



Non-equilibrium Effective Field Theory for Gravity

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Doctoral Dissertation

Non-equilibrium
Effective Field Theory for Gravity
(重力に対する非平衡有効場理論)

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Abstract

Black hole thermodynamics has been extensively studied as a field that captures the properties of gravitational theory and its quantum aspects. Particularly, the second law of thermodynamics is intriguing. Various attempts have been made to prove the entropy increase law for the generalized entropy, which is the sum of the black hole's surface area and the entropy of the matter outside the black hole.

In this thesis, motivated by the black hole thermodynamics in dynamical spacetimes, we aim to construct a gravitational theory that describes non-equilibrium scenarios. For this purpose, we propose a recipe for constructing an effective field theory (EFT) in the Schwinger-Keldysh formalism for dynamical gravitational systems. The EFT is formulated based on symmetry and quantum unitarity. However, a naive attempt to incorporate effects such as dissipation into the effective action would break part of the diffeomorphism symmetries, leading to inconsistencies with gravitational theory. This issue arises from the property that gravity couples to all degrees of freedom. To resolve this, Stückelberg fields are introduced to restore symmetry. These fields, for instance, are modeled as an EFT of hydrodynamics, playing the role of an environment sector. Finally, using this model that incorporates both gravity and hydrodynamics, we discuss the generalized second law for a dynamical black hole surrounded by a fluid. Specifically, using the entropy of a dynamical black hole defined by the area of the apparent horizon and the entropy of the fluid, it is shown that the change of the generalized entropy is mainly contributed by the temperature gradient between the black hole temperature and the local temperature of the fluid.

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Chapter 1

Introduction

A black hole is a solution to classical gravity theories. In classical settings, the black hole can only absorb infalling matter, and its mass never decreases. However, when treated semi-classically, incorporating the effects of quantum fluctuations, a black hole radiates energy and loses mass through Hawking radiation [1]. In this context, a black hole can be regarded as a non-equilibrium thermodynamic system, with its entropy defined in terms of the area of its “horizon” [2–4]. Indeed, when the horizon area is interpreted as entropy, the mass as energy, and the conserved charge as particle number with an associated chemical potential, the relationship among these quantities aligns with the first law of thermodynamics:

$$T \frac{\delta A}{4G} = \delta M - \Omega_H \delta Q, \quad (1.1)$$

where A is the area of the black hole, M is the black hole mass, Q is a conserved charge, Ω_H is potential and G is the Newton constant. Also, the temperature of a black hole T is determined by its surface gravity. Furthermore, if a black hole absorbs only ordinary matter, the area of its horizon—and hence its entropy—continues to increase, satisfying the second law of thermodynamics $\frac{\delta A}{4G} \geq 0$. However, important extensions to these results remain. For example, when matter falls into a black hole, the black hole’s area increases, but the entropy of the external matter S_{mat} decreases. This might appear to challenge the validity of the entropy-increase law for the matter outside. Moreover, when the black hole emits thermal radiation, its mass and area decrease, potentially violating the null energy condition and the second law. To address these issues, it becomes necessary to consider a generalized second law of thermodynamics that incorporates the entropy of the matter outside the black hole.

$$\delta S_{\text{mat}} + \frac{\delta A}{4G} \geq 0, \quad (1.2)$$

where S_{mat} is the entropy of matter outside the black hole. This sum of entropies $S_{\text{mat}} + \frac{A}{4G}$ is called the “generalized entropy”. Various attempts have been made to prove the entropy increase law for the generalized entropy [5–22]. The second extension is to generalize the consideration of black hole thermodynamics, which is typically studied for stationary black holes for simplicity,

to dynamical cases. For stationary black holes, the concept of the event horizon is used to understand the thermodynamic relations explained above. However, since the event horizon is determined by information from the infinite past to the infinite future, it cannot capture the local time evolution of the black hole. Therefore, it becomes difficult to consider non-equilibrium processes such as the absorption and evaporation of matter by the black hole, so it seems more appropriate to characterize the entropy with the area of the apparent horizon (A_{app}) which is a local concept [20, 23–31].

In this thesis, we address this problem by considering the Schwinger-Keldysh (SK) formalism of effective field theory (EFT) for dynamical gravity. The SK formalism is a powerful tool for describing non-equilibrium processes, such as real-time evolution in finite-temperature systems and open systems. However, applying the SK formalism to construct an open system in dynamical gravity theories poses issues related to diffeomorphism symmetry. For instance, in open systems, energy conservation does not generally hold, whereas in dynamical gravity theories, the Bianchi identity, which stems from diffeomorphism symmetry, ensures the conservation law of the energy-momentum tensor. In fact, when attempting to construct a theory that includes dissipative terms, it can be shown that part of the diffeomorphism symmetry is broken and the energy-momentum tensor is not conserved. In fact, when attempting to construct a theory that includes dissipative terms, it can be shown that part of the diffeomorphism symmetry is broken. This issue is resolved using the Stückelberg trick. By introducing Stückelberg fields, the broken diffeomorphism symmetry is restored. Treating these Stückelberg fields dynamically, they typically behave like a fluid and play the role of an environment that receives the energy loss due to dissipation. This stems from the nature of gravity, which couples to all degrees of freedom.

Through these procedures, it becomes possible to construct an SKEFT that describes situations with a fluid surrounding a dynamical black hole, while maintaining symmetry and unitarity. Furthermore, by imposing the dynamical Kubo-Martin-Schwinger (KMS) symmetry on the effective action, the introduced entropy current (See [32–35]) enables us to define the generalized entropy. Here, the entropy of the dynamical black hole is characterized not by the event horizon but by the apparent horizon. And then, by calculating the time evolution of the generalized entropy defined in this way, one can examine the conditions under which the generalized second law holds.

In this thesis, we construct gravitational theory with a fluid in the SK formalism and discuss the generalized second law in a black hole spacetime with a fluid, based on our paper [36]. This thesis is organized as follows.

- In Chapter 2, we review the black hole thermodynamics. First, after introducing Einstein’s theory of gravity, we consider the Schwarzschild solution as the simplest black hole solution

and see how the black hole is characterized by the horizon (Sec. 2.1). Then, using the Killing vector and Killing horizon, we confirm that the causal structure of such spacetimes can be understood through spacetime symmetries. Finally, in Sec. 2.2, we verify that the causal structure based on these symmetries can be understood in terms of thermodynamics by deriving the first and second laws of black hole thermodynamics, and comment on the issues in the thermodynamics of stationary black holes from the perspective of the generalized second law and the apparent horizon.

- In Chapter 3, we introduce the SK formalism. In Sec. 3.1, we first provide an overview of the SK formalism and the general structure of its EFT using a toy model. In Sec. 3.2, as preparation for constructing dynamical gravity theories in the SK formalism, we discuss the properties of diffeomorphism symmetry on the closed-time-path. Here, we explain how the doubled diffeomorphism symmetry separates into diagonal (physical diffeomorphism) and off-diagonal (noise diffeomorphism) parts in the classical limit. In Sec. 3.3, as an example, we construct a dynamical gravity theory in the SK formalism using the theory of a real scalar field. At this point, we explain the breaking of noise diffeomorphism symmetry due to dissipation and the use of the Stückelberg trick to resolve this issue. By treating the Stückelberg fields dynamically, we demonstrate how an environment sector, such as a fluid, is naturally introduced. Sec. 3.2 and 3.3 are part of the main content of this thesis.
- Chapter 4 is the main part of the thesis, where we study dissipative gravity theories. First, in Sec. 4.1, we explain how the effects of dissipation in an effective field theory of gravity, including dissipation, can be characterized using a dissipative fluid introduced as an environmental sector. As an application, in Sec. 4.2, we consider a scenario where a black hole is surrounded by a fluid. In this context, the EFT of hydrodynamics defines a local temperature and an entropy current based on the dynamical KMS symmetry. As a result, we describe how the change of the generalized entropy are characterized by the temperature gradient between the black hole's temperature and the fluid's local temperature. Furthermore, considerations are made for both cases: ordinary fluids (satisfying the null energy condition) being accreted by the black hole and exotic matter (violating the null energy condition), such as Hawking radiation, escaping from the black hole. It is shown that the temperature gradient is enhanced by the blueshift factor determined by the direction of the fluid, ensuring that the generalized entropy increases in both scenarios. This chapter and the previous chapter are based on our paper [36].
- Chapter 5 is devoted to the conclusion and the prospects.

In this thesis, we use natural units with $\hbar = c = 1$, and restore them as necessary.

Chapter 2

Black hole thermodynamics

A black hole, a solution of the Einstein equations, serves as a fundamental object for exploring the properties of gravitational theories. When spacetime is significantly warped by intense gravitational fields, a causal boundary known as the “black hole horizon” emerges. As the black hole’s mass increases, the strength of gravity grows, leading to an expansion of this horizon. This behavior aligns with well-established thermodynamic principles, where the black hole’s mass corresponds to internal energy, the horizon area represents entropy, and the conserved charges are analogous to chemical potential. Additionally, in the framework of quantum field theory on curved spacetime, a temperature can be associated with the black hole, further cementing its thermodynamic interpretation. Thus, “black hole thermodynamics” has become a crucial field for deepening our understanding of gravitational theories and their quantum aspects.

The physical quantities of the black hole, corresponding to the aforementioned thermodynamic quantities, including temperature, are deeply related to the symmetries of spacetime. These symmetries are described by a vector field known as the Killing vector. Furthermore, using this Killing vector, the Killing horizon of a stationary black hole can be characterized.

In this chapter, we review the thermodynamic properties of stationary black holes. Sec. 2.1 introduces the characteristics of black hole horizons, the symmetries of black hole solutions, and their associated Killing vectors. Building on these foundations, Sec. 2.2 derives the first and second laws of black hole thermodynamics, emphasizing their connection to the symmetries outlined in Sec. 2.1. Finally, we conclude with a discussion of unresolved questions in the thermodynamics of stationary black holes.

2.1 Black holes and symmetries

A black hole represents a solution to the equations of motion in gravitational theories. In this section, we explore the causal structure of black holes through the Schwarzschild solution, the simplest example of such solutions. Additionally, we examine how this causal structure connects to spacetime symmetries, with a focus on the role of Killing vectors.

2.1.1 Einstein gravity

The action, which describes the dynamics of gravity and is called the Einstein-Hilbert action, is given by

$$S_{\text{EH}} = \frac{1}{16\pi G} \int d^4x \sqrt{-g} (R - 2\Lambda) , \quad (2.1)$$

where G is the Newton constant, Λ is the cosmological constant, and R is the Ricci scalar. The action, including matter, is given by the following:

$$S = S_{\text{EH}} + S_{\text{matter}} . \quad (2.2)$$

By varying this action with respect to the metric $g_{\mu\nu}$, the Einstein equations are obtained.

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} - \Lambda g_{\mu\nu} , \quad (2.3)$$

where the Einstein tensor and the stress tensor are defined as the following:

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} , \quad T_{\mu\nu} \equiv -\frac{2}{\sqrt{-g}} \frac{\delta S_{\text{matter}}}{\delta g^{\mu\nu}} . \quad (2.4)$$

The Einstein equations are generally non-linear differential equations and are difficult to solve, but several solutions are known in cases with sufficiently high symmetries.

