2.1.3), the rotation curve is constant at the outside of the galaxy's disk. To explain this observational result using Equation (2.1.1), $M(r) \propto r$ is required, which implies $\rho(r) \propto r^{-2}$. Here, r being considered corresponds to regions well beyond the radius of the galactic disk and areas that appear almost completely empty when observed in visible light. In other words, the rotation curve of galaxies suggests the existence of some "invisible" matter, namely dark matter, distributed around the visible parts of galaxies. This dark matter forms a surrounding structure known as the halo, where the majority of the galaxy's mass resides. For example, in the case of the Milky Way galaxy, the mass contributed by stars is approximately $10^{10}M_{\odot}$, whereas the total mass of the galaxy is around $10^{12}M_{\odot}$, and a difference is about two orders of magnitude [23].

2.2 Observation of galaxy cluster

Galaxy clusters are astrophysical objects that are composed from a number of galaxies and gas emitting X-ray. We have another clues of the existence of dark matter from observations of galaxy clusters.

2.2.1 Estimation of mass of galaxy cluster using virial theorem

Mass of a galaxy cluster can be estimated using the virial theorem.

Let us consider a bounded system moving under a central force such as gravity. Then, the long-term average of the kinetic energy $\bar{T}$ and the potential energy $\bar{V}$ satisfies the following

relationship:

$$2\bar{T} + \bar{V} = 0. \quad (2.2.1)$$

This equation is called the virial theorem. If the system is in the steady state, the kinetic energy without the average T and the potential energy V satisfies

$$2T + V = 0. \quad (2.2.2)$$

We want to relate T and V of galaxy clusters to observational quantities and apply (2.2.2). Firstly, the kinetic energy can be written as follows:

$$T = \sum_i \frac{1}{2} m_i v_i^2 \quad (2.2.3)$$

$$= \frac{1}{2} M \langle v^2 \rangle, \quad (2.2.4)$$

where m_i , v_i are masses and velocities of each portion which compose the galaxy cluster. $M := \sum_i m_i$ is the total mass of the galaxy cluster. The mean square velocity is defined as

$$\langle v^2 \rangle := \sum_i m_i v_i^2 / \sum_i m_i. \quad (2.2.5)$$

On the other hand, the potential energy is given by

$$V = -\frac{1}{2} \sum_i \sum_{j \neq i} \frac{G m_i m_j}{r_{ij}} \quad (2.2.6)$$

$$= -GM^2/R_G, \quad (2.2.7)$$

where the gravitational radius R_G is defined as

$$R_G = 2 \left(\sum_i m_i \right)^2 \left(\sum_i \sum_{j \neq i} \frac{m_i m_j}{r_{ij}} \right)^{-1}, \quad (2.2.8)$$

which is about the size of the galaxy cluster. R_G can be inferred by measuring positions of galaxies under the assumption that galaxies distribute equally. Substituting (2.2.4) and (2.2.7) into the virial theorem (2.2.2), we have

$$M = \frac{R_G \langle v^2 \rangle}{G}. \quad (2.2.9)$$

Therefore, the total mass of the galaxy cluster can be estimated from the observation of positions and velocities of each galaxy.

Let us apply this equation to the Coma cluster, a galaxy cluster located approximately 100 Mpc away from Earth as a demonstration. $\langle v^2 \rangle$ can be obtained by measuring the Doppler shift of galaxies as

$$\langle v^2 \rangle = 2.32 \times 10^{12} \text{ m}^2 \text{ s}^{-2}, \quad (2.2.10)$$

whereas the size of the Coma cluster is

$$R_G \approx 1.5 \text{Mpc} \approx 4.6 \times 10^{22} \text{ m}. \quad (2.2.11)$$

Therefore, the mass of the Coma cluster can be estimated as

$$\begin{aligned} M_{\text{Coma}} &= \frac{\langle v^2 \rangle R_G}{G} \approx \frac{(2.32 \times 10^{12} \text{ m}^2 \text{ s}^{-2}) (4.6 \times 10^{22} \text{ m})}{(6.67 \times 10^{-11} \text{ m}^3 \text{ s}^{-2} \text{ kg}^{-1})} \\ &\approx 4 \times 10^{45} \text{ kg} \approx 2 \times 10^{15} M_\odot. \end{aligned} \quad (2.2.12)$$

Now, to what extent is this mass accounted for by components that emit visible light, such as stars? The proportion of visible matters can be inferred from the luminosity L . Let us compare the mass to luminosity ratio M/L of the Coma cluster to that of the Sun. The luminosity of the Coma cluster is measured to be $L_{\text{Coma}} \approx 5 \times 10^{12} L_\odot$, and the mass to luminosity ratio becomes

$$\frac{M_{\text{Coma}}}{L_{\text{Coma}}} \sim 400 \frac{M_\odot}{L_\odot}. \quad (2.2.13)$$

Therefore, it is concluded that most of the mass of galaxy cluster is composed of matter that does not emit visible light, that is, dark matter.

2.2.2 Baryon fraction from x-ray gas

Most of baryons present in galaxy clusters are contained in the intracluster medium(ICM), which is the heated plasma and emits X-rays. In this subsection, we discuss how X-ray measurements indicate the existence of dark matter.

Firstly, let us assume that the ICM is under hydrostatic equilibrium. Then, the following equation is satisfied:

$$\frac{dP_{\text{gas}}}{dr} = -\frac{GM(r)\rho_{\text{gas}}(r)}{r^2}, \quad (2.2.14)$$

where P_{gas} and ρ_{gas} are the pressure and the mass density of the ICM. $M(r)$ represents the total mass within a radius r , including not only the ICM but also galaxies and dark matter.

On the other hand, the pressure of gas can be given by the equation of state of the ideal gas

$$P_{\text{gas}} = \frac{k_B \rho_{\text{gas}} T_{\text{gas}}(r)}{\mu}, \quad (2.2.15)$$

where k_B is the boltzmann constant, T_{gas} is the temperature of gas, and $\mu \sim 0.6m_H$ is the average mass of particles composing the gas (m_H is the mass of Hydrogen), which is determined by the chemical composition of the gas. Combining (2.2.14) and (2.2.15), we have

$$M(r) = \frac{kT_{\text{gas}}(r)r}{G\mu} \left[-\frac{d \ln \rho_{\text{gas}}(r)}{d \ln r} - \frac{d \ln T_{\text{gas}}(r)}{d \ln r} \right]. \quad (2.2.16)$$

This equation means that the total mass $M(r)$ can be obtained if $\rho_{\text{gas}}(r)$ and $T_{\text{gas}}(r)$ is given as functions of a radius r . $\rho_{\text{gas}}(r)$, $T_{\text{gas}}(r)$ are reconstructed from the X-ray luminosity and spectrum, respectively. Applying this method to the Coma cluster, one finds that $M_{\text{Coma}} \approx 1.3 \times 10^{15} M_\odot$. This matches well with the estimation using the virial theorem and provides another evidence of the existence of dark matter.

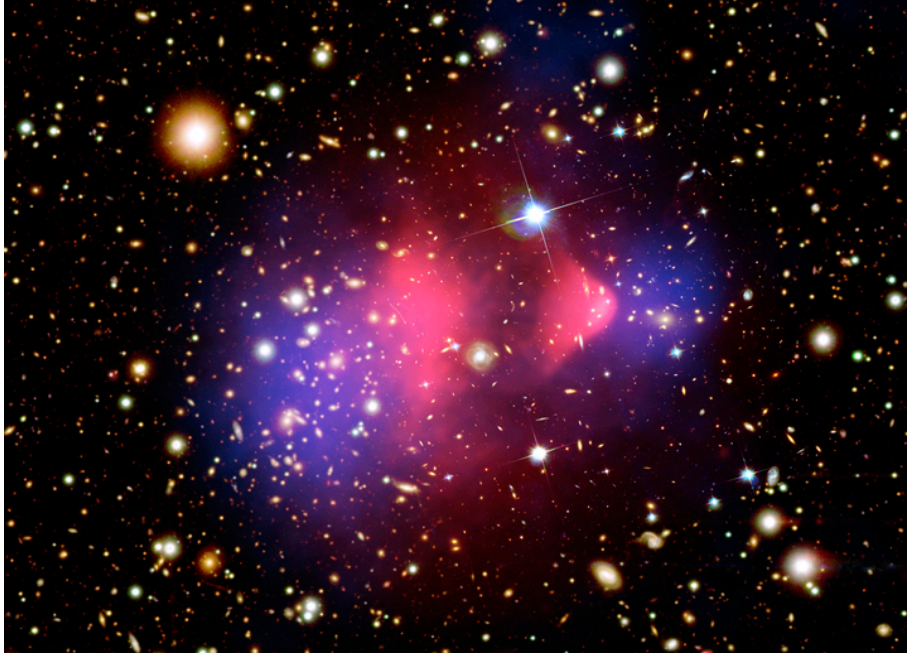


Figure 2.2: The Bullet cluster, cited from [24].

2.2.3 Gravitational lensing

In the presence of a mass distribution between a light source and an observer, light emitted by the light source is bent by gravity as if it passes through a lens. This phenomenon is called gravitational lensing. In the case of a spherically symmetric mass distribution, the deflection angle $\hat{\alpha}$ of light is given by

$$\hat{\alpha} = \frac{4GM}{c^2 b} = 1''75 \left(\frac{M}{M_\odot} \right) \left(\frac{b}{R_\odot} \right)^{-1}, \quad (2.2.17)$$

where b is the impact parameter, and $R_\odot$ is the solar radius. The deflection angle is proportional to the mass of the lensing object. Therefore, by measuring the deflection angle, we can estimate the mass of the lensing object.

Eddington's observation of the deflection of light from stars in the Hyades cluster by the Sun is well known as early evidence of general relativity. In this observation, the mass of the gravitational source (the Sun) was known, and based on that, the deflection angle of light (2.2.17) was predicted using general relativity.

On the other hand, it is possible to reconstruct the mass distribution of the lensing object by measuring the deflection angle of light from a distant light source, which can be used to show the existence of dark matter. The notable example of the evidence of dark matter using gravitational lensing is the Bullet Cluster [25], which is shown in figure 2.2. The Bullet Cluster is a galaxy cluster that is formed by two colliding galaxy clusters. The mass distribution of the ICM observed by X-ray, is indicated in red. On the other hand, the total mass distribution inferred from the gravitational lensing is shown in blue. The ICM of each galaxy cluster collide

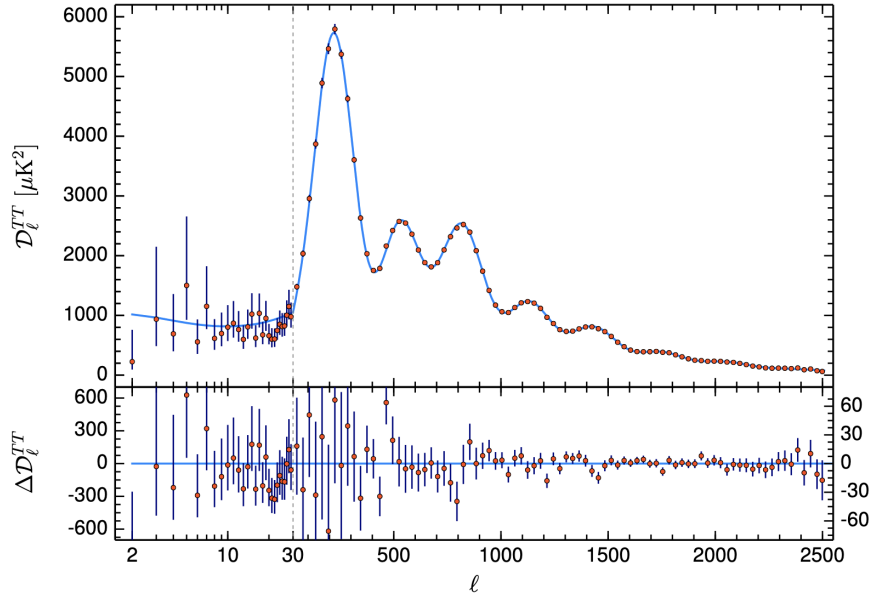


Figure 2.3: The temperature fluctuation of the CMB, cited from [26]. The vertical axis represents the magnitude of the temperature fluctuation, and the horizontal axis represents the angular scale.

each other and lost their velocities, whereas the dark matter does not interact and pass through each other. Therefore, the mass distribution of the dark matter is separated from the ICM, which is a strong evidence of the existence of dark matter.

2.3 Cosmic microwave background

In the early universe, protons and electrons did not form neutral atoms but existed as hot plasma. They are in thermally equilibrium with photons due to the Thomson scattering. Therefore, the universe is filled with photon-baryon fluid, which is hot fluid composed of proton, electron, and photon. As the universe expands, the temperature of the universe decreases and protons and electrons combine to form neutral atoms. This process is called recombination. Because interactions between neutral atoms and photons are small, photons became freely streaming. These photons are observed as the blackbody radiation with temperature 2.7K, which is called the cosmic microwave background (CMB). The observed CMB is highly isotropic, which is known as the strong evidence of the cosmological principle i.e. isotropy and homogeneity of the universe.

Actually, the CMB is not perfectly isotropic and there are small fluctuations in the temperature of the CMB, which provide another evidence of dark matter. The CMB temperature fluctuation measured by Planck satellite is shown in figure 2.3. The peaks in the temperature fluctuation correspond to the acoustic oscillation of the photon-baryon fluid. The gravitational potential generated by components of universe affects this oscillation so that the fraction of

each components can be inferred from fitting the observed temperature fluctuation by the theoretical model. From the observation by Planck satellite, the density parameter of dark matter is estimated as [26]

$$\Omega_{\chi}h^2 = 0.120 \pm 0.001. \quad (2.3.1)$$

Chapter 3

Ultra-light dark matter models

One of the most prominent scenario of dark matter is the weakly interacting massive particle (WIMP) scenario. In this scenario, dark matter is a thermal relic of the early universe and its mass is around the weak scale. However, there is no direct evidence of WIMP dark matter, and the parameter space of WIMP dark matter is becoming more constrained by the direct detection experiments. Therefore, it is important to consider other scenarios of dark matter.

Recently, there is a growing interest in ultra-light dark matter models. In this scenario, dark matter is very light and its mass is much smaller than the weak scale. In this chapter, we review ultra-light dark matter models and their observational constraints based on [27].

3.1 Dark photon

A light vector particle, called the dark photon, is considered as a candidate for ultra-light dark matter. The dark photon is a gauge boson of a new $U(1)$ gauge symmetry, which is called the dark $U(1)$ symmetry [28] (see [27] for a review). We refer to the usual electromagnetic $U(1)$ symmetry as $U(1)_{\text{EM}}$ and the dark $U(1)$ symmetry as $U(1)_d$. The dark photon can be massive because of the Higgs mechanism or the Stückelberg mechanism. The strength of the interaction between the dark photon and the Standard Model particles is naturally suppressed by the "kinetic mixing parameter". Therefore, The dark photon can be a good candidate for dark matter. In this section, we review the dark photon dark matter and its observational constraints.

Let us consider two $U(1)$ gauge symmetries $U(1)_a, U(1)_b$ and their gauge bosons A_a, A_b . Their kinetic terms are given By

$$\mathcal{L}_0 = -\frac{1}{4}F_{a\mu\nu}F_a^{\mu\nu} - \frac{1}{4}F_{b\mu\nu}F_b^{\mu\nu} - \frac{\epsilon}{2}F_{a\mu\nu}F_b^{\mu\nu}, \quad (3.1.1)$$

where $F_{a\mu\nu}$ and $F_{b\mu\nu}$ are the field strength tensors of A_a and A_b , respectively, and ϵ is a constant parameter called the kinetic mixing parameter. The kinetic mixing can be generated by the loop effect of heavy particles charged under both $U(1)_a$ and $U(1)_b$ symmetries.

Let us suppose that A_a corresponds to the dark photon and A_b corresponds to the ordinary photon. Then, $U(1)_a = U(1)_d$ and $U(1)_b = U(1)_{\text{EM}}$. The photon A_b is massless and the mass term are given by

$$\mathcal{L}_m = -\frac{1}{2}M_a^2 A_{a\mu} A_a^\mu. \quad (3.1.2)$$

Moreover, suppose that the ordinary photon A_b couples to the Standard Model current J_μ and the dark photon A_a coupled to J'_μ . Then, the interaction terms are given by

$$\mathcal{L} = eJ_\mu A_b^\mu + e'J'_\mu A_a^\mu \quad (3.1.3)$$

Let us introduce new fields A, A' defined as

$$\begin{pmatrix} A_a^\mu \\ A_b^\mu \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{1-\epsilon^2}} & 0 \\ -\frac{\epsilon}{\sqrt{1-\epsilon^2}} & 1 \end{pmatrix} \begin{pmatrix} A'^\mu \\ A^\mu \end{pmatrix}. \quad (3.1.4)$$

Then, the Lagrangian is diagonalized as

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}F'_{\mu\nu}F'^{\mu\nu} - \frac{1}{2}m_{A'}^2 A'_\mu A'^\mu + \left[\frac{e'}{\sqrt{1-\epsilon^2}}J'_\mu + \frac{e\epsilon}{\sqrt{1-\epsilon^2}}J_\mu \right] A'^\mu + eJ_\mu A^\mu, \quad (3.1.5)$$

where $m_{A'} = M_a$ is the mass of the dark photon. A is an ordinary massless photon and couples to the Standard Model $U(1)_{\text{EM}}$ current J . On the other hand, the dark photon A' has the mass $m_{A'}$ and couples to the dark current J' as well as the Standard Model current J with the strength proportional to ϵ .

3.1.1 Dark photon dark matter

The mass of the dark photon is typically considered to be tiny. If such light particles are thermally produced, they erase the small-scale structure of the universe, which is inconsistent with the observation. Therefore, the dark photon does not seem to be a candidate for thermal relic dark matter. However, we show that a coherently oscillating dark photon has zero pressure and thus behaves as cold dark matter [29].

The kinetic term of the dark photon is given by

$$\mathcal{L} = -\frac{1}{4}F'_{\mu\nu}F'^{\mu\nu} - \frac{1}{2}m_{A'}^2 A'_\mu A'^\mu. \quad (3.1.6)$$

The equation of motion is

$$\partial_\mu \left(\sqrt{-g} F'^{\mu\nu} \right) = \sqrt{-g} m_{A'}^2 A'^\nu. \quad (3.1.7)$$

Assuming homogeneity,

$$\partial_i A'_\mu = 0. \quad (3.1.8)$$

the equation of motion for the dark photon in a FRW universe $ds^2 = -dt^2 + a(t)^2 (dx^2 + dy^2 + dz^2)$ reduces to

$$\ddot{A}^i + 3H\dot{A}^i + m_{A'}^2 A^i = 0, \quad (3.1.9)$$

where $H(t) = \dot{a}/a$ is the Hubble parameter. Therefore, the spatial component of the massive vector field obeys the same equation of motion as a scalar field in the expanding universe. This equation describes a harmonic oscillator with a time-dependent damping term $H(t)$. In the early universe, $H(t) \gg m_{A'}$ and the field remains constant. As the universe expands, the Hubble parameter decreases. When $H(t) \sim m_{A'}$, the field begins to oscillate around the minimum of the potential. At later times, the Hubble parameter is negligible $H(t) \ll m_{A'}$. Then, the solution of the equation of motion is given by

$$A^i(t) \propto \cos m_{A'} t. \quad (3.1.10)$$

The spatial component of the energy-momentum tensor is given by

$$\begin{aligned} T_{ij} &= g^{\rho\sigma} F'_{i\rho} F'_{j\sigma} + m_{A'}^2 A'_i A'_j + g_{ij} \left(-\frac{1}{4} F'_{\rho\sigma} F'^{\rho\sigma} - \frac{1}{2} m_{A'}^2 A'_\rho A'^\rho \right) \\ &\propto \cos^2 m_{A'} t - \sin^2 m_{A'} t = \cos 2m_{A'} t. \end{aligned} \quad (3.1.11)$$

Since the Hubble time is much longer than the period of the oscillation, the spatial component of the energy momentum tensor averages to zero, effectively behaving as a pressureless fluid. Consequently, the dark photon behaves as cold dark matter.

3.1.2 Observational constraints on dark photon

Observational and experimental constraints on ϵ are shown in figure 3.1. Axion detection experiments such as haloscope and helioscope have sensitivity to the dark photon as well. For example, ADMX experiment, one of haloscope experiments give bound of $\epsilon \lesssim 10^{-14}$ in the mass range of $m_{A'} \sim 10^{-6}$ eV.

For the dark photon of $m_{A'} > 1$ eV, XENON experiment [31] and solar data [32] give the most stringent bound. The bound from solar data means that giving a lower bound on the energies which is carried away by the dark photon from the sun by the observation of the solar neutrino.

For the mass range lower than 10^{-4} eV, the cosmological observation gives the most stringent bound. For example, the CMB observation gives the bound for the dark matter dark photon in [33]. Photons in the early universe, which is filled with the plasma, have effective mass $m_A = \left(\frac{4\pi n_e \alpha}{m_e} \right)^{1/2}$. Here, n_e is the number density of electrons, α is the fine structure constant, and m_e is the electron mass. The effective mass gradually decreases as the universe expands and the number density of the electron decreases. When the dark photon mass coincides with the

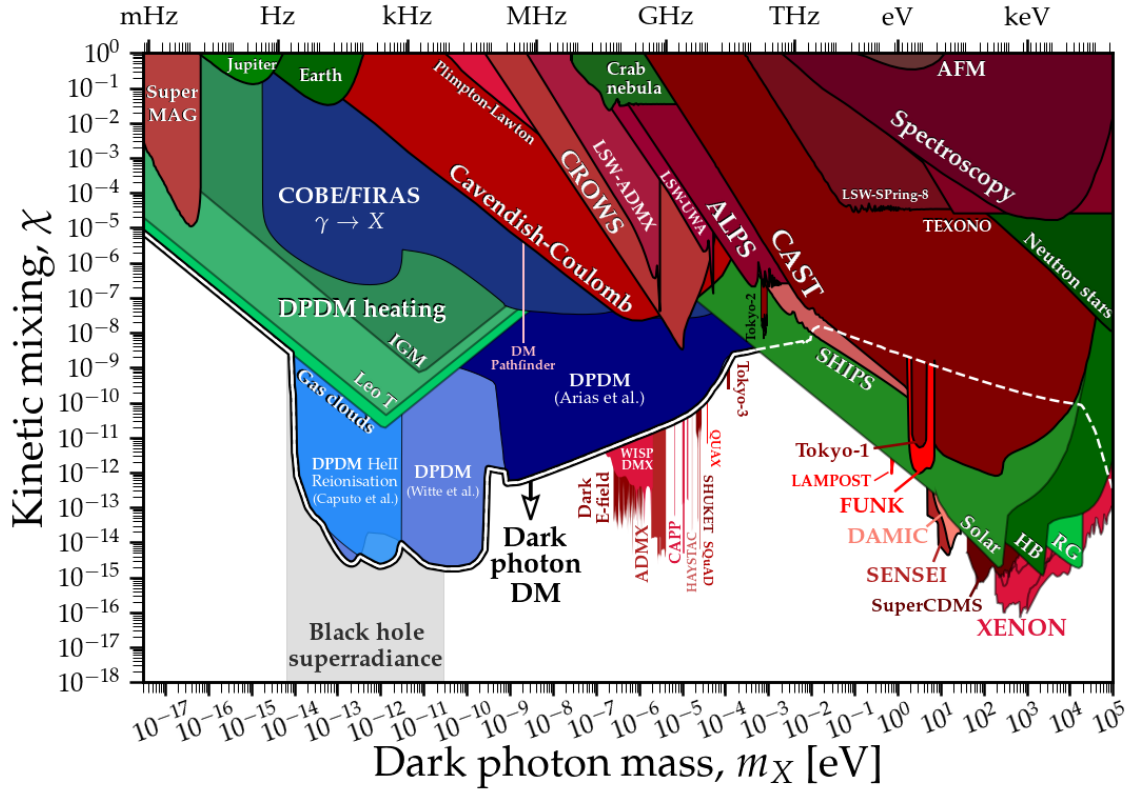


Figure 3.1: Constraints on the dark photon model, cited from [30].

effective photon mass, the conversion rate of the dark photon to the ordinary photon becomes large due to the resonance effect. This conversion affect to the CMB photon and it gives the lower bound of $\epsilon \lesssim 10^{-12}$ in [33].

Chapter 4

Scattering amplitude

4.1 Basics of scattering amplitudes

In the scattering theory, we consider a physical process where the state $|\alpha\rangle_{-\infty}$ which is prepared at the asymptotic past $t = -\infty$ transitions to the state $|\beta\rangle_{\infty}$ defined at the asymptotic future $t = \infty$. The matrix element of the S-matrix $\hat{S}$ corresponding to this transition is defined as follows:

$$S_{\beta\alpha} := {}_{\infty}\langle\beta|\alpha\rangle_{-\infty}. \quad (4.1.1)$$

The S-matrix describes the time evolution of the state in the quantum theory. Therefore, it is natural to demand the S-matrix to be unitary so that the sum of the transition probability is 1. The unitarity condition is given by

$$S_{\alpha\beta}^* S_{\beta\gamma} = \delta_{\alpha\gamma}, \quad (4.1.2)$$

or equivalently

$$\hat{S}^\dagger \hat{S} = \hat{1}. \quad (4.1.3)$$

where $\hat{1}$ is the identity operator.

S-matrix operator can be divided into two parts as follows:

$$\hat{S} \equiv \hat{1} + i\hat{T}, \quad (4.1.4)$$

where the identity operator $\hat{1}$ represents the time evolution where no scattering process occurs and $\hat{T}$, called T-matrix, describes the time evolution with scatterings.

In the following, we consider a $2 \rightarrow 2$ scattering process. Let the mass of all particles be m for simplicity. Firstly, we prepare a two-particle initial state $|p_1, p_2\rangle$, where p_1 and p_2 are momenta of incident particles. These particles scatter off each other and evolve to a final state $|p_3, p_4\rangle$. Mandelstam invariants for this process are defined as

$$s := (p_1 + p_2)^2 = 4(m^2 + |\vec{p}|^2), \quad (4.1.5)$$

$$t := (p_1 - p_3)^2 = -2|\vec{p}|^2(1 - \cos \theta), \quad (4.1.6)$$

$$u := (p_1 - p_4)^2 = -2|\vec{p}|^2(1 + \cos \theta), \quad (4.1.7)$$

where $\vec{p}$ is the momentum of particle 1 in the center-of-momentum(COM) frame. Using Mandelstam variables, the scattering angle in the COM frame $\cos \theta = \mathbf{p}_1 \cdot \mathbf{p}_3 / |\mathbf{p}_1| |\mathbf{p}_3|$ can be written as

$$\cos \theta = 1 + \frac{2t}{s - 4m^2}. \quad (4.1.8)$$

The kinematics of particles of a $2 \rightarrow 2$ scattering in a Lorentz invariant theory can be completely determined by the total energy in the COM frame and the scattering angle. Therefore, there are only two out of three Mandelstam invariants are independent and there is a relation

$$s + t + u = 4m^2. \quad (4.1.9)$$

A scattering amplitude $\mathcal{M}(s, t)$ is defined as the matrix element of a T-matrix as

$$\langle p_3, p_4 | \hat{T} | p_2, p_1 \rangle = (2\pi)^d \delta^d(p_1 + p_2 - p_3 - p_4) \mathcal{M}(s, t) \quad (4.1.10)$$

4.1.1 Unitarity

the unitarity condition of a S-matrix $\hat{S} \cdot \hat{S}^\dagger = \hat{\mathbf{1}}$ can be expressed using T-matrix as

$$\begin{aligned} \frac{1}{i} \langle p_3, p_4 | \hat{T} - \hat{T}^\dagger | p_2, p_1 \rangle &= \langle p_3, p_4 | \hat{T} \cdot \hat{T}^\dagger | p_2, p_1 \rangle \\ &= \sum_{n=1}^{\infty} \int d\mu(q_1, \dots, q_n) \langle p_3, p_4 | \hat{T} | \{q_i\}_{i=1}^n \rangle \langle \{q_i\}_{i=1}^n | \hat{T}^\dagger | p_2, p_1 \rangle, \end{aligned} \quad (4.1.11)$$

where $d\mu(q_1 \cdots q_n) = d\mu(q_1) \cdots d\mu(q_n)$, $d\mu(q) = \frac{d^{d-1}q}{(2\pi)^{d-1}2E}$ is the Lorentz invariant measure and a complete basis of asymptotic states $\sum_{n=1}^{\infty} |\{q_i\}_{i=1}^n\rangle \langle \{q_i\}_{i=1}^n|$ is inserted in the second line.

We consider an elastic process in the following. In the language of the S-matrix theory, we can define an elastic scattering as a scattering where T-matrix only has non-zero matrix elements in $2 \rightarrow 2$ scattering when the initial state is a two particle state. Then, the sum in (4.1.11) is limited to $n = 2$ and we have

$$2 \text{Disc}_s \mathcal{M}(s, t) = \int d\mu(q') \int d\mu(q'') (2\pi)^d \delta^d(p_1 + p_2 - q' - q'') \mathcal{M}^{(+)}(s, t') \mathcal{M}^{(-)}(s, t''), \quad (4.1.12)$$

where we define

$$\mathcal{M}^{(\pm)} \equiv \lim_{\epsilon \rightarrow 0} \mathcal{M}(s \pm i\epsilon, t), \quad \text{Disc}_s \mathcal{M}(s, t) \equiv \frac{1}{2i} (\mathcal{M}^{(+)}(s, t) - \mathcal{M}^{(-)}(s, t)), \quad (4.1.13)$$

and $t' = -(p_1 - q')^2, t'' = -(p_3 - q'')^2$.

The integral of RHS of (4.1.12) can be calculated as

$$\begin{aligned}
2\mathcal{M}_s(s, t) &= \int d\mu(q') \int d\mu(q'') (2\pi)^d \delta^d(p_1 + p_2 - q' - q'') \mathcal{M}^{(+)}(s, t') \mathcal{M}^{(-)}(s, t'') \\
&= \frac{1}{(2\pi)^{d-2}} \frac{1}{4E_p^2} \int d^{d-1}q' \delta(2E_p - 2E_{q'}) \mathcal{M}^{(+)}(s, t') \mathcal{M}^{(-)}(s, t'') \\
&= \frac{p^{d-2}}{(2\pi)^{d-2}} \frac{1}{4E_p^2} \frac{E_p}{2p} \int d^{d-2}\Omega_{\vec{n}} \mathcal{M}^{(+)}(s, t') \mathcal{M}^{(-)}(s, t''). \tag{4.1.14}
\end{aligned}$$

If $t = 0$, $p_1 = p_3$ so that $t' = t''$ and we get

$$2\mathcal{M}_s(s, 0) = \frac{p^{d-2}}{(2\pi)^{d-2}} \frac{1}{4E_p^2} \frac{E_p}{2p} \int d^{d-2}\Omega_{\vec{n}} |\mathcal{M}^{(+)}(s, t')|^2 > 0. \tag{4.1.15}$$

Using the Schwartz reflection principle, $\mathcal{M}_s(s, t)$ is equal to the imaginary part:

$$\begin{aligned}
\mathcal{M}_s(s, t) &= \frac{1}{2i} (\mathcal{M}^{(+)}(s, t) - \mathcal{M}^{(-)}(s, t)) \\
&= \frac{1}{2i} (\mathcal{M}^{(+)}(s, t) - (\mathcal{M}^{(+)}(s, t))^* (s, t)) \\
&= \text{Im } \mathcal{M}(s, t). \tag{4.1.16}
\end{aligned}$$

Therefore, (4.1.15) can also be written as

$$\text{Im } \mathcal{M}(s, 0) > 0. \tag{4.1.17}$$

This is called the optical theorem.

We define a function $\mathcal{P}_d(z, z', z'')$ as

$$\mathcal{P}_d(z, z', z'') = \frac{2\pi^{\frac{d-3}{2}}}{\Gamma(\frac{d-3}{2})} (1 - z^2)^{\frac{4-d}{2}} \frac{\Theta(1 - z^2 - z'^2 - z''^2 + 2zz'z'')}{(1 - z^2 - z'^2 - z''^2 + 2zz'z'')^{\frac{d-5}{2}}}. \tag{4.1.18}$$

Using $\mathcal{P}_d(z, z', z'')$, the integral over the $d - 2$ dimensional solid angle can be written as

$$\int d^{d-2}\Omega_{\vec{n}} \equiv \int_{-1}^1 dz' \int_{-1}^1 dz'' \mathcal{P}_d(z, z', z''). \tag{4.1.19}$$

Then, we can write (4.1.14) as

$$\mathcal{M}_s(s, t) = \frac{(s - 4m^2)^{\frac{d-3}{2}}}{4(4\pi)^{d-2}\sqrt{s}} \int_{-1}^1 dz' \int_{-1}^1 dz'' \mathcal{P}_d(z, z', z'') \mathcal{M}^{(+)}(s, t'(z')) \mathcal{M}^{(-)}(s, t''(z'')). \tag{4.1.20}$$

4.1.2 Partial wave expansion

In this section, we consider $d = 4$ case only for simplicity. Firstly, define a vector $|E, l, m\rangle$ as

$$|E, l, m\rangle = \sqrt{\frac{4\pi E}{p}} \int d\mu(p_1) d\mu(p_2) (2\pi)^4 \delta(E_1 + E_2 - E) \delta^3(\vec{p}_1 + \vec{p}_2) Y_{l,m}^*(\hat{p}_1) |p_1, p_2\rangle, \tag{4.1.21}$$

where $Y_{l,m}(\hat{n})$ is the spherical harmonics. This vector is a two particle state with a definite angular momentum, characterized by a total angular momentum l and a magnetic quantum number m . The parity of the spherical harmonics depends on l as follows:

$$Y_{l,m}(-\hat{n}) = (-1)^l Y_{l,m}(\hat{n}). \quad (4.1.22)$$

Therefore, $|E, l, m\rangle$ vanishes for odd l .

$|E, l, m\rangle$ with different quantum number are orthogonal:

$$\begin{aligned} \langle E, l, m | E', l', m' \rangle' &= \frac{8\pi E}{p} \int d\mu(p_1) d\mu(p_2) (2\pi)^4 \delta(E_1 + E_2 - E) \delta^3(\vec{p}_1 + \vec{p}_2) Y_{l,m}(\hat{p}_1) Y_{l',m'}^*(\hat{p}_1) \\ &= \frac{8\pi E}{p} \int \frac{d^3 p_1}{(2\pi)^6 (2E_1)^2} (2\pi)^4 \delta(2E_1 - E) Y_{l,m}(\hat{p}_1) Y_{l',m'}^*(\hat{p}_1) \\ &= \int \frac{d\Omega}{4\pi} Y_{l,m}(\hat{p}_1) Y_{l',m'}^*(\hat{p}_1) \\ &= \delta_{l,l'} \delta_{m,m'} \end{aligned} \quad (4.1.23)$$

where the prime indicates that the energy-conservation delta function is omitted. The inner product with a definite momentum state is given by

$$\langle p_1, p_2 | E, l, m \rangle = \frac{1}{2\pi^2 E^2} \sqrt{\frac{4\pi E}{p}} Y_{l,m}^*(\hat{p}_1). \quad (4.1.24)$$

A matrix element of a T-matrix with respect to $|E, l, m\rangle$ can be represented using a function $a_l(s)$, which is independent of m , as

$$\langle l, m | T | l', m' \rangle \propto a_l(s) \delta_{ll'} \delta_{mm'} \quad (4.1.25)$$

Therefore, a scattering amplitude can be written as

$$\begin{aligned} \mathcal{M}(s, \cos \theta) &= \frac{1}{(2\pi)^4} \langle p_3, p_4 | T | p_1, p_2 \rangle' \\ &= \frac{1}{(2\pi)^4} \sum_{l,m,l',m'} \langle p_3, p_4 | l, m \rangle' \langle l, m | T | l', m' \rangle \langle l', m' | p_1, p_2 \rangle' \\ &= 16\pi \sum_l (2l+1) a_l(s) P_l(\cos \theta), \end{aligned} \quad (4.1.26)$$

where the overall normalization 16π is chosen for the later convenience. This representation is called the partial wave expansion, and the coefficients $a_l(s)$ is called a partial amplitude.

In general dimension d , the partial wave expansion is obtained in a similar manner:

$$\mathcal{M}(s, t) = \sum_{l=0}^{\infty} n_l^{(d)} a_l(s) P_l^{(d)}(\cos \theta), \quad (4.1.27)$$

where $n_l^{(d)}$ is a constant given by (4.1.32). $P_l^{(d)}(z)$ is a polynomial of degree l and satisfies the following differential equation:

$$\left[(1-z^2)^{\frac{4-d}{2}} \frac{d}{dz} (1-z^2)^{\frac{d-2}{2}} \frac{d}{dz} + l(l+d-3) \right] P_l^{(d)}(z) = 0. \quad (4.1.28)$$

The explicit form of $P_l^{(d)}(z)$ is given using the hypergeometric function as

$$P_l^{(d)}(z) = {}_2F_1 \left(-l, l + d - 3, \frac{d-2}{2}, \frac{1-z}{2} \right). \quad (4.1.29)$$

The orthogonality relation

$$\frac{1}{2} \int_{-1}^1 dz (1-z^2)^{\frac{d-4}{2}} P_l^{(d)}(z) P_{\bar{l}}^{(d)}(z) = \frac{\delta_{l\bar{l}}}{\mathcal{N}_d n_l^{(d)}} \quad (4.1.30)$$

and the addition theorem

$$\mathcal{P}_d(z, z', z'') = (4\pi)^{d-2} \mathcal{N}_d^2 (1-z'^2)^{\frac{d-4}{2}} (1-z''^2)^{\frac{d-4}{2}} \sum_{l=0}^{\infty} n_l^{(d)} P_l^{(d)}(z) P_l^{(d)}(z') P_l^{(d)}(z'') \quad (4.1.31)$$

are useful. Here,

$$\mathcal{N}_d = \frac{(16\pi)^{\frac{2-d}{2}}}{\Gamma\left(\frac{d-2}{2}\right)}, \quad n_l^{(d)} = \frac{(4\pi)^{\frac{d}{2}} (d+2l-3) \Gamma(d+l-3)}{\pi \Gamma\left(\frac{d-2}{2}\right) \Gamma(l+1)}. \quad (4.1.32)$$

Especially, for 4-dimensional spacetime, it is given by

$$\mathcal{N}_4 = \frac{1}{16\pi}, \quad n_l^{(4)} = 16\pi(2l+1), \quad (4.1.33)$$

and (4.1.30) becomes

$$\frac{1}{2} \int_{-1}^1 dz P_l(z) P_{\bar{l}}(z) = \frac{\delta_{l\bar{l}}}{2l+1} \quad (4.1.34)$$

Substituting this partial wave representation into (4.1.20),

$$(\text{LHS}) = \mathcal{M}_s(s, t) \quad (4.1.35)$$

$$= \sum_{l=0}^{\infty} n_l^{(d)} \text{Im}(a_l(s)) P_l^{(d)}(\cos \theta) \quad (4.1.36)$$

$$\begin{aligned} (\text{RHS}) &= \frac{(s-4m^2)^{\frac{d-2}{2}}}{4(4\pi)^{d-3} \sqrt{s}} \int_{-1}^1 dz' \int_{-1}^1 dz'' \mathcal{P}_d(z, z', z'') \mathcal{M}^{(+)}(s, t'(z')) \mathcal{M}^{(-)}(s, t''(z'')) \\ &= \frac{(s-4m^2)^{\frac{d-2}{2}}}{4(4\pi)^{d-3} \sqrt{s}} \int_{-1}^1 dz' \int_{-1}^1 dz'' (4\pi)^{d-2} \mathcal{N}_d^2 (1-z'^2)^{\frac{d-4}{2}} (1-z''^2)^{\frac{d-4}{2}} \\ &\quad \times \sum_{l=0}^{\infty} n_l^{(d)} P_l^{(d)}(z) P_l^{(d)}(z') P_l^{(d)}(z'') \sum_{l'=0}^{\infty} n_{l'}^{(d)} a_{l'}(s+i\varepsilon) P_{l'}^{(d)}(z') \sum_{l''=0}^{\infty} n_{l''}^{(d)} a_{l''}(s-i\varepsilon) P_{l''}^{(d)}(z'') \\ &= \frac{(s-4m^2)^{\frac{d-2}{2}}}{4(4\pi)^{d-3} \sqrt{s}} (4\pi)^{d-2} \sum_{l=0}^{\infty} n_l^{(d)} P_l^{(d)}(z) \\ &\quad \times \left(\int_{-1}^1 dz' (1-z'^2)^{\frac{d-4}{2}} \mathcal{N}_d P_l^{(d)}(z') \sum_{l'=0}^{\infty} n_{l'}^{(d)} a_{l'}(s+i\varepsilon) P_{l'}^{(d)}(z') \right) \end{aligned}$$

$$\begin{aligned}
& \times \left(\int_{-1}^1 dz'' (1 - z''^2)^{\frac{d-4}{2}} \mathcal{N}_d P_l^{(d)}(z'') \sum_{l''=0}^{\infty} n_{l''}^{(d)} a_{l''}(s - i\varepsilon) P_{l''}^{(d)}(z'') \right) \\
& = \frac{(s - 4m^2)^{\frac{d-3}{2}}}{\sqrt{s}} \sum_{l=0}^{\infty} n_l^{(d)} |a_l(s)|^2 P_l^{(d)}(z)
\end{aligned} \tag{4.1.37}$$

Therefore, the unitarity condition of a scattering amplitude on an angular momentum eigenstate is given by

$$\text{Im}(a_l(s)) = \frac{(s - 4m^2)^{\frac{d-3}{2}}}{\sqrt{s}} |a_l(s)|^2. \tag{4.1.38}$$

Defining the partial wave element of S-matrix as

$$S_l(s) \equiv 1 + i \frac{2(s - 4m^2)^{\frac{d-3}{2}}}{\sqrt{s}} a_l(s), \tag{4.1.39}$$

the condition can be represented as

$$|S_l(s)| = 1. \tag{4.1.40}$$

In this form, it is transparent that the probability should add up to 1.

4.2 Froissart bound

In this section, we discuss on the Froissart bound [34, 35], which is a condition satisfied by a high energy behavior of a scattering amplitude of a unitary theory, based on [36]. We concentrate on $d = 4$.

Firstly, we enumerate the assumptions used in the derivation of the Froissart bound [37]:

1. Lorentz invariance

The kinematics of the scattering amplitude is determined by the Mandelstam invariants, and the scattering amplitude is the function of s, t : $\mathcal{M}(s, t)$. Therefore, the scattering amplitude is a Lorentz invariant object.

2. Unitarity

S-matrix, which defines the scattering, satisfies the unitary condition:

$$\hat{S}^\dagger \hat{S} = \hat{1}. \tag{4.2.1}$$

3. The analyticity in the Lehmann ellipse

Using parameter $\alpha > 0$ and a real variable $\varphi \in [-\pi, \pi]$, we define an ellipse on the complex plane

$$z = \cosh(\alpha + i\varphi), \tag{4.2.2}$$

which is called the Lehmann ellipse. Then, the scattering amplitude $\mathcal{M}(s, t(s, \cos \theta))$ is analytic within the Lehmann ellipse as a function of $z = \cos \theta$.

4. Polynomial boundedness

The scattering amplitude diverges at most as a polynomial of s in the high-energy limit $s \rightarrow \infty$. That is, using $n \in \mathbb{N}$,

$$\lim_{|s| \rightarrow \infty} \left| \frac{\mathcal{M}(s, t)}{s^n} \right| = 0. \quad (4.2.3)$$

Under these assumptions, we will see that there is an upper bound on the growth of the scattering amplitude in the high-energy regime.

Let us expand the amplitude into the partial waves with different normalization:

$$\mathcal{M}(s, t) = \frac{16\pi\sqrt{s}}{q_s} \sum_{l=0}^{\infty} a_l(s) (2l+1) P_l(\cos \theta), \quad (4.2.4)$$

where $q_s = \sqrt{s - 4m^2}$. In this normalization, partial amplitudes is given by

$$a_l(s) = \frac{1}{32\pi} \sqrt{\frac{q_s^2}{s}} \int_{-1}^1 d\cos \theta \mathcal{M}(s, \cos \theta) P_l(\cos \theta), \quad (4.2.5)$$

and the unitarity condition (4.1.38) with $d = 4$ becomes

$$|a_l(s)| \leq 1. \quad (4.2.6)$$

Let us define the Legendre function of the second kind as

$$Q_l(x) = -\frac{1}{2} \int_{-1}^1 \frac{P_l(s)}{s-x} ds. \quad (4.2.7)$$

For $x \in [-1, 1]$, there is an identity

$$\frac{1}{2i} (Q_l(x + i\varepsilon) - Q_l(x - i\varepsilon)) = -\frac{\pi}{2} P_l(x). \quad (4.2.8)$$

Using Q_l , a_l can be written as

$$a_l(s) = \frac{1}{32i\pi^2} \sqrt{\frac{q_s^2}{s}} \oint_{\mathcal{C}} dz' \mathcal{M}(s, z') Q_l(z'), \quad (4.2.9)$$

where the contour $\mathcal{C}$ is defined using $\varphi \in [-\pi, \pi]$ and the real parameter $\alpha > 0$ as

$$z' = \cosh(\alpha + i\varphi') \quad (4.2.10)$$

$$dz' = i \sinh(\alpha + i\varphi') d\varphi'. \quad (4.2.11)$$

This contour represents the ellipse

$$\left(\frac{\operatorname{Re} z}{\cosh \alpha} \right)^2 + \left(\frac{\operatorname{Im} z}{\sinh \alpha} \right)^2 = 1. \quad (4.2.12)$$

Then, (4.2.9) becomes

$$a_l(s) = \frac{1}{32\pi^2} \sqrt{\frac{q_s^2}{s}} \int_{-\pi}^{\pi} d\varphi' \mathcal{M}(s, z') Q_l(z') \sinh(\alpha + i\varphi). \quad (4.2.13)$$

Let us investigate the behavior of $Q_l(z)$ in large l limit. Firstly, there is a following representation of $Q_l(z)$ using the hypergeometric function:

$$Q_l(\cosh \beta) = \sqrt{\pi} \cdot \frac{\Gamma(l+1)}{\Gamma(l+\frac{3}{2})} \cdot \frac{e^{-\beta(l+1)}}{\sqrt{1-e^{-2\beta}}} \cdot {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; l+\frac{3}{2}; \frac{1}{1-e^{2\beta}}\right). \quad (4.2.14)$$

From the Stirling formula $\lim_{x \rightarrow \infty} \Gamma(x) \sim x^x$,

$$\lim_{l \rightarrow \infty} \frac{\Gamma(l+1)}{\Gamma(l+\frac{3}{2})} \sim \frac{1}{\sqrt{l}}, \quad (4.2.15)$$

in the large l region. Furthermore, using the power series representation of the hypergeometric function ${}_2F_1(a, b; c; x) = \sum_n \frac{(a)_n (b)_n}{n! (c)_n} x^n$,

$$\lim_{c \rightarrow \infty} {}_2F_1(a, b; c; x) = 1 + \mathcal{O}(c^{-1}), \quad (4.2.16)$$

holds. Using these results, we have

$$\sinh(\alpha + i\varphi') \cdot Q_l(z')|_{l \rightarrow +\infty} = e^{-(\alpha+i\varphi')(l+1/2)} \cdot \sqrt{\frac{\pi}{2l}} \sinh(\alpha + i\varphi') \cdot [1 + \mathcal{O}(l^{-1})]. \quad (4.2.17)$$

Moreover, there is an upper bounds $|\sinh(\alpha + i\varphi')| \leq \cosh \alpha$ because

$$\begin{aligned} |\sinh(\alpha + i\varphi)|^2 &= \frac{1}{2} (\cosh 2\alpha + \cos 2\varphi) \\ \cosh^2 \alpha &= \frac{1}{2} (\cosh 2\alpha + 1). \end{aligned} \quad (4.2.18)$$

Therefore, we have a following upper bound:

$$|\sinh(\alpha + i\varphi') Q_l(z')|_{l \rightarrow +\infty} < e^{-\alpha(l+1/2)} \sqrt{\frac{\pi}{2l}} \cosh \alpha. \quad (4.2.19)$$

Applying this upper bound on (4.2.13), we get an upper bound on partial waves $a_l(s)$ in the large l region as

$$|a_l| < \sqrt{\frac{q_s^2}{s}} B_\alpha(s) \sqrt{e^{-\alpha} \cosh \alpha} \frac{e^{-\alpha l}}{\sqrt{l}}, \quad (4.2.20)$$

where

$$B_\alpha(s) = \frac{1}{32\sqrt{2}\pi^{3/2}} \int_{-\pi}^{\pi} |\mathcal{M}(s, z')| d\varphi', \quad z' = \cosh(\alpha + i\varphi'). \quad (4.2.21)$$

Let us reconsider the assumption that the analyticity of $\mathcal{M}(s, z)$ as a function of $z = \cosh(\alpha + i\phi')$ with fixed s . This assumption is not satisfied for an arbitrary value of α . Let $\sqrt{t_0}$ be the mass of the lightest particle which is exchanged at the t-channel. Using (4.1.8), we have

$$z = \cosh(\alpha) = 1 + \frac{2t}{q_s^2}. \quad (4.2.22)$$

$t = t_0$ is the on-shell condition, and α should be smaller than the corresponding value. In the following, we define α as $\cosh(\alpha) = 1 + \frac{2t_0}{q_s^2}$.

In the high-energy limit $s \rightarrow \infty$, (4.2.22) approaches to $\alpha \sim 2\sqrt{t/s}$. Then,

$$\sqrt{e^{-\alpha} \cosh \alpha} = \sqrt{1 - e^{-\alpha} \sinh \alpha} \quad (4.2.23)$$

is smaller than 1 and approaches to 1 in the high-energy limit. Therefore, this part in (4.2.20) can be replaced by 1.

Therefore, we have shown that there are two upper bounds on the partial amplitude

$$|a_l| \leq 1 \quad \text{and} \quad |a_l| \leq \sqrt{\frac{q_s^2}{s}} B_\alpha(s) \frac{e^{-\alpha l}}{\sqrt{l}}. \quad (4.2.24)$$

The first one is derived from the unitarity of the theory and holds for all l . On the other hand, the second bound holds for l which is large enough.

For large l , the second bound is stronger than the first one. We define L as the first angular momentum for which the second bound is stronger. That is, L is the smallest number which satisfies

$$\frac{q_s}{\sqrt{s}} B_\alpha(s) \frac{e^{-\alpha L}}{\sqrt{L}} < 1, \quad (4.2.25)$$

and also satisfies (4.2.24).

Using this L , we have an upper bound on the absolute value of the scattering amplitude as

$$|\mathcal{M}(s, \cos \theta)| \leq 16\pi \frac{\sqrt{s}}{q_s} \left[\sum_{l=0}^{L-1} (2l+1) |a_l(s)| \cdot |P_l(\cos \theta)| + \sum_{l=L}^{\infty} (2l+1) |a_l(s)| \cdot |P_l(\cos \theta)| \right] \quad (4.2.26)$$

$$= 16\pi \frac{\sqrt{s}}{q_s} \left[\sum_{l=0}^{L-1} (2l+1) |P_l(\cos \theta)| + \sum_{l=L}^{\infty} (2l+1) \frac{q_s}{\sqrt{s}} B_\alpha(s) \frac{e^{-\alpha l}}{\sqrt{l}} \cdot |P_l(\cos \theta)| \right]. \quad (4.2.27)$$

For the forward scattering $\cos \theta = 1$, $|P_l(1)| = 1$ and we have

$$|\mathcal{M}(s, 1)| \leq 16\pi \frac{\sqrt{s}}{q_s} \sum_{l=0}^{L-1} (2l+1) + 16\pi B_\alpha(s) \sum_{l=L}^{\infty} (2l+1) \frac{e^{-\alpha l}}{\sqrt{l}} \quad (4.2.28)$$

$$= 16\pi \frac{\sqrt{s}}{q_s} L^2 + 16\pi B_\alpha(s) \sum_{l=L}^{\infty} (2l+1) \frac{e^{-\alpha l}}{\sqrt{l}}. \quad (4.2.29)$$

Defining $l' := l - L$, the second term is

$$16\pi B_\alpha(s) \sum_{l'=0}^{\infty} (2l+1) \frac{e^{-\alpha l}}{\sqrt{l}} = 16\pi B_\alpha(s) e^{-\alpha L} \sum_{l'=0}^{\infty} e^{-\alpha l'} \left[2L \left(1 + \frac{l'}{L}\right)^{\frac{1}{2}} + \left(1 + \frac{l'}{L}\right)^{-\frac{1}{2}} \right] \quad (4.2.30)$$

$$\leq 16\pi \frac{\sqrt{s}}{q_s} \sum_{l'=0}^{\infty} e^{-\alpha l'} \left[2L \left(1 + \frac{l'}{L}\right)^{\frac{1}{2}} + \left(1 + \frac{l'}{L}\right)^{-\frac{1}{2}} \right] \quad (4.2.31)$$

$$\approx 16\pi \frac{\sqrt{s}}{q_s} \frac{1}{\alpha} \int_0^\infty e^{-y} \left[2L \left(1 + \frac{y}{Y} \right)^{\frac{1}{2}} + \left(1 + \frac{y}{Y} \right)^{-\frac{1}{2}} \right] dy. \quad (4.2.32)$$

In the second line, we use (4.2.25). In the third line, we define $y = \alpha l'$, $Y = \alpha L$ and approximate the sum by the integral. Then, (4.2.29) becomes

$$|\mathcal{M}(s, 1)| \leq 16\pi \frac{\sqrt{s}}{q_s} [L^2 + I(s)], \quad (4.2.33)$$

where we define

$$I(s) := \frac{1}{\alpha} \int_0^\infty e^{-y} \left[2L \left(1 + \frac{y}{Y} \right)^{\frac{1}{2}} + \left(1 + \frac{y}{Y} \right)^{-\frac{1}{2}} \right] dy. \quad (4.2.34)$$

In the following, we write the energy-dependence of L explicitly as $L(s)$. The bound of (4.2.33) depend on the behavior of $L(s)$ at the high energy regime. When $L(s)$ is asymptotically constant, (4.2.33) imposes a constraint where the scattering amplitude is bounded by some constant value. Next, consider the case where both $L(s)$ and $Y(s) = \alpha L(s)$ increase with s . For large L, Y , (4.2.34) is approximated by

$$I(s) = \frac{1}{\alpha^2} [2Y + 1 + \mathcal{O}(1/Y, \alpha)], \quad (4.2.35)$$

and for the high energy limit, (4.2.33) becomes

$$|\mathcal{M}(s, 1)| < 16\pi \left(\frac{Y+1}{\alpha} \right)^2 \quad (4.2.36)$$

$$\approx 4\pi \frac{q_s^2}{t_0} (Y+1)^2. \quad (4.2.37)$$

In the second line, we approximate $\alpha \approx 2\sqrt{t_0/s}$. Therefore, the forward scattering amplitude is bounded by $\mathcal{O}(sY^2)$ in the high energy limit.

Finally, we discuss the energy dependence of Y . The scattering amplitude is bounded by polynomials from the assumption. Then, $B_\alpha(s)$ is also bounded by polynomials and $B_\alpha(s) \approx C \cdot s^n$ holds. In the high energy limit, we can express (4.2.25) as

$$Y \approx \log B_\alpha + \frac{1}{2} \log \alpha - \frac{1}{2} \log Y < \left(n - \frac{1}{4} \right) \log s, \quad (4.2.38)$$

therefore, $Y = \mathcal{O}(\log s)$ at the high energy regime. Using this, (4.2.37) becomes

$$|A(s, 1)| < \mathcal{O}(s \log^2 s). \quad (4.2.39)$$

This upper bound is called the Froissart bound.

4.3 Regge behavior

In the previous section, we observed that the analytic continuation of the scattering amplitude, as a function of the Mandelstam variable, is a powerful tool for uncovering properties of scattering processes. In this section, we see how the analytic continuation of the partial wave amplitude a_l to the complex l -plane, called Regge theory [17], reveals an additional interesting property of scattering, which is the relationship between the spectrum of t -channel exchanges and the behavior of high-energy scattering based on [38].

In (4.2.20), we get an upper bound on the partial wave $a_l(s)$ in the (real) large l limit. However, it is necessary to have an upper bound on a_l in the whole l plane to show the uniqueness of the analytic continuation of l .

Firstly, let us derive a different expression for the partial wave amplitude. It is convenient to make a partial wave decomposition in the t -channel. Let us define the scattering angle in the t -channel as

$$z_t := \cos \theta_t = 1 + \frac{2s}{t - 4m^2}. \quad (4.3.1)$$

Then, the partial wave decomposition in the t -channel is given by

$$\mathcal{M}(s, t) = 16\pi \sum_{l=0}^{\infty} (2l+1) a_l(t) P_l(z_t). \quad (4.3.2)$$

In this section, we use the usual normalization. From (4.1.34), the partial wave amplitude is given by

$$a_l(t) = \frac{1}{16\pi} \frac{1}{2} \int_{-1}^1 dz_t P_l(z_t) \mathcal{M}(s(z_t, t), t). \quad (4.3.3)$$

Using (4.2.8), we get

$$\begin{aligned} a_l(t) &= -\frac{1}{16\pi^2} \frac{1}{2i} \int_{-1}^1 dz_t \text{Disc}_{z_t}(Q_l(z_t)) \mathcal{M}(s(z_t, t)) \\ &= \frac{1}{16\pi^2} \frac{1}{2i} \int_{\mathcal{C}} dz_t Q_l(z_t) \mathcal{M}(s(z_t, t)), \end{aligned} \quad (4.3.4)$$

where $\text{Disc}_{z_t}[f(z_t)] = f(z_t + i\varepsilon) - f(z_t - i\varepsilon)$. The contour $\mathcal{C}$ is given in figure 4.1. By deforming the contour from $\mathcal{C}$ to $\mathcal{C}'$, we have

$$\begin{aligned} a_l(t) &= \frac{1}{16\pi^2} \frac{1}{2i} \int_{\mathcal{C}'} dz_t Q_l(z_t) \mathcal{M}(s(z_t, t), t) \\ &= \frac{1}{16\pi^2} \int_{z_T}^{\infty} D_s(s, t) Q_l(z_t) dz_t + \frac{1}{16\pi^2} \int_{-\infty}^{-z_T} D_u(u, t) Q_l(z_t) dz_t \\ &= \frac{1}{16\pi^2} \int_{z_T}^{\infty} (D_s(s, t) + e^{-i\pi l} D_u(s, t)) Q_l(z_t) dz_t. \end{aligned} \quad (4.3.5)$$

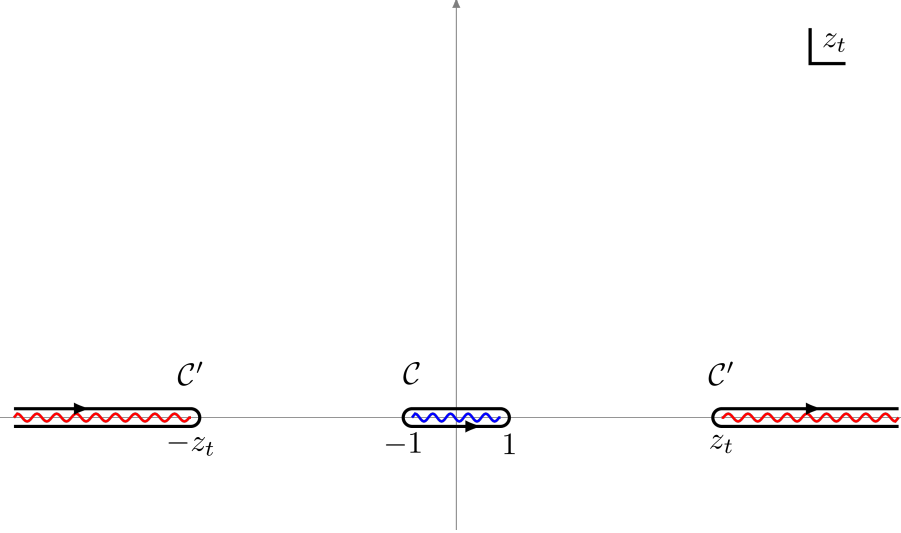


Figure 4.1: The contour $\mathcal{C}$ and $\mathcal{C}'$ in the complex z_t plane. The blue line represents the branch cut of $Q_l(z_t)$ and the red line represents the branch cut of $\mathcal{M}(s(z_t, t), t)$.

In the last equality, (A.5.14) is used. Note that $z_t \leftrightarrow -z_t$ corresponds to $s \leftrightarrow u$ exchange. The contour $\mathcal{C}'$ encircles the branch cut of $\mathcal{M}(s(z_t, t), t)$ in the complex z_t plane. The cut in the right corresponds to the s-channel on-shell exchange and the cut in the left corresponds to the u-channel on-shell exchange. The s-channel cut starts from $s = m_{\text{th}}$, where m_{th} is the mass of the lightest particle which is exchanged at the s-channel. In z_t plane, it corresponds to a point z_T defined as

$$z_T := 1 + \frac{2m_{\text{th}}^2}{t - 4m^2}. \quad (4.3.6)$$

The u-channel cut starts from $u = m_{\text{th}}$, and it corresponds to a point $-z_T$ in the z_t plane because

$$\begin{aligned} 1 + \frac{2(-t - m_{\text{th}}^2)}{t - 4m^2} &= -1 - \frac{2m_{\text{th}}^2}{t - 4m^2} \\ &= -z_T. \end{aligned} \quad (4.3.7)$$

D_s , D_u , which are discontinuities, are defined as

$$D_s(s, t) := \frac{\mathcal{M}(s(z_t, t) + i\varepsilon, t) - \mathcal{M}(s(z_t, t) - i\varepsilon, t)}{2i}, \quad (4.3.8)$$

$$\begin{aligned} D_u(u, t) &:= \frac{\mathcal{M}(s(z_t, t) - i\varepsilon, t) - \mathcal{M}(s(z_t, t) + i\varepsilon, t)}{2i} \\ &= \frac{\mathcal{M}(4m^2 - t - u(z_t, t) - i\varepsilon, t) - \mathcal{M}(4m^2 - t - u(z_t, t) + i\varepsilon, t)}{2i}. \end{aligned} \quad (4.3.9)$$

This expression is called the Froissart-Gribov projection [34, 39].

From (4.2.14), the large $|l|$ limit of Q_l is given by

$$Q_l(z) \underset{|l| \rightarrow \infty}{\sim} l^{-\frac{1}{2}} e^{-(l+\frac{1}{2})\zeta(z)}, \quad \zeta(z) := \log \left[z + \sqrt{z^2 - 1} \right]. \quad (4.3.10)$$

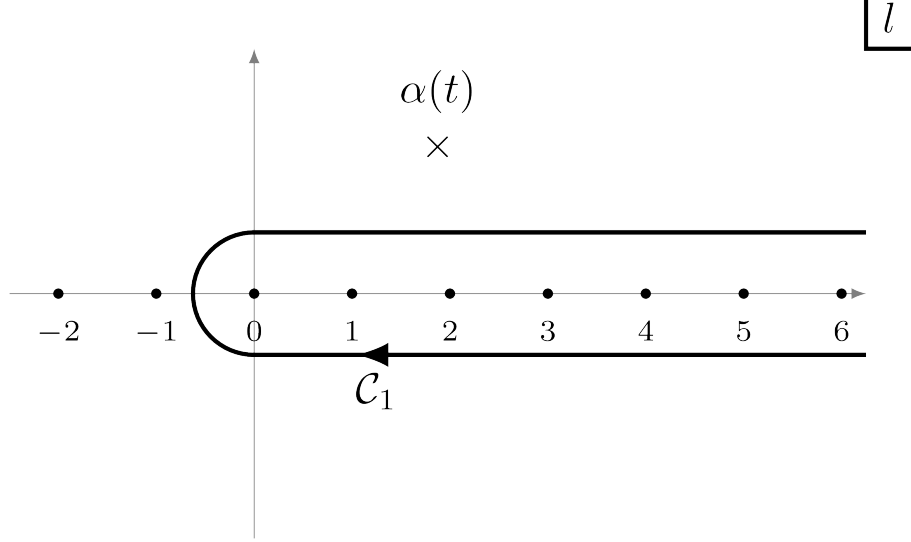


Figure 4.2: The contour $\mathcal{C}_1$ in the complex l plane. $\mathcal{C}_1$ embraces zero and positive integers, which are poles of $1/\sin \pi l$. $\alpha(t)$ is a pole of $a_l^{\mathcal{S}}(t)$.

D_s , D_u show polynomial growth. Therefore, the asymptotic behavior of the first term in (4.3.5), which corresponding to the s-channel cut, is given by

$$a_l^{(s)} \xrightarrow{l \rightarrow \infty} f(t) e^{-l\zeta(z_T)}. \quad (4.3.11)$$

However, the second term involves the factor and the asymptotic behavior is given by

$$a_l^{(u)} \xrightarrow{l \rightarrow \infty} f(t) e^{-l\zeta(z_T)} e^{-i\pi l}. \quad (4.3.12)$$

The factor $e^{-i\pi l}$ prohibits us from performing the contour deformation we are going to discuss.

Therefore, it is convenient to introduce the partial wave amplitude of "definite signature". Firstly, we define

$$D_s^{\mathcal{S}}(s, t) := D_s(s, t) + \mathcal{S} D_u(s, t), \quad (4.3.13)$$

where $\mathcal{S} = \pm 1$ is called the signature. Using $D_s^{\mathcal{S}}(s, t)$, the partial wave amplitude of definite signature is defined as

$$a_l^{\mathcal{S}}(t) := \frac{1}{16\pi^2} \int_{z_T}^{\infty} D_s^{\mathcal{S}}(s, t) Q_l(z_t) dz_t. \quad (4.3.14)$$

Comparing with (4.3.5), we find that

$$a_l(t) = \begin{cases} a_l^+(t) & (l : \text{even}) \\ a_l^-(t) & (l : \text{odd}) \end{cases} \quad (4.3.15)$$

The asymptotic behavior of (4.3.14) is given by

$$a_l^{\mathcal{S}} \xrightarrow{l \rightarrow \infty} f(t) e^{-l\zeta(z_T)}, \quad (4.3.16)$$

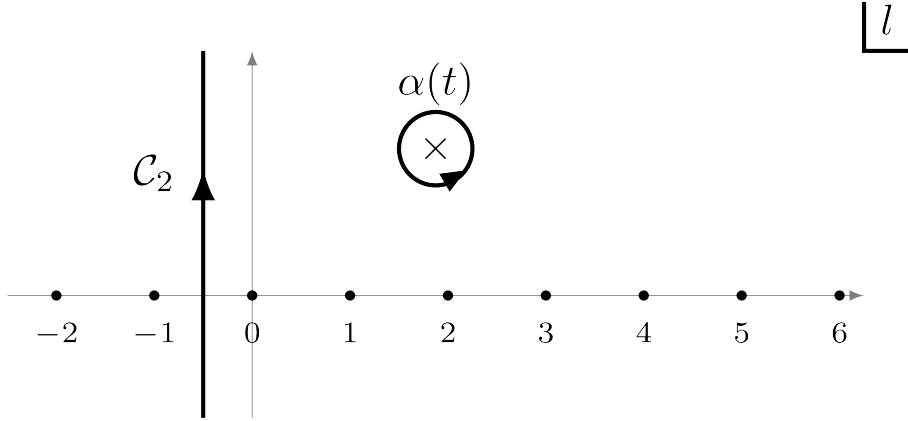


Figure 4.3: The contour of the Sommerfeld-Watson representation. C_2 is a line integral taken over a vertical line in the complex plane, whose real part is $-1/2$.

where no dangerous factor $e^{-i\pi l}$ appears.