2.1.2 Schwarzschild black hole

Let us start from the simplest black hole solution. The Schwarzschild metric is the most general spherically symmetric vacuum solution of the Einstein equations:

$$ds^2 = - \left(1 - \frac{2GM}{r} \right) dt^2 + \frac{dr^2}{1 - \frac{2GM}{r}} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) , \quad (2.5)$$

where M is a positive constant and corresponds to the black hole mass. This solution has a singularity at $r = 0$ and that can be checked by calculating the Kretschmann invariant scalar, which is given by

$$R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} = \frac{48G^2 M^2}{r^6} . \quad (2.6)$$

Then, the metric has a coordinate singularity at $r = 2GM$ where the component of the metric g_{rr} diverges and g_{tt} goes to zero, but it can be removed by a coordinate transformation.

Causal structures. In the above Schwarzschild metric, the coordinate singularity appears, but it disappears through a coordinate transformation. First, let us consider the following coordinate:

$$u = t + r^* , \quad v = t - r^* , \quad (2.7)$$

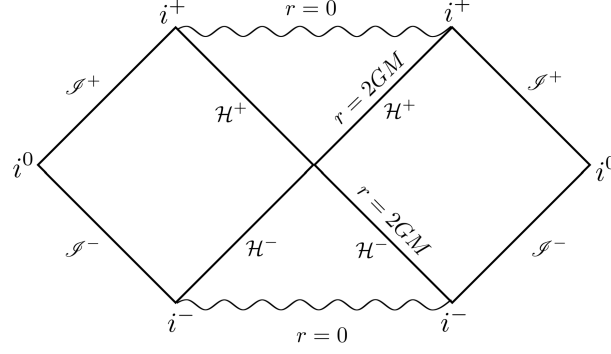


Figure 2.1: This figure shows Penrose–Carter diagram of the Schwarzschild spacetime depicting the causal structure. The vertical axis represents the time direction, and i^0 and $\mathcal{I}^\pm$ are the locations where $r = \infty$. i^+ (i^-) is the location where r is finite and $t = \infty$ ($t = 0$). The hypersurface at $r = 2GM$ forms the future (past) event horizon $\mathcal{H}^+$ ($\mathcal{H}^-$). As can be seen from this diagram, light rays emitted from the region where $r < 2GM$ in the future region cannot escape beyond the future event horizon $\mathcal{H}^+$ and eventually reach the singularity at $r = 0$.

$$r^* = \int \frac{dr}{1 - \frac{2GM}{r}} = r + 2MG \ln \left| \frac{r}{2GM} - 1 \right| \quad (2.8)$$

Then, by using $r^* \simeq 2MG \left| \frac{r}{2GM} - 1 \right|$ on a small neighbourhood of the surface $r = 2GM$, we can obtain

$$U = \mp \exp \left(-\frac{u}{4GM} \right), \quad V = \exp \left(\frac{v}{4GM} \right), \quad (2.9)$$

where the upper sign of U corresponds to the part of $r > 2MG$, while the lower sign corresponds to $r < 2MG$. The coordinates U and V are called null Kruskal coordinates. By using this coordinate, a regular metric on $r = 2GM$ can be obtained:

$$ds^2 = -\frac{32M^3G^3}{r} e^{-r/2GM} dU dV + r^2 (d\theta^2 + \sin^2 \theta d\phi^2). \quad (2.10)$$

The causal structure of this spacetime becomes clearer by depicting the spacetime in a finite region using the following coordinates:

$$\tilde{U} = \arctan U, \quad \tilde{V} = \arctan V, \quad (2.11)$$

From Fig 2.1, we can see that $r = 2GM$ surface provides the event horizon $\mathcal{H}^\pm$. A person at $r < 2GM$ cannot send a signal to outside of the event horizon, and the signal reach to $r = 0$ at future infinity.

Apparent horizons. The apparent horizon is a concept that represents a boundary in space-time, separating regions where outgoing light rays are still capable of expanding and those where they are not. Unlike the event horizon, which is a global and absolute boundary marking the

point beyond which no information can escape from a black hole, the apparent horizon is a local feature that depends on the observer's position and the moment in time.

More formally, the apparent horizon is the surface where the expansion of outgoing null geodesics (light rays) vanishes. In dynamic spacetimes, especially in non-stationary situations like during gravitational collapse or mergers, the apparent horizon may change over time. It can move depending on the energy distribution in spacetime.

2.1.3 Killing vectors and Killing horizons

Killing vectors. Transformation rules for tensor components by diffeomorphisms $x^\mu \rightarrow x^\mu + \xi^\mu(x)$ are given by the Lie derivative

$$\mathcal{L}_\xi T^{\mu_1 \dots \mu_k}_{\nu_1 \dots \nu_l} = \xi^\alpha \nabla_\alpha T^{\mu_1 \dots \mu_k}_{\nu_1 \dots \nu_l} - \sum_{i=1}^k T^{\mu_1 \dots \beta \dots \mu_k}_{\nu_1 \dots \nu_l} \nabla_\beta \xi^{\mu_i} + \sum_{i=1}^l T^{\mu_1 \dots \mu_k}_{\nu_1 \dots \beta \dots \nu_l} \nabla_{\nu_i} \xi^\beta \quad (2.12)$$

If a metric is invariant under some diffeomorphisms,

$$\mathcal{L}_\xi g_{\mu\nu} = 0, \quad (2.13)$$

this symmetry is called isometry. This ξ^μ is called Killing vector and it satisfies the Killing equation:

$$\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0. \quad (2.14)$$

Killing horizons. Let us consider the isometry and the Killing vector of the Schwarzschild metric as an example. The metric is invariant under a constant a shift of time $t \rightarrow t + a$, clearly. A corresponding time-like Killing vector t^μ is

$$t^\mu \partial_\mu = \partial_t. \quad (2.15)$$

The inner product of this Killing vector is given by

$$g_{\mu\nu} t^\mu t^\nu = - \left(1 - \frac{2GM}{r} \right), \quad (2.16)$$

so t^μ is null on the event horizon. In this sense, the null hypersurface which is orthogonal to the Killing vector is called Killing horizon. In the Schwarzschild spacetime, the event horizon and the Killing horizon coincide.

Surface gravities. Let us consider the thermodynamics of black holes. The surface gravity κ plays an important role, and it is given on the Killing horizon by

$$\xi^\mu \nabla_\mu \xi^\nu = \kappa \xi^\nu. \quad (2.17)$$

The normalization of κ can be decided so that $\xi^\mu \xi_\mu \rightarrow -1$ as $r \rightarrow \infty$. Alternatively, the following equations give the equivalent surface gravity in a stationary black hole¹:

$$\nabla_\nu(\xi^\mu \xi_\mu) = -2\kappa \xi_\nu, \quad (2.19)$$

and

$$\kappa^2 = -\frac{1}{2}(\nabla_\mu \xi_\nu)(\nabla^\mu \xi^\nu). \quad (2.20)$$

From (2.15), it can be derived that

$$\mathcal{L}_\xi \kappa = 0, \quad (2.21)$$

which means the surface gravity is a constant along with the orbit of the Killing vector. This result corresponds to the zeroth law of thermodynamics, that a system in thermal equilibrium has a uniform temperature.

The surface gravity for the Schwarzschild solution κ_{Sch} is

$$\kappa_{\text{Sch}} = \frac{1}{4GM}. \quad (2.22)$$

Using the surface gravity, one can construct the generator of the Killing horizon. In fact, if the vector k^μ is defined as in

$$\begin{aligned} \xi^\mu &= \kappa e^{-\kappa v} k^\mu \\ &\equiv \kappa \lambda k^\mu, \end{aligned} \quad (2.23)$$

where v is defined to satisfy $\xi^\mu \nabla_\mu v = 1$ and λ is an affine parameter of k^μ . This k^μ will be null on the Killing horizon and satisfy the geodesic equation:

$$k^\mu \nabla_\mu k^\nu = 0. \quad (2.24)$$

Black hole temperature and zeroth law. The surface gravity is very important for the black hole thermodynamics because black holes have a well-defined temperature T determined by the surface gravity:

$$T = \frac{\kappa}{2\pi}. \quad (2.25)$$

¹It should also be noted that (2.17) can be expressed as

$$\xi^\mu \nabla_\mu \xi_\nu = \xi^\mu \nabla_{[\mu} \xi_{\nu]} = \frac{\xi^\mu}{2} (\nabla_\mu \xi_\nu - \nabla_\nu \xi_\mu). \quad (2.18)$$

In Sec. 4.2, this is related to the Kodama vector.

This can also be understood from the perspective of periodicity in the time direction. As an example, let us consider the region near the horizon $r = 2GM + \varepsilon$ ($\varepsilon \ll 2GM$) of Euclidean the Schwarzschild spacetime ($t = -it_E$) (2.5):

$$ds^2 \simeq d\rho^2 + \left(\frac{1}{4GM}\right)^2 \rho^2 dt_E^2 + (2GM)^2 (d\theta^2 + \sin^2 \theta d\phi^2), \quad \rho \equiv \sqrt{8GM\varepsilon}. \quad (2.26)$$

In this case, for t_E to determine the temperature without introducing a conical singularity, it must be periodic as in

$$t_E \sim t_E + 2\pi \left(\frac{1}{4GM}\right)^{-1}. \quad (2.27)$$

Thus, it can be understood that the temperature of the Schwarzschild black hole is interpreted as

$$T_{\text{Sch}} = \frac{\kappa_{\text{Sch}}}{2\pi}. \quad (2.28)$$

Following, it is explained why the surface gravity is interpreted as temperature through the derivation of the laws of black hole thermodynamics.

2.2 The laws of black hole thermodynamics

In this section, we derive the first and second laws of black hole thermodynamics using the quantities defined in the previous section 2.1. We explain that, while the first law is derived from the calculations as an identity, the second law requires imposing reasonable conditions, such as the null energy condition. Finally, we comment on the generalized second law and the apparent horizon as the dynamical black hole entropy.

2.2.1 First law

Now let us consider a situation that a black hole in a stationary state initially is perturbed by a small matter described by the stress tensor $T_{\mu\nu}$ [2]. We assume that the black hole is returned to another stationary state after the process. The most genral Killing vector for stationallly black holes is given by

$$\xi^\mu = t^\mu + \Omega_H \psi^\mu, \quad (2.29)$$

where Ω_H is a constant angular momentum, When $\Omega_H \neq 0$, it corresponds to an axisymmetric rotating black hole.

By using the Killing vector, the energy momentum vector is defined as

$$E^\mu \equiv -T^{\mu\nu} t_\nu, \quad (2.30)$$

and the angular momentum vector is

$$J^\mu \equiv -T^{\mu\nu}\psi_\nu. \quad (2.31)$$

When a perturbation is introduced, the Killing vector of the original space-time is no longer a precise Killing vector. However, we assume here that it remains a Killing vector from the perspective of the order of perturbation. Using these quantities, the energy and angular momentum that flowed through the horizon can be calculated as follows.

$$\delta M = - \int_{\mathcal{N}} d\lambda d^2x \sqrt{a} k_\mu E^\mu, \quad (2.32)$$

$$\delta J = - \int_{\mathcal{N}} d\lambda d^2x \sqrt{a} k_\mu J^\mu, \quad (2.33)$$

where $\mathcal{N}$ is the null hypersurface on the future event horizon $\mathcal{H}^+$ and $d^2x\sqrt{a}$ is the area element.