Using $a_l^{\mathcal{S}}(t)$, we define the scattering amplitude of definite signature as

$$\mathcal{M}^{\mathcal{S}}(s, t) := 16\pi \sum_{l=0}^{\infty} (2l+1) a_l^{\mathcal{S}}(t) P_l(z_t). \quad (4.3.17)$$

Using $P_l(-z) = (-1)^l P_l(z)$ for the integer l , we can recover the original scattering amplitude as

$$\mathcal{M}(s, t) = \frac{1}{2} (\mathcal{M}^+(z_t, t) + \mathcal{M}^+(-z_t, t) + \mathcal{M}^-(z_t, t) - \mathcal{M}^-(-z_t, t)). \quad (4.3.18)$$

In the following, we perform analytic continuation of (4.3.17) in angular momentum. For simplicity, we assume that $a_l^{\mathcal{S}}(t)$ has the single pole $l = \alpha(t)$ in the complex l plane (See figure 4.2) with the residue $\beta(t)$. That is,

$$a_l^{\mathcal{S}}(t) \sim \frac{\beta(t)}{l - \alpha(t)}, \quad (4.3.19)$$

and analytic elsewhere. This pole is called the Regge pole.¹

The sum over l of (4.3.17) can be written as the following contour integral:

$$\mathcal{M}^{\mathcal{S}}(s, t) = -\frac{16\pi^2}{2\pi i} \int_{C_1} (2l+1) a_l^{\mathcal{S}}(t) \frac{P_l(-z_t)}{\sin \pi l} dl, \quad (4.3.20)$$

where the contour C_1 is shown in figure 4.2. It embraces zero and positive integers, which are the poles of $\frac{1}{\sin \pi l}$.

By displacing the contour as shown in figure 4.3, we have the following expression of $\mathcal{M}^{\mathcal{S}}(s, t)$ called the Sommerfeld-Watson representation:

$$\mathcal{M}^{\mathcal{S}}(s, t) = -\frac{16\pi^2}{2\pi i} \int_{C_2} (2l+1) a_l^{\mathcal{S}}(t) \frac{P_l(-z_t)}{\sin \pi l} dl - 16\pi^2 (2\alpha(t) + 1) \beta(t) \frac{P_{\alpha(t)}(-z_t)}{\sin \pi \alpha(t)}. \quad (4.3.21)$$

¹In general, $a_l^{\mathcal{S}}(t)$ has several poles in the complex l plane. The Regge pole that is furthest to the right in the complex plane is called the leading Regge pole, and the others are called sub-leading Regge poles.

The second term in (4.3.21) is the contribution of the Regge pole. Its contribution to the scattering amplitude of definite signature is

$$\mathcal{M}^{\mathcal{R}}(s, t) := -16\pi^2(2\alpha(t) + 1)\beta(t)\frac{P_{\alpha(t)}(-z_t)}{\sin \pi\alpha(t)}. \quad (4.3.22)$$

Firstly, let us consider the contribution of $\mathcal{M}^{\mathcal{R}}$ to the partial wave amplitude. Using (A.5.7) and (A.5.8), the contribution to the s-channel discontinuity takes the form

$$\begin{aligned} D_s^{\mathcal{R}}(s, t) &= \frac{\text{Disc } \mathcal{M}^{\mathcal{R}}(s, t)}{2i} \\ &= 16\pi^2(2\alpha(t) + 1)\beta(t)P_{\alpha(t)}(z_t) \quad z_t > 1. \end{aligned} \quad (4.3.23)$$

Substituting in (4.3.14), we have

$$\begin{aligned} a_l^{\mathcal{J}}(t) &= \frac{1}{16\pi^2} \int_1^\infty D_s^{\mathcal{J}}(s, t) Q_l(z_t) dz_t \\ &= \int_1^\infty (2\alpha(t) + 1)\beta(t)P_{\alpha(t)}(z_t) Q_l(z_t) dz_t \\ &= \frac{(2\alpha(t) + 1)\beta(t)}{(l - \alpha(t))(l + \alpha(t) + 1)}, \end{aligned} \quad (4.3.24)$$

where we use (A.5.15) in the last line. $a_l^{\mathcal{J}}(t)$ has the pole at $l = \alpha(t)$ with the residue $\beta(t)$ as expected.

Let us consider the meaning of $\alpha(t)$. We can separate $\alpha(t)$ into the real part $\alpha_R(t)$ and the imaginary part $\alpha_I(t)$. We define a point t_l such that $\alpha_R(t_l) = l$. Then, $\alpha(t)$ can be expanded about t_l as

$$\alpha(t) = l + \alpha'_R(t_l)(t - t_l) + i\alpha_I(t_l) + i\alpha'_I(t_l)(t - t_l) + \dots. \quad (4.3.25)$$

The partial wave is then approximated as

$$\begin{aligned} a_l^{\mathcal{J}}(t) &\approx \frac{\beta(t_l)}{-\alpha'_R(t_l)(t - t_l) - i\alpha_I(t_l) - i\alpha'_I(t_l)(t - t_l)} \\ &\approx \frac{\beta(t_l)/\alpha'_R(t_l)}{t - t_l - i\alpha_I(t_l)/\alpha'_R(t_l)}. \end{aligned} \quad (4.3.26)$$

In general, $\alpha_R(t)$ is an increasing function of t , and (4.3.26) implies that the Regge pole $\alpha(t)$ a tower of particles if $\alpha_I(t_l) = 0$ or resonances if $\alpha_I(t_l) \neq 0$ with increasing masses and spins.² The mass is given by $M_l = \sqrt{t_l}$ and the total width is given by

$$\Gamma = \frac{\alpha_I(t_l)}{\alpha'_R(t_l)M_r}. \quad (4.3.27)$$

²When there are several poles in the complex l plane, the leading Regge pole corresponds to the tower of the lightest particles for a given spin.

Next, let us consider the implication of the Regge pole to the high-energy behavior of the scattering amplitude. Firstly, let us consider the first term in (4.3.21), which is called the "background integral". The asymptotic behavior of the Legendre function is given by

$$P_l(z) \rightarrow \pi^{-\frac{1}{2}} \frac{\Gamma(l + \frac{1}{2})}{\Gamma(l + 1)} (2z)^l, \quad z \rightarrow \infty. \quad (4.3.28)$$

The contour of the background integral is taken along $\text{Re } l = -1/2$ so that the background integral is $\sim s^{-\frac{1}{2}}$ in the limit of $s \rightarrow \infty$, and vanishes.

Therefore, the high-energy behavior of the scattering amplitude is determined by the contribution of the Regge pole. From (4.3.18), the contribution of the Regge pole to the physical amplitude is given by

$$\mathcal{M}^R(s, t) = -16\pi^2(2\alpha(t) + 1)\beta(t) \frac{P_{\alpha(t)}(-z_t) + \mathcal{S}P_{\alpha(t)}(z_t)}{\sin \pi\alpha(t)}. \quad (4.3.29)$$

Using (A.5.13), it becomes

$$\begin{aligned} \mathcal{M}^R(s, t) = -16\pi^2(2\alpha(t) + 1)\beta(t) & \left[(1 + \mathcal{S}e^{-i\pi\alpha(t)}) \frac{P_{\alpha(t)}(-z_t)}{\sin \pi\alpha(t)} \right. \\ & \left. - \mathcal{S} \frac{2}{\pi} \sin \pi\alpha(t) Q_{\alpha(t)}(-z_t) \right]. \end{aligned} \quad (4.3.30)$$

The last term is asymptotically negligible in the limit of $s \rightarrow \infty$ because of (A.5.12) and we can write

$$\mathcal{M}^R(s, t) = -16\pi^2(2\alpha(t) + 1)\beta(t) (1 + \mathcal{S}e^{-i\pi\alpha(t)}) \frac{P_{\alpha(t)}(-z_t)}{\sin \pi\alpha(t)}. \quad (4.3.31)$$

From this expression, we find that there is the family of the physical particles and the resonances of (4.3.26) with even spins for $\mathcal{S} = +$ or odd spins for $\mathcal{S} = -$.

We can also obtain information for an asymptotic behavior of the scattering amplitude in the limit of $s \rightarrow \infty$, t : fixed using (4.3.31). The s -dependence of $\mathcal{M}^R(s, t)$ lie in the Legendre function $P_{\alpha(t)}(-z_t)$. Using (A.5.8), the large- s limit can be calculated as

$$\mathcal{M}^R(s, t) \underset{s \rightarrow \infty}{\approx} f(t)s^{\alpha(t)}, \quad (4.3.32)$$

where $f(t)$ is some function which is independent from s . This shows that the high-energy scattering in the limit of $s \rightarrow \infty$, t : fixed is given by the power behavior, and the power $\alpha(t)$ is a function of the momentum transfer.³

These two considerations lead us that there is a relation between the mass spectrum of a tower of particles which is transferred in the t -channel and the asymptotic behavior of the high-energy scattering through the Regge pole $\alpha(t)$.

³For multiple Regge poles, the scattering amplitude is dominated by the pole with the largest $\alpha(t)$, that is the leading Regge pole. This explains why the leading Regge pole is called "leading".

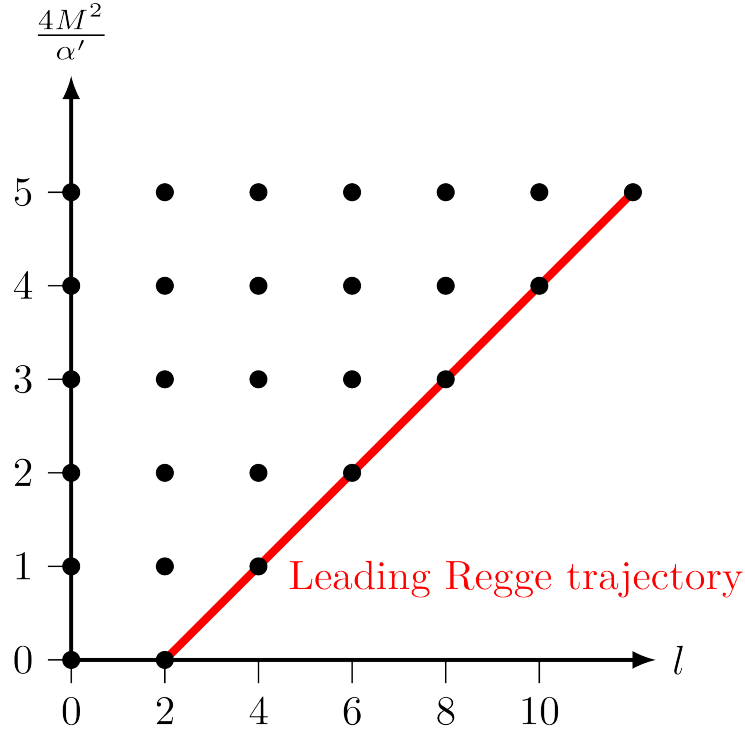


Figure 4.4: The particle spectrum and the leading Regge trajectory which contributes to the scattering of NS-NS bosons in the type-II superstring theory. The vertical axis is the mass squared in the unit of $\alpha'/4$, and the horizontal axis is the spin.

4.3.1 Example of the Regge pole: type-II superstring theory

Here, we review one concrete example of the Regge pole, that is, the Regge pole which appear in a scattering amplitude of the type-II superstring theory. The string theory is a consistent theory of quantum gravity and it is expected to be the UV completion of the gravity. The string theory is a theory of the extended object, the string, and the particle spectrum is determined by the vibrational modes of the string, which has the mass spectrum of towers of particles (See figure 4.4). The tower which is composed of particles of the lightest mass for a given spin is the leading Regge trajectory and it determines the leading high-energy behavior of the scattering amplitude.

The scattering amplitude for NS-NS bosons, which are massless bosons, in type-II superstring theory [16] is given by

$$\mathcal{M}(s, t) = -P(s, t) \frac{\Gamma(-\frac{\alpha' s}{4}) \Gamma(-\frac{\alpha' t}{4}) \Gamma(-\frac{\alpha' u}{4})}{\Gamma(1 + \frac{\alpha' s}{4}) \Gamma(1 + \frac{\alpha' t}{4}) \Gamma(1 + \frac{\alpha' u}{4})}, \quad (4.3.33)$$

with $s + t + u = 0$. The polarization factor $P(s, t)$ is define as

$$P(s, t) = A(s^2 u^2 + t^2 u^2 + s^2 t^2), \quad (4.3.34)$$

where $A > 0$ is a constant whose explicit expression is irrelevant in the following analysis.

Firstly, let us consider physical t-channel region and the mass spectrum of particles which are exchanged in this scattering. The mass of the t-channel exchanged particles corresponds to the location of the t-pole of the scattering amplitude. If the number of the exchanged particles are finite at the same mass level, the residue is the polynomial of s , whose degree gives the highest spin at that mass level. Explicitly, the scattering amplitude with the exchanged particle with mass M and spin l takes the form

$$\mathcal{M}(s, t) \underset{t \rightarrow M}{\sim} \frac{\#s^l}{M^2 - t} + \mathcal{O}(s^{l-1}). \quad (4.3.35)$$

Let us study the pole and the residue of (4.3.33). The t-channel poles come from the term $\Gamma(-\frac{\alpha't}{4})$ and locate at $t := t_n = \frac{4}{\alpha'}n$, $n = 0, 1, 2, \dots$. Using the formula of the Gamma function

$$\lim_{z \rightarrow -n} \Gamma(z) = \frac{(-1)^n n!}{z + n}, \quad (4.3.36)$$

the residue at $t = t_n$ can be calculated as

$$\begin{aligned} \text{Res}_{t=t_n} \mathcal{M}(s, t) &= -P(s, t_n) \frac{\Gamma(-\frac{\alpha's}{4})\Gamma(n + \frac{\alpha's}{4})}{\Gamma(1 + \frac{\alpha's}{4})\Gamma(-(n-1) - \frac{\alpha's}{4})} \frac{1}{\Gamma(n+1)} (-1)^{n+1} n! \frac{4}{\alpha'} \\ &= -P(s, t_n) \left(1 + \frac{\alpha's}{4}\right)_{n-1}^2 \frac{4}{\alpha'}, \end{aligned} \quad (4.3.37)$$

where $(a)_n = \Gamma(a+n)/\Gamma(a)$ is the Pochhammer symbol. Here, $P(s, t)$ is the polynomial of the degree 4 and $(1 + \frac{\alpha's}{4})_k$ is the polynomial of the degree k . This shows that the highest spin exchanged at the mass level $t = t_n$ is given by $l = 2n + 2$.

Rewriting $M_l^2 := t_n$, the mass-spin spectrum of the exchanged particle is given by $M_l^2 = 2(l-2)/\alpha'$, where the spin runs the even integer $l = 2, 4, 6, \dots$. This spectrum suggests that the string theory amplitude have the Regge pole with the form of

$$\alpha(t) = 2 + \frac{\alpha'}{2}t. \quad (4.3.38)$$

For example, the residue of the massless particle exchange $t_0 = 0$ is given by

$$\begin{aligned} \text{Res}_{t=0} \mathcal{M}(s, t) &= -P(s, 0) \left(1 + \frac{\alpha's}{4}\right)_{-1}^2 \frac{4}{\alpha'} \\ &= -\frac{64As^2}{\alpha'^3}, \end{aligned} \quad (4.3.39)$$

which shows that the exchanged particle is a massless spin-2. It corresponds to the graviton.

Then, let us see the high-energy behavior of (4.3.33). Using Euler's reflection formula

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x}, \quad (4.3.40)$$

and Stirling's formula

$$\Gamma(x) \approx \sqrt{\frac{2\pi}{x}} \left(\frac{x}{e}\right)^x, \quad (4.3.41)$$

the high-energy limit $s \rightarrow \infty$ with t : fixed of (4.3.33) is calculated as

$$\begin{aligned}
\mathcal{M}(s, t) &= -P(s, t) \frac{\Gamma(-\frac{\alpha's}{4})\Gamma(-\frac{\alpha't}{4})\Gamma(-\frac{\alpha'u}{4})}{\Gamma(1+\frac{\alpha's}{4})\Gamma(1+\frac{\alpha't}{4})\Gamma(1+\frac{\alpha'u}{4})} \\
&= -P(s, t) \frac{\Gamma(-\frac{\alpha't}{4})}{\Gamma(1+\frac{\alpha't}{4})} \frac{\sin(1+\frac{\alpha'u}{4})}{\sin(\frac{\alpha's}{4})} \times \frac{\Gamma(-\frac{\alpha's}{4})^2}{\Gamma(1+\frac{\alpha'u}{4})^2} \\
&\approx f(t) \left(\frac{\alpha's}{4}\right)^4 \frac{(\frac{\alpha's}{4})^{-\frac{\alpha's}{2}}}{(\frac{\alpha's}{4})^{2-\frac{\alpha'}{2}(s+t)}} \\
&= f(t) \left(\frac{\alpha's}{4}\right)^{2+\frac{\alpha't}{2}}, \tag{4.3.42}
\end{aligned}$$

where

$$f(t) = -A \left(\frac{4}{\alpha'}\right)^4 \frac{\Gamma(-\alpha't/4)}{\Gamma(1+\alpha't/4)} \tag{4.3.43}$$

This behavior precisely matches with the Regge analysis with a linear Regge trajectory (4.3.38).

The full mass spectrum and the leading Regge trajectory considered here is shown in figure 4.4.

Chapter 5

Gravitational positivity bounds

In this chapter, we summarize the gravitational positivity bounds of [16]. We outline the basic assumptions, definitions of quantities to be computed, bounds, and their interpretations. We begin with the non-gravitational case and then include gravity. We will close with some remarks needed for applications to realistic phenomenological model building.

5.1 Positivity bounds without gravity

5.1.1 Setup and assumptions

Consider an scattering of $AB \rightarrow AB$ type in a given low-energy EFT, where scattering amplitude is $\mathcal{M}_{\text{IR}}(s, t)$. s, t, u are the standard Mandelstam variables that satisfy $s + t + u = 2(m_A^2 + m_B^2)$, and m_A and m_B are the masses of the external particles A and B , respectively.

We assume that the low-energy EFT are defined with the UV cutoff Λ_{UV} . Therefore, the scattering amplitude $\mathcal{M}_{\text{IR}}(s, t)$ approximate well the scattering amplitude $\mathcal{M}_{\text{UV}}(s, t)$, which is the scattering amplitude of $AB \rightarrow AB$ evaluated in the would-be UV complete theory, in the region $s, t < \Lambda_{\text{UV}}$.

We will use $\mathcal{M}(s, t)$ as an abbreviation of $\mathcal{M}_{\text{UV}}(s, t)$ for the notational simplicity. We assume that the scattering amplitude $\mathcal{M}(s, t)$ satisfies the following properties:

1. Analyticity: The amplitude $\mathcal{M}(s, t)$ is analytic on the physical sheet¹ of the complex s -plane except for poles and discontinuities on the real axis required by unitarity. The analyticity of the scattering amplitude is known to be closely connected to the causality of the theory. While the derivation of analyticity has been discussed in the context of axiomatic QFT [40], we simply assume it in this work.
2. Unitarity: The imaginary part of the forward amplitude $\mathcal{M}(s, t)$ is non-negative, i.e., $\text{Im } \mathcal{M}(s, t = 0) \geq 0$.

¹There are in general unstable resonances, whose associated poles are however located in other sheets in the complex s -plane.

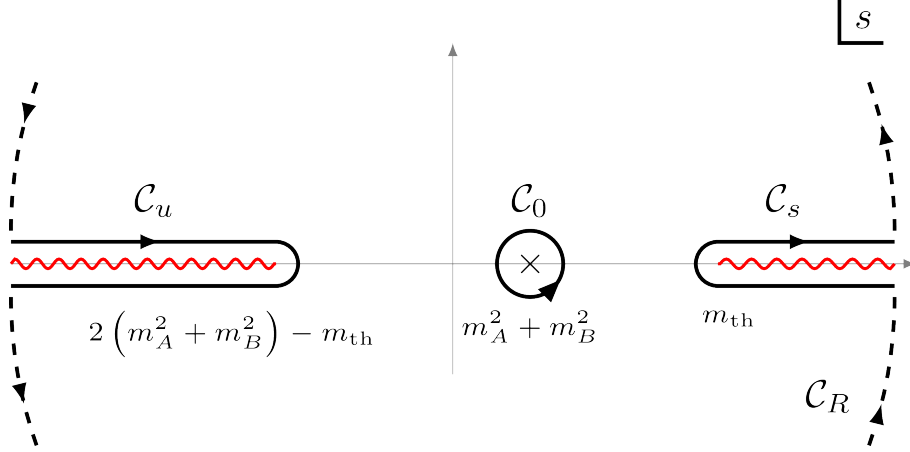


Figure 5.1: The contour $\mathcal{C}_0$, $\mathcal{C}_s$, $\mathcal{C}_u$, $\mathcal{C}_R$ in the complex s -plane.

3. s^2 -boundedness: The amplitude is bounded by s^2 at high energy, i.e., $\lim_{|s| \rightarrow \infty} \mathcal{M}(s, 0)/s^2 = 0$. This third condition is guaranteed in local quantum field theories thanks to the Froissart bound (4.2.39) and the Phragmén-Lindelöf theorem.
4. Crossing symmetry: The amplitude satisfies s - u crossing symmetry:

$$\mathcal{M}(s, t) = \mathcal{M}(u = 2(m_A^2 + m_B^2) - s - t, t). \quad (5.1.1)$$

Positivity bounds. Using these four properties, we can derive an infinite set of consistency relations among scattering amplitudes evaluated at the UV and IR. Let us define

$$a_{2n} := \left[\frac{\partial^{2n} \mathcal{M}(s, t=0)}{\partial s^{2n}} \right]_{s=m_A^2+m_B^2} \quad (n = 1, 2, \dots). \quad (5.1.2)$$

a_{2n} are IR object in the sense that they are evaluated at $s, t < \Lambda_{\text{UV}}$ and we can approximate these quantities using the IR scattering amplitude $\mathcal{M}_{\text{IR}}(s, t)$:

$$a_{2n} \approx \left[\frac{\partial^{2n} \mathcal{M}_{\text{IR}}(s, t=0)}{\partial s^{2n}} \right]_{s=m_A^2+m_B^2}. \quad (5.1.3)$$

A higher derivative term $\mathcal{O}_{2n}$ of the low-energy EFT, which has the mass dimension $2n$, would contribute to a_{2n} .

The analyticity of $\mathcal{M}(s, t)$ allows us to use the Cauchy's integral expression and the defining equation (5.1.2) can be evaluated as the following contour integral (see figure 5.1)

$$\begin{aligned} a_{2n} &= \left[\frac{\partial^{2n} \mathcal{M}(s, t=0)}{\partial s^{2n}} \right]_{s=m_A^2+m_B^2} \\ &= (2n)! \oint_{\mathcal{C}_0} \frac{ds'}{2\pi i} \frac{\mathcal{M}(s', 0)}{(s' - m_A^2 - m_B^2)^{2n+1}} \\ &=: (2n)! \oint_{\mathcal{C}_0} \frac{ds'}{2\pi i} \mathcal{R}_{2n}(s'), \end{aligned} \quad (5.1.4)$$

where we define $\mathcal{R}_{2n}(s)$ as an integrand. The contour $\mathcal{C}_0$ is taken around the point $s = m_A^2 + m_B^2$ in the counterclockwise direction. $\mathcal{R}_{2n}(s)$ have the following properties as a complex function on the s -plane:

1. $\mathcal{R}_{2n}(s)$ has a pole at $m_A^2 + m_B^2$ due to the denominator $(s - m_A^2 - m_B^2)^{2n+1}$.
2. $\mathcal{R}_{2n}(s)$ has poles and branch cuts, that corresponding to the intermediate one particle states and multi particle states respectively. The singularities corresponding to the s -channel intermediate states begin at $s = m_{\text{th}}$, while these corresponding to the u -channel start at $s = 2(m_A^2 + m_B^2) - m_{\text{th}}$, or equivalently, at $u = m_{\text{th}}$.
3. $\mathcal{R}_{2n}(s)$ is analytic other region in the complex s -plane.
4. $\mathcal{R}_{2n}(s)$ is bounded by s^{-2n+1} in large $|s|$ limit:

$$\lim_{|s| \rightarrow \infty} \frac{\mathcal{R}_{2n}(s)}{s^{-2n+1}} = 0. \quad (5.1.5)$$

The property 2 and 3 allows us to perform the contour deformation:

$$a_{2n} = (2n)! \left(\oint_{\mathcal{C}_s} \frac{ds'}{2\pi i} + \oint_{\mathcal{C}_u} \frac{ds'}{2\pi i} + \oint_{\mathcal{C}_R} \frac{ds'}{2\pi i} \right) \mathcal{R}_{2n}(s'), \quad (5.1.6)$$

where $\mathcal{C}_s$ is taken around the s -channel cuts, $\mathcal{C}_u$ is taken around the u -channel cuts, and $\mathcal{C}_R$ is a circle at infinity. Each contour integral can be evaluated as

$$(2n)! \oint_{\mathcal{C}_s} \frac{ds'}{2\pi i} \mathcal{R}_{2n}(s') = \frac{(2n)!}{2\pi i} \int_{m_{\text{th}}^2}^{\infty} ds' \frac{\text{Disc } \mathcal{M}(s', 0)}{(s' - m_A^2 - m_B^2)^{2n+1}}, \quad (5.1.7)$$

$$\begin{aligned} (2n)! \oint_{\mathcal{C}_u} \frac{ds'}{2\pi i} \mathcal{R}_{2n}(s') &= \frac{(2n)!}{2\pi i} \int_{-\infty}^{-m_{\text{th}}^2 + 2(m_A^2 + m_B^2)} ds' \frac{\text{Disc } \mathcal{M}(s', 0)}{(s' - m_A^2 - m_B^2)^{2n+1}} \\ &= \frac{(2n)!}{2\pi i} \int_{m_{\text{th}}^2}^{\infty} du' \frac{\text{Disc } \mathcal{M}(u', 0)}{(u' - m_A^2 - m_B^2)^{2n+1}}, \end{aligned} \quad (5.1.8)$$

where we used the crossing symmetry in the second equality.

$$(2n)! \oint_{\mathcal{C}_R} \frac{ds'}{2\pi i} \mathcal{R}_{2n}(s') = 0, \quad (5.1.9)$$

because the integrand is bounded by s^{-2n+1} in the large $|s|$ limit.

Therefore, the following dispersion relation holds from the analyticity and s^2 -boundedness of the amplitude:

$$a_{2n} = \frac{2 \cdot (2n)!}{\pi} \int_{m_{\text{th}}^2}^{\infty} ds \frac{\text{Im } \mathcal{M}(s, 0)}{(s - m_A^2 - m_B^2)^{2n+1}} \quad (n = 1, 2, \dots), \quad (5.1.10)$$

where $\text{Disc } \mathcal{M}(s, t) = 2i \text{Im } \mathcal{M}(s, t)$ is used. This equation relates the IR quantity a_{2n} , which is calculated in the small energy region, and the UV integral whose properties are constrained by basic principles such as unitarity. Especially, the integrand of (5.1.10) is positive due to the unitarity. As a result, we have inequalities

$$a_{2n} \geq 0, \quad (5.1.11)$$

which are called positivity bounds.

Improved positivity bounds. The EFT is defined with a UV cutoff which we denote by Λ_{UV} . The actual cutoff of the EFT, namely the scale of new physics, is unknown from the low-energy perspective. We thus introduce a reference scale Λ below which the EFT is assumed to be valid, i.e., Λ is assumed to satisfy $\Lambda < \Lambda_{\text{UV}}$.

Let us define [4, 6, 41]

$$B_{2n}(\Lambda) := a_{2n} - \frac{2 \cdot (2n)!}{\pi} \int_{m_{\text{th}}^2}^{\Lambda^2} ds \frac{\text{Im } \mathcal{M}(s, t=0)}{(s - m_A^2 - m_B^2)^{2n+1}} \quad (n = 1, 2, \dots). \quad (5.1.12)$$

We emphasize that $B_{2n}(\Lambda)$ is defined in terms of the amplitude at the IR scale below Λ , and therefore it is well approximated within the EFT:

$$B_{2n}(\Lambda) \approx \left[\frac{\partial^{2n} \mathcal{M}_{\text{IR}}(s, t=0)}{\partial s^{2n}} \right]_{s=m_A^2+m_B^2} - \frac{2 \cdot (2n)!}{\pi} \int_{m_{\text{th}}^2}^{\Lambda^2} ds \frac{\text{Im } \mathcal{M}_{\text{IR}}(s, t=0)}{(s - m_A^2 - m_B^2)^{2n+1}}. \quad (5.1.13)$$

Then, the dispersion relation (5.1.10) implies

$$B_{2n}(\Lambda) = \frac{2 \cdot (2n)!}{\pi} \int_{\Lambda^2}^{\infty} ds \frac{\text{Im } \mathcal{M}(s, t=0)}{(s - m_A^2 - m_B^2)^{2n+1}} \quad (n = 1, 2, \dots), \quad (5.1.14)$$

which provides a consistency relation between the IR physics below Λ (LHS) and the UV physics above Λ (RHS). Furthermore, unitarity implies that the RHS is non-negative, so that

$$B_{2n}(\Lambda) \geq 0 \quad \text{for all } \Lambda \text{ such that } \Lambda < \Lambda_{\text{UV}} \quad (n = 1, 2, \dots). \quad (5.1.15)$$

These bounds are called the improved positivity bounds. Note that $B_{2n}(\Lambda)$ is a monotonically non-increasing function of Λ because the integrand in (5.1.12) is positive. The typical behavior is shown in Fig. 5.2. When the bounds (5.1.15) are satisfied at a certain scale, they also hold at any scale below it. The positivity bound (5.1.15) is thus stronger for larger Λ .

Interpretations. In a given EFT, one can compute the forward amplitude and evaluate $B_{2n}(\Lambda)$ defined in Eq. (5.1.12) in terms of parameters of the EFT and a scale Λ . The positivity bounds (5.1.15) then provide certain inequalities among the EFT couplings and Λ , leading to two complementary interpretations:

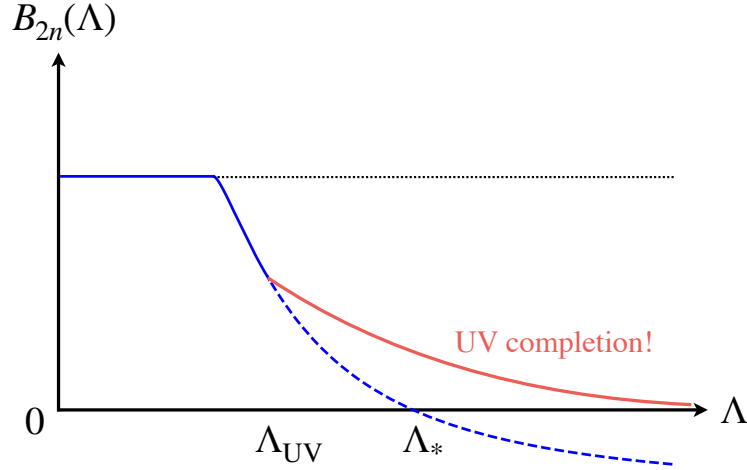


Figure 5.2: The expression (5.1.12) of $B_{2n}(\Lambda)$ is a monotonically non-increasing function as a function of Λ , and crosses zero at the scale $\Lambda = \Lambda_*$. For a well-defined UV completion, we need $B_{2n}(\Lambda) \geq 0$, which defines an upper bound Λ_* on the scale Λ .

1. Bounds on EFT couplings.

Suppose that the low-energy EFT describes a system below an energy scale E . We can then identify Λ with the energy scale of interest E to find the bounds on the EFT couplings. These are the necessary conditions for the EFT to have a standard UV completion and be valid below the scale $E = \Lambda$. (In here we are agnostic on the value of the cutoff scale Λ_{UV} .)

2. Bounds on the UV cutoff.

In some cases, one may want to assume certain values of the EFT couplings—for example, coupling constants are already fixed by experiments or they need to take certain values for phenomenological purposes (e.g., so that they can be searched in a particular experiment). One can then ask up to which scale the EFT can be valid, i.e., at which scale the new physics comes in. As shown in Fig. 5.2, $B_{2n}(\Lambda)$ may become negative at some energy scale Λ_* , in which case we can derive an upper bound $\Lambda_{UV} < \Lambda_*$ on the UV cutoff of the EFT.

The condition (5.1.15) is based on the assumptions on UV, but care must be taken to ensure because we do not know *the* UV theory. If a violation of the inequality (5.1.15) is observed in the experiment, it may indicate a violation of the original assumptions. In this sense, we can also use the positivity bounds (5.1.15) to probe UV from IR.