By using (2.23), one can obtain

$$\delta M - \Omega_H \delta J = \kappa \int_{\mathcal{N}} d\lambda d^2x \sqrt{a} \lambda T_{\mu\nu} k^\mu k^\nu. \quad (2.34)$$

From the Raychaudhuri equation (See (A.6) in Appendix A), the r.h.s of (2.34) can be calculated as

$$\begin{aligned} \int_{\mathcal{N}} d\lambda d^2x \sqrt{a} T_{\mu\nu} \xi^\mu k^\nu &\simeq -\frac{\kappa}{8\pi G} \int_{\mathcal{N}} d\lambda d^2x \sqrt{a} \lambda \frac{d\theta}{d\lambda} \\ &\simeq \frac{\kappa}{8\pi G} \int_{\mathcal{N}} d\lambda (\sqrt{a}\theta) - \frac{\kappa}{8\pi G} (\lambda\sqrt{a}\theta) \Big|_{\lambda=0}^{\lambda=\infty} \\ &= \frac{\kappa}{8\pi G} \delta A. \end{aligned} \quad (2.35)$$

where in the first and second line, we ignore higher order perturbations. In the third line, we used the assumption that the black hole eventually becomes a stationary black hole as $\lambda \rightarrow \infty$ and $\lambda\theta \rightarrow 0$.

As a result, the first law of black hole thermodynamics is obtained:

$$\delta M - \Omega_H \delta J = \frac{\kappa}{8\pi G} \delta A. \quad (2.36)$$

where it can be seen here that $\frac{A}{4G}$ can be interpreted as the black hole's entropy and $\frac{\kappa}{2\pi}$ as the temperature.

2.2.2 Second law

When the surface area of the black hole is taken as the entropy, the entropy increase law is satisfied if the null energy condition $T_{\mu\nu}k^\mu k^\nu \geq 0$ is satisfied²:

$$\frac{\kappa}{8\pi G} \delta A = \kappa \int_{\mathcal{N}} d\lambda d^2x \sqrt{a} \lambda T_{\mu\nu} k^\mu k^\nu \geq 0. \quad (2.37)$$

²Strictly speaking, Penrose's theorem is required.

Generalized second law. On the other hand, when matter is thrown into the black hole from the outside, the entropy of the black hole increases, but the entropy of the matter S_{mat} outside decreases. This seems to contradict the usual thermodynamics. Furthermore, if the black hole emits thermal radiation at the temperature we have defined so far, its mass decreases, violating the null energy condition and the second law. Therefore, by considering the generalized entropy $\frac{A}{4G} + S_{\text{mat}}$ we expect that the second law will hold for this. In this thesis, we will investigate the conditions under which this generalized second law of thermodynamics holds in situations where there is a fluid outside the black hole.

Event horizon vs. apparent horizon. For stationary black holes, the event horizon, apparent horizon, and Killing horizon all coincide, hence there is no ambiguity in the area law. However, extending the concept of the area law to dynamical black holes remains an open question. A reasonable approach is to use the apparent horizon's area, as it can be determined without requiring the complete time evolution of the black hole, unlike the event horizon.

Defining the event horizon requires information from the infinite past and future, so it seems more appropriate to characterize the entropy with the area of the apparent horizon (A_{app}) which is a local concept. Under this framework, one can consider the second law of thermodynamics for the generalized entropy (S_{gen}), defined as the sum of the BH entropy $\left(\frac{A_{\text{app}}}{4G}\right)$ and the entropy of the matter outside the black hole (S_{mat}):

$$\delta S_{\text{mat}} + \frac{\delta A_{\text{app}}}{4G} \geq 0. \quad (2.38)$$

In Sec. 4.2, we consider the generalized second law of black holes from these perspectives. In particular, we examine the case where the matter around the black hole is approximated as a fluid, and explore how the temperature gradient contributes to the increase in generalized entropy.

Chapter 3

EFT in the Schwinger-Keldysh formalism for Dynamical Gravity

The Schwinger-Keldysh (SK) formalism is a powerful tool for analyzing nonequilibrium systems, such as real-time evolution in finite-temperature systems and quantum open systems. Effective field theories (EFTs) constructed with this formalism explicitly include fluctuations and dissipation, making it suitable for describing realistic situations. Recent advancements in non-equilibrium systems based on the SK formalism have been remarkable [32–35, 37–58]. For instance, active research has been conducted on the EFT of hydrodynamics (HydroEFT) based on the dynamical Kubo-Martin-Schwinger (KMS) symmetry that characterizes local equilibrium in [32–35, 37–40]. Spontaneous symmetry breaking in open systems and EFTs thereof are studied in [41–49]. More recently, following these developments, application to the open system type EFT of cosmic inflation, the open system EFT of inflation, has also been initiated [42, 52] along the line of the EFT of inflation [59]¹, even though current studies are limited to EFTs on a fixed background spacetime without dynamical gravity.

It is a natural idea to extend this framework to cases where gravity is dynamical. However, in such cases, the action must preserve local symmetries like diffeomorphism invariance. In the SK formalism, the doubled symmetries split into diagonal (physical diffeomorphism) and off-diagonal (noise diffeomorphism) parts under the classical limit. Breaking the noise diffeomorphism symmetry leads to energy dissipation, violating energy-momentum conservation and contradicting the Bianchi identity. To resolve this issue, Stueckelberg fields are introduced, restoring the noise diffeomorphism symmetry and defining an “environment” sector to capture dissipative effects. This arises from the fact that gravity is a field that couples with all degrees of freedom. In this chapter, we discuss these mechanisms in detail using specific models.

This chapter is organized as follows: In Sec. 3.1, we begin with a review of the general framework of the SK formalism, using a toy oscillator model for illustration. In Sec. 3.2, we then provide a detailed discussion of the diffeomorphism symmetries within the closed-time-path

¹See [60] for a pioneering work on dissipation in EFT of inflation that does not use the SK formalism. For discussions on the EFT of open systems in stochastic inflation, see [61].

contour. In Sec. 3.3, we elaborate on the above recipe of the SKEFT for dissipative systems coupled to dynamical gravity using a scalar matter field as an illustrative example. Sec. 3.2 and 3.3 are part of the main content of this thesis.

3.1 A Review of the Schwinger-Keldysh formalism

We begin by a review of the Schwinger-Keldysh (SK) formalism (also known as the closed-time-path formalism) and its application to dissipative systems, using a simple model of harmonic oscillators. The aim of this section is to share the necessary physical intuitions and background technical information of the present paper in a rather compact and self-contained manner. For more details of the SK formalism, we refer the readers to, for example, Refs. [35, 62, 63].

3.1.1 Microscopic description

The SK formalism offers a framework for evaluating correlation functions in non-equilibrium systems. A basic quantity is the closed-time-path generating functional $W[J_1, J_2]$ defined in the operator formalism as

$$e^{iW[J_1, J_2]} = \text{Tr} \left[\rho_0 U_{J_2}^\dagger(t_f, t_i) U_{J_1}(t_f, t_i) \right], \quad (3.1)$$

where $U_J(t_2, t_1)$ is the time-evolution operator from t_1 to t_2 in the presence of external sources collectively denoted by J . ρ_0 is the initial density operator inserted at t_1 . In the path-integral form, $W[J_1, J_2]$ is given by

$$e^{iW[J_1, J_2]} = \int \mathcal{D}\Phi_1 \mathcal{D}\Phi_2 e^{i(S[\Phi_1; J_1] - S[\Phi_2; J_2])}, \quad (3.2)$$

where $S[\Phi; J]$ is the microscopic action of dynamical degrees of freedom, collectively denoted by Φ . The SK formalism uses a pair of path-integral variables, $\{\Phi_1, \Phi_2\}$ for each dynamical degree of freedom Φ , with each variable assigned to a (horizontal) section of the contour in Fig. 3.1. As a result, the number of degrees of freedom is doubled compared to the in-out formalism. The boundary conditions at the initial time t_i are specified by the initial density operator ρ_0 . On the other hand, the boundary conditions at the final time t_f are $\Phi_1(t_f) = \Phi_2(t_f)$. Notice that dynamical variables are doubled compared to the in-out formalism and also that the same action is used for time-evolution along the two contours since the time-evolution operator is the same in the microscopic theory.

r -a basis. Let us define $(\Phi_s, J_s) = (\{\mathcal{O}_{s1}, \mathcal{O}_{s2}, \dots, \mathcal{O}_{sn}\}, \{j_{s1}, j_{s2}, \dots, j_{sn}\})$ ($s = 1, 2$) for the purposes of this discussion. For a better physical interpretation of the action, it is convenient to define a new pair of variables

$$j_{ri} = \frac{1}{2} (j_{1i} + j_{2i}), \quad j_{ai} = j_{1i} - j_{2i}, \quad (3.3)$$

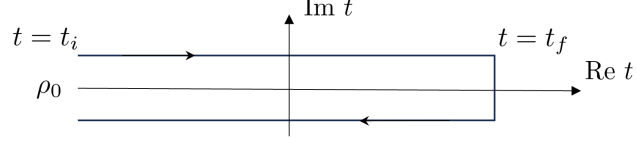


Figure 3.1: The closed-time-path on which the generating functional is defined.

$$\mathcal{O}_{ri} = \frac{1}{2} (\mathcal{O}_{1i} + \mathcal{O}_{2i}) , \quad \mathcal{O}_{ai} = \mathcal{O}_{1i} - \mathcal{O}_{2i} , \quad (3.4)$$

Physically speaking, r -variables describe the physical profile, whereas a -variables capture fluctuations. In the following, we call these the r - a basis.

Green functions. By using this formulation, correlation functions in the r - a basis are defined as

$$\begin{aligned} G_{\alpha_1 \dots \alpha_N}(x_1, \dots, x_N) &\equiv \frac{1}{i^{N_r}} \frac{\delta^N W}{\delta j_{\bar{\alpha}_1}(x_1) \dots \delta j_{\bar{\alpha}_N}(x_N)} \Big|_{j_{a,r}=0} \\ &= i^{N_a} \langle \mathcal{P} \mathcal{O}_{\alpha_1}(x_1) \dots \mathcal{O}_{\alpha_N}(x_N) \rangle , \end{aligned} \quad (3.5)$$

where $\alpha_1, \dots, \alpha_N \in (a, r)$ and $\alpha = a, r$ for $\bar{\alpha} = r, a$. Here N_r and N_a are the number of r and a -index $N_r + N_a = N$. One can now obtain several Green's functions as the following:

$$G_{nm}^F(x_1, x_2) = i \langle T \mathcal{O}_n(x_1) \mathcal{O}_m(x_2) \rangle , \quad \tilde{G}_{nm}^F(x_1, x_2) = i \langle \tilde{T} \mathcal{O}_n(x_1) \mathcal{O}_m(x_2) \rangle \quad (3.6)$$

$$G_{nm}^+(x_1, x_2) = \langle \mathcal{O}_n(x_1) \mathcal{O}_m(x_2) \rangle , \quad G_{nm}^-(x_1, x_2) = \langle \mathcal{O}_m(x_2) \mathcal{O}_n(x_1) \rangle \quad (3.7)$$

$$\Delta_{nm}(x_1, x_2) = \langle [\mathcal{O}_n(x_1), \mathcal{O}_m(x_2)] \rangle , \quad G_{nm}^S(x_1, x_2) = \frac{1}{2} \langle \mathcal{O}_n(x_1) \mathcal{O}_m(x_2) + \mathcal{O}_m(x_2) \mathcal{O}_n(x_1) \rangle \quad (3.8)$$

$$G_{nm}^R(x_1, x_2) = i\theta(t_1 - t_2) \Delta_{mn}(x_1, x_2) , \quad G_{nm}^A(x_1, x_2) = -i\theta(t_2 - t_1) \Delta_{nm}(x_1, x_2) , \quad (3.9)$$

where the subscripts R , A and S each represent retarded, advanced and symmetric respectively.