5.2 Positivity bounds with gravity

We next explain how the positivity bounds are extended to theories with gravity. The main new ingredient is that the t -channel graviton exchange contributes a new singularity to the low-energy scattering amplitudes as $\sim -s^2/(M_{Pl}^2 t)$ ($M_{Pl} \simeq 2.4 \times 10^{18}$ GeV being the reduced Planck

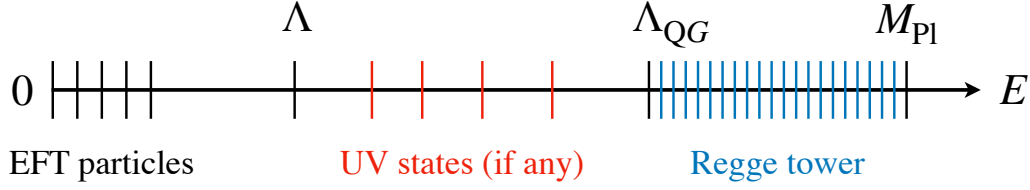


Figure 5.3: Typical scenario: a UV completion of gravity is achieved well below the Planck scale by an infinite tower of higher-spin states (Regge tower). There may also exist new UV states beyond the EFT that are described within QFT model building and that are not directly related to the UV completion of gravity.

mass), invalidating the definition (5.1.2) of a_2 and hence the positivity argument for $B_2(\Lambda)$.² In this subsection, we summarize how the positivity bounds are formulated in the presence of gravity.

Assumptions. The basic assumptions are the same as before, namely analyticity, unitarity, and the s^2 -boundedness $\lim_{|s| \rightarrow \infty} |\mathcal{M}(s, t)/s^2| = 0$ for small negative t .

While the argument below can be applied to any UV complete theory with these three properties, a motivating candidate of such a UV completion is a perturbative string theory discussed in 4.3.1. A typical particle spectrum of such theory are shown in Fig. 5.3.

Gravitational positivity bounds. We define a_2 by subtracting the graviton t -pole from the amplitude and taking the forward limit as³

$$a_2 := \lim_{t \rightarrow -0} \left[\frac{\partial^2 \mathcal{M}(s, t)}{\partial s^2} + \frac{2}{M_{\text{Pl}}^2 t} \right]_{s=m_A^2+m_B^2}. \quad (5.2.1)$$

With a_2 defined in this manner, we introduce

$$B(\Lambda) := a_2 - \frac{4}{\pi} \int_{m_{\text{th}}^2}^{\Lambda^2} ds \frac{\text{Im } \mathcal{M}(s, t=0)}{(s - m_A^2 - m_B^2)^3}. \quad (5.2.2)$$

Here and in what follows we suppress the subscript 2 of $B_2(\Lambda)$ for notational simplicity. The dispersion relation (5.1.14) is then modified as

$$B(\Lambda) = \lim_{t \rightarrow -0} \left[\frac{2}{M_{\text{Pl}}^2 t} + \frac{4}{\pi} \int_{\Lambda^2}^{\infty} ds \frac{\text{Im } \mathcal{M}(s, t)}{(s + \frac{t}{2} - m_A^2 - m_B^2)^3} \right]. \quad (5.2.3)$$

Due to the first term in the square bracket (which is negative for $t < 0$), the sign of $B(\Lambda)$ is undetermined. However, one can show that the positivity of $B(\Lambda)$ holds at least approximately.

²Note that the positivity argument for B_{2n} ($n = 2, 3, \dots$) is unchanged. In the following, we focus on B_2 .

³For simplicity, we focus on the amplitude up to $\mathcal{O}(M_{\text{Pl}}^{-2})$ and only subtract the graviton t -pole as a divergent term in the forward limit $t \rightarrow -0$ in the definition of a_2 . See [42, 43] for arguments with the $\mathcal{O}(M_{\text{Pl}}^{-4})$ terms.

The idea is that we split the integral of the dispersion relation into the quantum gravity part and the other,

$$B(\Lambda) = \frac{4}{\pi} \int_{\Lambda^2}^{\Lambda_{\text{QG}}^2} ds \frac{\text{Im } \mathcal{M}(s, 0)}{(s - m_A^2 - m_B^2)^3} + \lim_{t \rightarrow -0} \left[\frac{2}{M_{\text{Pl}}^2 t} + \frac{4}{\pi} \int_{\Lambda_{\text{QG}}^2}^{\infty} ds \frac{\text{Im } \mathcal{M}(s, t)}{(s + \frac{t}{2} - m_A^2 - m_B^2)^3} \right]. \quad (5.2.4)$$

When we take the forward limit $t \rightarrow -0$, the l.h.s of (5.2.4) is finite because it is defined by subtracting the t -pole term. Then, although the t -pole term in the r.h.s diverges in the forward limit $t \rightarrow -0$, the sum in the square bracket should be finite as a result of a cancellation with the quantum gravity part:

$$\lim_{t \rightarrow -0} \left[\frac{2}{M_{\text{Pl}}^2 t} + \frac{4}{\pi} \int_{\Lambda_{\text{QG}}^2}^{\infty} ds \frac{\text{Im } \mathcal{M}(s, t)}{(s + \frac{t}{2} - m_A^2 - m_B^2)^3} \right] =: \frac{\sigma}{M_{\text{Pl}}^2 M^2} \quad (\sigma = 0, \pm 1), \quad (5.2.5)$$

where we defined the sign σ and the mass scale M , which depend on details of the cancellation mechanism. (The case $\sigma = 0$ corresponds to the exact cancellation.) As a result, we find

$$B(\Lambda) \geq \frac{\sigma}{M_{\text{Pl}}^2 M^2} \quad \text{for all } \Lambda \text{ s.t. } \Lambda < \Lambda_{\text{UV}} < \Lambda_{\text{QG}}, \quad (5.2.6)$$

which we call the *gravitational positivity bound*.

The right-hand side of (5.2.6) is suppressed by powers of M_{Pl} , and hence disappears in the decoupling limit $M_{\text{Pl}} \rightarrow \infty$, where the gravitational positivity bound reduces to the positivity bound without gravity.

While the derivation here works irrespective of the details of the cancellation mechanism, the most plausible scenario is the Regge behavior discussed in 4.3⁴: the gravitational amplitude is modified as $\text{Im } \mathcal{M} \sim s^{2+\alpha' t+\dots}$ with $\alpha' \sim \Lambda_{\text{QG}}^{-2}$ by an infinite tower of higher-spin states at $s \geq \Lambda_{\text{QG}}^2$ (see Fig. 5.3). The perturbative string theory is a typical example of this scenario in which the quantum gravity scale Λ_{QG} corresponds to the string scale. The cancellation of the t -pole can be explicitly shown under the assumption of Regge behavior [16].

Interpretations. In contrast to the non-gravitational positivity bound (5.1.15), the gravitational positivity bound (5.2.6) contains not only the IR data $B(\Lambda)$ determined by the EFT parameters up to the scale Λ , but also the UV data (M, σ) of gravitational Regge amplitudes (or, generically speaking, quantum gravity amplitudes). Therefore, there are three possible interpretations, depending on the context [19, 46–49]:

1. Quantum gravity constraints on IR physics.

If we specify/assume a quantum gravity scenario and the parameters (σ, M) of gravitational Regge amplitudes, we can think of the bounds (5.2.6) as quantum gravity constraints

⁴See also [44, 45] for connections between Reggeization and causality.

on the low-energy EFT at a scale $\Lambda = E$. Such constraints are useful to carve out the parameter space of phenomenological models in the spirit of the Swampland Program⁵.

2. Constraints on the scale of new physics.

As in the non-gravitational case, we can interpret (5.2.6) as the bounds giving the maximum cutoff Λ_* when we assume the parameters of the EFT and the UV data (σ, M) . When the bounds (5.2.6) are violated, one can try to increase the cutoff of the EFT by adding new state(s) to EFT like Fig. 5.2. If the bounds (5.2.6) are inevitably violated at $\Lambda > \Lambda_*$, this is a sign of the necessity for a UV completion of gravity. In this case, Λ_* is not a maximum scale of just new physics, but the upper bound on the quantum gravity scale Λ_{QG} .

3. IR constraints on quantum gravity.

Once the parameters and the validity of the model are identified by experiments at a scale $\Lambda = E$, we can use the bounds (5.2.6) to constrain the parameters (σ, M) of the gravitational Regge amplitudes required for UV completion of gravity. Such constraints are useful as a necessary condition for a quantum gravity theory to describe our real world.

As we mentioned in Sec. 5.1, the violation of original assumptions on UV theory leads to the violation of (5.2.6). Differently from the non-gravitational case, however, experimental tests of the violation of (5.2.6) require knowledge of the UV data (M, σ) .

Comments on (M, σ) . As we explain more explicitly later, nontrivial constraints on the IR physics are obtained in the spirit of Interpretation 1 when $B_{\text{grav}} - \sigma/(M_{\text{Pl}}^2 M^2) < 0$. Here, B_{grav} is the gravitational part of B which will be defined in the next section. This condition is satisfied either when $\sigma = 0, +1$ or when M is sufficiently large, typically $M \gg m_{\text{light}}$ with m_{light} being the mass scale of light particles. This shows that details of the UV data (M, σ) are important in discussing phenomenological implications of the gravitational positivity [19].

Notably, recent studies proceed in the direction to carve out the parameter space of (M, σ) itself and sharpen the gravitational positivity bound (5.2.6). For scattering of identical scalars, the best bound obtained so far is schematically of the form [51, 52]⁶ (see also [53, 54]),

$$B(\Lambda) \geq \frac{-\mathcal{O}(1)}{M_{\text{Pl}}^2 m^2}, \quad (5.2.7)$$

⁵See [14, 15, 47, 50] for possible connections between positivity and the Swampland conjecture.

⁶There are two complementary approaches to this problem depending on how to treat the graviton t -channel pole. First, the one initiated in [51] works in finite impact parameter to make the graviton contribution finite. While it gives a universal bound that does not rely on details of UV completion, the obtained bound is trivialized in spacetime four dimensions due to a logarithmic IR divergence. On the other hand, the one proposed in [52] uses the sum rule of gravitational Regge amplitudes to give a bound on the parameter space (M, σ) . While this approach assumes Reggeization of gravitational amplitudes, it is applicable even in spacetime four dimensions.

where m denotes the threshold energy, i.e., the energy of the lightest intermediate state. For tree-level amplitudes, this scale is essentially the mass of the lightest higher-spin particle (spin 2 or higher). For loop amplitudes, this corresponds to the lowest energy of the intermediate multi-particle states. Note that this bound is a necessary condition for the effective theory to have a standard UV completion, rather than a sufficient condition. Therefore, it is still a nontrivial question if there exists a consistent quantum gravity theory that accommodates a negative $B(\Lambda)$,⁷ more specifically that with $\sigma = -1$ and $M \sim m_{\text{light}}$. Further studies in this direction are encouraged for phenomenological applications of the gravitational positivity.

5.2.1 Recipe for gravitational positivity

We are now ready to discuss procedures to derive gravitational positivity bounds for EFTs. The gravitational positivity bound is an inequality for the quantity $B(\Lambda)$. When we wish to apply the gravitational positivity bound, we run into one practical problem which is already mentioned: we can almost never compute an exact scattering amplitude $\mathcal{M}(s, t)$ in an EFT. In fact, if we know the exact expression for $\mathcal{M}(s, t)$, we can analytically continue the expression to the whole complex plane, and we have effectively solved for the quantum gravity already. The best we can do is to compute an approximate expression $\mathcal{M}_{\text{IR}}(s, t)$ for $\mathcal{M}(s, t)$, by including only a finite number of parameters out of an infinite number of higher-dimensional operators. The differences between the two are small in the regions of the validity of EFT (at scales $\Lambda \ll \Lambda_{\text{UV}}$). Hence, the computation of the *approximate amplitude* $\mathcal{M}_{\text{IR}}$ is sufficient for evaluating the quantity $B(\Lambda)$ which only requires information below Λ . In the following, we will discuss EFT scattering amplitude $\mathcal{M}_{\text{IR}}$ and explain a practical way to calculate $B(\Lambda)$ from the EFT amplitude $\mathcal{M}_{\text{IR}}$: for notational simplicity we will simply denote this approximate amplitude as $\mathcal{M}$ in the rest of this thesis.

We provide a step-by-step procedure for deriving the gravitational positivity bound. The basic task is simple: we need to evaluate $B(\Lambda)$:

$$B(\Lambda) := a_2 - \frac{4}{\pi} \int_{m_{\text{th}}^2}^{\Lambda^2} ds \frac{\text{Im } \mathcal{M}(s, t=0)}{(s - m_A^2 - m_B^2)^3}, \quad (5.2.8)$$

where m_{th} is the threshold energy, i.e., the mass of the lightest intermediate state.

Step 1: Calculate $\mathcal{M}(s, t)$

We compute the $2 \rightarrow 2$ scattering amplitude $\mathcal{M}(s, t)$ up to order $\mathcal{O}(M_{\text{Pl}}^{-2})$:

$$\mathcal{M}(s, t) = \underbrace{\mathcal{M}_{\text{non-grav}}(s, t)}_{\mathcal{O}(M_{\text{Pl}}^0)} + \underbrace{\mathcal{M}_{\text{grav}}(s, t)}_{\mathcal{O}(M_{\text{Pl}}^{-2})} + \underbrace{\cdots}_{\mathcal{O}(M_{\text{Pl}}^{-4})}. \quad (5.2.9)$$

⁷To our best knowledge, there is no known tree-level string amplitude that accommodates $\sigma = -1$, even though $\sigma = -1$ is not prohibited by the consistency of scalar scattering alone. It would be interesting to clarify in what class of quantum gravity theories (if exist) $\sigma = -1$ can be realized.

Here $\mathcal{M}_{\text{non-grav}}(s, t)$ is the contribution from diagrams with no graviton exchange, while $\mathcal{M}_{\text{grav}}(s, t)$ is the contribution from diagrams with one graviton exchange and is of the order $\mathcal{O}(M_{\text{Pl}}^{-2})$. The remaining terms are the order of $\mathcal{O}(M_{\text{Pl}}^{-4})$ and are neglected in the rest of the analysis.

Step 2: Calculate $B_{\text{non-grav}}(\Lambda)$

We compute the non-gravitational contribution $B_{\text{non-grav}}(\Lambda)$ to $B(\Lambda)$:

$$B_{\text{non-grav}}(\Lambda) := a_{2,\text{non-grav}} - \frac{4}{\pi} \int_{m_{\text{th}}^2}^{\Lambda^2} ds \frac{\text{Im } \mathcal{M}_{\text{non-grav}}(s, t=0)}{(s - m_A^2 - m_B^2)^3}, \quad (5.2.10)$$

In the following chapters, we consider cases where theories of our interest is UV complete and therefore renormalizable in the gravity decoupling limit $M_{\text{Pl}} \rightarrow \infty$. In such cases, $\mathcal{M}_{\text{non-grav}}$, which is $\mathcal{O}(M_{\text{Pl}}^0)$ part of the amplitude, satisfies the analyticity and the s^2 -boundedness when the expression is analytically continued throughout the complex plane, even outside the regions of validity of the EFT. The non-gravitational part then satisfies the twice-subtracted dispersion relation, giving a formula

$$B_{\text{non-grav}}(\Lambda) \simeq \frac{4}{\pi} \int_{\Lambda^2}^{\infty} ds \frac{\text{Im } \mathcal{M}_{\text{non-grav}}(s, t=0)}{(s - m_A^2 - m_B^2)^3}. \quad (5.2.11)$$

Therefore, $B_{\text{non-grav}}(\Lambda)$ can be represented using $\mathcal{M}_{\text{non-grav}}(s, t=0)$ at large s region although it is defined in low-energy regime. This formula is practically useful since we only need to compute the imaginary part of the forward amplitude.

Step 3: Calculate $B_{\text{grav}}(\Lambda)$

We next compute the gravitational contribution $B_{\text{grav}}(\Lambda)$ to $B(\Lambda)$. While this contains many different Feynman diagrams in general, we will argue in Appendix B that it is enough to consider the contribution from t -channel graviton exchange diagrams. In this case, the integral part of (5.2.2) is absent for this contribution because $\mathcal{M}_{\text{grav},t\text{-channel}}(s, t)$ does not have the s -channel or u -channel discontinuity at the non-gravitational one-loop order. Therefore the gravitational part of $B(\Lambda)$ is simplified to a_2 defined in (5.2.1), evaluated for the gravity t -channel amplitude:

$$B_{\text{grav}}(\Lambda) \simeq a_{2,\text{grav},t\text{-channel}} := \lim_{t \rightarrow -0} \left[\frac{\partial^2 \mathcal{M}_{\text{grav},t\text{-channel}}(s, t)}{\partial s^2} + \frac{2}{M_{\text{Pl}}^2 t} \right]_{s=m_A^2+m_B^2}. \quad (5.2.12)$$

Step 4: Write down the inequality for total $B(\Lambda)$

The constraints on EFT are obtained as the inequality (5.2.6) for the combined expression $B(\Lambda) = B_{\text{non-grav}}(\Lambda) + B_{\text{grav}}(\Lambda)$. There are three interpretations of this inequality, as we discussed in Sec. 5.2.

The explicit calculation shows that the gravitational contribution $B_{\text{grav}}(\Lambda)$ is negative. Consequently, (5.2.6) with $\sigma = 0$ becomes

$$B(\Lambda) = B_{\text{non-grav}}(\Lambda) - |B_{\text{grav}}(\Lambda)| > 0. \quad (5.2.13)$$

This implies that, when interpreted as quantum gravity constraints on the EFT parameter space under the assumption $\sigma = 0$, the gravitational positivity bounds require the non-gravitational contribution to exceed the gravitational one.

Let us conclude this chapter by reviewing previous research. The concept of the positivity bound was first introduced in [3], where it was argued that parameters of EFTs are tied to the causality of the theory, and how to use analyticity and unitarity to derive constraints on these parameters. It was further proposed that applying positivity bounds to theories that include gravity provides non-trivial implications for the quantum nature of gravity [14, 15]. In these works, the authors considered photon-photon scattering in a EFT that includes only massless particles: photons and gravitons. The operators relevant to scattering are determined by the lightest charged particle with mass m and charge e in the spectrum of the full theory, which is integrated out to construct an effective theory involving photons and gravitons. The positivity bounds impose constraints on these operators, provided the t-channel pole is subtracted. This analysis leads to the bound:

$$\frac{m}{M_{\text{Pl}}} \lesssim e. \quad (5.2.14)$$

Interestingly, this bound, known as the weak gravity conjecture, can also be derived from other considerations, such as black hole physics and string theory. It is widely regarded as a nontrivial constraints arising from the consistency of quantum gravity.

As reviewed in this section, the authors of [16] argue that Regge behavior is a plausible mechanism to subtract the t-channel pole. In [47], this technique is applied to photon-photon scattering within quantum electrodynamics (QED) coupled to gravity, where the charged matter is explicitly included compared to the setup of [14, 15]. Because QED is UV complete (except for the Landau pole issue), the improved version of positivity bound can be used. Consequently, the authors obtain an upper bound on the EFT cutoff given by $\Lambda \lesssim \sqrt{emM_{\text{Pl}}} \sim 10^8 \text{GeV}$. In the last equality, the mass and charge are identified with those of the electron. In [48], the authors analyze photon-photon scattering within the Standard Model matter content, and obtain a bound of $\Lambda \lesssim 10^{16} \text{GeV}$. Then, it is argued that the Standard Model is consistent with (5.2.6) with $\sigma = 0$. In [46], the gravitational positivity bound is evaluated in the context of scalar field theory and its implications for inflation models are investigated.

These studies demonstrate that nontrivial bounds can be placed on the cutoff Λ of given theories assuming $\sigma = 0$ in (5.2.6). From the perspective of Interpretation 3, however, positivity bounds (5.2.6) do not provide possible method to assess the UV parameters (M, σ) based on IR observables, because very large values of Λ is still possible even if $\sigma = 0$. In the following

chapters, we will see that the situation changes when the dark sector is taken into account.

Chapter 6

Positivity bounds on dark photon-dark photon scattering

A gauge interaction is one of the cornerstones of the quantum field theory. Gauge symmetries and their breakings are one of the most crucial ingredients of the Standard Model, and many BSM models also introduce new gauge symmetries. In particular, a light gauge boson is one of the candidates for dark matter [29, 33, 55].

In this chapter, we work out the details of the gravitational positivity recipe for the dark U(1) gauge boson. Here we study the Abelian Higgs mechanism of the simplest gauge theory, the U(1) theory. Firstly, we discuss practical recipe to derive gravitational positivity on dark U(1) gauge boson. We discuss theoretical constraints obtained by step-by-step computations spelled out in the previous section.

6.1 Recipe for gravitational positivity on dark U(1) gauge boson

6.1.1 Kinematics

In this chapter, we consider a $2 \rightarrow 2$ scattering process of the gauge bosons $VV \rightarrow VV$. The mass of the gauge boson is m_V . Firstly, we prepare a two-particle initial state $|p_1, p_2\rangle$, where p_1 and p_2 are momenta of incident particles. These particles scatter off each other and evolve to a final state $|p_3, p_4\rangle$. Mandelstam invariants for this process are defined as

$$s := (p_1 + p_2)^2 = 4(m_V^2 + |\vec{p}|^2), \quad (6.1.1)$$

$$t := (p_1 - p_3)^2 = -2|\vec{p}|^2(1 - \cos \theta), \quad (6.1.2)$$

$$u := (p_1 - p_4)^2 = -2|\vec{p}|^2(1 + \cos \theta), \quad (6.1.3)$$

where $\vec{p}$ is the momentum of particle 1 in the center-of-momentum(COM) frame. Using Mandelstam variables, the scattering angle in the COM frame $\cos \theta = \mathbf{p}_1 \cdot \mathbf{p}_3 / |\mathbf{p}_1| |\mathbf{p}_3|$ can be written

as

$$\cos \theta = 1 + \frac{2t}{s - 4m_V^2}. \quad (6.1.4)$$

There is a relation between the Mandelstam variables:

$$s + t + u = 4m_V^2. \quad (6.1.5)$$

6.1.2 Generalizations for realistic theories

When one tries to apply the gravitational positivity bounds (5.1.15) to realistic phenomenological models, the basic idea is the same as in the case of scalar scatterings of the previous section. However, one encounters several new subtleties to be carefully addressed, which we list in the following.

Spin. The positivity bound can be extended to a scattering of particles with spin, see [4, 5].¹ There are extra technical complications, however, and here we highlight the issue of the kinematic singularity, which is relevant for our discussion of gauge bosons in Sec. 6.1.

Let us consider the two-to-two scattering amplitudes of massive spin-1 particles with mass m_V . The kinematic singularity refers to the singularity of a scattering amplitude $\mathcal{M}$, which arises from the pole at $s = 4m_V^2$ in the (6.1.4), which is contained in the polarization tensor. The kinematic singularity disappears in the forward limit $t \rightarrow 0$ if the scattering amplitude does not contain the massless t -pole. However, in our setup, the kinematic singularity appears from t -channel graviton exchange diagrams. We can deal with this singularity by defining a kinematic singularity-free amplitude to be [5] (see also [57, 58]):²

$$\widetilde{\mathcal{M}}(s, t) := s^2 (s - 4m_V^2)^2 \mathcal{M}(s, t). \quad (6.1.6)$$

We can then derive the positivity bound by essentially the same arguments, as long as we replace the amplitude $\mathcal{M}$ by $\widetilde{\mathcal{M}}$. In practice, this is the same as ignoring the contribution of kinematic singularity to the t -channel graviton exchange diagrams.

Massless particles. The general properties of the scattering amplitudes have been established only in gapped systems, and not when massless particles are present. Even worse, the traditional non-perturbative S-matrix does not exist for massless particles, due to the issue of the IR divergence. To avoid this IR issue, we estimate the contributions from IR divergent diagrams to the positivity bounds by introducing an IR cutoff in Appendix B. We find that they will be

¹For this purpose, it is useful to diagonalize the crossing relation using the transversity formalism of [56]. For the use of this formalism in the context of positivity bounds, see [5].

²Although the s^2 factor is not necessary to cancel the kinematic singularity, it simplifies the subsequent expressions by retaining $s \leftrightarrow u$ crossing symmetry in $t \rightarrow 0$ limit.

subdominant, and hence we expect that the subtleties associated with the IR divergence will not affect our main conclusion. However, this may not be the case in other models, in which case an appropriate prescription for massless particles is required to derive the bounds.

Unstable particles. If a particle has a decay channel, such an unstable particle does not appear in the asymptotic states and the standard scattering amplitudes cease to exist for the particle [59]. However, unstable-particle scattering amplitudes can be unambiguously defined by residues of higher-point amplitudes (see e.g. [60]) and their properties can be studied in the S-matrix theory.³ Unitarity of the S-matrix leads to certain constraints on the unstable-particle scattering amplitudes, and in particular, there exists a positivity constraint on the imaginary part [61], suggesting the existence of positivity bounds even from scattering of unstable particles. In the present paper, however, we will only focus on the scattering of stable particles and leave a precise treatment of unstable particles for a future study.

Anomalous threshold singularities. Even for stable particles, the analytic structure of a scattering of a heavier particle is not as simple as that of a lighter particle. It is known that the scattering of the heavier particle can have a new singularity called an anomalous threshold, whose existence does not immediately follow from unitarity. While general knowledge of anomalous threshold is still missing to the best of our knowledge (see [62, 63] for recent discussions), we expect that the anomalous thresholds appear only in the IR regime, i.e., they are detectable and controlled within EFT at least when the heavy particle is stable. If this is indeed the case, the anomalous thresholds will give additional technical complications but will not spoil the success of positivity bounds, as long as these thresholds are taken into account in the derivation of the dispersion relations.

Recipe for gravitational positivity on dark $U(1)$ gauge boson. We are now ready to discuss practical recipes for deriving gravitational positivity bounds for EFTs. Instead of making the presentation fully general, we focus on the case of $2 \rightarrow 2$ scattering $VV \rightarrow VV$ of spin 1 particle V with mass m_V . This will be the case relevant for this section.

We consider the scattering amplitude of the massive gauge bosons $VV \rightarrow VV$. We choose the coordinate system where the scattering occurs in the z - x plane. Then, we can take the momentum of the gauge boson as

$$p^\mu = \begin{pmatrix} E \\ |\vec{p}| \sin \theta \\ 0 \\ |\vec{p}| \cos \theta \end{pmatrix}, \quad (6.1.7)$$

³There are still IR divergences, which need to be addressed separately.

where $E = \sqrt{m_V^2 + |\vec{p}|^2}$ is the energy of the gauge boson. For this p^μ , the transverse polarization vector is defined as

$$\varepsilon_\pm^\mu = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ \cos \theta \\ \pm i \\ -\sin \theta \end{pmatrix}, \quad (6.1.8)$$

and the longitudinal polarization vector is defined as

$$\varepsilon_L^\mu = \frac{1}{m_V} \begin{pmatrix} |\vec{p}| \\ E \sin \theta \\ 0 \\ E \cos \theta \end{pmatrix}. \quad (6.1.9)$$

Corresponding to polarizations of the gauge boson, we define the following combinations of helicity amplitudes. For the scattering of transverse modes,

$$\widetilde{\mathcal{M}}_{TT}(s, t) := \frac{1}{4} \left[\widetilde{\mathcal{M}}(1^+ 2^+ 3^+ 4^+) + \widetilde{\mathcal{M}}(1^+ 2^- 3^+ 4^-) + \widetilde{\mathcal{M}}(1^- 2^- 3^- 4^-) + \widetilde{\mathcal{M}}(1^- 2^+ 3^- 4^+) \right], \quad (6.1.10)$$

where superscripts $\pm$ denote helicities and $\widetilde{\mathcal{M}}$ is the kinematic singularity-free amplitude defined in (6.1.6). For the scattering of the transverse mode and the longitudinal mode,

$$\widetilde{\mathcal{M}}_{TL}(s, t) := \frac{1}{2} \left[\widetilde{\mathcal{M}}(1^+ 2^L 3^+ 4^L) + \widetilde{\mathcal{M}}(1^- 2^L 3^- 4^L) \right], \quad (6.1.11)$$

where superscripts L denote longitudinal polarizations. For the scattering of longitudinal modes,

$$\widetilde{\mathcal{M}}_{LL}(s, t) := \widetilde{\mathcal{M}}(1^L 2^L 3^L 4^L). \quad (6.1.12)$$

These amplitudes are defined such that they exhibit $s \leftrightarrow u$ symmetry (up to subtleties associated with the kinematical singularity discussed around (6.1.6), which are not relevant to the following discussion).

We need to evaluate the cutoff-dependent quantity $B(\Lambda)$ defined by a spin-1 generalization of (5.2.3):

$$B(\Lambda) := a_2 - \frac{4}{\pi} \int_{m_{\text{th}}^2}^{\Lambda^2} ds \frac{\text{Im } \widetilde{\mathcal{M}}(s, t=0)}{(s - 2m_V^2)^7}, \quad (6.1.13)$$

where m_{th} is the threshold energy, i.e., the mass of the lightest intermediate state, and a_2 is basically defined as in (5.2.1), but with gravitational corrections included (as we will soon see in (6.1.17)).

Because of the use of $\widetilde{\mathcal{M}}$, the definition of $B_{\text{non-grav}}(\Lambda)$ changes as well:

$$B_{\text{non-grav}}(\Lambda) := a_{2, \text{non-grav}} - \frac{4}{\pi} \int_{m_{\text{th}}^2}^{\Lambda^2} ds \frac{\text{Im } \widetilde{\mathcal{M}}_{\text{non-grav}}(s, t=0)}{(s - 2m_V^2)^7}. \quad (6.1.14)$$

When the EFT of interest is renormalizable in the gravity decoupling limit $M_{\text{Pl}} \rightarrow \infty$, $B_{\text{non-grav}}$ have the expression

$$B_{\text{non-grav}}(\Lambda) = \frac{4}{\pi} \int_{\Lambda^2}^{\infty} ds \frac{\text{Im} \widetilde{\mathcal{M}}_{\text{non-grav}}(s, t=0)}{(s - 2m_V^2)^7}, \quad (6.1.15)$$

from the same discussion of the previous chapter. Further simplification comes in the limit $\Lambda \gg m_V$:

$$B_{\text{non-grav}}(\Lambda) \simeq \frac{4}{\pi} \int_{\Lambda^2}^{\infty} ds \frac{\text{Im} \mathcal{M}_{\text{non-grav}}(s, t=0)}{(s - 2m_V^2)^3}. \quad (6.1.16)$$

In other words, in this limit, we can in practice neglect the subtleties associated with the use of the kinematic-singularity-free amplitude $\widetilde{\mathcal{M}}$, and use the original amplitude $\mathcal{M}$ for the evaluation of $B_{\text{non-grav}}(\Lambda)$ in (6.1.16) (which coincides (5.1.14) with $n = 1$ and $m_A = m_B = m_V$).

The expression of the gravitational contribution $B_{\text{grav}}(\Lambda)$ also modified to

$$B_{\text{grav}}(\Lambda) \simeq a_{2,\text{grav},t\text{-channel}} := \lim_{t \rightarrow -0} \left[\frac{2}{6!} \frac{\partial^6 \widetilde{\mathcal{M}}_{\text{grav},t\text{-channel}}(s, t)}{\partial s^6} + \frac{2}{M_{\text{Pl}}^2 t} \right]_{s=2m_V^2}. \quad (6.1.17)$$

For $\Lambda \gg m_V$, this expression is further simplifies to

$$B_{\text{grav}}(\Lambda) \simeq \lim_{t \rightarrow -0} \left[\frac{\partial^2 \mathcal{M}_{\text{grav},t\text{-channel}}(s, t)}{\partial s^2} + \frac{2}{M_{\text{Pl}}^2 t} - (\text{kinematic singularity}) \right]_{s=2m_V^2}. \quad (6.1.18)$$

6.1.3 Graviton propagator

The gravitational part of the positivity bound $B_{\text{grav}}(\Lambda)$ is obtained from graviton exchange diagrams (See figure 6.1b). In this section, we briefly review the derivation of the graviton propagator.

The graviton propagator can be obtained by expanding the metric in the Einstein-Hilbert term $\frac{M_{\text{Pl}}^2}{2} \sqrt{-g} R$ around the flat metric $\eta^{\mu\nu} = \text{diag}(1, -1, -1, -1)$ as

$$g^{\mu\nu} = \eta^{\mu\nu} + \frac{2}{M_{\text{Pl}}^2} h^{\mu\nu}. \quad (6.1.19)$$

There are gauge degrees of freedom in the metric, which are infinitesimal version of the general coordinate transformations:

$$g_{\mu\nu} \rightarrow g_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu, \quad (6.1.20)$$

where ξ is an arbitrary vector field. We need to fix the gauge to calculate the graviton propagator. We use the harmonic gauge defined as

$$h^{\mu\nu}{}_{,\nu} - \frac{1}{2}\eta^{\mu\nu}h_{,\nu} = 0, \quad (6.1.21)$$

where $h = \eta^{\mu\nu}h_{\mu\nu}$. Then, expanding the Einstein-Hilbert term up to the quadratic order in $h_{\mu\nu}$, we obtain graviton kinetic term as

$$\mathcal{L}_h = \frac{1}{2}h_{\mu\nu} \left(\eta^{\mu\rho}\eta^{\nu\sigma} - \frac{1}{2}\eta^{\mu\nu}\eta^{\rho\sigma} \right) \partial^2 h_{\rho\sigma}. \quad (6.1.22)$$

Then, the propagator of the graviton is given by

$$iD_{\mu\nu,\rho\sigma}(p) = \frac{i(\eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho} - \eta_{\mu\nu}\eta_{\rho\sigma})}{p^2}, \quad (6.1.23)$$

where p is the momentum of the graviton.

6.2 Higgs contribution

In the Higgs mechanism, the gauge boson gets a mass when a charged scalar field Φ gets a non-zero vacuum expectation value (VEV) v . The renormalizable Lagrangian of the Higgs mechanism to generate a mass for a U(1) gauge boson is

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + |D_\mu\Phi|^2 - \frac{\lambda}{4}(|\Phi|^2 - v^2)^2, \quad (6.2.1)$$

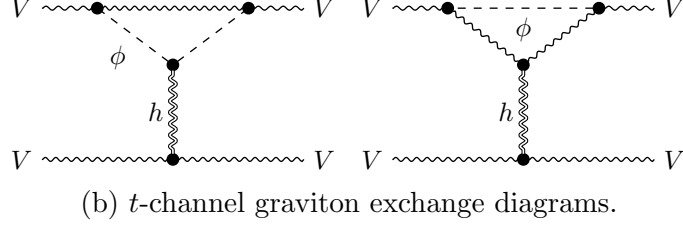
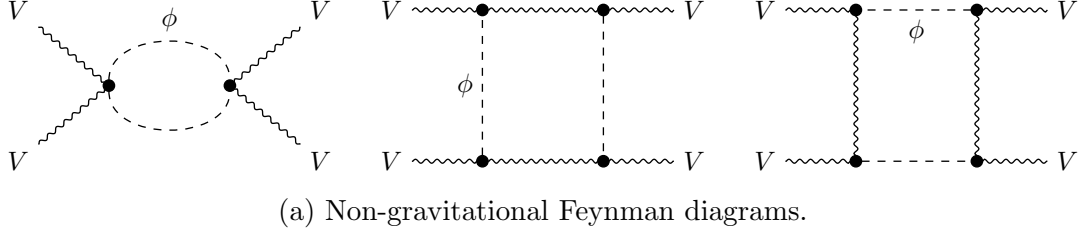
where $F_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu$ is the field strength, $D_\mu = \partial_\mu - ig_\Phi V_\mu$ is the covariant derivative, g_Φ is a charge of Φ and $\lambda > 0$. After $\Phi = v + \phi/\sqrt{2} + iG$ develops the VEV, the Goldstone component G is absorbed into the longitudinal component of the gauge boson. There are a massive gauge boson V and a real scalar ϕ after the gauge symmetry breaking, and the gauge boson gets a mass $m_V = \sqrt{2}g_\Phi v$ and a real scalar ϕ has a mass $m_\phi = \sqrt{\lambda}v$. The interaction of the Higgs and gauge bosons are given by

$$\mathcal{L} \ni \left(\frac{g_\Phi^2}{2}\phi^2 + g_\Phi m_V \phi \right) V^\mu V_\mu - \lambda \left(\frac{v}{2\sqrt{2}}\phi^3 + \frac{1}{16}\phi^4 \right). \quad (6.2.2)$$

The gravitational interaction can be found by expanding the Lagrangian (6.2.1) in terms of the canonical gravitational field $h_{\mu\nu}$ as

$$\mathcal{L}_{\text{grav}} = -\frac{h_{\mu\nu}T^{\mu\nu}}{M_{\text{Pl}}} + \mathcal{O}(M_{\text{Pl}}^{-2}); \quad T^{\mu\nu} \equiv -\frac{2}{\sqrt{-g}} \frac{\partial(\sqrt{-g}\mathcal{L})}{\partial g_{\mu\nu}}. \quad (6.2.3)$$

The Higgs-gauge boson interaction (6.2.2) does not make the gauge boson V unstable since it has a $\mathbb{Z}_2$ symmetry $V \leftrightarrow -V$.

Figure 6.1: Some examples of the Higgs loop contribution to $B(\Lambda)$.

6.2.1 Calculation of $B(\Lambda)$

We calculate $B(\Lambda)$ at the one-loop order using the recipe described in Sec. 6.1. In Figs. 6.1a and 6.1b, we show some examples of the gravitational and non-gravitational processes. The one-loop diagrams are calculated by using the Mathematica packages FeynRules [64–66], FeynArts [67], FeynCalc [68–70] and Package-X [71].

Non-gravitational contributions

Fig. 6.1a show some examples of the non-gravitational Feynman diagrams involving the Higgs boson ϕ . From these (and other) diagrams we evaluate the non-gravitational part of $B(\Lambda)$ to be

$$B_{\text{non-grav}}^{TT} = \frac{g_\Phi^4}{4\pi^2\Lambda^4} + \frac{g_\Phi^4 \left[m_\phi^4 (12 \log(\frac{m_\phi}{\Lambda}) + 1) + m_\phi^2 m_V^2 (12 \log(\frac{\Lambda^3 m_V}{m_\phi^4}) - 27) + 24m_V^4 \right]}{18\pi^2\Lambda^6 m_\phi^2} + \dots, \quad (6.2.4)$$

$$B_{\text{non-grav}}^{TL} = \frac{g_\Phi^4}{2\pi^2\Lambda^4} + \frac{g_\Phi^4 \left[-3 \left(m_\phi^3 - 2m_\phi m_V^2 \right)^2 \log(\frac{m_\phi m_V}{\Lambda^2}) + 17m_\phi^4 m_V^2 - 20m_\phi^2 m_V^4 - 8m_\phi^6 + 12m_V^6 \right]}{9\pi^2\Lambda^6 m_\phi^2 m_V^2} + \dots, \quad (6.2.5)$$

$$B_{\text{non-grav}}^{LL} = \frac{g_\Phi^4}{\pi^2\Lambda^2 m_V^2} + \frac{g_\Phi^4 \left(4m_\phi^2 m_V^2 \log(\frac{m_V^2}{\Lambda^2}) - 8m_\phi^2 m_V^2 + 5m_\phi^4 + 6m_V^4 \right)}{8\pi^2\Lambda^4 m_V^4} + \dots. \quad (6.2.6)$$

Here $\dots$ represents contributions suppressed by higher powers of Λ , which scale is assumed to be much greater than m_ϕ and m_V . We have also numerically estimated the two-loop contributions and confirm that these effects are smaller than the one-loop contributions in the parameter region of present interest.

Gravitational contributions

As the next step, we compute the one-loop diagrams of the gravitational contributions (Fig. 6.1b):

$$B_{\text{grav}}^{TT} = \frac{-g_{\Phi}^2}{72\pi^2 m_{\phi}^2 M_{\text{Pl}}^2} g_{TT} \left(\frac{m_V}{m_{\phi}} \right), \quad (6.2.7)$$

$$B_{\text{grav}}^{TL} = \frac{-g_{\Phi}^2}{144\pi^2 m_V^2 M_{\text{Pl}}^2} g_{TL} \left(\frac{m_V}{m_{\phi}} \right), \quad (6.2.8)$$

$$B_{\text{grav}}^{LL} = \frac{-g_{\Phi}^2}{72\pi^2 m_V^2 M_{\text{Pl}}^2} g_{LL} \left(\frac{m_V}{m_{\phi}} \right), \quad (6.2.9)$$

where the functions $g_{TT,TL,LL}$ are given by

$$\begin{aligned} g_{TT}(x) &= \frac{2}{x^8 (1 - 4x^2)^2} \\ &\times \left[-6 (2x^2 - 1) (4x^4 - 5x^2 + 1)^2 \log(x) + (200x^8 - 346x^6 + 230x^4 - 63x^2 + 6) x^2 \right. \\ &\quad \left. - 6 \sqrt{\frac{1}{x^2} - 4} (4x^{10} - 32x^8 + 52x^6 - 35x^4 + 10x^2 - 1) x \log \left(\frac{1}{2} \left(\sqrt{\frac{1}{x^2} - 4} + \frac{1}{x} \right) \right) \right], \end{aligned} \quad (6.2.10)$$

$$\begin{aligned} g_{TL}(x) &= \frac{-1}{x^6 (1 - 4x^2)^2} \\ &\times \left[6 \sqrt{\frac{1}{x^2} - 4} (51x^2 + 2 (8x^6 - 92x^4 + 147x^2 - 93) x^4 - 5) \log \left(\frac{1}{2} \left(\sqrt{\frac{1}{x^2} - 4} + \frac{1}{x} \right) \right) \right. \\ &\quad \left. + (4x^2 - 1) (-344x^8 + 416x^6 - 201x^4 + 30x^2 + 6 (32x^8 - 128x^6 + 114x^4 - 41x^2 + 5) \log(x)) \right] \end{aligned} \quad (6.2.11)$$

$$\begin{aligned} g_{LL}(x) &= \frac{-244x^6 + 268x^4 - 123x^2 + 18}{x^4 - 4x^6} + \frac{6x \left(\frac{28x^4 - 19x^2 + 3}{x^6} - 2 \right) \left(\frac{1}{x^2} - 2 \right)^2 \log \left(\frac{1}{2} \left(\sqrt{\frac{1}{x^2} - 4} + \frac{1}{x} \right) \right)}{\left(\frac{1}{x^2} - 4 \right)^{3/2}} \\ &\quad + \frac{6 (-4x^6 + 20x^4 - 13x^2 + 3) \log(x)}{x^6}. \end{aligned} \quad (6.2.12)$$

We plot the functions $g_{TT,TL,LL}$ in Fig. 6.2. These functions are real and positive for any real values of m_V and m_{ϕ} , and we find that the gravitational contributions from the Higgs to $B(\Lambda)$ are negative.

In the limit of $m_V/m_{\phi} \rightarrow 0$ and $\rightarrow \infty$, the above expressions reduce to

$$B_{\text{grav}}^{TT} = \begin{cases} -\frac{g_{\Phi}^2}{72\pi^2 m_{\phi}^2 M_{\text{Pl}}^2} & (m_V/m_{\phi} \rightarrow 0), \\ -\frac{g_{\Phi}^2}{24\pi m_V m_{\phi} M_{\text{Pl}}^2} & (m_V/m_{\phi} \rightarrow \infty), \end{cases} \quad (6.2.13)$$

$$B_{\text{grav}}^{TL} = \begin{cases} -\frac{g_{\Phi}^2}{144\pi^2 m_V^2 M_{\text{Pl}}^2} & (m_V/m_{\phi} \rightarrow 0), \\ -\frac{g_{\Phi}^2}{24\pi m_V m_{\phi} M_{\text{Pl}}^2} & (m_V/m_{\phi} \rightarrow \infty), \end{cases} \quad (6.2.14)$$

$$B_{\text{grav}}^{LL} = \begin{cases} -\frac{g_{\Phi}^2}{72\pi^2 m_V^2 M_{\text{Pl}}^2} & (m_V/m_{\phi} \rightarrow 0), \\ -\frac{g_{\Phi}^2}{24\pi m_V m_{\phi} M_{\text{Pl}}^2} & (m_V/m_{\phi} \rightarrow \infty). \end{cases} \quad (6.2.15)$$

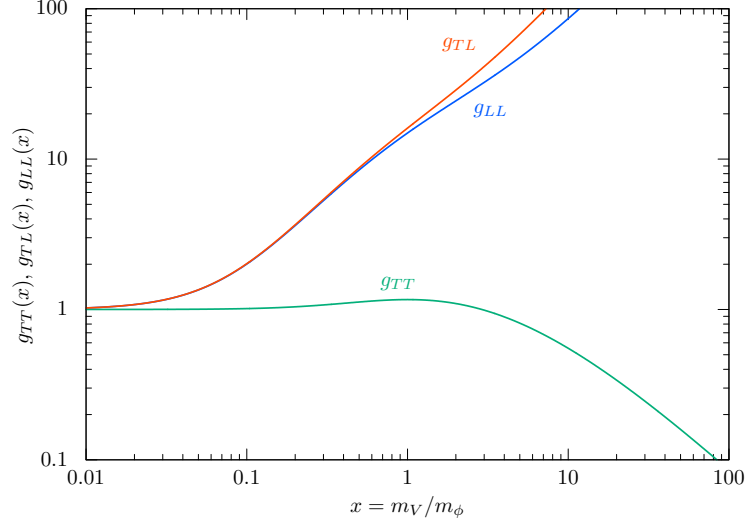


Figure 6.2: The functions $g_{TT,TL,LL}(x)$. These functions are real and positive for $x > 0$ and therefore the gravitational contributions are negative for all the masses of the Higgs boson and the gauge boson.

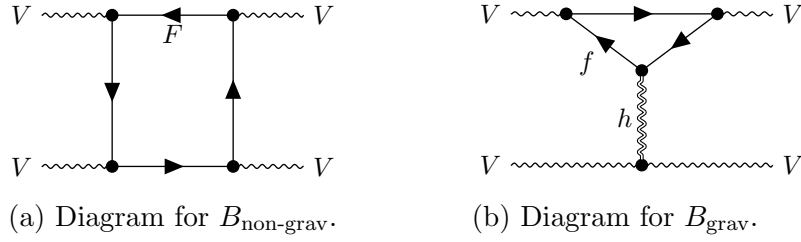


Figure 6.3: The fermion loop contributions to B .

6.3 Fermion contribution

Next, let us introduce a dark U(1) charged Dirac fermion ψ as a matter field, whose interaction is given by

$$\mathcal{L}_\psi = i\bar{\psi}\not{D}\psi - m_F\bar{\psi}\psi, \quad (6.3.1)$$

where $D_\mu = \partial_\mu - ig_F V_\mu$ is the covariant derivative. Note that this interaction can destabilize the gauge boson, by allowing $V \rightarrow \psi\bar{\psi}$ decay if $m_V > 2m_F$ and decay into gravitons.

If the mass of the charged particles is large ($m_F \gg \Lambda$), these charged particles are integrated out in the EFT and yield higher-derivative corrections to the EFT action. These effects to $B(\Lambda)$ can be estimated as $\mathcal{O}(g_F^4/m_F^4)$ and we can ignore them compared to contributions (6.2.4) from the Higgs field. Hence, our interest is the charged particle within the regime of EFT ($m_F < \Lambda$).

6.3.1 Calculation of $B(\Lambda)$

We calculate the contributions of fermion-loop diagrams to $B(\Lambda)$ for the dark photon scattering amplitudes at one-loop order.

Non-gravitational contributions

The diagram for a non-gravitational fermion loop is shown in Fig. 6.3a. The one-loop contributions are given by

$$B_{\text{non-grav},F}^{TT} = \frac{g_F^4 \left(2 \log \frac{\Lambda^2}{m_F^2} + 1 \right)}{4\pi^2 \Lambda^4} + \dots, \quad (6.3.2)$$

$$B_{\text{non-grav},F}^{TL} = \frac{4g_F^4 m_V^2}{3\pi^2 \Lambda^6} + \dots, \quad (6.3.3)$$

$$B_{\text{non-grav},F}^{LL} = \frac{g_F^4 m_V^4 \left(4 \log \frac{\Lambda^2}{m_F^2} - 7 \right)}{\pi^2 \Lambda^8} + \dots \quad (6.3.4)$$

for $\Lambda \gg m_{V,F}$. These contributions of LL and TL vanish in the limit $m_V/\Lambda \rightarrow 0$. Recall that $B_{\text{non-grav}}$ is determined by the high-energy limit of the forward limit amplitude thanks to the dispersion relation. The behaviors $B_{\text{non-grav},F}^{TL}, B_{\text{non-grav},F}^{LL} \rightarrow 0$ as $m_V/\Lambda \rightarrow 0$ can be understood by the decoupling of the longitudinal sector in the high-energy limit, which will be discussed in 7.4.4. The contributions of the charged particles to $B_{\text{non-grav}}^{TL}$ and $B_{\text{non-grav}}^{LL}$ can be ignored in the limit $\Lambda \gg m_V$.

We thus focus on the TT scattering. The charged spin-1/2 loops give positive contributions to the non-gravitational diagram of the order $B_{\text{non-grav},F}^{TT} \sim g_F^4/\Lambda^4$, which is of the same order as the Higgs counterpart (6.2.4). Therefore, the inclusion of charged spin-1/2 particles does not drastically change the non-gravitational part. We reach a similar conclusion even in the presence of charged spin-0 particles [47].

In contrast, as discussed in the next chapter, charged spin-1 loops with the mass $m_{\text{spin-1}}$ and the charge $g_{\text{spin-1}}$ lead to a different asymptotic behavior $B_{\text{non-grav}}^{TT} \sim g_{\text{spin-1}}^4/(m_{\text{spin-1}}^2 \Lambda^2)$ which dominates over other contributions in the limit $\Lambda \gg m_{\text{spin-1}}$. One may then wonder about contributions from higher-spin particles. The scaling in Λ is determined by the high-energy behavior of the imaginary part of the amplitude in the forward limit, i.e., the total cross-section. Roughly, spin-0, 1/2 loops give $\text{Im } \mathcal{M} \propto s^0$ while spin-1 loops yield $\text{Im } \mathcal{M} \propto s/m_{\text{spin-1}}^2$ in high-energy limit. The faster growth in s provides a larger contribution to $B_{\text{non-grav}}(\Lambda)$. However, the asymptotic growth of the cross-section in s is bounded by the Froissart bound [34] as $\text{Im } \mathcal{M} < s \ln^2 s$ in gapped theories, implying that the spin-1 contribution almost saturates the bound. Higher-spin particles, if they are described by gapped theories like QCD, may not give drastically different contributions compared to spin-1 particles. For instance, in the case of light-by-light scattering, the hadronic contribution can be estimated as $\text{Im } \mathcal{M} \propto s^{1.08}$ by employing the vector meson dominance model [72, 73].

In short, if a charged spin-1 or higher-spin particle appears at m , the non-gravitational contribution $B_{\text{non-grav}}^{TT}$ needs to be modified in $\Lambda > m$. By contrast, spin-0 or spin-1/2 particles give additional contributions but the qualitative behavior of $B_{\text{non-grav}}^{TT}$ is the same.

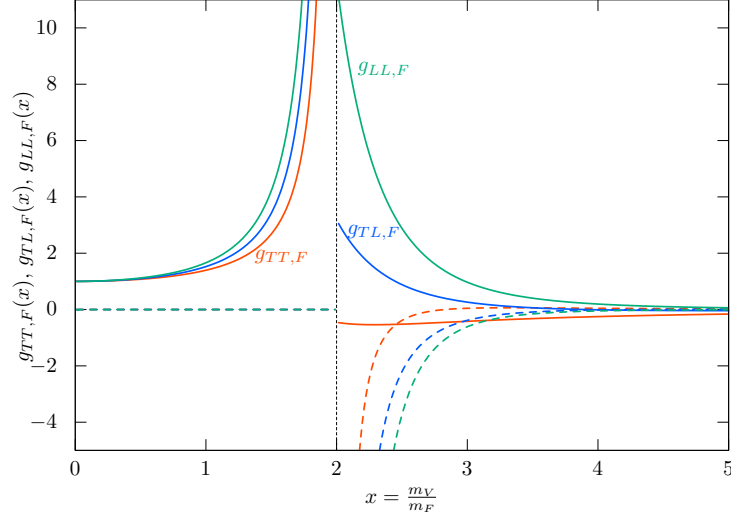


Figure 6.4: The functions $g_{TT,F}(x)$ (red), $g_{TL,F}(x)$ (green) and $g_{LL,F}(x)$ (blue). The solid and dashed lines represent the real and imaginary parts, respectively.

Gravitational contributions

Next, we discuss the gravitational parts (Fig. 6.3b), which are given by

$$B_{\text{grav},F}^{TT} = -\frac{11g_F^2}{360\pi^2 M_{\text{Pl}}^2 m_F^2} g_{TT,F} \left(\frac{m_V}{m_F} \right), \quad (6.3.5)$$

$$B_{\text{grav},F}^{TL} = -\frac{11g_F^2}{720\pi^2 M_{\text{Pl}}^2 m_F^2} g_{TL,F} \left(\frac{m_V}{m_F} \right), \quad (6.3.6)$$

$$B_{\text{grav},F}^{LL} = -\frac{g_F^2 m_V^2}{420\pi^2 M_{\text{Pl}}^2 m_F^4} g_{LL,F} \left(\frac{m_V}{m_F} \right), \quad (6.3.7)$$

where

$$g_{TT,F}(x) = \frac{60(x^2 - 8)(x^2 - 2) \log\left(\frac{1}{2}x(\sqrt{x^2 - 4} - x) + 1\right) - 20x\sqrt{x^2 - 4}(2x^4 - 11x^2 + 24)}{11x^5(x^2 - 4)^{3/2}}, \quad (6.3.8)$$

$$g_{TL,F}(x) = \frac{60(x^4 - 22x^2 + 56) \log\left(\frac{1}{2}x(\sqrt{x^2 - 4} - x) + 1\right) - 20x\sqrt{x^2 - 4}(x^4 - 19x^2 + 84)}{11x^5(x^2 - 4)^{3/2}}, \quad (6.3.9)$$

$$g_{LL,F}(x) = \frac{70((120 - 36x^2) \log\left(\frac{1}{2}x(\sqrt{x^2 - 4} - x) + 1\right) + x\sqrt{x^2 - 4}(x^4 + 8x^2 - 60))}{3x^7(x^2 - 4)^{3/2}}. \quad (6.3.10)$$

We plot the functions $g_{AB,F}(x)$ ($A, B = T, L$) in Fig. 6.4.

Let us first discuss the limit $m_V \rightarrow 0$, which yields

$$B_{\text{grav},F}^{TT} = -\frac{11g_F^2}{360\pi^2 M_{\text{Pl}}^2 m_F^2}, \quad (6.3.11)$$

$$B_{\text{grav},F}^{TL} = -\frac{11g_F^2}{720\pi^2 M_{\text{Pl}}^2 m_F^2}, \quad (6.3.12)$$

$$B_{\text{grav},F}^{LL} = -\frac{g_F^2 m_V^2}{420\pi^2 M_{\text{pl}}^2 m_F^4}. \quad (6.3.13)$$

If the Higgs and the gauge boson are lighter than the charged particles ($m_\phi, m_V \ll m_F$), we can neglect these contributions to the gravitational process B_{grav} in comparison to the contributions of the gauge boson sector. Note that the LL part has an additional suppression m_V^2/m_F^2 because the longitudinal mode should decouple from the charged particle in the limit $m_V \rightarrow 0$. However, since the longitudinal mode does not decouple from gravity, the TL mode does not have such a suppression factor in the diagram where the longitudinal mode attaches to the graviton line.

By contrast, a peculiar behavior appears in a heavy mass range of the gauge boson. The functions $g_{AB,F}$ are singular at $m_V = 2m_F$ and become complex numbers in $m_V > 2m_F$. The threshold $m_V = 2m_F$ corresponds to the value at which the decay of the gauge boson to the charged particles starts to be kinematically allowed. The singularity at $m_V = 2m_F$ is understood as an anomalous threshold. Let us consider the triangle diagram of Fig. 6.3b, which gives rise to the gravitational form factor of the gauge boson. The triangle diagram has a normal threshold at $t = 4m_F^2$. In addition, the anomalous threshold comes up on the first sheet of the complex t -plane in $m_V > \sqrt{2}m_F$ and the position of the singularity is at

$$t = \frac{m_V^2}{m_F^2}(4m_F^2 - m_V^2). \quad (6.3.14)$$

Therefore, the form factor at $t = 0$ is singular when $m_V = 2m_F$, generating the singularity of $B_{\text{grav},F}^{AB}$. In Fig. 6.4, we have plotted the functions in the unstable range $m_V > 2m_F$ as well. Recall that $B(\Lambda)$ has to be real according to the dispersion relation (5.2.3) if all the mentioned properties are satisfied. This implies that at least one of the properties does not hold when the gauge boson decays. In fact, amplitudes exhibit peculiar behaviors if the mass of the external particle is extrapolated to the unstable region [61, 62]. The conventional positivity bounds need to be modified for unstable particles.

We expect that the subtleties associated with the decay are negligible if the particle is long-lived ($g_F \ll 1$). In other words, although the fermion contribution to $B_{\text{grav}}(\Lambda)$ is quite subtle in the mass range $m_V > 2m_F$, this subtle contribution can be smaller than the contribution from the Higgs sector and could be simply negligible. In the following, we shall adopt this optimistic expectation when discussing models in which the gauge boson decays into other particles. The issue of unstable particles will be studied elsewhere.

6.4 Constraint on dark gauge boson

In Table 6.1, we summarize our estimation of $B(\Lambda)$ for a light gauge boson. We apply the gravitational positivity bound (5.2.6) to these amplitudes:

$$B(\Lambda) = B_{\text{non-grav}}(\Lambda) + B_{\text{grav}}(\Lambda) \geq \frac{\sigma}{M_{\text{pl}}^2 M^2}. \quad (6.4.1)$$

Table 6.1: Summary of the gravitational and non-gravitational contributions to B for $m_V \ll m_{\phi,F} \ll \Lambda$ and $m_\phi \ll \sqrt{\Lambda m_V}$.

	Higgs loop		Fermion loop	
	$B_{\text{non-grav}}$	B_{grav}	$B_{\text{non-grav}}$	B_{grav}
TT	$\frac{g_\Phi^4}{4\pi^2\Lambda^4}$	$-\frac{g_\Phi^2}{72\pi^2 M_{\text{Pl}}^2 m_\phi^2}$	$\frac{g_F^4 \left(2 \log \frac{\Lambda^2}{m_F^2} + 1\right)}{4\pi^2\Lambda^4}$	$-\frac{11g_F^2}{360\pi^2 M_{\text{Pl}}^2 m_F^2}$
TL	$\frac{g_\Phi^4}{2\pi^2\Lambda^4}$	$-\frac{g_\Phi^2}{144\pi^2 M_{\text{Pl}}^2 m_V^2}$	$\frac{4g_F^4 m_V^2}{3\pi^2\Lambda^6}$	$-\frac{11g_F^2}{720\pi^2 M_{\text{Pl}}^2 m_F^2}$
LL	$\frac{g_\Phi^4}{\pi^2\Lambda^2 m_V^2}$	$-\frac{g_\Phi^2}{72\pi^2 M_{\text{Pl}}^2 m_V^2}$	$\frac{g_F^4 m_V^4 \left(4 \log \frac{\Lambda^2}{m_F^2} - 7\right)}{\pi^2\Lambda^8}$	$-\frac{g_F^2 m_V^2}{420\pi^2 M_{\text{Pl}}^2 m_F^4}$

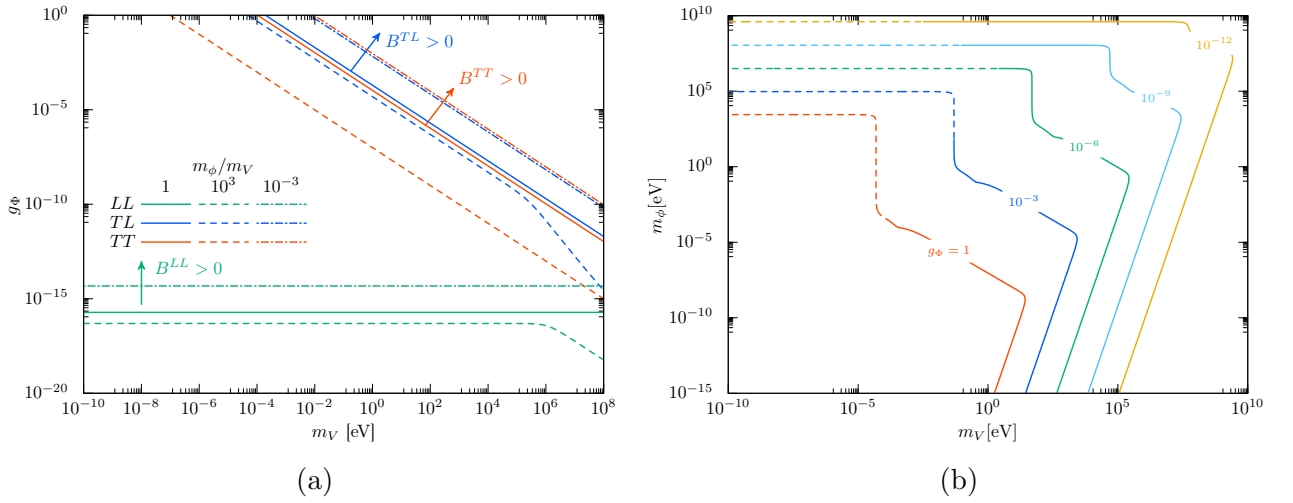


Figure 6.5: The constraints on the dark gauge boson parameters in the Higgs mechanism. (a): Lower limit of g_Φ as a function of m_V . The positivity bound is satisfied in the region above the lines. (b): Contour of the lower bound of g_Φ on $m_V - m_\phi$ plane from the positivity bound. The dashed lines indicate the Higgs quartic coupling is large $\lambda > \pi$. We choose $\Lambda = 1$ TeV in both figures.

For simplicity of the presentation, in the following, we will discuss the constraints on the dark gauge boson parameters for $\sigma = 0$ with a fixed value of Λ (Interpretation 1 in Sec. 5.2). One should recall, however, that the actual bound is (6.4.1). The bound becomes stronger for $\sigma = +1$ and weaker for $\sigma = -1$.

Abelian Higgs without matter fields

Let us first look at the case where there is no matter field and only gauge and Higgs fields exist. For a very light gauge boson $m_V \ll m_{\phi,F} \ll \Lambda$ and $m_\phi \ll \sqrt{\Lambda m_V}$, the following conditions can be obtained from each helicity

$$B_{\text{non-grav}}^{TT} + B_{\text{grav}}^{TT} \geq 0 \quad \longrightarrow \quad m_\phi \geq \frac{\Lambda^2}{3\sqrt{2}g_\Phi M_{\text{Pl}}}, \quad (6.4.2)$$

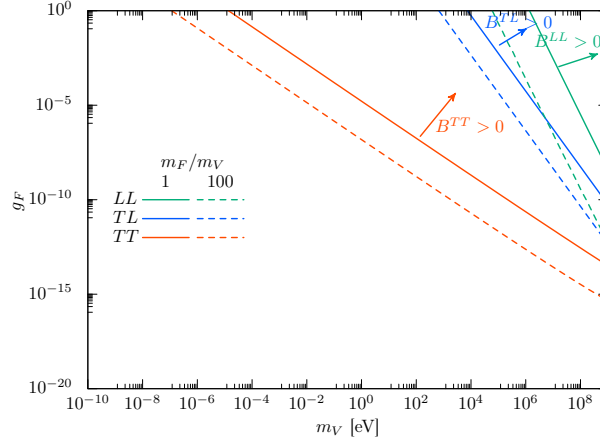


Figure 6.6: The constraints on the dark gauge boson parameters for the Stückelberg U(1) gauge theory with a fermion field. The positivity bound is satisfied in the region above the lines. We choose $\Lambda = 1$ TeV.

$$B_{\text{non-grav}}^{TL} + B_{\text{grav}}^{TL} \geq 0 \quad \longrightarrow \quad m_V \geq \frac{\Lambda^2}{6\sqrt{2}g_\Phi M_{\text{Pl}}}, \quad (6.4.3)$$

$$B_{\text{non-grav}}^{LL} + B_{\text{grav}}^{LL} \geq 0 \quad \longrightarrow \quad g_\Phi \geq \frac{\Lambda}{6\sqrt{2}M_{\text{Pl}}}. \quad (6.4.4)$$

In Fig. 6.5, we show the positivity conditions. Note that the higher terms of Λ for the non-gravitational contributions can be dominant if $m_\phi \gtrsim \sqrt{m_V \Lambda}$ or $m_\phi \lesssim m_V^2/\Lambda$.

Stückelberg gauge boson with matter field

Next, let us discuss the case where the Higgs field ϕ is absent in the EFT, as in the case of the Stückelberg mechanism. We find

$$B_{\text{non-grav}}^{TT} + B_{\text{grav}}^{TT} \geq 0 \quad \longrightarrow \quad g_F \gtrsim 0.2 \frac{\Lambda^2}{m_F M_{\text{Pl}} \sqrt{\log(\Lambda m_F^{-1})}}, \quad (6.4.5)$$

$$B_{\text{non-grav}}^{TL} + B_{\text{grav}}^{TL} \geq 0 \quad \longrightarrow \quad m_V \gtrsim 0.1 \frac{\Lambda^3}{g_F m_F M_{\text{Pl}}}, \quad (6.4.6)$$

$$B_{\text{non-grav}}^{LL} + B_{\text{grav}}^{LL} \geq 0 \quad \longrightarrow \quad m_V \gtrsim 0.02 \frac{\Lambda^4}{g_F m_F^2 M_{\text{Pl}} \sqrt{\log(\Lambda m_F^{-1})}}. \quad (6.4.7)$$

In Fig. 6.6, we show the positivity conditions on $m_V - g_F$ plane. Note that for $m_V > 2m_F$, delicate arguments are required and the positivity conditions used previously cannot be applied as they are.

More generally, it is possible to incorporate both Higgs and fermions. The helicity configurations involving longitudinal modes provide stronger constraints. As discussed earlier, the Higgs contribution is dominant when $g_\Phi \sim g_F$ and $m_V \ll m_{\phi,F}$, in which case the positivity conditions will be similar to those of the Higgs scenarios.

Comparison with other swampland constraints for Stückelberg gauge bosons

Let us compare our swampland constraints with another swampland constraint on the gauge boson mass [74] (see also [75]) motivated by the swampland distance conjecture [76]. This bound states that the UV cutoff Λ_{UV} of an EFT with a gauge boson should obey

$$\Lambda_{\text{UV}} \lesssim \min((m_V M_{\text{Pl}}/g)^{1/2}, g^{1/3} M_{\text{Pl}}), \quad (6.4.8)$$

where (as before) m_V is the gauge boson mass and g is the gauge coupling constant. In Fig. 6.7 we compare our bounds and the bound (6.4.8) for sample values of parameters.