Thermal equilibrium and KMS conditions Let us consider the properties derived from the above formalism when the initial state ρ_0 is in global equilibrium:

$$\rho_0 = \frac{e^{-\beta_0 H}}{Z_0} , \quad Z_0 = \text{Tr}(e^{-\beta_0 H}) \quad (3.10)$$

where β_0 is the inverse temperature of the initial state. In this case, it can be seen that the generating function satisfies the following Kubo-Martin-Schwinger (KMS) condition:

$$\begin{aligned} e^{W[J_1, J_2]} &= \frac{1}{Z_0} \text{Tr} \left[e^{-\beta_0 H} \left(\tilde{T} e^{-i \int \Phi_2 J_2} \right) \times e^{\beta_0 H} e^{-\beta_0 H} \left(T e^{i \int \Phi_1 J_1} \right) \right] \\ &= \frac{1}{Z_0} \text{Tr} \left[e^{-\beta_0 H} \left(T e^{i \int \Phi_1 J_1} \right) \times \left(\tilde{T} e^{-i \int \Phi_2 J_2(t-i\beta_0)} \right) \right] \end{aligned} \quad (3.11)$$

$$\equiv e^{W_T[J_1, J_2(t-i\beta_0)]} \quad (3.12)$$

where $\tilde{T}$ is the anti-time-ordered operator. In this case, one can find the standard fluctuation-dissipation theorem (FDT) for two-point functions which in momentum space:

$$G_{nm}^S(k) = \frac{1}{2} \coth \frac{\beta_0 \omega}{2} \Delta_{nm}(k). \quad (3.13)$$

Now, let us assume that the microscopic system is invariant under the Z_2 transformation Θ , which is one of the Z_2 transformation $T/PT/CPT$, where C , P and T represents the charge conjugation, parity and time reversal, respectively. Then the generating function W is invariant under the following:

$$W[J_1(x), J_2(x)] = W[\tilde{J}_1(x), \tilde{J}_2(x)], \quad (3.14)$$

where

$$\tilde{J}_1(x) = \Theta J_1(t, \vec{x}), \quad \tilde{J}_2(x) = \Theta J_2(t + i\beta_0, \vec{x}). \quad (3.15)$$

Then, the condition (3.14) with (3.15) implies the Onsager relation

$$G_{ij}^R(k) = G_{ji}^R(k), \quad \text{and} \quad G_{ij}^S(k) = G_{ij}^S(-k). \quad (3.16)$$

A toy model. Many aspects of the SKEFT can be captured by a simple model of a harmonic oscillator X coupled to an environment modeled by a set of harmonic oscillators Y_n ($n = 1, \dots, N$) with the following microscopic action²:

$$S[X, Y_n; J] = \int dt \left[\left(\frac{1}{2} \dot{X}^2 - \frac{1}{2} \omega_0^2 X^2 \right) + \sum_{n=1}^N \left(\frac{1}{2} \dot{Y}_n^2 - \frac{1}{2} \omega_n^2 Y_n^2 \right) + \sum_{n=1}^N g_n X Y_n + (\text{source terms}) \right], \quad (3.17)$$

where the overhead dot represents the time derivative $\dot{f} \equiv \frac{d}{dt}f$, ω_n ($n = 0, \dots, N$) are the frequencies of the oscillators and g_n are the couplings between the system oscillator X and the set of environment oscillators Y_n . Details of the source terms denoted by the last term are not important in our discussion and are kept implicit.

The corresponding path-integral description is

$$e^{iW[J_1, J_2]} = \int \mathcal{D}X_1 \mathcal{D}X_2 \mathcal{D}Y_{n1} \mathcal{D}Y_{n2} e^{iS_{\text{micro}}[X_1, Y_{n1}, X_2, Y_{n2}; J_1, J_2]}, \quad (3.18)$$

with appropriate boundary conditions at $t = t_i$ and $t = t_f$. Here we have introduced a pair of path-integral variables $\{(X_1, Y_{n1}), (X_2, Y_{n2})\}$ for the set of dynamical variables $\{(X, Y_n)\}$ appearing in (3.17). Also, the path-integral weight is

$$S_{\text{micro}}[X_1, Y_{n1}, X_2, Y_{n2}; J_1, J_2] = S[X_1, Y_{n1}; J_1] - S[X_2, Y_{n2}; J_2]. \quad (3.19)$$

²This model is often used as an introduction to the SKEFT. See, e.g., Refs. [42, 62] for more details.

We emphasize that the microscopic SK action S_{micro} is factorized into the $(X_1, Y_{n1}; J_1)$ part and the $(X_2, Y_{n2}; J_2)$ part, whereas the boundary conditions mix the two sectors.

3.1.2 Green's functions of the environment oscillators

Next we would like to write down the effective theory of the system oscillator X . In our illustrative example, where only terms up to quadratic order are present, it is straightforward to integrate out the environment oscillators Y_{n1} and Y_{n2} to obtain the following effective action in the Fourier space:

$$iS_{\text{eff}} = -\frac{1}{2} \sum_{s,s'=1,2} \int \frac{d\omega}{2\pi} \left[X_s(-\omega) D_{ss'}(\omega; \omega_0^2) X_{s'}(\omega) + \sum_{n=1}^N g_n^2 X_s(-\omega) G_{ss'}(\omega; \omega_n^2) X_{s'}(\omega) \right], \quad (3.20)$$

where we have suppressed the source terms for visual clarity. $D_{ss'}(\omega; \Omega^2)$ is the 2×2 kinetic matrix of the harmonic oscillator with frequency Ω and $G_{ss'}(\omega; \Omega^2)$ are the Green's functions given by the inverse matrix of $D_{ss'}(\omega; \Omega^2)$.

Off-shell Green's functions. To elaborate on the effects of the environment, let us take a closer look at the Green's functions. First, the kinetic matrix $D_{ss'}(\omega; \omega_n)$ is of the form,

$$D_{ss'}(\omega; \omega_n^2) = \begin{pmatrix} -i(\omega^2 - \omega_n^2) & 0 \\ 0 & i(\omega^2 - \omega_n^2) \end{pmatrix}. \quad (3.21)$$

For now, we will neglect the $i\epsilon$ -prescription typically required to address the on-shell dynamics at $\omega = \omega_0$. The corresponding Green's functions $G_{ss'}(\omega)$ are

$$G_{ss'}(\omega; \omega_n^2) = (D_{ss'}(\omega; \omega_n^2))^{-1} = \begin{pmatrix} \frac{i}{\omega^2 - \omega_n^2} & 0 \\ 0 & -\frac{i}{\omega^2 - \omega_n^2} \end{pmatrix} \quad \text{for } \omega \neq \omega_n. \quad (3.22)$$

Note that these off-shell Green's functions are diagonal due to the diagonal nature of the kinetic matrix $D_{ss'}$, so that the X_1 sector and the X_2 sector are factorizable even in the effective action, as long as there are no on-shell contributions of the environment.

In the field theory language, this means that the effective theory has no mixing between the two sets of path-integral variables as long as on-shell particle production in the coarse-grained environment sector does not occur. In this situation, one can simply use the in-out effective action to compute correlation functions in the system sector. For example, the EFT of inflation [59, 64] captures effective interactions sourced by heavy fields whose particle production during inflation is exponentially suppressed by the Boltzmann factor.

Full Green's functions. On the other hand, to derive the full Green's functions including the on-shell component, we need to identify the initial conditions and also be careful about the $i\epsilon$ -prescription. For example, when the initial density operator is a thermal state with an inverse temperature β , the full Green's functions read (for details, see, e.g., the textbook [65])³

$$G_{11}(\omega; \omega_n^2) = \mathbf{P} \frac{i}{\omega^2 - \omega_n^2} + 2\pi \operatorname{sgn}(\omega) \delta(\omega^2 - \omega_n^2) \left[\frac{1}{2} + n_B(\omega) \right], \quad (3.23)$$

$$G_{12}(\omega; \omega_n^2) = 2\pi \operatorname{sgn}(\omega) \delta(\omega^2 - \omega_n^2) n_B(\omega), \quad (3.24)$$

$$G_{21}(\omega; \omega_n^2) = 2\pi \operatorname{sgn}(\omega) \delta(\omega^2 - \omega_n^2) [1 + n_B(\omega)], \quad (3.25)$$

$$G_{22}(\omega; \omega_n^2) = (G_{11}(\omega; \omega_n^2))^*, \quad (3.26)$$

where $\mathbf{P}$ denotes the principal value, $\operatorname{sgn}(\omega)$ is the sign function, and $n_B(\omega)$ is the Bose-Einstein distribution $n_B(\omega) = (e^{\beta\omega} - 1)^{-1}$. Notice that the non-zero off-diagonal components only contribute on-shell $\omega = \omega_n$. This is essentially why we need the SK effective action, rather than the in-out effective action, for effective descriptions of dissipations and noises associated with on-shell particle creation. Also, for the same reason, the effective theory after coarse graining is in general non-unitary with a complex effective action.

3.1.3 Effective theory

Generality. Now let us discuss properties of the effective theory (3.20):

$$iS_{\text{eff}} = -\frac{1}{2} \sum_{s,s'=1,2} \int \frac{d\omega}{2\pi} [X_s(-\omega) D_{ss'}(\omega; \omega_0^2) X_{s'}(\omega) + X_s(-\omega) \mathcal{G}_{ss'}(\omega) X_{s'}(\omega)], \quad (3.27)$$

where we defined the effective coupling matrix $\mathcal{G}_{ss'}(\omega)$ by

$$\mathcal{G}_{ss'}(\omega) = \sum_{n=1}^N g_n^2 G_{ss'}(\omega; \omega_n^2). \quad (3.28)$$

In general, $\mathcal{G}_{ss'}(\omega)$ is non-analytic with respect to ω because of the delta functions, i.e., the on-shell contributions, in the Green's functions (3.23)-(3.26). Note that Eqs. (3.23)-(3.26) are for the initial thermal state, but it is a general statement that non-analyticity appears through on-shell contributions. This non-analyticity creates a non-local effect in the effective action (3.27). This non-locality is especially prominent if the number of the environment oscillators, N , is not very large. In the field theory intuition, this simply means that the effect of particle productions in the environment sector appears as non-local effects in the effective description of the system.

³ Here we implicitly assumed that Green's functions of the environment oscillators Y_n are approximated by the free theory propagator at finite temperature. In other words, we assumed that the dynamics of the system oscillator X does not affect properties (e.g., temperature) of the environment thermal bath.

Local effective theory. On the other hand, this non-local effect can be smoothed out when N is large, for which it is convenient to introduce a spectral function $J(\omega_*^2)$ such that

$$\mathcal{G}_{ss'}(\omega) = \int_0^\infty d\omega_*^2 J(\omega_*^2) G_{ss'}(\omega; \omega_*^2) \quad \text{with} \quad J(\omega_*^2) = \sum_{n=1}^N g_n^2 \delta(\omega_*^2 - \omega_n^2). \quad (3.29)$$

Note that the sum over n can be approximated by an integral if N is large, hence non-analyticity may be resolved depending on the distributions of (ω_n^2, g_n) . While the spectral function $J(\omega_*)$ depends on details of the environment sector and its coupling to the system sector, the effective theory indeed becomes local if the spectral function $\text{sign}(\omega_*)J(\omega_*^2)$ is analytic at $\omega_* = 0$. For the so-called Ohmic bath modeling such situations, the effective coupling matrix $\mathcal{G}_{ss'}(\omega)$ is approximated for small ω as follows⁴ (see, e.g., [42, 62] for details):

$$\mathcal{G}_{11}(\omega) \simeq 2\gamma T, \quad \mathcal{G}_{12}(\omega) \simeq -\gamma\omega + 2\gamma T, \quad \mathcal{G}_{21}(\omega) \simeq \gamma\omega + 2\gamma T, \quad \mathcal{G}_{22}(\omega) \simeq 2\gamma T, \quad (3.30)$$

where we have assumed that the initial density operator is a thermal state with a temperature $T = 1/\beta$. Also, positivity of $J(\omega_*^2)$ requires $\gamma > 0$. The corresponding effective action in the position space is

$$iS_{\text{eff}}[X_1, X_2] = \frac{i}{2} \int dt \left[\left(\dot{X}_1^2 - \omega_0^2 X_1^2 \right) - \left(\dot{X}_2^2 - \omega_0^2 X_2^2 \right) - \gamma(X_1 \dot{X}_2 - \dot{X}_1 X_2) + 2i\gamma T(X_1^2 + X_2^2 - 2X_1 X_2) \right]. \quad (3.31)$$

As advertised, we find local effective interactions that mix X_1 and X_2 after integrating out the environment oscillators, which originate from the on-shell part of the Green's functions.

r -a basis. In the r -a basis, the effective action (3.31) reads

$$iS_{\text{eff}}[X_r, X_a] = i \int dt \left[-(\ddot{X}_r + \gamma \dot{X}_r + \omega_0^2 X_r) X_a + i\gamma T X_a^2 \right]. \quad (3.32)$$

The equations of motion (EOMs) for X_r and X_a imply

$$\ddot{X}_r + \gamma \dot{X}_r + \omega_0^2 X_r = 0, \quad X_a = 0, \quad (3.33)$$

which reproduce the Brownian motion with a dissipation coefficient γ . Also, in the computation of correlation functions, the last term in Eq. (3.32) gives Gaussian random noise. In this manner, dissipation and noise are induced by integrating out the environment sector.

3.1.4 Consistency of the effective theory

It is worth noticing that the effective action (3.31) enjoys the following properties:

$$S_{\text{eff}}[X_1 = X, X_2 = X] = 0, \quad (3.34)$$

⁴We performed renormalization of the frequency ω_0 of the system oscillator X to simplify presentation.

$$S_{\text{eff}}[X_2, X_1] = - (S_{\text{eff}}[X_1, X_2])^* , \quad (3.35)$$

$$\text{Im } S_{\text{eff}}[X_1, X_2] \geq 0 , \quad (3.36)$$

or equivalently, in the r - a basis,

$$S_{\text{eff}}[X_r, X_a = 0] = 0 , \quad (3.37)$$

$$S_{\text{eff}}[X_r, -X_a] = - (S_{\text{eff}}[X_r, X_a])^* , \quad (3.38)$$

$$\text{Im } S_{\text{eff}}[X_r, X_a] \geq 0 . \quad (3.39)$$

These are actually generic properties of the SKEFT that follow from unitarity of the microscopic theory and the self-adjointness of the initial density operator. To see this, it is convenient to go back to the definition (3.1) of the closed-time-path generating functional $W(J_1, J_2)$ as we will elaborate below.

First, unitarity of the microscopic theory implies that the time-evolution operator U_J in Eq. (3.1) satisfies

$$U_J^\dagger(t_2, t_1) U_J(t_2, t_1) = 1 , \quad (3.40)$$

which leads to the following condition on the closed-time-path generating functional:

$$W[J_1 = J, J_2 = J] = 0 . \quad (3.41)$$

Also, norm positivity together with the Cauchy-Schwarz inequality applied to Eq. (3.1) leads to

$$\text{Im } W[J_1, J_2] \geq 0 , \quad (3.42)$$

which simply means that probability decreases after tracing out positive-norm states. Finally, the self-adjointness $\rho_0^\dagger = \rho_0$ implies

$$W[J_1, J_2] = - (W[J_2, J_1])^* . \quad (3.43)$$

The conditions (3.34)-(3.39) follow in the same manner, essentially because we can think of the system variables are kept fixed like the sources when integrating out the environment variables.

Comments on the Kubo-Martin-Schwinger (KMS) condition. In the present analysis, we have assumed that the initial state is thermal, for which the correlation functions are related to each other, e.g., as Eqs. (3.23)-(3.26) show. This condition is known as the KMS condition. This relation in turns correlates coefficients of the dissipation term (the second term) and the noise term (the last term) in the effective action (3.32), hence the name fluctuation-dissipation relation (FDR). Although the KMS condition is typically presented as a condition for the generating functional, it can be generalized to a symmetry transformation acting on the

dynamical fields at the action level. From this symmetry prospective, the FDR can be captured by requiring the action to be invariant under the transformations,

$$X'_r(t) = X_r(-t), \quad X'_a(t) = X_a(-t) + i\beta\partial_{-t}X_r(-t). \quad (3.44)$$

This symmetry of the effective action is called the dynamical KMS symmetry [34, 35, 37]. More details on its properties will be explained in Appendix B. We emphasize that this symmetry is a consequence of the choice of the initial state, so it is not a universal property of the SKEFT. Indeed, if we choose other initial states, coefficients of the dissipation and noise terms enjoy other types of relations reflecting the properties of the Green's functions in the environment sector.

3.1.5 General properties of the Schwinger-Keldysh EFT

Various properties of the harmonic oscillator model explained in the previous subsections hold in field theories too. Below we summarize general properties of the Schwinger-Keldysh EFT (see, e.g., a review article [35] for details):

1. Dynamical variables, collectively denoted by χ , are described by a pair of path-integral variables (χ_1, χ_2) , reflecting the closed-time-path nature of the formalism.
2. The EFT is in general non-unitary. In particular, the non-unitarity is due to mixing of the 1, 2 sectors, which originates from on-shell particle creations in the environment sector and results in, e.g., dissipation and noise effects.
3. Consistency such as unitarity of the microscopic theory behind the EFT implies

$$S_{\text{eff}}[\chi_1 = \chi, \chi_2 = \chi] = 0, \quad S_{\text{eff}}[\chi_2, \chi_1] = -(S_{\text{eff}}[\chi_1, \chi_2])^*, \quad \text{Im } S_{\text{eff}}[\chi_1, \chi_2] \geq 0. \quad (3.45)$$

Similarly to Eq. (3.3), it is convenient to define

$$\chi_r = \frac{\chi_1 + \chi_2}{2}, \quad \chi_a = \chi_1 - \chi_2. \quad (3.46)$$

In this r - a basis, the conditions (3.45) are rephrased as

$$S_{\text{eff}}[\chi_r, \chi_a = 0] = 0, \quad S_{\text{eff}}[\chi_r, -\chi_a] = -(S_{\text{eff}}[\chi_r, \chi_a])^*, \quad \text{Im } S_{\text{eff}}[\chi_r, \chi_a] \geq 0. \quad (3.47)$$

4. The SKEFT depends not only on the Hamiltonian of the environment sector, but also details of the initial state, since the mixing of the 1-2 sector arises through on-shell particle creation. For example, if the initial state is thermal, the SKEFT should satisfy the dynamical KMS condition, a generalization of the fluctuation-dissipation relation. This type of information can also be taken into account in the EFT construction, once the initial state is specified.

3.2 Diffeomorphism symmetries in gravitational Schwinger-Keldysh EFTs

In this section we elaborate on diffeomorphism symmetries in the SKEFT. We first introduce the two types of diffeomorphism symmetries corresponding to the doubled path-integral variables in the SK formalism in Sec. 3.2.1, based on the microscopic description. We then discuss their impacts on the SKEFT in Sec. 3.2.2.

3.2.1 Microscopic description

For illustration, let us consider a scalar field ϕ coupled to a dynamical metric $g_{\mu\nu}$ and an environment sector described by microscopic dynamical fields collectively denoted by Σ :

$$S[\phi, g_{\mu\nu}, \Sigma] = \int d^4x \sqrt{-g} \left[\frac{M_{\text{Pl}}^2}{2} R - \frac{1}{2} (\partial_\mu \phi)^2 - V(\phi) + \mathcal{L}_{\text{env}}[\phi, g_{\mu\nu}, \Sigma] \right], \quad (3.48)$$

where M_{Pl} is the reduced Planck mass and $\mathcal{L}_{\text{env}}$ is the microscopic Lagrangian density describing the dynamics of the environment fields Σ as well as their couplings to ϕ and $g_{\mu\nu}$. In the language of the path-integral form (3.2) of the SK formalism, we have $\Phi = \phi, g_{\mu\nu}, \Sigma$ as the microscopic dynamical fields. For simplicity, we switch off the source terms as they are not important in the present paper, but it is straightforward to reintroduce them if necessary. We assume that the gravitational microscopic action (3.48) enjoys the diffeomorphism symmetries. We emphasize that this setup is chosen for illustrative purposes, but the following discussion is not restricted to it. The discussion remains universal as long as the underlying microscopic theory is general covariant.

Doubled diffeomorphism symmetries. The corresponding SK action, i.e., the exponent of the path-integral weight in Eq. (3.2) is factorized as

$$S_{\text{micro}}[\Phi_1, \Phi_2] = S[\Phi_1] - S[\Phi_2], \quad (3.49)$$

where we have introduced a pair of path-integral variables $\Phi_s = \phi_s, g_{s\mu\nu}, \Sigma_s$ ($s = 1, 2$). This implies that S_{micro} enjoys two types of diffeomorphism symmetries, each of which acts on Φ_1 and Φ_2 separately. More concretely, we define the first type of diffeomorphism symmetries that we call diffs_1 by the following transformation rule:

$$\phi'_1(x) = \phi_1(x - \xi_1(x)), \quad (3.50a)$$

$$g'_{1\mu\nu}(x) = \frac{\partial(x - \xi_1(x))^\rho}{\partial x^\mu} \frac{\partial(x - \xi_1(x))^\sigma}{\partial x^\nu} g_{1\rho\sigma}(x - \xi_1(x)), \quad (3.50b)$$

$$\phi'_2(x) = \phi_2(x), \quad (3.50c)$$

$$g'_{2\mu\nu}(x) = g_{2\mu\nu}(x), \quad (3.50d)$$

with a local transformation parameter $\xi_1^\mu(x)$, and similarly for the environment fields Σ_s . Note that the first set of variables Φ_1 transforms in the standard manner, whereas the second set Φ_2 is unchanged. We also define the second type of diffeomorphism symmetries that we call difs_2 in a similar fashion which only acts non-trivially on Φ_2 but Φ_1 .