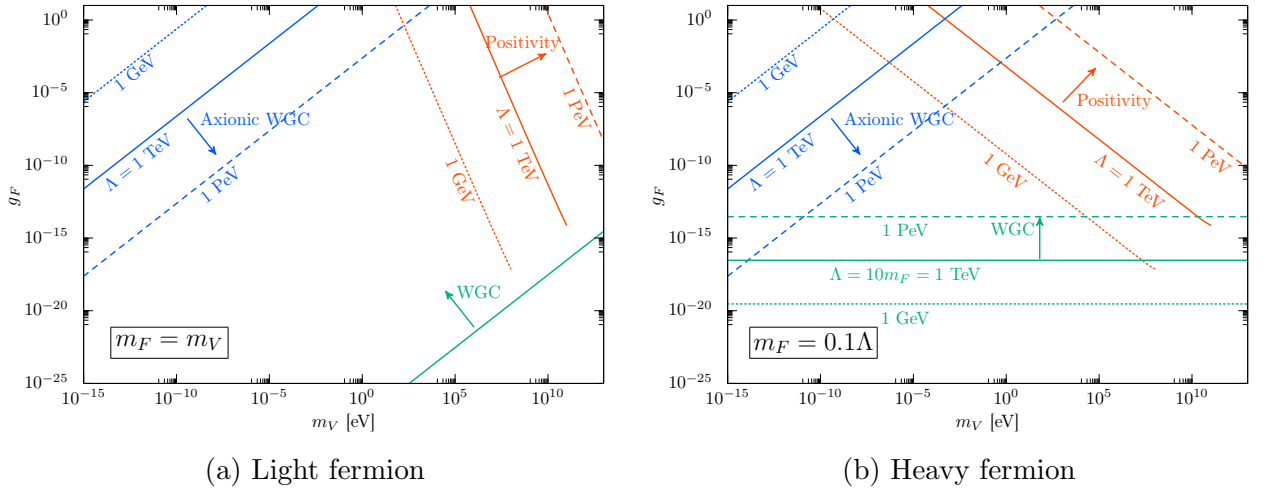


Figure 6.7: Comparison between the swampland constraints for Stückelberg gauge bosons, the gravitational positivity bound (in red) and other swampland bounds (6.4.8) (in blue) and the weak gravity conjectures (in green). We take the cutoff scale Λ is 1 GeV (dotted lines), 1 TeV (solid lines) and 1 PeV (dashed lines). (a): Light fermion case $m_F = m_V$ and (b): Heavy fermion case $m_F = 0.1\Lambda$. In this plot, the lines are chopped in the parameter space where $m_V > 0.1\Lambda$.

One can see in Fig. 6.7 that overall our bound is stronger than the bound (6.4.8). Let us quickly point out, however, that the two bounds are derived under a different set of assumptions/arguments, and one needs to be careful in any meaningful comparison.

The gravitational positivity bound is derived by general properties of Reggeized scattering amplitudes of a weakly-coupled UV theory. The assumptions are believed to be standard, but the bound has some ambiguities due to the unknown UV mass scale/sign (M, σ) in (5.2.5). By contrast, the bound (6.4.8) is derived from a set of swampland conjectures. The $(m_V M_{\text{Pl}}/g)^{1/2}$ bound in (6.4.8) is derived by noting that the Stückelberg mass is obtained when the gauge boson eats a fundamental axion, for which we can apply an axionic version [74, 75] of the weak gravity conjecture [77]; the $g^{1/3} M_{\text{Pl}}$ bound in (6.4.8) is derived from a combination of the (sub)lattice/tower weak gravity conjecture [78, 79] (see also [50, 80]) and the species bound [81–83].

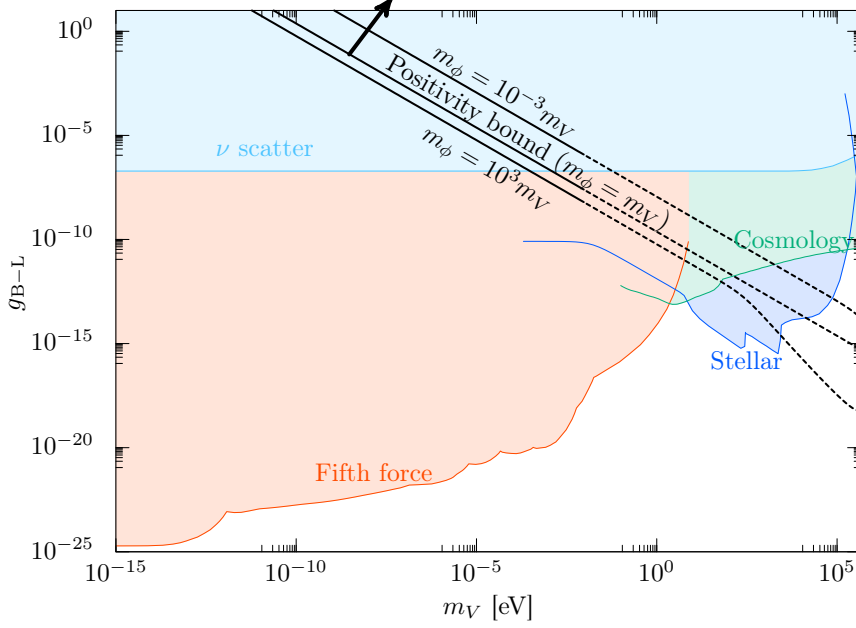


Figure 6.8: Implication to the $U(1)_{B-L}$ gauge boson from the gravitational positivity bound with $\sigma = 0$. Here we adopt $\Lambda = 1$ GeV. The positivity constraint excludes the parameter region below the black lines. The dotted lines indicate that the gauge boson mass is greater than neutrino masses and our analysis cannot be directly applied. We also show several experimental constraints from the fifth force searches [84–92], stellar cooling [93], cosmological constraints on the neutrino properties [94, 95], and XENONnT constraint on the $e - \nu$ scattering [96, 97]. We assume that right-handed neutrinos are heavy and not relevant to the constraints.

Let us also note that our bound applies to gauge boson masses both in the Higgs mechanism and the Stückelberg mechanism, while the bound (6.4.8) applies only to those in the Stückelberg mechanism. Our bound is more general in this respect.

In Fig. 6.7 we also plotted the bound from the weak gravity conjecture [77], which requires an existence of a particle with charge g and mass m such that $\sqrt{2}g \geq m/M_{\text{Pl}}$.

6.5 Comparison with observational constraints

The dark $U(1)$ gauge sectors discussed so far are theoretical constructs. However, any new physics describing our universe must necessarily coexist with the Standard Model. Since some important subtleties arise at this point, we will first briefly discuss them and then discuss the specific model (B–L gauge model) as a possible application of the gravitational positivity.

If we apply the discussion of gravitational positivity bounds to, for example, the scattering of photons and dark gauge bosons, the dark sector necessarily has interactions beyond those described by the Standard Model and gravity as discussed in next chapter. In general, these interactions allow the dark gauge boson to decay into photons and SM fermions and also decay into gravitons.

In this case, the dark gauge boson has a finite decay width, and hence care must be taken

when applying the positivity constraint. Despite those caveats, the decay rate of light and weakly interacting gauge bosons is extremely small, and we expect no major practical problems.

As an example, we consider the $U(1)_{B-L}$ extension of the Standard Model. We assume the gauge charge of the B–L breaking Higgs is the same as that of the SM leptons. In Fig. 6.8, we show the positivity bound (6.4.1) on the $U(1)_{B-L}$ model and current experimental constraints. Here we take $\sigma = 0$ and $\Lambda = 1$ GeV. In this plot, we assume that the B–L Higgs contributions are dominated over the SM fermions and neglect the SM contributions. The dashed lines in the figure show the parameter region where the dark gauge boson can decay into neutrinos. The stringent bound from the positivity constraint comes from TL scattering mode. Note that the constraint is more severe for the higher cutoff, as seen in the TL constraint (6.4.3). The gravitational positivity bound requires the larger gauge coupling for smaller masses and has strong tensions with the experimental searches.

Chapter 7

Positivity bounds on photon-dark photon scattering

7.1 Dark photon model

As described in the chapter 3, the dark photon is the gauge boson A_a^μ of an extra $U(1)$ symmetry in the dark sector to which we refer as $U(1)_d$. The kinetic mixing of A_a and the ordinary photon A_b is renormalizable, so that there is no reason to exclude it ¹. Indeed, it appears naturally if there exist heavy fields which are charged under both of the electromagnetic $U(1)_{\text{EM}}$ and the hidden $U(1)_d$. Also, A_a is massive in general. We then consider the massive dark photon model described by

$$\mathcal{L} = \frac{M_{\text{Pl}}^2}{2} R + \mathcal{L}_{\text{SM}}[A_b] + \mathcal{L}_{\text{DS}}[A_a], \quad (7.1.1)$$

where $\mathcal{L}_{\text{SM}}$ is the Standard Model Lagrangian and $\mathcal{L}_{\text{DS}}$ is the dark sector Lagrangian. We only denote the $U(1)$ gauge fields dependence explicitly.

In this work, we consider the following form of $\mathcal{L}_{\text{DS}}$:

$$\mathcal{L}_{\text{DS}}[A_a] = -\frac{1}{4}(F_{a,\mu\nu})^2 - \frac{1}{2}m_{A'}^2(A_{a,\mu})^2 - \frac{\epsilon}{2}F_{a,\mu\nu}F_b^{\mu\nu} + \mathcal{L}_{\text{hm}}[A_a]. \quad (7.1.2)$$

where $\mathcal{L}_{\text{hm}}$ denotes Lagrangian of matters in the dark sector. We implicitly assume that there is no light particle which is charged under both $U(1)_{\text{EM}}$ and $U(1)_d$ in the given EFT by using (7.1.1).

In the section 3.1, we see that the kinetic terms of $U(1)$ gauge bosons are diagonalized as

$$\mathcal{L}_{U(1)} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}F'_{\mu\nu}F'^{\mu\nu} - \frac{1}{2}m_{A'}^2(A'_\mu)^2, \quad (7.1.3)$$

in terms of new fields (A, A') which are defined by

$$\begin{pmatrix} A_a \\ A_b \end{pmatrix} = \begin{pmatrix} 1/\sqrt{1-\epsilon^2} & 0 \\ -\epsilon/\sqrt{1-\epsilon^2} & 1 \end{pmatrix} \begin{pmatrix} A' \\ A \end{pmatrix}. \quad (7.1.4)$$

¹To be precise, the dark photon kinetically mixes with the hyper-charge gauge boson. In this case, the dark photon couples to the weak neutral current as well. However, this coupling is suppressed as $\mathcal{O}(m_{A'}^2/m_Z^2)$ and irrelevant in the parameter space studied in the present paper.

The rotation (7.1.4) generates tiny couplings between A' and the Standard Model charged particles. The interaction term is given by replacing A_b of $\mathcal{L}_{\text{SM}}$ with $A_b = A - \epsilon/\sqrt{1-\epsilon^2}A'$:

$$\mathcal{L}_{\text{SM}} \left[A - \frac{\epsilon}{\sqrt{1-\epsilon^2}} A' \right] \approx \mathcal{L}_{\text{SM}} [A - \epsilon A']. \quad (7.1.5)$$

As apparent from this form, interactions between the dark photon and the Standard Model charged particles are the same form as that of the ordinary photon, and the strength of the interaction is suppressed by the kinetic mixing parameter ϵ . We will review the explicit form of the interaction in the next section.

By contrast, interactions between the dark photon and hidden matters $\mathcal{L}_{\text{hm}}[A_a]$ do not induce couplings between A and hidden matters by this rotation, because $A_a = \frac{1}{\sqrt{1-\epsilon^2}}A'$. Therefore, we do not need to specify the matter content in the dark sector to analyze the photon-dark photon scattering $\gamma\gamma' \rightarrow \gamma\gamma'$.

7.2 Explicit form of Lagrangian

In this section, we will see the explicit form of the Lagrangian (7.1.1) considered in this chapter. As we have seen in the previous section, the dark photon interacts with the Standard Model particles in the same way as the ordinary photon. Therefore, it is enough to consider the interaction between the ordinary photon and the Standard Model particles and perform the substitution $A \rightarrow A - \epsilon A'$ to analyze the photon-dark photon scattering. We also discuss the interaction between graviton and other fields.

7.2.1 The Lagrangian of the electroweak sector

In the Standard Model, QCD sector is strongly coupled at low energy. We consider the photon-dark photon scattering in $t \rightarrow 0$ limit to calculate the positivity bounds. However, the QCD sector is strongly coupled in the t-channel, and the perturbative calculation is not valid. In our work, we neglect the QCD sector and the estimation including the QCD sector is left for future work. Therefore, we focus on the electroweak sector Lagrangian $\mathcal{L}_{\text{EW}} \subset \mathcal{L}_{\text{SM}}$ below.

Among the Lagrangian of the electroweak sector, the relevant terms for the photon-dark photon scattering are [98]:

$$\mathcal{L}_{\text{EW}} = \mathcal{L}_A + \mathcal{L}_W[A] + \mathcal{L}_L[A] \quad ; \quad (7.2.1)$$

$$\mathcal{L}_A = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \quad (7.2.2)$$

$$\begin{aligned} \mathcal{L}_W[A] = & -\frac{1}{2}(\partial_\mu W_\nu^+ - \partial_\nu W_\mu^+)(\partial^\mu W^{-\nu} - \partial^\nu W^{-\mu}) - m_W^2 W_\mu^+ W^{-\mu} - ie\partial^\mu A^\nu (W_\mu^+ W_\nu^- - W_\nu^+ W_\mu^-) \\ & - ieA^\nu (-W_\mu^+ \partial_\nu W^{-\mu} + W^{-\mu} \partial_\nu W_\mu^+ + W_\mu^+ \partial^\mu W_\nu^- - W_\mu^- \partial^\mu W_\nu^+) \\ & + e^2 (A^\mu W_\mu^+ A^\nu W_\nu^- - A_\mu A^\mu W_\nu^+ W^{-\nu}), \end{aligned} \quad (7.2.3)$$

$$\mathcal{L}_L[A] = \sum_{k=e,\mu,\tau} i\bar{\psi}_k \gamma^\mu (\partial_\mu - ieA_\mu) \psi_k - m_k \bar{\psi}_k \psi_k. \quad (7.2.4)$$

$\mathcal{L}_A$ is the kinetic term of the electromagnetic field, $\mathcal{L}_W[A]$ is the Lagrangian of the W boson, and $\mathcal{L}_L[A]$ is the Lagrangian of the lepton fields, where $[A]$ denotes the A dependence. e is the electromagnetic coupling constant, and m_W is the mass of the W boson. Z boson and neutrinos are not relevant to the photon-dark photon scattering. Couplings between leptons and W boson can also be neglected because they are not relevant too.

7.2.2 The interaction between graviton and other fields

As discussed in the previous chapter, the graviton kinetic term is given by expanding the Einstein-Hilbert term around the flat metric in the harmonic gauge as

$$\mathcal{L}_h = \frac{1}{2} h_{\mu\nu} \left(\eta^{\mu\rho} \eta^{\nu\sigma} - \frac{1}{2} \eta^{\mu\nu} \eta^{\rho\sigma} \right) \partial^2 h_{\rho\sigma}, \quad (7.2.5)$$

and the propagator of the graviton is given by

$$iD_{\mu\nu,\rho\sigma}(p) = \frac{i(\eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho} - \eta_{\mu\nu}\eta_{\rho\sigma})}{p^2}. \quad (7.2.6)$$

Next, let us consider how graviton couples to other fields. Interaction terms between graviton and the SM particles is given by expanding the metric in (7.2.1) around the flat space time metric.

Let us start from vector fields, that is, photon, dark photon and W -boson. We define the dark photon kinetic term as

$$\mathcal{L}_{A'} = -\frac{1}{4}(F'_{\mu\nu})^2 - \frac{1}{2}m_{A'}^2(A'_\mu)^2. \quad (7.2.7)$$

The interaction term is given by

$$\mathcal{L}_{i,\text{grav}} = -\frac{h_{\mu\nu}T_i^{\mu\nu}}{M_{\text{Pl}}} + \mathcal{O}(M_{\text{Pl}}^{-2}), \quad (7.2.8)$$

where $i = A, A', W$ and $T_i^{\mu\nu}$ is the energy-momentum tensor defined as

$$T_{i,\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\partial \sqrt{-g} \mathcal{L}_i}{\partial g^{\mu\nu}} \Big|_{g_{\mu\nu} = \eta_{\mu\nu}}. \quad (7.2.9)$$

The explicit form of the energy-momentum tensor is given by

$$T_{A,\mu\nu} = F_{\mu\alpha}F_\nu^\alpha - \frac{\eta_{\mu\nu}}{4}F^{\alpha\beta}F_{\alpha\beta}, \quad (7.2.10)$$

$$\begin{aligned} T_{W,\mu\nu}[A] = & [(\partial_\mu W_\alpha^+ - \partial_\alpha W_\mu^+) (\partial_\nu W^{-\alpha} - \partial^\alpha W_\nu^-) \\ & + 2ie\partial_\mu A^\alpha (W_\nu^+ W_\alpha^- - W_\alpha^+ W_\nu^-) + 2ie\partial^\alpha A^\mu (W_\alpha^+ W_\nu^- - W_\nu^+ W_\alpha^-) \end{aligned}$$

$$\begin{aligned}
& -2e^2 (A_\mu W_\nu^+ A^\alpha W_\alpha^- - A_\mu A_\nu W_\alpha^+ W^{-\alpha}) - 2e^2 (A^\alpha W_\alpha^+ A_\mu W_\nu^- - A_\alpha A^\alpha W_\nu^+ W_\mu^-) \\
& + 2ieA^\alpha (-W_\mu^+ \partial_\alpha W_\nu^- + W_\mu^- \partial_\alpha W_\nu^+ + W_\mu^+ \partial_\nu W_\alpha^- - W_\mu^- \partial_\nu W_\alpha^+) \\
& + 2ieA_\mu (-W_\alpha^+ \partial_\nu W^{-\alpha} + W^{-\alpha} \partial_\nu W_\alpha^+ + W_\alpha^+ \partial^\alpha W_\nu^- - W_\alpha^- \partial^\alpha W_\nu^+) \Big] + (\mu \leftrightarrow \nu) \\
& + 2m_W^2 W_\mu^+ W_\nu^-,
\end{aligned} \tag{7.2.11}$$

$$T_{A',\mu\nu} = F'_{\mu\alpha} F'^\alpha_\nu + m_{A'}^2 A'_\mu A'_\nu + \eta_{\mu\nu} \left(-\frac{1}{4} F'^{\alpha\beta} F'_{\alpha\beta} - \frac{1}{2} m_{A'}^2 A'_\alpha A'^\alpha \right). \tag{7.2.12}$$

The Lagrangian of leptons should be re-written to the form which is suitable to the curved spacetime. Introducing the vierbein field e^μ_A , the Lagrangian of the fermion fields in the curved space-time is given by [99]

$$\sqrt{-g} [\bar{\psi} \gamma^A e^\mu_A (i\nabla_\mu + eA_\mu) \psi - m\bar{\psi}\psi], \tag{7.2.13}$$

where

$$\nabla_\mu = \partial_\mu + \frac{1}{2} \omega_{\mu AB} J^{AB}, \tag{7.2.14}$$

$$J^{AB} = \frac{1}{4} [\gamma^A, \gamma^B], \tag{7.2.15}$$

$$\omega_{\mu B}^A = -e_B^\nu (\partial_\mu e_\nu^A - \Gamma_{\mu\nu}^\lambda e_\lambda^A). \tag{7.2.16}$$

Around the flat spacetime, the vierbein field is expanded as

$$e^\mu_A = \delta^\mu_A - \frac{1}{2M_{\text{Pl}}} h^\mu_A. \tag{7.2.17}$$

Therefore, interaction terms between graviton and the lepton fields are given by

$$V_{hL}[A] = \sum_{k=e,\mu,\tau} \left[-\frac{i}{2M_{\text{Pl}}} h_A^\mu \bar{\psi}_k \gamma^A D_\mu \psi_k + \frac{1}{2M_{\text{Pl}}} h \bar{\psi}_k (i\not{D} - m_k) \psi_k + \frac{i}{2} \bar{\psi}_k (\gamma^\mu J^{AB} \omega_{\mu AB}) \psi_k \right], \tag{7.2.18}$$

where $D_\mu = \partial_\mu - ieA_\mu$, $J^{AB} = \frac{1}{4} [\gamma^A, \gamma^B]$, $\omega_{\mu AB} = \frac{1}{2M_{\text{Pl}}} (\partial_A h_{B\mu} - \partial_B h_{A\mu})$.

Combining everything, we obtain the explicit form of the Lagrangian (7.1.1) as

$$\begin{aligned}
\mathcal{L} = & \mathcal{L}_h + \left(-\frac{1}{4} F'_{\mu\nu} F'^{\mu\nu} - \frac{1}{2} m_{A'}^2 A'_\mu A'^\mu \right) - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \mathcal{L}_W[A - \epsilon A'] + \mathcal{L}_L[A - \epsilon A'] \\
& - \frac{h^{\mu\nu}}{M_{\text{Pl}}} T_{A,\mu\nu} - \frac{h^{\mu\nu}}{M_{\text{Pl}}} T_{A',\mu\nu} - \frac{h^{\mu\nu}}{M_{\text{Pl}}} T_{W,\mu\nu}[A - \epsilon A'] + V_{hL}[A - \epsilon A'] \Big|_{g^{\mu\nu}=\eta_{\mu\nu}}.
\end{aligned} \tag{7.2.19}$$

7.3 Kinematics and amplitudes

In this chapter, we consider a $2 \rightarrow 2$ photon-dark photon scattering $\gamma\gamma' \rightarrow \gamma\gamma'$. The photon is massless whereas the mass of the dark photon is $m_{A'}$. We denote momenta of incident particles

as p_1, p_2 and momenta of scattered particles as p_3 and p_4 , as usual. Mandelstam invariants for this process are defined as

$$s := (p_1 + p_2)^2, \quad (7.3.1)$$

$$t := (p_1 - p_3)^2 = -2|\vec{p}|^2(1 - \cos \theta), \quad (7.3.2)$$

$$u := (p_1 - p_4)^2 = -2|\vec{p}|^2(1 + \cos \theta), \quad (7.3.3)$$

where $\vec{p}$ is the 3-momentum of particle 1 in the center-of-momentum(COM) frame. Using Mandelstam variables, the scattering angle in the COM frame $\cos \theta = \mathbf{p}_1 \cdot \mathbf{p}_3 / |\mathbf{p}_1| |\mathbf{p}_3|$ can be written as

$$\cos \theta = 1 + \frac{2t}{s - 2m_{A'}^2}. \quad (7.3.4)$$

The relation between the Mandelstam variables is given by

$$s + t + u = 2m_{A'}^2. \quad (7.3.5)$$

As in the previous chapter, we have to deal with the kinematic singularity, and we introduce a kinematic singularity free amplitude $\widetilde{\mathcal{M}}(s, t)$ defined by

$$\widetilde{\mathcal{M}}(s, t) := s^2 (s - 2m_{A'}^2)^2 \mathcal{M}(s, t), \quad (7.3.6)$$

which maintain the $s \leftrightarrow u$ crossing symmetry in the forward limit.

We apply the gravitational positivity bound to the scattering amplitude of the photon-dark photon scattering. The positivity bound is given by

$$B(\Lambda) := a_2 - \frac{4}{\pi} \int_{m_{\text{th}}^2}^{\Lambda^2} ds \frac{\text{Im } \widetilde{\mathcal{M}}(s, t=0)}{(s - m_{A'}^2)^7}, \quad (7.3.7)$$

where m_{th} is the threshold energy. In the case of the photon-dark photon scattering considered here, the threshold energy is determined by the mass of the electron, which is the lightest charged particle in the Standard Model. Explicitly, the threshold energy is given by $m_{\text{th}}^2 = 4m_e^2$. a_2 is defined at $s = m_{A'}^2$:

$$a_2 := \lim_{t \rightarrow -0} \left[\frac{\partial^2 \mathcal{M}(s, t)}{\partial s^2} + \frac{2}{M_{\text{Pl}}^2 t} \right]_{s=m_{A'}^2}. \quad (7.3.8)$$

As in the previous chapter, $B(\Lambda)$ is composed of the non-gravitational and gravitational contributions, $B_{\text{non-grav}}(\Lambda)$ and $B_{\text{grav}}(\Lambda)$, respectively. $B_{\text{non-grav}}(\Lambda)$ is given by

$$B_{\text{non-grav}}(\Lambda) := a_{2, \text{non-grav}} - \frac{4}{\pi} \int_{4m_e^2}^{\Lambda^2} ds \frac{\text{Im } \widetilde{\mathcal{M}}_{\text{non-grav}}(s, t=0)}{(s - m_{A'}^2)^7}. \quad (7.3.9)$$

The interaction term considered in this chapter is renormalizable in the gravity decoupling limit $M_{\text{Pl}} \rightarrow \infty$, therefore $B_{\text{non-grav}}$ can be expressed as

$$B_{\text{non-grav}}(\Lambda) = \frac{4}{\pi} \int_{\Lambda^2}^{\infty} ds \frac{\text{Im } \widetilde{\mathcal{M}}_{\text{non-grav}}(s, t=0)}{(s - m_{A'}^2)^7}. \quad (7.3.10)$$

In the limit $\Lambda \gg m_{A'}$, we have

$$B_{\text{non-grav}}(\Lambda) \simeq \frac{4}{\pi} \int_{\Lambda^2}^{\infty} ds \frac{\text{Im } \mathcal{M}_{\text{non-grav}}(s, t=0)}{(s - m_{A'}^2)^3}. \quad (7.3.11)$$

Therefore, it is sufficient to us the original amplitude $\mathcal{M}$ for the evaluation of $B_{\text{non-grav}}(\Lambda)$.

The expression of the gravitational contribution $B_{\text{grav}}(\Lambda)$ is given by

$$B_{\text{grav}}(\Lambda) \simeq a_{2,\text{grav},t\text{-channel}} := \lim_{t \rightarrow -0} \left[\frac{2}{6!} \frac{\partial^6 \widetilde{\mathcal{M}}_{\text{grav},t\text{-channel}}(s, t)}{\partial s^6} + \frac{2}{M_{\text{Pl}}^2 t} \right]_{s=m_{A'}^2}. \quad (7.3.12)$$

For $\Lambda \gg m_{A'}$, this expression is simplified to

$$B_{\text{grav}}(\Lambda) \simeq \lim_{t \rightarrow -0} \left[\frac{\partial^2 \mathcal{M}_{\text{grav},t\text{-channel}}(s, t)}{\partial s^2} + \frac{2}{M_{\text{Pl}}^2 t} - (\text{kinematic singularity}) \right]_{s=m_{A'}^2}. \quad (7.3.13)$$

7.3.1 Helicity configurations

We define the polarization vector of the photon and the dark photon in the same manner in the previous chapter, (6.1.8) and (6.1.9). There are two $s \leftrightarrow u$ symmetric amplitudes depending on the helicity configurations of dark photons. For the process with transversely polarized dark photon, we consider

$$\begin{aligned} \widetilde{\mathcal{M}}^T(s, t) := & \frac{1}{4} [\widetilde{\mathcal{M}}(1^+ 2^+ 3^+ 4^+) + \widetilde{\mathcal{M}}(1^+ 2^- 3^+ 4^-) \\ & + \widetilde{\mathcal{M}}(1^- 2^- 3^- 4^-) + \widetilde{\mathcal{M}}(1^- 2^+ 3^- 4^+)], \end{aligned} \quad (7.3.14)$$

where 1 and 2 (3 and 4) are the ingoing (outgoing) photon and dark photon, respectively, and the superscript $\pm$ denotes the helicity. For the process with longitudinal modes, we consider

$$\widetilde{\mathcal{M}}^L(s, t) := \frac{1}{2} [\widetilde{\mathcal{M}}(1^+ 2^L 3^+ 4^L) + \widetilde{\mathcal{M}}(1^- 2^L 3^- 4^L)], \quad (7.3.15)$$

where L means the longitudinal polarization. For these amplitudes, the crossing symmetry implies the $s \leftrightarrow u$ permutation invariance.

Below, we refer to $(B_{\text{non-grav}}, B_{\text{grav}})$ for $\widetilde{\mathcal{M}}^T$ and $\widetilde{\mathcal{M}}^L$ as $(B_{\text{non-grav}}^T, B_{\text{grav}}^T)$ and $(B_{\text{non-grav}}^L, B_{\text{grav}}^L)$, respectively.

The one-loop diagrams are calculated by using the Mathematica packages FeynRules [64–66], FeynArts [67], FeynCalc [68–70] and Package-X [71].

7.4 Calculation of $B(\Lambda)$

From (7.2.19), it can be seen that the Standard Model charged particles mediate the photon-dark photon scattering. In this section, we summarize calculations of the gravitational positivity bounds $B_{\text{non-grav}}$, B_{grav} on the photon-dark photon scattering.

Here, we consider the light dark photon and assume that $m_{A'}$ is much smaller than other scales in the theory. Then, we can use the formulas (7.3.11) and (7.3.13) to calculate $B_{\text{non-grav}}$, B_{grav} . Therefore, it is practically enough to calculate the standard scattering amplitude $\mathcal{M}$, not the kinematic singularity subtracted amplitude $\widetilde{\mathcal{M}}$.

The form of the contribution of each charged particles to $B_{\text{non-grav}}$, B_{grav} depend on their spins. Although only fermions and W-boson directly couples to the photon in the Standard Model, we also discuss the contribution of a charged scalar particle for completeness.

As discussed in the section 7.2, the kinetic term of the graviton h , the photon A and the dark photon A' is given by

$$\mathcal{L}_h = \frac{1}{2} h_{\mu\nu} \left(\eta^{\mu\rho} \eta^{\nu\sigma} - \frac{1}{2} \eta^{\mu\nu} \eta^{\rho\sigma} \right) \partial^2 h_{\rho\sigma}, \quad (7.4.1)$$

$$\mathcal{L}_A = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}, \quad (7.4.2)$$

$$\mathcal{L}_{A'} = -\frac{1}{4} (F'_{\mu\nu})^2 - \frac{1}{2} m_{A'}^2 (A'_\mu)^2. \quad (7.4.3)$$

The graviton propagator in the harmonic gauge is given by

$$iD_{\mu\nu,\rho\sigma}(p) = \frac{i(\eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho} - \eta_{\mu\nu}\eta_{\rho\sigma})}{p^2}. \quad (7.4.4)$$

The covariant derivative is defined as

$$D_\mu = \partial_\mu - ieA_\mu, \quad (7.4.5)$$

where e is a coupling constant. The fine-structure constant is denoted as

$$\alpha = \frac{e^2}{4\pi}. \quad (7.4.6)$$

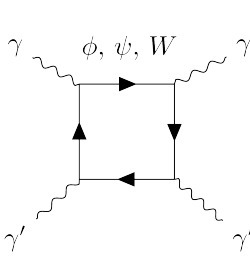
7.4.1 Scalar

The part of the Lagrangian which contain the kinetic term and the interaction term of a charged scalar ϕ is given by

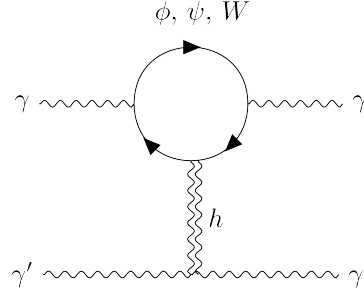
$$\mathcal{L}_S[A] = \frac{1}{M_{\text{Pl}}} h_{\mu\nu} T_S^{\mu\nu}, \quad (7.4.7)$$

where $\mathcal{L}_S[A]$ is given by

$$\mathcal{L}_S[A] = |D_\mu \phi|^2 - m_S^2 |\phi|^2, \quad (7.4.8)$$



(a) Non-gravitational Feynman diagram.



(b) t-channel graviton exchange diagram.

Figure 7.1: Examples of Feynman diagrams considered in this section. Charged particles ϕ, ψ, W mediate the photon-dark photon scattering.

and the energy momentum tensor $T_{S,\mu\nu}$ is defined as

$$T_{S,\mu\nu} = 2D_\mu\phi D_\nu\phi - \eta_{\mu\nu} [|D_\alpha\phi|^2 + m_S^2|\phi|^2]. \quad (7.4.9)$$

The full Lagrangian is given by

$$\mathcal{L} = \mathcal{L}_h + \mathcal{L}_A + \mathcal{L}_{A'} + \mathcal{L}_S[A - \epsilon A'] - \frac{1}{M_{\text{Pl}}} h_{\mu\nu} T_S^{\mu\nu}. \quad (7.4.10)$$

Using the above Lagrangian, we can calculate the contribution of charged scalar loops to the scattering amplitude $\mathcal{M}_S^T, \mathcal{M}_S^L$.

Transverse

Assuming that $s \geq \Lambda \gg m_S > m_{A'}, \mathcal{O}(M_{\text{Pl}}^0)$ part of the scattering amplitude of the transverse polarization is approximately given by

$$\mathcal{M}_{S,\text{non-grav}}^T \simeq 2\alpha^2\epsilon^2 \left(\ln \frac{m_S^2}{s} + \ln \frac{m_S^2}{-s} + 2 \right). \quad (7.4.11)$$

The imaginary part is

$$\text{Im } \mathcal{M}_{S,\text{non-grav}} \simeq 2\pi\alpha^2\epsilon^2. \quad (7.4.12)$$

Then, $B_{S,\text{non-grav}}^T$ is given by

$$B_{S,\text{non-grav}}^T = \frac{4}{\pi} \int_{\Lambda^2}^{\infty} \frac{2\pi\alpha^2\epsilon^2}{\mu^3} d\mu = \frac{4\alpha^2\epsilon^2}{\Lambda^4}. \quad (7.4.13)$$

On the other hand, the gravitational part $B_{S,\text{grav}}^T$ is given by

$$B_{S,\text{grav}}^T = -\frac{\alpha}{90\pi m_S^2 M_{\text{Pl}}^2} \quad (7.4.14)$$

Longitudinal

Assuming that $s \geq \Lambda \gg m_S > m_{A'}$, $\mathcal{O}(M_{\text{Pl}}^0)$ part of the scattering amplitude of the longitudinal polarization is approximately given by

$$\mathcal{M}_{S,\text{non-grav}}^L \simeq -\frac{2\alpha^2\epsilon^2 m_{A'}^2}{3m_S^2} + \frac{2\alpha^2\epsilon^2 m_S^2}{s} \left(\ln^2 \frac{m_S^2}{s} - \ln^2 \frac{m_S^2}{-s} + 6 \ln \frac{m_S^2}{s} - 6 \ln \frac{m_S^2}{-s} \right), \quad (7.4.15)$$

and the imaginary part is

$$\text{Im}\mathcal{M}_{S,\text{non-grav}}^L \simeq 4\pi\alpha^2\epsilon^2 \left(\ln \frac{s}{m_S^2} - 3 \right) \frac{m_{A'}^2}{s}. \quad (7.4.16)$$

Therefore, $B_{S,\text{non-grav}}^L$ is given by

$$B_{S,\text{non-grav}}^L \simeq \frac{32\alpha^2\epsilon^2 m_{A'}^2}{3\Lambda^6} \ln \frac{\Lambda}{m_S} - \frac{128\alpha^2\epsilon^2 m_{A'}^2}{9\Lambda^6}. \quad (7.4.17)$$

$B_{S,\text{grav}}$ is given by

$$B_{S,\text{grav}} \simeq -\frac{\alpha}{90\pi m_S^2 M_{\text{Pl}}^2}. \quad (7.4.18)$$

7.4.2 Fermion

The Lagrangian which contain the kinetic term and the interaction term of a charged fermion ψ is given by

$$\mathcal{L}_F[A] = i\bar{\psi}D_\mu\psi - m_F\bar{\psi}\psi + V_{fh}[A]. \quad (7.4.19)$$

The interaction term of ψ and the graviton $V_{fh}[A]$ is given by

$$V_{fh}[A] = -\frac{i}{2M_{\text{Pl}}} h_A^\mu \bar{\psi} \gamma^A D_\mu \psi + \frac{1}{2M_{\text{Pl}}} h \bar{\psi} (i\not{D} - m_f) \psi + \frac{i}{2} \bar{\psi} (\gamma^\mu J^{AB} \omega_{\mu AB}) \psi, \quad (7.4.20)$$

where $J^{AB} = \frac{1}{4} [\gamma^A, \gamma^B]$ and $\omega_{\mu AB} = \frac{1}{2M_{\text{Pl}}} (\partial_A h_{B\mu} - \partial_B h_{A\mu})$.

Transverse

Assuming that $s \geq \Lambda \gg m_F > m_{A'}$, the $\mathcal{O}(M_{\text{Pl}}^0)$ part of the scattering amplitude of the transverse polarization is approximately given by

$$\mathcal{M}_{F,\text{non-grav}}^T \simeq -2\alpha^2\epsilon^2 \left(\ln^2 \frac{-s}{m_F^2} + \ln^2 \frac{s}{m_F^2} - 2 \ln \frac{-s}{m_F^2} - 2 \ln \frac{s}{m_F^2} + 4 \right), \quad (7.4.21)$$

and the imaginary part is

$$\text{Im}\mathcal{M}_{F,\text{non-grav}}^T \simeq 4\pi\alpha^2\epsilon^2 \left(\ln \frac{s}{m_F^2} - 1 \right). \quad (7.4.22)$$

Then, $B_{F,\text{non-grav}}^T$ is

$$\begin{aligned} B_{F,\text{non-grav}}^T &= \frac{4}{\pi} \int_{\Lambda^2}^{\infty} \frac{4\pi\alpha^2\epsilon^2 \left(\ln \frac{s}{m_F^2} - 1 \right)}{\mu^3} d\mu \\ &= \frac{4\alpha^2\epsilon^2 \left(\ln \frac{\Lambda^4}{m_F^4} - 1 \right)}{\Lambda^4}. \end{aligned} \quad (7.4.23)$$

On the other hand, the gravitational part is given by

$$B_{F,\text{grav}}^T = -\frac{11\alpha}{180\pi m_F^2 M_{\text{Pl}}^2}. \quad (7.4.24)$$

Longitudinal

Assuming that $s \geq \Lambda \gg m_F > m_{A'}$, the $\mathcal{O}(M_{\text{Pl}}^0)$ part of the scattering amplitude of the longitudinal polarization is approximately given by

$$\mathcal{M}_{F,\text{non-grav}}^L \simeq -\frac{8\alpha^2\epsilon^2 m_{A'}^2}{3m_F^2} + \frac{16\alpha^2\epsilon^2 m_{A'}^2}{s} \left(\ln \frac{s}{m_F^2} + \ln \frac{m_F^2}{-s} \right), \quad (7.4.25)$$

and the imaginary part is

$$\text{Im} \mathcal{M}_{F,\text{non-grav}}^L \simeq 16\pi\alpha^2\epsilon^2 \frac{m_{A'}^2}{s} \quad (7.4.26)$$

Then, $B_{F,\text{non-grav}}^L$ is

$$\begin{aligned} B_{F,\text{non-grav}}^L &= \frac{4}{\pi} \int_{\Lambda^2}^{\infty} \frac{16\pi\alpha^2\epsilon^2 m_{A'}^2 / \mu}{\mu^3} d\mu \\ &= \frac{64\alpha^2\epsilon^2 m_{A'}^2}{3\Lambda^6} \end{aligned} \quad (7.4.27)$$

On the other hand, the gravitational part is given by

$$B_{F,\text{grav}}^L = -\frac{11\alpha}{180\pi m_F^2 M_{\text{Pl}}^2}. \quad (7.4.28)$$

7.4.3 Vector

The kinetic term and the interaction term with the photon and the graviton of the Standard Model W-boson is given by

$$\mathcal{L}_W - \frac{1}{M_{\text{Pl}}} h_{\mu\nu} T_W^{\mu\nu}, \quad (7.4.29)$$

where $\mathcal{L}_W$ is given in (7.2.3) and $T_{W,\mu\nu}$ is given in (7.2.11). Although the interaction terms between the W-boson and the photon in (7.2.3) are explicitly obtained within the electroweak symmetry breaking $\text{SU}(2) \times \text{U}(1) \rightarrow \text{U}(1)$, it can be shown that this is the unique combination of 3-point and 4-point vertices which maintain the perturbative unitarity of a photon-photon scattering.

Transverse

Assuming that $s \geq \Lambda \gg m_W > m_{A'}$, the $\mathcal{O}(M_{\text{Pl}}^0)$ part of the scattering amplitude of the transverse polarization is approximately given by

$$\mathcal{M}_{W,\text{non-grav}}^T \simeq \frac{8\alpha^2\epsilon^2 s}{m_W^2} \left(\ln \frac{s}{m_W^2} + \ln \frac{m_W^2}{-s} \right), \quad (7.4.30)$$

and the imaginary part is

$$\text{Im} \mathcal{M}_{W,\text{non-grav}}^T \simeq \frac{8\pi\alpha^2\epsilon^2 s}{m_W^2}. \quad (7.4.31)$$

Then, $B_{W,\text{non-grav}}^T$ is given by

$$B_{W,\text{non-grav}}^T \simeq \frac{32\alpha^2\epsilon^2}{m_W^2\Lambda^2} \quad (7.4.32)$$

On the other hand, the gravitational part $B_{V,\text{grav}}^T$ is given by

$$B_{W,\text{grav}}^T \simeq -\frac{47\alpha}{90\pi M_{\text{Pl}}^2 m_W^2}. \quad (7.4.33)$$

Longitudinal

Assuming that $s \geq \Lambda \gg m_W > m_{A'}$, the scattering amplitude is given by

$$\mathcal{M}_{W,\text{non-grav}}^L \simeq \frac{2\alpha^2\epsilon^2 m_{A'}^2 s}{m_W^4} \left(\ln \frac{s}{m_W^2} + \ln \frac{m_W^2}{-s} \right), \quad (7.4.34)$$

and the imaginary part is

$$\text{Im} \mathcal{M}_{W,\text{non-grav}}^L \simeq 2\pi\alpha^2\epsilon^2 \frac{m_{A'}^2}{m_W^4} s. \quad (7.4.35)$$

$B_{W,\text{non-grav}}^L$ is given by

$$B_{W,\text{non-grav}}^L \simeq \frac{8\alpha^2\epsilon^2 m_{A'}^2}{\Lambda^2 m_W^4}. \quad (7.4.36)$$

$B_{W,\text{grav}}^L$ is given by

$$B_{W,\text{grav}}^L \simeq -\frac{47\alpha}{90\pi m_W^2 M_{\text{Pl}}^2}. \quad (7.4.37)$$

7.4.4 Decoupling of longitudinal mode

From the above results, we can see that $B_{\text{non-grav}}$ for the longitudinal mode of the dark photon is suppressed by $m_{A'}^2$, compared to the transverse mode. This suppression can be explained by the decoupling of the longitudinal mode from charged particles in the limit where $m_{A'} \rightarrow 0$.

For a dark photon with the momentum

$$p^\mu = \begin{pmatrix} E \\ |\vec{p}| \sin \theta \\ 0 \\ |\vec{p}| \cos \theta \end{pmatrix}, \quad (7.4.38)$$

where $E = \sqrt{m_{A'}^2 + |\vec{p}|^2}$, the longitudinal polarization vector is given by

$$\varepsilon_L^\mu = \frac{1}{m_{A'}} \begin{pmatrix} |\vec{p}| \\ E \sin \theta \\ 0 \\ E \cos \theta \end{pmatrix}. \quad (7.4.39)$$

If the momentum of the particle is much larger than the dark photon mass $\vec{p} \gg m_{A'}$, the energy can be expanded as

$$E = |\vec{p}| + \mathcal{O}(m_{A'}^2). \quad (7.4.40)$$

Then, the polarization vector can be written as

$$\varepsilon_L^\mu = \frac{p^\mu}{m_{A'}} + \mathcal{O}(m_{A'}). \quad (7.4.41)$$

Let us consider the scattering amplitude involving a longitudinally polarized dark photon $\mathcal{M} = \varepsilon_L^\mu \mathcal{M}_\mu$. When the scattering energy is large, $\mathcal{M}$ can be written as

$$\mathcal{M} = \varepsilon_L^\mu \mathcal{M}_\mu = \frac{p^\mu}{m_{A'}} \mathcal{M}_\mu + \mathcal{O}(m_{A'}). \quad (7.4.42)$$

However, the Ward identity $p^\mu \mathcal{M}_\mu = 0$ indicates that the first term should vanish. Therefore, we conclude that

$$\mathcal{M} = \mathcal{O}(m_{A'}), \quad (7.4.43)$$

and the longitudinal mode decouples from the charged particle in $m_{A'} \rightarrow 0$ limit.

The scattering amplitude $\widetilde{\mathcal{M}}^L(s, t)$ defined in (7.3.15) involves two longitudinal dark photons. Therefore, it should be suppressed by $m_{A'}^2$ compared to the transverse mode.

7.5 Gravitational positivity bounds on dark photons

Let us apply the results of the previous section to the model (7.2.19). It is the simplest setup where the contributions of heavy particles including those in the dark sector to $B(\Lambda)$ are negligibly small. In this setup, the scattering process $\gamma\gamma' \rightarrow \gamma\gamma'$ is mediated only by the Standard Model particles. Since new particles coupled to photon below 1 TeV have not been found, we choose $\Lambda \geq 1$ TeV.

The results of the previous section show that non-gravitational contribution $B_{\text{non-grav}}(\Lambda)$ depend on the spin as well as the mass of the charged particle. The contributions of spin 1 particles are less suppressed by Λ than that of a spin 0 or 1/2 particles for both transverse and longitudinal polarization configurations. Therefore, the dominant contribution to $B^T_{\text{non-grav}}(\Lambda)$, $B^L_{\text{non-grav}}(\Lambda)$ is given by the W-boson loop.

On the other hand, a contribution to the gravitational part only depends on the mass m of the charged particle, and proportional to m^{-2} . Therefore, the dominant contributions to $B^T_{\text{grav}}(\Lambda)$, $B^L_{\text{grav}}(\Lambda)$ come from the electron loop, which is the lightest charged particle in the Standard Model.

The results are,

$$\begin{aligned} B^T_{\text{non-grav}}(\Lambda) &\simeq \frac{32\alpha^2\epsilon^2}{m_W^2\Lambda^2}, B^L_{\text{non-grav}}(\Lambda) \simeq \frac{8\alpha^2\epsilon^2 m_{A'}^2}{\Lambda^2 m_W^4}, \\ B^T_{\text{grav}}(\Lambda) &\simeq B^L_{\text{grav}}(\Lambda) \simeq -\frac{11\alpha}{180\pi m_e^2 M_{\text{Pl}}^2}, \end{aligned} \quad (7.5.1)$$

where $\alpha \simeq 1/137$ denotes the fine-structure constant.

Let us discuss the implication of (7.5.1). As discussed in chapter 5, the gravitational positivity bound (5.2.6) involves the set of UV parameters (M, σ) which is deeply tied with the properties of the UV completion of gravity and cannot be assessed theoretically in the current technique. Although we cannot make a confident prediction on the dark photon model due to this UV ambiguity, the implication of (7.5.1) highlights aspects that have not found in prior researches.

Firstly, let us consider the case where we adopt the perspective of Interpretation 1 and set $\sigma = 0$. Then, (5.2.6) implies a lower bound on ϵ : from $\mathcal{M}_T$ we obtain

$$\epsilon \geq \sqrt{\frac{11}{5760\pi\alpha}} \frac{m_W\Lambda}{m_e M_{\text{Pl}}} \simeq 1.9 \times 10^{-11} \times \left(\frac{\Lambda}{1 \text{ TeV}} \right), \quad (7.5.2)$$

while we obtain a stronger constraint from $\mathcal{M}_L$ as

$$\begin{aligned} \epsilon &\geq \sqrt{\frac{11}{1440\pi\alpha}} \frac{m_W^2\Lambda}{m_{A'} m_e M_{\text{Pl}}} \\ &\simeq 3.0 \times 10^{-3} \times \left(\frac{\Lambda}{1 \text{ TeV}} \right) \left(\frac{1 \text{ keV}}{m_{A'}} \right). \end{aligned} \quad (7.5.3)$$

The bound (7.5.3) also shows the existence of a lower bound on $m_{A'}$ for given ϵ and Λ . Note that (7.5.3) only applies to the massive dark photon case. The origin of the lower bound on $m_{A'}$ is the suppression factor $(m_{A'}/m_W)^2$ contained in $B_{\text{L,non-grav}}^{(2)}$. This factor appears because longitudinal modes of dark photons are decoupled from W-bosons in the limit $m_{A'} \rightarrow 0$, which is discussed in 7.4.4.

On the other hand, (7.5.2) and (7.5.3) can be regarded as a possibility to experimentally constrain the properties of quantum gravity from the perspective of Interpretation 3. If a dark

photon is discovered within the parameter space outside of (7.5.2) and (7.5.3), it inevitably imply $\sigma < 0$. Moreover, this would allow us to place a lower bound on M . Such opportunity is not explored in setups of previous studies [46–48].

We have neglected QCD contributions so far. Since the QCD scale is lower than the electroweak scale, it may give relevant contributions to $B_{\text{non-grav}}^{(2)}$. Indeed, QCD effects, particularly the Pomeron exchange, provide the leading contribution to $B_{\text{non-grav}}^{(2)}$ in the Standard Model analysis of [48]. If we extend the analysis to transversely polarized dark photons simply by multiplying the factor ϵ to the coupling between photon and Pomeron, the bound (7.5.2) is slightly relaxed as $\epsilon \gtrsim 10^{-13}$ for $\Lambda \sim 1$ TeV.

It would be interesting to investigate how much the bound (7.5.3) is relaxed when QCD is taken into account. In general, the QCD contributions to $\mathcal{M}_L$ will be suppressed by the dark photon mass $m_{A'}$ at least as long the mass is the Stückelberg mass. We thus expect that the existence of a lower bound on $m_{A'}$ like (7.5.3) is a robust result, leaving further studies for future work.

7.5.1 Remarks on higher-dimensional operators

In the above analysis, we assume that heavy particles in the dark sector are not charged under the Standard Model $U(1)_{\text{EM}}$ symmetry. In general, however, we can also consider dark photon models in which new particles other than γ' are also present whose mass scales are well below the quantum gravity scales such as M_{Pl} or M_s . Let us discuss the validity of our results (7.5.2) and (7.5.3) on the kinetic mixing when new particles are introduced.

Let us introduce a new particle Y with mass m_Y for illustrations. Below the mass scale m_Y , we can take into account the contribution from Y to the process $\gamma\gamma' \rightarrow \gamma\gamma'$ by adding the higher-dimensional operators to the Lagrangian: operators which are relevant for the discussion of positivity bounds at the leading order are,

$$\frac{C_1}{m_Y^4} F^2 F'^2, \quad \frac{C_2}{m_Y^6} F_{\mu\nu} F'^\nu{}_\rho \partial^\mu F'_{\alpha\beta} \partial^\rho F'^{\alpha\beta}. \quad (7.5.4)$$

Here, Wilson coefficients C_1 and C_2 are typically given by the couplings between Y and (γ, γ') (See figure 7.2). The contributions of the leading higher-dimensional operators (7.5.4) to $B_{\text{T,non-grav}}(\Lambda)$ and $B_{\text{L,non-grav}}(\Lambda)$ are estimated as

$$B_{\text{T,non-grav}}(\Lambda)|_{\text{operators (7.5.4)}} \sim \frac{C_1}{m_Y^4}, \quad (7.5.5a)$$

$$B_{\text{L,non-grav}}(\Lambda)|_{\text{operators (7.5.4)}} \sim \frac{C_2 m_{A'}^2}{m_Y^6}, \quad (7.5.5b)$$

where we have $\Lambda < m_Y$ because the cutoff scale cannot be heavier than the mass of Y which is integrated out. When these contributions are larger than those from particles in the Standard Model (7.5.1), the implications of the positivity bound (5.2.6) will be different from (7.5.2) and

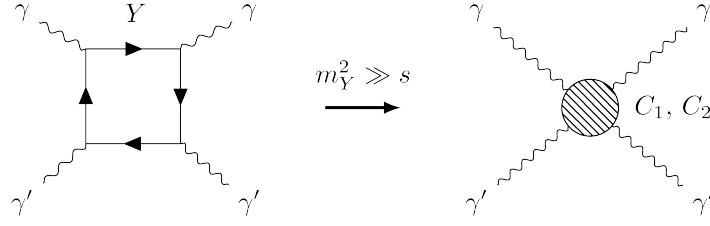


Figure 7.2: If the scattering energy is small compared to the mass of the charged particle Y in the dark sector, loop diagrams of Y can be approximated by effective operators.

(7.5.3).² According to eqs. (7.5.1) and (7.5.5), the contributions from higher-dimensional operators to $B_{\text{T,non-grav}}(\Lambda)$ and $B_{\text{T,non-grav}}(\Lambda)$ are negligible when the following condition is satisfied, respectively:

$$m_Y \gtrsim \left(\frac{|C_1|}{\epsilon^2 \alpha^2} \right)^{\frac{1}{4}} \sqrt{m_W \Lambda}, \quad (7.5.6a)$$

$$m_Y \gtrsim \left(\frac{|C_2|}{\epsilon^2 \alpha^2} \right)^{\frac{1}{6}} (m_W^2 \Lambda)^{\frac{1}{3}}, \quad (7.5.6b)$$

where we impose $(m_W <) \Lambda < m_Y$.

C_1 and C_2 are not suppressed by ϵ in general if Y is charged under both of $U(1)$ and $U(1)_d$. In particular, we can consider the scenario in which $|C_1|, |C_2| \gg \epsilon^2$ such that the conditions (7.5.6) are violated even for very heavy m_Y . Therefore, it will be possible to construct models which are consistent with the positivity bound (5.2.6) with $\sigma = 0$ regardless of the value of ϵ , by introducing heavy bi-charged particles.

²The contribution from Y to $B_{\text{T,grav}}$ and $B_{\text{L,grav}}$ will be typically $\mathcal{O}(M_{\text{Pl}}^{-2} m_Y^{-2})$ which is negligible compared to those from electron (7.5.1).

Chapter 8

Conclusion

In this thesis, we explored gravitational positivity bounds on massive $U(1)$ gauge boson models. For non-gravitational theories, positivity bounds are derived from fundamental properties of scattering amplitudes, which encode consistency conditions of quantum field theories such as unitarity. The dispersion relation connects the IR and UV regimes of the scattering amplitude and imposes constraints on the coefficients of low-energy expansion of amplitudes, which can be translated into bounds on EFT parameters.

For theories with gravity, however, the t -channel pole arising from graviton exchange obstructs the naive derivation of positivity bounds. Nevertheless, this t -channel pole can be subtracted by assuming that the scattering amplitudes exhibit Regge behavior in the high-energy regime. Once the t -channel subtracted dispersion relation is established, it is straightforward to derive positivity bounds for gravitational theories.

The gravitational positivity bound (5.2.6) establishes a connection between EFT amplitudes and the set of the UV parameters (M, σ) . The implication of these bounds depend on M and σ , which parameterize the UV behavior of scattering amplitudes. When $\sigma = 0, +1$ or M is sufficiently large, it can be considered as constraints on the parameter space of the EFT, reflecting the nature of quantum gravity. Schematically, this bound takes the form of $B_{\text{non-grav}} > |B_{\text{grav}}|$, where $B_{\text{non-grav}}$ and B_{grav} represents non-gravitational and gravitational contributions, respectively. From this relation, lower bounds on non-gravitational interactions can be derived. On the other hand, (5.2.6) can be viewed as IR constraints on quantum gravity. From the data that can be accessible in IR, it is possible to infer the information about the UV amplitude.

In this thesis, we formulated gravitational positivity bounds on the scattering amplitude of gauge bosons and derived constraints on parameter spaces of theories that include a massive $U(1)$ gauge boson. First, we discussed the gravitational positivity bounds for a massive $U(1)$ gauge boson V by considering the $VV \rightarrow VV$ scattering process. This scattering is mediated by either a Higgs boson or charged particles depending on the mass generation mechanism. In the case of Stückelberg mechanism, we derived lower bounds on the gauge coupling of the charged particle and the mass of the gauge boson. We compared our bounds with other quantum gravity

constraints found that our bound is the strongest, especially when the gauge boson mass is small. For the Higgs mechanism, the scattering is mediated by the Higgs particle. We obtained lower bounds on the gauge coupling of Higgs, the mass of Higgs and the gauge boson. We also compare our bounds with observational constraints on the $U(1)_{B-L}$ gauge boson. It turns out that the positivity bounds are sufficiently strong to be competitive with these observational constraints.

We also study the dark photon dark matter model in chapter 7. By analyzing photon-dark photon scattering, we calculate $B^T(\Lambda)$, $B^L(\Lambda)$ within the simplest model where only the dark photon and the Standard Model particles are present. Assuming $\sigma = 0$, we found that gravitational positivity bounds impose lower limits on $m_{A'}$ and ϵ .

In particular, $B_{\text{non-grav}}^L(\Lambda)$ is suppressed by $m_{A'}^2$ due to the decoupling of longitudinal modes in the $m_{A'} \rightarrow 0$ limit, and make it difficult to reconcile small $m_{A'}$, ϵ and $\sigma = 0$ in this model. We also argued that the presence of a heavy bi-charged particle could provide a dominant contribution to the photon-dark photon scattering. In this case, tiny value of ϵ might be compatible with $\sigma = 0$.

We conclude this thesis with a discussions on potential future directions. First, it is crucial to deepen our understanding of the parameters (M, σ) to obtain concrete constraints. These parameters are determined from UV data, making it non-trivial to demonstrate that they possesses desirable features. One promising approach is to constrain UV parameters using unitarity techniques as discussed in [52, 53]. Another approach is to calculate the scattering amplitude in the high-energy region within specific setup, as discussed in [54, 100].

There are also some subtleties involved in applying our method to realistic models, which we did not touch in this work. First, we ignored issues arising from massless loops. Massless loops alter the analytic structure of scattering amplitudes, and it is necessary to choose different contours in deriving positivity bounds [42]. Additionally, massless loops present problems related to IR divergence. Although we estimated the contributions from massless loops by introducing IR cutoff by hand and discussed that these contributions can be ignored, it would be preferable to formulate positivity bounds on scattering amplitudes with IR divergence properly addressed using technique such as the dressed state formalism [101]. There is also a problem in applying positivity bounds to decaying particles, as these particles are not asymptotic states and, strictly speaking, the S-matrix does not exist for them. There is discussion on formulating positivity bounds for decaying particles [61], and it would be interesting to apply this formulation within our setup.

As mentioned at the end of chapter 7, it is interesting to consider a theory in which heavy bi-charged particles mediate photo-dark photon scattering. In such a model, $\sigma = 0$ and tiny value of ϵ might be compatible. However, there are a lot of experimental constraints on a new particle charged under $U(1)_{EM}$. For example, long-lived charged particles have been searched for in collider experiments. In particular, the ATLAS collaboration has placed bounds on

muon-like, long-lived, multi-charged articles using $\sqrt{s} = 13$ TeV proton-proton collision data. These bounds range from 1050 GeV to 1570 GeV, depending on the particle's charge [102]. Furthermore, if heavy charged particles Y were present during the time of nucleosynthesis, some protons in nuclei could have been replaced by Y , resulting in the formation of anomalously heavy hydrogen. By examining deep-sea water samples, the authors of [103] have placed an upper limit on the fraction of Y as $Y/p < 4 \times 10^{-17}$ for the mass range $5u < m_Y < 1600u$. Therefore, it is crucial to establish a concrete setup and scrutinize the relevant experimental results before drawing definitive conclusion.

Appendix A

Special function

A.1 Hypergeometric function

The (Gaussian) hypergeometric function ${}_2F_1(a, b; c; z)$ is defined by

$${}_2F_1(a, b; c; z) = \sum_{n=1}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}, \quad (\text{A.1.1})$$

where $(x)_n$ is the Pochhammer symbol

$$(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)}. \quad (\text{A.1.2})$$

The hypergeometric function is a solution of hypergeometric differential equation

$$z(1-z) \frac{d^2 {}_2F_1(a, b; c; z)}{dz^2} + [c - (a+b+1)] \frac{d {}_2F_1(a, b; c; z)}{dz} - ab {}_2F_1(a, b; c; z) = 0. \quad (\text{A.1.3})$$

The hypergeometric function has the integral expression:

$${}_2F_1(a, b; c; z) = \frac{1}{B(a, c-a)} \int_0^1 t^{a-1} (1-t)^{c-a-1} (1-zt)^{-b} dt \quad \text{Re}(c) > \text{Re}(a) > 0. \quad (\text{A.1.4})$$

This equality is proved by expanding $(1-zt)^{-b}$ and integrating term by term. Especially,

$$\begin{aligned} {}_2F_1(a, b; c; 1) &= \frac{1}{B(a, c-a)} \int_0^1 t^{a-1} (1-t)^{c-a-b-1} dt \\ &= \frac{B(a, c-a-b)}{B(a, c-a)} \\ &= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}. \end{aligned} \quad (\text{A.1.5})$$

${}_2F_1(a, b; c; z)$ has a branch cut along $z \in [1, \infty]$ and the discontinuity is given by

$$\begin{aligned} \text{Disc } {}_2F_1(a, b; c; z) &:= {}_2F_1(a, b; c; z + i\varepsilon) - {}_2F_1(a, b; c; z - i\varepsilon) \\ &= \frac{-2\pi i \Gamma(c)}{\Gamma(1+c-a-b)\Gamma(a)\Gamma(b)} (z-1)^{c-a-b} {}_2F_1(c-a, c-b; c-a-b+1; 1-z) \end{aligned} \quad (\text{A.1.6})$$

This is the consequence of an linear transformation formula

$${}_2F_1(a, b; c; z) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} {}_2F_1(a, b; a+b-c+1; 1-z) + \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} (1-z)^{c-a-b} {}_2F_1(c-a, c-b; c-a-b+1; 1-z). \quad (\text{A.1.7})$$

A.2 Rodrigues formula

In this subsection, we will review the Rodrigues formula, which gives polynomial solutions of an equation of hypergeometric type, based on [104].

Let us consider the differential equation

$$\sigma(z)y'' + \tau(z)y' + \lambda y = 0, \quad (\text{A.2.1})$$

where $\sigma(z)$ is a polynomial of degree at most 2, and τ is a polynomial of degree at most 1. A differential equation of this form is called an equation of hypergeometric type. More general differential equation

$$u'' + \frac{\tilde{\tau}(z)}{\sigma(z)}u' + \frac{\tilde{\sigma}(z)}{\sigma^2(z)}u = 0, \quad (\text{A.2.2})$$

where $\sigma(z), \tilde{\sigma}(z)$ are polynomials of degree at most 2, and $\tilde{\tau}$ is a polynomial of degree at most 1, can be reduced to an equation of hypergeometric type through a change of variables.

We will discuss conditions for (A.2.1) having an polynomial solution. Let us assume $y(z)$ to be a solution of (A.2.1) which is a polynomial of degree n . Using $v_1(z) \equiv y'(z)$, the differentiation of (A.2.1) is:

$$\sigma(z)v_1'' + \tau_1(z)v_1' + \mu_1 v_1 = 0, \quad (\text{A.2.3})$$

where

$$\begin{aligned} \tau_1(z) &\equiv \tau(z) + \sigma'(z), \\ \mu_1 &\equiv \lambda + \tau'(z), \end{aligned} \quad (\text{A.2.4})$$

and (A.2.3) is an equation of hypergeometric type which have a solution of a polynomial of order $n-1$. By repeating this operation k times, we obtain

$$\sigma(z)v_k'' + \tau_k(z)v_k' + \mu_k v_k = 0, \quad (\text{A.2.5})$$

where $v_k \equiv \frac{d^k}{dz^k} y(z)$ is a polynomial of degree $n-k$ and

$$\begin{aligned} \tau_k(z) &\equiv \tau(z) + k\sigma'(z), \\ \mu_k &\equiv \lambda + k\tau'(z) + \frac{k(k-1)}{2}\sigma''(z). \end{aligned} \quad (\text{A.2.6})$$

Especially, considering $k = n$, μ_n should be zero so that (A.2.5) have a constant solution v_n . In other words, a condition for (A.2.1) having an polynomial solution is given by

$$\lambda = -n\tau' - \frac{1}{2}n(n-1)\sigma'' . \quad (\text{A.2.7})$$

Then, we will obtain this polynomial solution explicitly. Let $\rho(z)$ be a function which satisfies

$$[\sigma(z)\rho(z)]' = \tau(z)\rho(z) . \quad (\text{A.2.8})$$

Using $\rho(z)$, (A.2.1) can be written as

$$\frac{1}{\rho(z)} [\rho(z)\sigma(z)y'(z)]' + \lambda y(z) = 0 . \quad (\text{A.2.9})$$

$\rho(z)$ is chosen according to degree of $\sigma(z)$ as

$$\rho(z) = \begin{cases} (b-z)^\alpha(z-a)^\beta & \text{for } \sigma(z) = (b-z)(z-a) , \\ (z-a)^\alpha e^{\beta z} & \text{for } \sigma(z) = z-a , \\ e^{\alpha z^2 + \beta z} & \text{for } \sigma(z) = 1 . \end{cases} \quad (\text{A.2.10})$$

Similarly for (A.2.5), defining $\rho_k(z)$ as

$$[\sigma(z)\rho_k(z)]' = \tau_k(z)\rho_k(z) , \quad (\text{A.2.11})$$

we obtain

$$\frac{1}{\rho_k(z)} [\rho_k(z)\sigma(z)v'_k(z)]' + \mu_k v_k(z) = 0 \quad (\text{A.2.12})$$

From (A.2.6), (A.2.8), and (A.2.11),

$$\frac{\rho'_k}{\rho_k} = \frac{\rho'}{\rho} + k \frac{\sigma'}{\sigma} , \quad (\text{A.2.13})$$

and consequently we have

$$\rho_k(z) = \rho(z)\sigma^k(z) . \quad (\text{A.2.14})$$

Using this equation and $v_{k+1} = v'_k$, (A.2.12) becomes

$$\rho_k v_k = -\frac{1}{\mu_k} (\rho_{k+1} v_{k+1})' . \quad (\text{A.2.15})$$

From this equation, it is understood that $y^{(m)}(z)$ is given by differentiating $\rho_n v_n \propto \rho(z)\sigma^n(z)$:

$$y^{(m)}(z) = v_m(z) = \frac{A_m B}{\rho(z)\sigma^m(z)} [\rho(z)\sigma^n(z)]^{(n-m)} . \quad (\text{A.2.16})$$

Here, constants A_m, B is given by

$$A_m \equiv (-1)^m \prod_{k=0}^{m-1} \mu_k, \quad \mu_0 \equiv \lambda \quad (\text{A.2.17})$$

$$B \equiv \frac{1}{A_n} y^{(n)}(z) . \quad (\text{A.2.18})$$

Especially, the polynomial solution $y(z)$ is given by

$$y(z) = \frac{B}{\rho(z)} [\rho(z) \sigma^n(z)]^{(n)} , \quad (\text{A.2.19})$$

which is called the Rodrigues formula.