Comments on boundary conditions. The microscopic action S_{micro} in the SK formalism enjoys the doubled diffeomorphism symmetries, schematically written as

$$\text{difs}_1 \times \text{difs}_2, \quad (3.51)$$

since the action is factorized into the Φ_1 sector and the Φ_2 sector. In contrast, the boundary conditions imposed at the initial time t_i and the final time t_f of the SK path-integral mix the two sectors. For example, the boundary conditions at $t = t_f$ are

$$\Phi_1(t_f, \mathbf{x}) = \Phi_2(t_f, \mathbf{x}), \quad (3.52)$$

which break the doubled diffeomorphism symmetries on the boundary into the diagonal one satisfying

$$\xi_1^\mu(t_f, \mathbf{x}) = \xi_2^\mu(t_f, \mathbf{x}). \quad (3.53)$$

In a similar fashion, boundary conditions at the initial time break the doubled diffeomorphism symmetries. However, we emphasize that the two diffeomorphism symmetries in the bulk ($t_i < t < t_f$) are not affected by the boundary conditions, thanks to the local nature of diffeomorphism symmetries.

3.2.2 Schwinger-Keldysh EFT

We have explained that in the SK formalism, the microscopic theory enjoys the doubled diffeomorphism symmetries (except at $t = t_i, t_f$) thanks to the local nature of the symmetries. As a consequence, the SKEFT after integrating out the environment sector should also respect the doubled diffeomorphism symmetries. This is in sharp contrast to global symmetries. On the other hand, as we discussed in Sec. 3.1, the SKEFT accommodates mixing of the doubled path-integral variables Φ_1 and Φ_2 , as a consequence of on-shell particle creation that results in dissipation and noise effects. Actually, these mixing effects make the SKEFT construction somewhat non-trivial as we explain below.

Impacts of mixing effects. For illustration, let us imagine constructing a SK effective action of a pair of scalars ϕ_s and metrics $g_{s\mu\nu}$ ($s = 1, 2$). The SKEFT accommodates mixing of the 1, 2

sectors, so that one may consider, e.g., the following operator as a candidate for the effective coupling:

$$\int d^4x \sqrt{-g_1} \phi_1 \phi_2. \quad (3.54)$$

Obviously, this explicitly breaks the doubled diffeomorphism symmetries $\text{diffeo}_1 \times \text{diffeo}_2$ with the transformation parameters $\xi_1^\mu \neq \xi_2^\mu$, while the diagonal part $\xi_1^\mu = \xi_2^\mu$ is still preserved. More generally, we find that local effective couplings mixing the 1, 2 sectors break the non-diagonal diffeomorphism symmetries, as long as we use fields in linear representations of diffeomorphisms only. In other words, mixing couplings require Stückelberg fields that non-linearly realize the non-diagonal diffeomorphism symmetries. In Secs. 3.3-4.1, we will illustrate this property more explicitly with concrete examples.

Physical and noise diffs. Having said this, we reorganize the doubled diffeomorphism symmetries $\text{diffeo}_1 \times \text{diffeo}_2$ into diagonal and non-diagonal parts. First, we introduce the physical diffeomorphism symmetries with the transformation parameter $\xi^\mu(x)$ by

$$\xi_1^\mu(x) = \xi^\mu(x), \quad \xi_2^\mu(x) = \xi^\mu(x), \quad (3.55)$$

under which the 1, 2 variables transform in the same manner. On the other hand, we introduce the noise diffeomorphism symmetries with the transformation parameter $\xi_a^\mu(x)$ by the non-diagonal part of $\text{diffeo}_1 \times \text{diffeo}_2$ such that

$$\xi_1^\mu = \frac{1}{2} \xi_a^\mu(x), \quad \xi_2^\mu = -\frac{1}{2} \xi_a^\mu(x), \quad (3.56)$$

under which the 1, 2 variables transform in the opposite way.

r -a basis. To discuss the physical meaning of these symmetries, it is convenient to introduce the r -a variables as

$$\Phi = \frac{\Phi_1 + \Phi_2}{2}, \quad \Phi_a = \Phi_1 - \Phi_2. \quad (3.57)$$

Here and in what follows, we suppress the subscript r of r -variables for visual clarity. In particular, we define the r -a variables for the metrics by

$$g_{\mu\nu} = \frac{g_{1\mu\nu} + g_{2\mu\nu}}{2}, \quad g_{a\mu\nu} = g_{1\mu\nu} - g_{2\mu\nu}, \quad (3.58)$$

with lower indices and we raise and lower the indices by the physical metric $g_{\mu\nu}$ and its inverse $g^{\mu\nu}$. Under the physical diffeomorphism transformations, the r -a variables for tensor fields transform simply as tensors, e.g.,

$$\phi'(x) = \phi(x - \xi(x)), \quad (3.59)$$

$$\phi'_a(x) = \phi_a(x - \xi(x)), \quad (3.60)$$

$$g'_{\mu\nu}(x) = \frac{\partial(x - \xi(x))^\rho}{\partial x^\mu} \frac{\partial(x - \xi(x))^\sigma}{\partial x^\nu} g_{\rho\sigma}(x - \xi(x)), \quad (3.61)$$

$$g'_{a\mu\nu}(x) = \frac{\partial(x - \xi(x))^\rho}{\partial x^\mu} \frac{\partial(x - \xi(x))^\sigma}{\partial x^\nu} g_{a\rho\sigma}(x - \xi(x)). \quad (3.62)$$

On the other hand, the noise diffeomorphism transformations of the scalars are

$$\phi'(x) = \frac{\phi(x_+) + \phi(x_-)}{2} + \frac{\phi_a(x_+) - \phi_a(x_-)}{4}, \quad (3.63)$$

$$\phi'_a(x) = \frac{\phi_a(x_+) + \phi_a(x_-)}{2} + [\phi(x_+) - \phi(x_-)], \quad (3.64)$$

where $x_\pm^\mu = x^\mu \mp \frac{1}{2}\xi_a^\mu(x)$. Similarly, the transformation rule of the metrics reads

$$\begin{aligned} g'_{\mu\nu}(x) &= \frac{1}{2} \left[\frac{\partial x_+^\rho}{\partial x^\mu} \frac{\partial x_+^\sigma}{\partial x^\nu} g_{\rho\sigma}(x_+) + \frac{\partial x_-^\rho}{\partial x^\mu} \frac{\partial x_-^\sigma}{\partial x^\nu} g_{\rho\sigma}(x_-) \right] \\ &\quad + \frac{1}{4} \left[\frac{\partial x_+^\rho}{\partial x^\mu} \frac{\partial x_+^\sigma}{\partial x^\nu} g_{a\rho\sigma}(x_+) - \frac{\partial x_-^\rho}{\partial x^\mu} \frac{\partial x_-^\sigma}{\partial x^\nu} g_{a\rho\sigma}(x_-) \right], \end{aligned} \quad (3.65)$$

$$\begin{aligned} g'_{a\mu\nu}(x) &= \frac{1}{2} \left[\frac{\partial x_+^\rho}{\partial x^\mu} \frac{\partial x_+^\sigma}{\partial x^\nu} g_{a\rho\sigma}(x_+) + \frac{\partial x_-^\rho}{\partial x^\mu} \frac{\partial x_-^\sigma}{\partial x^\nu} g_{a\rho\sigma}(x_-) \right] \\ &\quad + \left[\frac{\partial x_+^\rho}{\partial x^\mu} \frac{\partial x_+^\sigma}{\partial x^\nu} g_{\rho\sigma}(x_+) - \frac{\partial x_-^\rho}{\partial x^\mu} \frac{\partial x_-^\sigma}{\partial x^\nu} g_{\rho\sigma}(x_-) \right]. \end{aligned} \quad (3.66)$$

3.2.3 Classical limit

The noise diffeomorphism transformations (3.63)-(3.66) in terms of the r - a variables look rather complicated, but they are simplified in the classical limit $\hbar \rightarrow 0$, where quantum fluctuations are negligible compared to the noise effects originating from the coupling to the environment sector. To see this, let us assign the scale $\hbar$ to the a -variables as

$$\Phi_r \rightarrow \Phi_r, \quad \Phi_a \rightarrow \hbar \Phi_a, \quad (3.67)$$

and define the classical limit by $\hbar \rightarrow 0$ while keeping Φ_r and Φ_a fixed⁵. Correspondingly, we assign $\hbar$ to the noise diffeomorphism transformation parameter ξ_a^μ as $\xi_a^\mu \rightarrow \hbar \xi_a^\mu$ and take the limit

⁵One might wonder that fluctuations around classical solutions disappear in the classical limit, since $\Phi_1 - \Phi_2 = \hbar \Phi_a$ goes to zero. However, it is not the case. To see this, let us consider the harmonic oscillator model (3.32) for illustration. First, recovering $\hbar$ gives the effective action of the form,

$$\frac{i}{\hbar} S_{\text{eff}} = \frac{i}{\hbar} \int dt \left[-X_a(\ddot{X}_r + \gamma \dot{X}_r + \omega_0^2 X_r) + i\gamma \frac{T}{\hbar} X_a^2 \right], \quad (3.68)$$

where the temperature T is replaced as $T \rightarrow T/\hbar$ as the dimensional analysis implies. Redefining the a -variable X_a as $X_a \rightarrow \hbar X_a$, we have

$$\frac{i}{\hbar} S_{\text{eff}} = i \int dt \left[-X_a(\ddot{X}_r + \gamma \dot{X}_r + \omega_0^2 X_r) + i\gamma T X_a^2 \right], \quad (3.69)$$

so that the fluctuation term, i.e., the second term survives even in the classical limit $\hbar \rightarrow 0$ (T fixed).

$\hbar \rightarrow 0$ while keeping ξ_a^μ fixed. Then, the noise diffeomorphism transformations (3.63)-(3.66) are simplified in the classical limit as

$$\phi'(x) = \phi(x), \quad (3.70a)$$

$$\phi'_a(x) = \phi_a(x) - \xi_a^\mu \partial_\mu \phi(x), \quad (3.70b)$$

$$g'_{\mu\nu}(x) = g_{\mu\nu}(x), \quad (3.70c)$$

$$g'_{a\mu\nu}(x) = g_{a\mu\nu}(x) - \nabla_\mu \xi_{a\nu} - \nabla_\nu \xi_{a\mu}. \quad (3.70d)$$

Here we emphasize that the above is for a finite transformation, rather than an infinitesimal one. Also, if needed, it is straightforward to perform the semiclassical expansion to include higher orders in $\hbar$, which corresponds to expansion in the number of a -variables.

3.3 Dissipative scalar coupled to dynamical gravity

In this section, we perform the EFT construction for a dissipative scalar coupled to dynamical gravity, as an illustrative example. In Sec. 3.3.1, we first explain how the dissipative effects are closely tied to the noise diffeomorphism symmetry. Then, in Sec. 3.3.2, we demonstrate that dynamical Stückelberg fields for this symmetry and the energy-momentum tensor of the environment are required to describe dissipation in the presence of dynamical gravity. As described in Sec 3.1, degrees of freedom in the SK formalism always come in a pair. The partner of the noise diffeomorphism Stückelberg field is that for the physical diffeomorphism, which naturally appears in HydroEFT as slow-variables. We then use HydroEFT to model the environment in Sec. 3.3.3. While Secs. 3.3.1-3.3.3 focus on the SK action at the linear order in a -variables that describe classical EOMs, extension to higher orders in a -variables capturing noise effects is discussed in Sec. 3.3.4. Finally, in Sec. 3.3.5, we quantify impacts of fluctuations in the environment sector on the scalar dynamics and demonstrate that they are negligible when the background energy of the environment is large enough compared to the typical energy of fluctuations (hence we call it the decoupling regime).