In the following, we denote $y_n(z)$ as a n -th order polynomial solution of (A.2.1), and $\lambda_n, B_n, \mu_{k;n}$ as corresponding λ, B, μ_k . Let us assume that

$$\sigma(x) \rho(x) x^k|_{x=a,b} = 0 \quad (k = 0, 1, \dots) \quad (\text{A.2.20})$$

is satisfied for some points a, b . Then, the orthogonality condition

$$\int_a^b y_n(x) y_m(x) \rho(x) dx = \begin{cases} \left(-\frac{\sigma''}{2}\right)^n \frac{\Gamma\left(n-1+\frac{2\tau'}{\sigma''}\right)}{\Gamma\left(2n-1+\frac{2\tau'}{\sigma''}\right)n!} \int_a^b \sigma^n(x) \rho(x) y_n^{(n)}(x) y_n^{(n)}(x) dx & (n = m) , \\ 0 & (n \neq m) , \end{cases} \quad (\text{A.2.21})$$

can be proved as follows.

Firstly, Let us consider $n \neq m$ case. Without loss of generality, we can assume $m < n$. Using notation here, (A.2.15) can be written as

$$\frac{d}{dx} [\rho_{k+1}(x) y_n^{(k+1)}(x)] + \mu_{k;n} \rho_k(x) y_n^{(k)}(x) = 0 . \quad (\text{A.2.22})$$

Multiplying this by $y_m^{(k)}$ and integrating from a to b ,

$$y_m^{(k)}(x) \rho_{k+1}(x) y_n^{(k+1)}(x) \Big|_a^b - \int_a^b dx \rho_{k+1}(x) y_m^{(k+1)}(x) y_n^{(k+1)}(x) + \mu_{k;n} \int_a^b dx \rho_k(x) y_m^{(k)}(x) y_n^{(k)}(x) = 0 . \quad (\text{A.2.23})$$

The first term vanishes due to (A.2.20) so that

$$\int_a^b \rho_{k+1}(x) y_m^{(k+1)}(x) y_n^{(k+1)}(x) dx = \mu_{k;n} \int_a^b \rho_k(x) y_m^{(k)}(x) y_n^{(k)}(x) dx \quad (\text{A.2.24})$$

is derived. By repeating the above procedure from $k = 0$ to $k = m + 1$, it is proved that

$$\int_a^b \rho(x) y_m(x) y_n(x) dx = \left(\prod_{k=0}^m \frac{1}{\mu_{k;n}} \right) \int_a^b \sigma^n(x) \rho(x) y_m^{(m+1)}(x) y_n^{(m+1)}(x) dx = 0 . \quad (\text{A.2.25})$$

Next, let us consider $n = m$ case. By repeating (A.2.24) from $k = 1$ to $k = n$, the norm of $y_n(x)$ can be calculated as

$$\int_a^b \rho(x) y_n(x) y_n(x) dx = \left(\prod_{k=0}^{n-1} \frac{1}{\mu_{k;n}} \right) \int_a^b \sigma^n(x) \rho(x) y_n^{(n)}(x) y_n^{(n)}(x) dx$$

$$= \left(\prod_{k=0}^{n-1} \frac{1}{\mu_{kn}} \right) \int_a^b \sigma^n(x) \rho(x) y_n^{(n)}(x) y_n^{(n)}(x) dx . \quad (\text{A.2.26})$$

Because

$$\begin{aligned} \mu_{k;n} &= \lambda_n + k\tau' + \frac{k(k-1)}{2} \sigma'' \\ &= k\tau' + \frac{k(k-1)}{2} \sigma'' - n\tau' - \frac{n(n-1)}{2} \sigma'' \\ &= -\frac{\sigma''}{2} (n-k) \left(k+n-1 + \frac{2\tau'}{\sigma''} \right) , \end{aligned} \quad (\text{A.2.27})$$

the product of $\mu_{k;n}$ is calculated as

$$\prod_{k=0}^{n-1} \mu_{k;n} = \left(-\frac{\sigma''}{2} \right)^n n! \frac{\Gamma(2n-1 + \frac{2\tau'}{\sigma''})}{\Gamma(n-1 + \frac{2\tau'}{\sigma''})} . \quad (\text{A.2.28})$$

Using this result, we finally obtain

$$\int_a^b \rho(x) y_n(x) y_n(x) dx = \left(-\frac{\sigma''}{2} \right)^n \frac{\Gamma(n-1 + \frac{2\tau'}{\sigma''})}{\Gamma(2n-1 + \frac{2\tau'}{\sigma''}) n!} \int_a^b \sigma^n(x) \rho(x) y_n^{(n)}(x) y_n^{(n)}(x) dx . \quad (\text{A.2.29})$$

A.3 Q function

Using Cauchy's integral formula, the Rodrigues formula (A.2.19) can be written as

$$y_n(z) = \frac{B_n n!}{\rho(z)} \oint_{\mathcal{C}} \frac{ds}{2\pi i} \frac{\sigma^n(s) \rho(s)}{(s-z)^{n+1}} , \quad (\text{A.3.1})$$

where $\mathcal{C}$ is a closed contour surrounding the point $s = z$.

For $z \notin [a, b]$, it is possible to define a function which is called a function of the second kind by changing a contour of (A.3.1) as follows:

$$\mathcal{Q}_n(z) = \frac{B_n n!}{\rho(z)} \int_a^b \frac{\sigma^n(s) \rho(s)}{(s-z)^{n+1}} ds . \quad (\text{A.3.2})$$

This function satisfies the same differential equation as $y_n(x)$. As is obvious from the definition, $\mathcal{Q}_n(z)$ has branch points at $z = a, b$.

It is possible to translate (A.3.2) by repeating integration by parts as

$$\begin{aligned} \mathcal{Q}_n(z) &= \frac{B_n}{\rho(z)} \int_a^b \frac{(d^n/ds^n) [\sigma^n(s) \rho(s)]}{s-z} ds \\ &= \frac{1}{\rho(z)} \int_a^b \frac{y_n(s) \rho(s)}{s-z} ds , \end{aligned} \quad (\text{A.3.3})$$

where the Rodrigues formula is used in the second line.

For $z \in [a, b]$, $\mathcal{Q}_n(z)$ has the discontinuity. Using

$$\text{Disc} \frac{1}{s-z} = \frac{1}{s-z-i\varepsilon} - \frac{1}{s-z+i\varepsilon}$$

$$= \pi \delta(s - z), \quad (\text{A.3.4})$$

$\mathcal{Q}_n(z)$ satisfies

$$\text{Disc } \mathcal{Q}_n(z) = \pi y_n(z) \quad (\text{A.3.5})$$

A.4 Gamma function

Gamma function is defined as

$$\Gamma(x) \equiv \int dt t^{x-1} e^{-t}. \quad (\text{A.4.1})$$

For $n \in \mathbb{N}$,

$$\frac{\Gamma(n)}{\Gamma\left(\frac{n}{2}\right) \Gamma\left(\frac{n+1}{2}\right)} = \frac{2^{n-1}}{\sqrt{\pi}}. \quad (\text{A.4.2})$$

A.5 Legendre polynomial

Legendre polynomial of order n is defined as

$$P_n(x) = \frac{1}{2^n n!} \left(\frac{d}{dx} \right)^n (x^2 - 1)^n. \quad (\text{A.5.1})$$

$P_n(x)$ satisfies the following equation:

$$\frac{d}{dx} \left[(1 - x^2) \frac{d}{dx} f(x) \right] + n(n+1) f(x) = 0. \quad (\text{A.5.2})$$

Let us calculate inner products of Legendre polynomials using (A.2.29). The functions which define Legendre polynomials are given by

$$\begin{aligned} \rho &= 1, \\ \sigma &= x^2 - 1, \\ \tau &= 2x, \\ \frac{d^n}{dx^2} P_n(x) &= \frac{(2n)!}{2^n n!}. \end{aligned} \quad (\text{A.5.3})$$

Then, the inner product is given by

$$\begin{aligned} \int_{-1}^1 P_n(x) P_m(x) dx &= \delta_{nm} \cdot \frac{\Gamma(n+1)}{\Gamma(2n+1) n!} \cdot \left[\frac{(2n)!}{2^n n!} \right]^2 \int_{-1}^1 (1-x^2)^n dx \\ &= \delta_{nm} \cdot \frac{1}{(2n)!} \cdot \left[\frac{(2n)!}{2^n n!} \right]^2 \cdot 2^{2n+1} \int_0^1 s^n (1-s)^n ds \\ &= \delta_{nm} \cdot \frac{1}{(2n)!} \cdot \left[\frac{(2n)!}{2^n n!} \right]^2 \cdot 2^{2n+1} \frac{\Gamma(n+1)^2}{\Gamma(2n+2)} \end{aligned}$$

$$= \frac{2}{2n+1} \delta_{nm} . \quad (\text{A.5.4})$$

Changing the variable from x to $y = \frac{1-x}{2}$ in (A.5.2), we have

$$y(1-y) \frac{d^2}{dy^2} P_n + (1-2y) \frac{d}{dy} P_n + n(n+1) P_n = 0. \quad (\text{A.5.5})$$

This is the hyper geometric differential equation with $\alpha = -l, \beta = l+1, \gamma = 1$. Therefore, the legendre polynomial can be written as

$$P_n(x) = {}_2F_1(-n, n+1; 1; (1-x)/2). \quad (\text{A.5.6})$$

Using this expression, $P_n(x)$ can be analytically continued to non-integer l . For $x < -1$, P_l have the discontinuity. Using (A.1.6), it is given by

$$\text{Disc} P_l(x) = -2i P_l(-z) \sin \pi l \quad (\text{A.5.7})$$

The asymptotic behavior of $P_l(z)$ in $|z| \rightarrow \infty$ is given by

$$P_l(z) \rightarrow \begin{cases} \pi^{-\frac{1}{2}} \frac{\Gamma(l+\frac{1}{2})}{\Gamma(l+1)} (2z)^l & \text{Re} l \geq -\frac{1}{2} \\ \pi^{-\frac{1}{2}} \frac{\Gamma(-l-\frac{1}{2})}{\Gamma(-l)} (2z)^{-l-1} & \text{Re} l \leq -\frac{1}{2} \end{cases} \quad (\text{A.5.8})$$

The legendre function of the second kind is given by

$$Q_l(x) = -\frac{1}{2} \int_{-1}^1 \frac{P_l(s)}{s-x} ds \quad (\text{A.5.9})$$

, where normalization is different from that of (A.3.3). $Q_l(z)$ can be expressed using the hypergeometric function as

$$Q_l(z) = \pi^{\frac{1}{2}} \frac{\Gamma(l+1)}{\Gamma(l+\frac{3}{2})} (2z)^{-l-1} F\left(\frac{1}{2}l+1, \frac{1}{2}l+\frac{1}{2}; \frac{1}{2}l+\frac{3}{2}; z^{-2}\right). \quad (\text{A.5.10})$$

When $x \in [-1, 1]$, the equation

$$\frac{1}{2i} (Q_l(x+i\varepsilon) - Q_l(x-i\varepsilon)) = -\frac{\pi}{2} P_l(x) \quad (\text{A.5.11})$$

is satisfied.

The asymptotic behavior of $Q_l(z)$ is given by

$$Q_l(z) \xrightarrow{z \rightarrow \infty} \pi^{\frac{1}{2}} \frac{\Gamma(l+1)}{\Gamma(l+\frac{3}{2})} (2z)^{-l-1} \quad (\text{A.5.12})$$

Using (A.5.6), (A.5.10), and the relation between hypergeometric functions, there is the relation between P_l and Q_l :

$$P_l(-z) = e^{-i\pi l} P_l(z) - \frac{2}{\pi} \sin(\pi l) Q_l(z). \quad (\text{A.5.13})$$

The reflection relation is given by

$$Q_l(-z) = -e^{-i\pi l} Q_l(z). \quad (\text{A.5.14})$$

There is integral relation

$$\int_1^\infty P_\alpha(z) Q_l(z) dz = \frac{1}{(l-\alpha)(l+\alpha+1)}. \quad (\text{A.5.15})$$

A.6 Spherical harmonics

Spherical harmonics is defined as

$$Y_l^m(\theta, \varphi) \equiv \sqrt{\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!}} P_l^m(\cos \theta) e^{im\varphi} , \quad (\text{A.6.1})$$

where $P_l^m(\cos \theta)$ is defined using Legendre function as

$$P_l^m(x) = (-1)^m (1-x^2)^{m/2} \frac{d^m}{dx^m} P_l(x) , \quad (\text{A.6.2})$$

and called associated Legendre polynomial.

There is an important relation called the spherical harmonic addition theorem:

$$P_n(\cos \gamma) = \frac{4\pi}{2n+1} \sum_{m=-n}^n (-1)^m Y_n^m(\theta_1, \varphi_1) Y_n^{-m}(\theta_2, \varphi_2) , \quad (\text{A.6.3})$$

where γ is the angle between the directions (θ_1, φ_1) and (θ_2, φ_2) :

$$\cos \gamma = \cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2 \cos(\varphi_1 - \varphi_2) . \quad (\text{A.6.4})$$

A.7 Gegenbauer polynomial

Gegenbauer polynomials are d-dimensional analog of legendre polynomials. In this section, we will review some fundamental properties of Gegenbauer polynomials.

Let us start from the metric of $d-1$ dimensional spherical coordinates:

$$\begin{aligned} ds^2 &= dr^2 + r^2 d\theta_1^2 + r^2 \sin^2 \theta_1 d\theta_2^2 + \cdots + (r^2 \sin^2 \theta_1 \cdots \sin^2 \theta_{d-1}) d\theta_{d-2}^2 \\ \sqrt{g} &= r^{d-2} (\sin \theta_1)^{d-3} \cdots \sin \theta_{d-3} . \end{aligned} \quad (\text{A.7.1})$$

Then, the Laplacian is given by

$$\nabla^2 = \frac{1}{\sqrt{g}} \partial_\mu (\sqrt{g} g^{\mu\nu} \partial_\nu) = \partial_r^2 + \frac{1}{r^2 (\sin \theta_1)^{d-3}} \partial_{\theta_1} \left[(\sin \theta_1)^{d-3} \partial_{\theta_1} \right] + \cdots . \quad (\text{A.7.2})$$

The eigenvalue equation of the differential operator, where r^2 is multiplied by the θ_1 component of the Laplacian, and $z = \cos \theta_1$ is substituted is:

$$\left[(1-z^2)^{\frac{4-d}{2}} \frac{d}{dz} (1-z^2)^{\frac{d-2}{2}} \right] f + \lambda f = 0 . \quad (\text{A.7.3})$$

This differential equation is a equation of hypergeometric type where

$$\sigma(z) = 1 - z^2 , \quad (\text{A.7.4})$$

$$\tau(z) = -(d-2)z , \quad (\text{A.7.5})$$

$$\rho(z) = (1 - z^2)^{\frac{d-4}{2}} . \quad (\text{A.7.6})$$

Using (A.2.7), it has l -dimensional polynomial solution $P_l^{(d)}(z)$ if

$$\lambda = \lambda_l \equiv l(l + d - 3) . \quad (\text{A.7.7})$$

Explicitly,

$$\left[(1 - z^2)^{\frac{4-d}{2}} \frac{d}{dz} (1 - z^2)^{\frac{d-2}{2}} \frac{d}{dz} + l(l + d - 3) \right] P_l^{(d)}(z) = 0 \quad (\text{A.7.8})$$

holds. This solution is called Gegenbauer polynomial. Using the Rodrigues formula (A.2.19), it can be written as

$$P_l^{(d)}(z) = \frac{B_l}{(1 - z^2)^{\frac{d-4}{2}}} \frac{d^l}{dz^l} \left[(1 - z^2)^{\frac{d-4}{2} + l} \right] , \quad (\text{A.7.9})$$

, where B_l is a normalization constant.

(A.7.8) equal to

$$x(1 - x) \frac{d^2}{dx^2} P_l^{(d)} + \frac{d-2}{2} (1 - 2x) \frac{d}{dx} P_l^{(d)} + l(l + d - 3) P_l^{(d)} = 0 \quad (\text{A.7.10})$$

by the change of variable $x = \frac{1-z}{2}$. This is the hypergeometric differential equation and Gegenbauer polynomial can be written using the hypergeometric function as:

$$P_l^{(d)}(z) \propto {}_2F_1 \left(-l, l + d - 3; \frac{d-2}{2}; \frac{1-z}{2} \right) . \quad (\text{A.7.11})$$

In the following, we calculate appropriate normalization factor B_l where Gegenbauer polynomial coincide with the hypergeometric function. Because

$$\frac{d}{dz^l} [(1 - z)^x (1 + z)^x] = \sum_{k=0}^l \binom{l}{k} (-1)^k \frac{\Gamma(x+1)^2}{\Gamma(x-k+1) \Gamma(x+k-l+1)} (1 - z)^{x-k} (1 + z)^{x+k-l} , \quad (\text{A.7.12})$$

the following relation is satisfied:

$$\begin{aligned} \frac{1}{(1 - z^2)^{\frac{d-4}{2}}} \frac{d^l}{dz^l} \left[(1 - z^2)^{\frac{d-4}{2} + l} \right] &= \sum_{k=0}^l \binom{l}{k} (-1)^k \frac{\Gamma\left(\frac{d-4}{2} + l + 1\right)^2 (1 - z)^{l-k} (1 + z)^k}{\Gamma\left(\frac{d-4}{2} + l - k + 1\right) \Gamma\left(\frac{d-4}{2} + k + 1\right)} \\ &= (-1)^l 2^l \frac{\Gamma\left(\frac{d-4}{2} + l + 1\right)}{\Gamma\left(\frac{d-4}{2} + 1\right)} + \mathcal{O}(1 - z) . \end{aligned} \quad (\text{A.7.13})$$

Then, choosing

$$B_l = (-1)^l \frac{\Gamma\left(\frac{d-4}{2} + 1\right)}{2^l \Gamma\left(\frac{d-4}{2} + l + 1\right)} , \quad (\text{A.7.14})$$

the desired result is obtained:

$$P_l^{(d)}(z) = {}_2F_1 \left(-l, l + d - 3; \frac{d-2}{2}; \frac{1-z}{2} \right). \quad (\text{A.7.15})$$

We will calculate the norm using (A.2.29) under this normalization. Firstly, the term proportional to z^l is given by

$$\begin{aligned} P_l^{(d)}(z) &= {}_2F_1 \left(-l, l + d - 3; \frac{d-2}{2}; \frac{1-z}{2} \right) \\ &= \sum_{k=0}^l \frac{(-l)_k (l + d - 3)_k}{k! \left(\frac{d-2}{2} \right)_k} \left(\frac{1-z}{2} \right)^k \\ &= (-1)^l \frac{\Gamma(2l + d - 3) / \Gamma(l + d - 3)}{\Gamma\left(\frac{d-2}{2} + l\right) / \Gamma\left(\frac{d-2}{2}\right)} \left(\frac{1-z}{2} \right)^l + \mathcal{O}(z^{l-1}) \\ &= \frac{\Gamma(2l + d - 3) \Gamma\left(\frac{d-2}{2}\right)}{2^l \Gamma\left(\frac{d-2}{2} + l\right) \Gamma(l + d - 3)} z^l + \mathcal{O}(z^{l-1}), \end{aligned} \quad (\text{A.7.16})$$

and $\frac{d^l}{dz^l} P_l^{(d)}(z)$ is given by

$$\frac{d^l}{dz^l} P_l^{(d)}(z) = \frac{l! \Gamma(2l + d - 3) \Gamma\left(\frac{d-2}{2}\right)}{2^l \Gamma\left(\frac{d-2}{2} + l\right) \Gamma(l + d - 3)}. \quad (\text{A.7.17})$$

Therefore, the norm can be calculated as

$$\begin{aligned} &\int_{-1}^1 (1 - z^2)^{\frac{d-4}{2}} P_l^{(d)}(z) P_k^{(d)}(z) dz \\ &= \delta_{lk} \times \frac{\Gamma(l + d - 3)}{\Gamma(2l + d - 3) l!} \times \left[\frac{l! \Gamma(2l + d - 3) \Gamma\left(\frac{d-2}{2}\right)}{2^l \Gamma\left(\frac{d-2}{2} + l\right) \Gamma(l + d - 3)} \right]^2 \times \int_{-1}^1 (1 - z^2)^{\frac{d-4}{2} + l} dz \\ &= \delta_{lk} \times \frac{l! \Gamma(2l + d - 3) \Gamma\left(\frac{d-2}{2}\right)^2}{2^{2l} \Gamma\left(\frac{d-2}{2} + l\right)^2 \Gamma(l + d - 3)} \times 2^{2l + d - 3} \int_0^1 x^{\frac{d-4}{2} + l} (1 - x)^{\frac{d-4}{2} + l} dx \\ &= \delta_{lk} \times \frac{2^{d-3} l! \Gamma(2l + d - 3) \Gamma\left(\frac{d-2}{2}\right)^2}{\Gamma\left(\frac{d-2}{2} + l\right)^2 \Gamma(l + d - 3)} \times \frac{\Gamma\left(\frac{d-2}{2} + l\right)^2}{\Gamma(2l + d - 2)} \\ &= \frac{2^{d-3} l! \Gamma\left(\frac{d-2}{2}\right)^2}{\Gamma(l + d - 3) (2l + d - 3)} \delta_{lk}. \end{aligned} \quad (\text{A.7.18})$$

Defining constant factors $\mathcal{N}_d, n_J^{(d)}$ as

$$\mathcal{N}_d = \frac{(16\pi)^{\frac{2-d}{2}}}{\Gamma\left(\frac{d-2}{2}\right)}, \quad n_J^{(d)} = \frac{(4\pi)^{\frac{d}{2}} (d + 2J - 3) \Gamma(d + J - 3)}{\pi \Gamma\left(\frac{d-2}{2}\right) \Gamma(J + 1)}, \quad (\text{A.7.19})$$

The norm can be written as

$$\frac{1}{2} \int_{-1}^1 dz (1 - z^2)^{\frac{d-4}{2}} P_J^{(d)}(z) P_{\tilde{J}}^{(d)}(z) = \frac{\delta_{J\tilde{J}}}{\mathcal{N}_d n_J^{(d)}}. \quad (\text{A.7.20})$$

There is a identity called the Gegenbauer addition formula [105]:

$$2^{4-d} \frac{\Gamma(d-3)}{\Gamma^2\left(\frac{d-3}{2}\right)} \int_{-1}^1 dz P_J^{(d)} \left(z_1 z_2 + z \sqrt{1-z_1^2} \sqrt{1-z_2^2} \right) (1-z^2)^{\frac{d-5}{2}} = P_J^{(d)}(z_1) P_J^{(d)}(z_2) . \quad (\text{A.7.21})$$

When $d = 4$, this is the integral of the addition theorem of spherical harmonics (A.6.3) . By changing the variable of the integral

$$y = z_1 z_2 + z \sqrt{1-z_1^2} \sqrt{1-z_2^2} , \quad (\text{A.7.22})$$

with integration limits

$$z_1 z_2 - \sqrt{1-z_1^2} \sqrt{1-z_2^2} < y < z_1 z_2 + \sqrt{1-z_1^2} \sqrt{1-z_2^2} . \quad (\text{A.7.23})$$

This can be written as

$$1 - z_1^2 - z_2^2 - y^2 + 2y z_1 z_2 > 0 . \quad (\text{A.7.24})$$

Using

$$\mathcal{P}_d(y, z_1, z_2) = \frac{2\pi^{\frac{d-3}{2}}}{\Gamma\left(\frac{d-3}{2}\right)} (1-y^2)^{\frac{4-d}{2}} \frac{\Theta(1-y^2-z'^2-z''^2+2yz'z'')}{(1-y^2-z'^2-z''^2+2yz'z'')^{\frac{5-d}{2}}} \quad (\text{A.7.25})$$

$$1-z^2 = \frac{1-z_1^2-z_2^2-y^2-2yz_1z_2}{(1-z_1^2)(1-z_2^2)} , \quad (\text{A.7.26})$$

we get

$$\begin{aligned} & 2^{4-d} \frac{\Gamma(d-3)}{\Gamma^2\left(\frac{d-3}{2}\right)} \int_{-1}^1 dz P_J^{(d)} \left(z_1 z_2 + z \sqrt{1-z_1^2} \sqrt{1-z_2^2} \right) (1-z^2)^{\frac{d-5}{2}} \\ &= 2^{4-d} \frac{\Gamma(d-3)}{\Gamma^2\left(\frac{d-3}{2}\right)} \int_{-1}^1 dy \frac{\Theta(1-y^2-z'^2-z''^2+2yz'z'')}{\sqrt{1-z_1^2} \sqrt{1-z_2^2}} P_J^{(d)}(y) \left(\frac{1-z_1^2-z_2^2-y^2-2yz_1z_2}{(1-z_1^2)(1-z_2^2)} \right)^{\frac{d-5}{2}} \\ &= (1-z_1^2)^{\frac{4-d}{2}} (1-z_2^2)^{\frac{4-d}{2}} \frac{2^{4-d} \Gamma(d-3)}{2\pi^{\frac{d-3}{2}} \Gamma\left(\frac{d-3}{2}\right)} \int_{-1}^1 dy \mathcal{P}_d(y, z_1, z_2) (1-y^2)^{\frac{d-4}{2}} P_J^{(d)}(y) \\ &= (1-z_1^2)^{\frac{4-d}{2}} (1-z_2^2)^{\frac{4-d}{2}} \frac{2^{4-d}}{2\pi^{\frac{d-3}{2}}} \frac{2^{d-4}}{\sqrt{\pi}} \Gamma\left(\frac{d-2}{2}\right) \int_{-1}^1 dy \mathcal{P}_d(y, z_1, z_2) (1-y^2)^{\frac{d-4}{2}} P_J^{(d)}(y) \\ &= (1-z_1^2)^{\frac{4-d}{2}} (1-z_2^2)^{\frac{4-d}{2}} \frac{1}{2} \pi^{\frac{2-d}{2}} \Gamma\left(\frac{d-2}{2}\right) \int_{-1}^1 dy \mathcal{P}_d(y, z_1, z_2) (1-y^2)^{\frac{d-4}{2}} P_J^{(d)}(y) . \quad (\text{A.7.27}) \end{aligned}$$

This shows that the kernel $\mathcal{P}_d(z, z', z'')$ can be written as

$$\mathcal{P}_d(z, z', z'') = (4\pi)^{d-2} \mathcal{N}_d^2 (1-z'^2)^{\frac{d-4}{2}} (1-z''^2)^{\frac{d-4}{2}} \sum_{J=0}^{\infty} n_J^{(d)} P_J^{(d)}(z) P_J^{(d)}(z') P_J^{(d)}(z'') . \quad (\text{A.7.28})$$

A.7.1 Derivation of kernel

Let us consider $d - 2$ -dimensional vectors $\vec{p}_1, \vec{p}_3$ (They represent momentum of 2-2 scattering amplitude) and $\vec{n}$. Let us define $z = \frac{\vec{p}_1 \cdot \vec{p}_3}{|\vec{p}_1| |\vec{p}_3|}$, $z' = \frac{\vec{p}_1 \cdot \vec{n}}{|\vec{p}_1|}$, $z'' = \frac{\vec{p}_3 \cdot \vec{n}}{|\vec{p}_3|}$. Then, $\vec{n}$ can be written using $\vec{p}_1, \vec{p}_3$ as

$$\vec{n} = \alpha \frac{\vec{p}_1}{|\vec{p}_1|} + \beta \frac{\vec{p}_3}{|\vec{p}_3|} + \vec{n}_\perp, \quad \vec{n}_\perp \cdot \vec{p}_1 = \vec{n}_\perp \cdot \vec{p}_3 = 0, \quad (\text{A.7.29})$$

where α, β are real coefficients. There are relations between α, β and z', z'' :

$$z' = \alpha + \beta z, \quad z'' = \beta + \alpha z, \quad \alpha = \frac{z' - z z''}{1 - z^2}, \quad \beta = \frac{z'' - z z'}{1 - z^2}. \quad (\text{A.7.30})$$

The integral of $d - 1$ dimensional solid angle is

$$\begin{aligned} \int d^{d-2} \Omega_{\vec{n}} &= 2 \int d^{d-1} \vec{n} \delta(\vec{n}^2 - 1) \\ &= 2\sqrt{1 - z^2} \int d\alpha d\beta d^{d-3} \vec{n}_\perp \delta(\vec{n}_\perp^2 + \alpha^2 + \beta^2 + 2\alpha\beta z - 1) \\ &= 2\sqrt{1 - z^2} \int d\alpha d\beta r_\perp^{d-4} dr_\perp d^{d-4} \Omega_{\vec{n}_\perp} \delta(r_\perp^2 + \alpha^2 + \beta^2 + 2\alpha\beta z - 1) \\ &= \sqrt{1 - z^2} \text{Vol}_{S^{d-4}} \int d\alpha d\beta \frac{\Theta(1 - \alpha^2 - \beta^2 - 2\alpha\beta z)}{(1 - \alpha^2 - \beta^2 - 2\alpha\beta z)^{\frac{5-d}{2}}} \\ &= \sqrt{1 - z^2} \text{Vol}_{S^{d-4}} \int \frac{1}{1 - z^2} dz' dz'' \frac{\Theta(1 - z^2 - z'^2 - z''^2 + 2zz'z'')}{(1 - z^2 - z'^2 - z''^2 + 2zz'z'')^{\frac{5-d}{2}}} (1 - z^2)^{\frac{5-d}{2}} \\ &= \int_{-1}^1 dz' \int_{-1}^1 dz'' \mathcal{P}_d(z, z', z''). \end{aligned} \quad (\text{A.7.31})$$

Here, we define

$$\mathcal{P}_d(z, z', z'') = \frac{2\pi^{\frac{d-3}{2}}}{\Gamma(\frac{d-3}{2})} (1 - z^2)^{\frac{4-d}{2}} \frac{\Theta(1 - z^2 - z'^2 - z''^2 + 2zz'z'')}{(1 - z^2 - z'^2 - z''^2 + 2zz'z'')^{\frac{5-d}{2}}}, \quad (\text{A.7.32})$$

and $\text{Vol}_{S^{d-4}} = \frac{2\pi^{(d-3)/2}}{\Gamma(\frac{d-3}{2})}$.

Appendix B

Gravitational Contributions Beyond Graviton t -channel

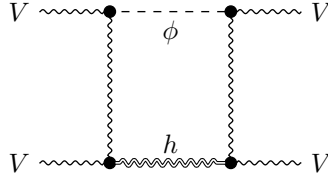


Figure B.1: An example of a gravitational contribution beyond t -channel exchange.

The gravitational contribution $B_{\text{grav}}(\Lambda)$ to $B(\Lambda)$ can be divided into $B_{\text{grav},t\text{-channel}}(\Lambda)$ from the graviton t -channel exchange and the remaining part $B_{\text{grav,others}}(\Lambda)$ (see Fig. B.1). In this Appendix we show that we can neglect $B_{\text{grav,others}}(\Lambda)$ in comparison with $B_{\text{grav},t\text{-channel}}(\Lambda)$.

Calculation of $B_{\text{grav,others}}(\Lambda)$ is challenging due to the presence of IR-divergent diagrams in $\widetilde{\mathcal{M}}_{\text{grav,others}}(s, t)$. While it is more desirable to define IR-finite amplitudes using the dressed state formalism, it is beyond the scope of this work and will be pursued in future research. To regularize this IR divergence, instead, we introduce an IR regulator μ_{IR} by deforming the graviton propagator as

$$\frac{iP_{\mu\nu\rho\sigma}}{q^2 + i\epsilon} \rightarrow \frac{iP_{\mu\nu\rho\sigma}}{q^2 - \mu_{\text{IR}}^2 + i\epsilon} , \quad (\text{B.0.1})$$

where $P_{\mu\nu\rho\sigma}$ is defined by $P_{\mu\nu\rho\sigma} = (\eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho} - \eta_{\mu\nu}\eta_{\rho\sigma})/2$ in the harmonic gauge. Using the deformed graviton propagator, we can confirm that $|\widetilde{\mathcal{M}}_{\text{grav,others}}(s, t)|$ is bounded by $\mathcal{O}(s^6)$ as $|s| \rightarrow \infty$. Along with the analyticity of $\widetilde{\mathcal{M}}_{\text{grav,others}}$, this makes it possible to derive the relation

$$B_{\text{grav, others}}(\Lambda) = \frac{4}{\pi} \int_{\Lambda^2}^{\infty} ds \frac{\text{Im } \widetilde{\mathcal{M}}_{\text{grav, others}}(s, t=0)}{(s - 2m_V^2)^7} . \quad (\text{B.0.2})$$

Moreover, since $\text{Im } \widetilde{\mathcal{M}}_{\text{grav,others}}(s, t)$ is also bounded by $\mathcal{O}(s^6)$, we obtain

$$B_{\text{grav, others}}(\Lambda) = \mathcal{O}\left(\frac{1}{M_{\text{Pl}}^2 \Lambda^2}\right) . \quad (\text{B.0.3})$$

This is much smaller compared to the contribution from the graviton t -channel exchange diagrams and can be safely neglected.

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