3.3.1 Dissipative scalar coupled to background metric

First, we consider a scalar field ϕ coupled to a non-dynamical background metric $g_{\mu\nu}$ to illustrate that the noise diffeomorphism symmetry can be utilized to distinguish dissipative operators from non-dissipative operators in the SKEFT.

Non-dissipative scalar. As we explained in Sec. 3.1, the SK action for non-dissipative systems is simply a difference of two copies of the in-out action. For a scalar ϕ coupled to a

background metric $g_{\mu\nu}$, we have

$$S_{\text{n.d.}} = \int d^4x \left[\sqrt{-g_1} \left(-\frac{1}{2} \partial_\mu \phi_1 \partial^\mu \phi_1 - V(\phi_1) \right) - \sqrt{-g_2} \left(-\frac{1}{2} \partial_\mu \phi_2 \partial^\mu \phi_2 - V(\phi_2) \right) \right], \quad (3.71)$$

which enjoys the two diffeomorphism symmetries given in (3.50) (and similarly for the one acting non-trivially on ϕ_2 and g_2). In the r - a basis, the corresponding action in the classical limit reads

$$S_{\text{n.d.}} = \int d^4x \sqrt{-g} \left[\frac{1}{2} T_{\mu\nu}^{(\phi)} g_a^{\mu\nu} + (\Box \phi - V'(\phi)) \phi_a \right], \quad (3.72)$$

where $V'(\phi)$ is the derivative of $V(\phi)$ with respect to ϕ and $T_{\mu\nu}$ is the energy-momentum tensor given by

$$T_{\mu\nu}^{(\phi)} = \partial_\mu \phi \partial_\nu \phi - \left[\frac{1}{2} \partial_\rho \phi \partial^\rho \phi + V(\phi) \right] g_{\mu\nu}. \quad (3.73)$$

In the following, unless otherwise stated, we take the classical limit for simplicity. Note that the variation in ϕ_a reproduces the standard EOM,

$$\Box \phi - V'(\phi) = 0. \quad (3.74)$$

Also, the noise diffeomorphism transformation (3.70) of the action gives

$$\Delta S_{\text{n.d.}} = \int d^4x \sqrt{-g} \left[\nabla^\mu T_{\mu\nu}^{(\phi)} - (\Box \phi - V'(\phi)) \partial_\nu \phi \right] \xi_a^\nu = 0, \quad (3.75)$$

where we performed integration by parts at the second equality. This expression manifests that the noise diffeomorphism invariance of the action guarantees that the energy-momentum tensor $T_{\mu\nu}^{(\phi)}$ is conserved on-shell.

Dissipative scalar. Next, we introduce dissipative effects to the action (3.72). The SK action now reads

$$S_\phi = \int d^4x \sqrt{-g} \left[\frac{1}{2} T_{\mu\nu}^{(\phi)} g_a^{\mu\nu} + (\Box \phi - \gamma u^\mu \partial_\mu \phi - V'(\phi)) \phi_a \right], \quad (3.76)$$

where u^μ is a time-like unit vector ($u_\mu u^\mu = -1$) that specifies the time-direction for dissipation and γ is the dissipation coefficient. The EOM for ϕ_a of the modified action is

$$\Box \phi - \gamma u^\mu \partial_\mu \phi - V'(\phi) = 0, \quad (3.77)$$

which describes a scalar with dissipation now. The noise diffeomorphism transformation of the modified action is

$$\Delta S_\phi = \int d^4x \sqrt{-g} \left[\xi_a^\nu \nabla^\mu T_{\mu\nu}^{(\phi)} - (\Box \phi - \gamma u^\mu \partial_\mu \phi - V'(\phi)) \xi_a^\mu \partial_\mu \phi \right] \quad (3.78)$$

$$= \int d^4x \sqrt{-g} \xi_a^\nu \gamma u^\mu \partial_\mu \phi \partial_\nu \phi, \quad (3.79)$$

which shows that the new dissipation term breaks the noise diffeomorphism invariance of the action, as expected from the general argument of the previous section. Note that one can explicitly show that the noise diffeomorphism symmetry cannot be recovered by any modification of the energy-momentum tensor $T_{\mu\nu}^{(\phi)}$. Also, Eqs. (3.78)-(3.79) show that the energy-momentum tensor $T_{\mu\nu}^{(\phi)}$ is no longer conserved due to violation of the noise diffeomorphism symmetry:

$$\nabla^\mu T_{\mu\nu}^{(\phi)} = \partial_\nu \phi [\Box \phi - V(\phi)] \stackrel{\text{eom}}{=} \gamma u^\mu \partial_\mu \phi \partial_\nu \phi, \quad (3.80)$$

where $\stackrel{\text{eom}}{=}$ means equality that holds on-shell, i.e., after using EOMs. In this manner, the noise diffeomorphism is useful in distinguishing dissipative terms from non-dissipative ones.

3.3.2 Lesson from a naive model with dynamical gravity

Next we elaborate on subtleties in coupling dissipative systems with dynamical gravity. For this, let us consider the following naive model where the dissipative scalar of Eq. (3.72) is simply coupled to the Einstein gravity⁶:

$$S_{\text{naive}} = \int d^4x \sqrt{-g} \left[\frac{1}{2} (-M_{\text{Pl}}^2 G_{\mu\nu} + T_{\mu\nu}^{(\phi)}) g_a^{\mu\nu} + (\Box \phi - \gamma u^\mu \partial_\mu \phi - V'(\phi)) \phi_a \right], \quad (3.83)$$

where $G_{\mu\nu}$ is the Einstein tensor constructed from the r -variable metric $g_{\mu\nu}$ and M_{Pl} is the reduced Planck mass.

Noise diffs and Stückelberg trick. As we discussed in the previous section, both of the physical and noise diffeomorphism symmetries have to be respected in the presence of dynamical gravity. On the other hand, the dissipation term in Eq. (3.83) breaks the noise diffeomorphism symmetry. A simple trick to recover this symmetry is to introduce the Stückelberg fields X_a^μ that nonlinearly realize the noise diffeomorphism symmetry. Practically, referring to (3.70b) and (3.70d), it is achieved by performing the replacement,

$$g_{a\mu\nu} \rightarrow \mathcal{G}_{a\mu\nu} \equiv g_{a\mu\nu} + \nabla_\mu X_{a\nu} + \nabla_\nu X_{a\mu}, \quad \phi_a \rightarrow \varphi_a \equiv \phi_a + X_a^\mu \partial_\mu \phi, \quad (3.84)$$

⁶ In the 1-2 basis, the SK action for the pure Einstein gravity is given by

$$S_{\text{pure}} = \int d^4x \frac{M_{\text{Pl}}^2}{2} [\sqrt{-g_1} (R[g_1] - \Lambda) - \sqrt{-g_2} (R[g_2] - \Lambda)], \quad (3.81)$$

where $R[g_s]$ ($s = 1, 2$) is the Ricci scalar constructed from the metric $g_{s\mu\nu}$ and Λ is the cosmological constant. In the classical limit, the corresponding action in the r - a basis reads

$$S_{\text{pure}} = \frac{M_{\text{Pl}}^2}{2} \int d^4x \sqrt{-g} (-G_{\mu\nu} - \Lambda g_{\mu\nu}) g_a^{\mu\nu}. \quad (3.82)$$

Here and in what follows we suppress metric indication when curvature tensors are constructed from the r -variable metric $g_{\mu\nu}$, e.g., $G_{\mu\nu} = G_{\mu\nu}[g]$. Up to the two-derivative order, Eq. (3.82) indeed gives the general SK action of $g_{\mu\nu}$ and $g_{a\mu\nu}$ that is invariant under both the physical and noise diffeomorphism symmetries. Eq. (3.83) is obtained by coupling this to the model (3.72) (Λ is absorbed into the potential $V(\phi)$).

and assigning a non-linear transformation rule of the noise diffeomorphism to X_a^μ as

$$X_a^\mu \rightarrow X'^\mu_a = X_a^\mu + \xi_a^\mu(x). \quad (3.85)$$

Note that X_a^μ transforms as a vector under the physical diffeomorphism. By construction, $\mathcal{G}_{a\mu\nu}$ and φ_a are noise diffeomorphism invariant. Then, the SK action after performing the Stückelberg trick reads

$$S_{\text{naive}} = \int d^4x \sqrt{-g} \left[\frac{1}{2} (-M_{\text{Pl}}^2 G_{\mu\nu} + T_{\mu\nu}^{(\phi)}) \mathcal{G}_a^{\mu\nu} + (\Box\phi - \gamma u^\mu \partial_\mu \phi - V'(\phi)) \varphi_a \right], \quad (3.86)$$

which is invariant under both of the physical and noise diffeomorphisms. Note that the action (3.83) can be thought of as an action in the unitary gauge, in which the Stückelberg fields X_a^μ are eaten by the noise metric $g_{a\mu\nu}$.

Necessity of the environment sector. Now let us take a closer look at the action (3.86) to show that the naive model fails to describe dissipation. For this, it is convenient to rewrite Eq. (3.86) into the following form:

$$S_{\text{naive}} = \int d^4x \sqrt{-g} \left[\frac{1}{2} (-M_{\text{Pl}}^2 G_{\mu\nu} + T_{\mu\nu}^{(\phi)}) g_a^{\mu\nu} + (\Box\phi - \gamma u^\mu \partial_\mu \phi - V'(\phi)) \phi_a - \gamma u^\mu \partial_\mu \phi \partial_\nu \phi X_a^\nu \right], \quad (3.87)$$

where we just performed integration by parts similarly to the second equality of Eq. (3.75). This expression manifests that only the γ term requires the Stückelberg field X_a^μ to recover the noise diffeomorphism symmetry. The EOMs for $g_{a\mu\nu}$ and ϕ_a give the Einstein equation and the dissipative scalar EOM:

$$M_{\text{Pl}}^2 G_{\mu\nu} = T_{\mu\nu}^{(\phi)}, \quad \Box\phi - \gamma u^\mu \partial_\mu \phi - V'(\phi) = 0. \quad (3.88)$$

On the other hand, the EOMs for X_a^μ are

$$\gamma u^\mu \partial_\mu \phi \partial_\nu \phi = 0. \quad (3.89)$$

Note that this can be obtained also by taking divergence of the Einstein equation and applying the dissipative scalar EOM, so that the unitary gauge action (3.83) leads to the same result. Notably, Eq. (3.89) requires that the dissipated energy quantified in Eq. (3.80) has to be zero, hence dissipation cannot be captured by this naive model.

The physical origin of this consequence is simple: Gravity couples to all degrees of freedom universally, so that the energy-momentum tensor appearing in the Einstein equation has to take into account the whole matter sector including the environment. However, the current naive model does not contain the environment sector, hence the energy of the scalar cannot escape

anywhere. It is also technically important to notice that Eq. (3.89) does not contain second-time-derivative of ϕ , so that it gives a constraint equation due to which initial conditions cannot be taken arbitrarily. In particular, Eq. (3.89) requires $u^\mu \partial_\mu \phi = 0$, so that time-evolution of the scalar ϕ is prohibited. In other words, X_a^μ play a role of auxiliary fields, which is a technical reason why this naive model failed to describe dissipation.

Summary of the lesson. To summarize, we need to take into account the energy-momentum tensor of the environment to describe dissipation in the presence of dynamical gravity, simply because gravity couples to all degrees of freedom universally. For the dissipative scalar, this is achieved by improving the action (3.86) into the following form:

$$S = \int d^4x \sqrt{-g} \left[\frac{1}{2} (-M_{\text{Pl}}^2 G_{\mu\nu} + T_{\mu\nu}^{(\phi)} + T_{\mu\nu}^{(\text{env})}) \mathcal{G}_a^{\mu\nu} + (\Box\phi - \gamma u^\mu \partial_\mu \phi - V'(\phi)) \varphi_a + \dots \right], \quad (3.90)$$

where r -variables of the model include those slow variables in the environment sector collectively denoted by Φ , in addition to ϕ and $g_{\mu\nu}$. Also, a -variables of the model are ϕ_a , $g_{a\mu\nu}$, X_a^μ and possibly other slow variables required to describe the environment sector collectively denoted by Φ_a . $T_{\mu\nu}^{(\text{env})}$ is the energy-momentum tensor of the environment sector that is composed of $g_{\mu\nu}$, Φ , and possibly ϕ as well. The time-like unit vector u^μ and the dissipation coefficient γ may have Φ -dependence reflecting properties of the environment. The dots stand for terms linear in Φ_a as well as $\mathcal{O}(\phi_a)$ terms for interactions between the scalar and the environment that are not displayed in the above. In particular, the EOMs for $g_{a\mu\nu}$ and X_a^μ are

$$M_{\text{Pl}}^2 G_{\mu\nu} = T_{\mu\nu}^{(\phi)} + T_{\mu\nu}^{(\text{env})}, \quad \nabla^\mu (T_{\mu\nu}^{(\phi)} + T_{\mu\nu}^{(\text{env})}) = 0, \quad (3.91)$$

which can capture the energy transfer from the scalar sector to the environment and also gravity sourced by the environment sector appropriately. In the next subsection, we argue that the EFT of hydrodynamics offers a simple and natural model for the environment.

3.3.3 Hydro model for the environment

Now we take into account the environment to which energy of the scalar can escape. As we discussed, the main technical issue of the naive model (3.87) was that the Stückelberg fields X_a^μ for the noise diffeomorphism played a role of auxiliary fields leading to the constraint equation (3.89) that forbids dissipation. A simple and natural way to resolve this issue is to introduce r -variables associated with X_a^μ and a kinetic term to give them dynamics. Based on the motivation explained below, we model the environment sector by HydroEFT to construct a self-consistent SKEFT of a dissipative scalar coupled to dynamical gravity.

Motivation for HydroEFT. Recall that the physical and noise diffeomorphisms are r - a components of the doubled diffeomorphisms. Then, it is natural to introduce r -variables associated with X_a^μ as Stückelberg fields for the physical diffeomorphism. Notably, these Stückelberg fields for the doubled diffeomorphism symmetries are used in HydroEFT [33, 34] as the slow-variables that represent hydrodynamic modes. We review this formalism in Appendix B to make the thesis self-contained (see also the original papers [33, 34] and the lecture note [35]), but below we give a minium review useful for following the main text.

Minimum review on HydroEFT. First, the SK action of HydroEFT takes the following general form at the linear order in a -variables:

$$S_{\text{hydro}} = \frac{1}{2} \int d^4x \sqrt{-g} T_{\mu\nu}^{(\text{hydro})} \mathcal{G}_a^{\mu\nu}, \quad (3.92)$$

where $\mathcal{G}_{a\mu\nu}$ is defined by Eq. (3.84). The dynamical a -variables of the SKEFT are the Stückelberg fields X_a^μ for the noise diffeomorphism and also the noise metric $g_{a\mu\nu}$ is introduced as an external field. $T_{\mu\nu}^{(\text{hydro})}$ represents the energy-momentum tensor of fluids, whose details will be explained shortly. The EOM for X_a^μ guarantees the conservation law:

$$\nabla^\mu T_{\mu\nu}^{(\text{hydro})} = 0. \quad (3.93)$$

The energy-momentum tensor $T_{\mu\nu}^{(\text{hydro})}$ is constructed from dynamical r -variables $\sigma^A(x)$ ⁷ ($A = \bar{0}, \bar{1}, \bar{2}, \bar{3}$) that describe a coordinate map from the physical spacetime (Eulerian) coordinates x^μ to the fluid spacetime (Lagrangian) coordinates σ^A , as well as $g_{\mu\nu}$.

For example, a concrete form of the energy-momentum tensor for ideal fluids is

$$T_{\mu\nu}^{(\text{hydro})} = T_{\mu\nu}^{(0)} \equiv \rho_0 u_\mu u_\nu + p_0 \Delta_{\mu\nu} \quad (\text{ideal fluids}), \quad (3.94)$$

where u^μ is a time-like unit vector representing the fluid velocity defined by

$$u^\mu = b^{-1} K_0^\mu \quad (u^\mu u_\mu = -1). \quad (3.95)$$

Here, $K_A^\mu(x) \equiv (\partial_\mu \sigma^A(x))^{-1}$ is the inverse of $\partial_\mu \sigma^A$ (such that $K_A^\mu \partial_\mu \sigma^B = \delta_A^B$) with $\partial_\mu \equiv \frac{\partial}{\partial x^\mu}$ and $b = \sqrt{-g_{\mu\nu} K_0^\mu K_0^\nu}$. The energy density ρ_0 and the pressure p_0 are functions of $\beta \equiv b\beta_0$ with a reference inverse temperature $\beta_0 = T_0^{-1}$. When fluids are locally in thermal equilibrium, β is identified as the local inverse temperature, even though the EFT construction is not limited to the locally thermal case. Higher derivative corrections to fluid dynamics can also be incorporated in a systematic manner.

⁷Note that although the HydroEFT is more naturally formulated in the fluid spacetime with the associated pair of Stückelberg fields $(X^\mu(\sigma), X_a^\mu(\sigma))$, it is more convenient for our purpose to rewrite the hydro action in the physical spacetime, x^μ , the conversion is provided by the inversion map of $X^\mu(\sigma)$, i.e. $\sigma^A(x)$. All the other dynamical fields, e.g. $X_a^\mu(\sigma(x))$ and $\beta(\sigma(x))$, are expressed as functions of $\sigma^A(x)$.

As a final remark, we summarize the doubled diffeomorphisms in HydroEFT and explain why $\sigma^A(x)$ is the variable used in HydroEFT expressed in physical spacetime. First, the physical diffeomorphism transformation of the Stückelberg field $X^\mu(\sigma)$ (dynamical variable in the fluid spacetime) (See Footnote 7 and Appendix B) is given by

$$X^\mu(\sigma') = X^\mu(\sigma) - \xi^\mu(X(\sigma)), \quad (3.96)$$

where $X^\mu(\sigma)$ is the inverse map of $\sigma^A(x)$ (dynamical variable in the physical spacetime). The connection between $\sigma^A(x)$ and $X^\mu(\sigma)$ shows that $\sigma^A(x)$ play a role of Stückelberg fields for the physical diffeomorphism in physical spacetime, as advertised. On the other hand, we require $\sigma^A(x)$ to be invariant under the noise diffeomorphism similarly to other r -variables. This is the reason why we say that HydroEFT offers a simple and natural framework to model the environment sector⁸. See Appendix B for more details of the EFT construction, including the symmetry principle thereof.

Dissipative scalar coupled to gravity and hydro. Now we use HydroEFT to model the environment sector of a dissipative scalar coupled to dynamical gravity. It is simply achieved by considering the following SK action:

$$\begin{aligned} S &= \int d^4x \sqrt{-g} \left[\frac{1}{2} (-M_{\text{Pl}}^2 G_{\mu\nu} + T_{\mu\nu}^{(\phi)} + T_{\mu\nu}^{(\text{hydro})}) \mathcal{G}_a^{\mu\nu} + (\Box\phi - \gamma u^\mu \partial_\mu \phi - V'(\phi)) \varphi_a \right], \\ &= \int d^4x \sqrt{-g} \left[\frac{1}{2} (-M_{\text{Pl}}^2 G_{\mu\nu} + T_{\mu\nu}^{(\phi)}) g_a^{\mu\nu} + \frac{1}{2} T_{\mu\nu}^{(\text{hydro})} \mathcal{G}_a^{\mu\nu} \right. \\ &\quad \left. + (\Box\phi - \gamma u^\mu \partial_\mu \phi - V'(\phi)) \phi_a - \gamma u^\mu \partial_\mu \phi \partial_\nu \phi X_a^\nu \right], \end{aligned} \quad (3.97)$$

where dynamical variables are ϕ , ϕ_a , $g_{\mu\nu}$, $g_{a\mu\nu}$, σ^A , and X_a . u^μ in the dissipation term of ϕ is identified with the one defined in Eq. (3.95). Also, it is compatible with the symmetry of HydroEFT to promote the dissipation coefficient γ to a function of β , just as ρ_0 and p_0 in Eq. (3.94)⁹. The EOMs for $g_a^{\mu\nu}$ and ϕ_a are

$$M_{\text{Pl}}^2 G_{\mu\nu} = T_{\mu\nu}^{(\phi)} + T_{\mu\nu}^{(\text{hydro})}, \quad \Box\phi - \gamma u^\mu \partial_\mu \phi - V'(\phi) = 0. \quad (3.98)$$

On the other hand, the EOM for X_a^μ is modified as

$$\nabla^\mu T_{\mu\nu}^{(\text{hydro})} = -\gamma u^\mu \partial_\mu \phi \partial_\nu \phi, \quad (3.99)$$

which captures the energy flow from the scalar sector to the environment sector modeled by HydroEFT. In the next subsection we include noise effects described by higher orders terms in a -variables. Also, in Sec. 3.3.5, we use our model to identify the condition under which fluctuations in the environment sector are irrelevant to the scalar dynamics.

⁸We are not claiming that HydroEFT is required as a general model for the environment sector, even though it offers a simple and natural framework to construct a local effective action in the presence of dissipation. It would be interesting to explore other possibilities too.

⁹It is also compatible with the symmetry to include an operator $\nabla_\mu u^\mu \varphi_a$ that is at the same order in derivatives as the dissipation term, but we ignore it for simplicity.

3.3.4 Noise effects

So far, we focused on the $\mathcal{O}(a)$ part of the SK action that describes classical EOMs, where the non-trivial features of dynamical gravity already appear at this order. However, it is straightforward to include higher order terms in a -variables that describe noise effects in the same fashion. For illustration, we consider coupling a dissipative scalar ϕ with the following SK action to dynamical gravity taking into account the environment:

$$S_\phi = \int d^4x \sqrt{-g} \left[\frac{1}{2} T_{\mu\nu}^{(\phi)} g_a^{\mu\nu} + (\Box\phi - \gamma u^\mu \partial_\mu \phi - V'(\phi)) \phi_a + \frac{i}{2} A \phi_a^2 \right]. \quad (3.100)$$

Here we have added the last term to the earlier action (3.76), which is second order in a -variables and captures Gaussian noise effects in the same manner as the harmonic oscillator model discussed in Sec. 3.1. Note that $A \geq 0$ follows from the unitarity condition (3.47).

Generality. We begin by a general argument without specifying details of the environment. For a general environment, the SK action (3.90) at the first order in a -variables can be extended to the second order as

$$S = \int d^4x \sqrt{-g} \left[\frac{1}{2} (-M_{\text{Pl}}^2 G_{\mu\nu} + T_{\mu\nu}^{(\phi)} + T_{\mu\nu}^{(\text{env})}) \mathcal{G}_a^{\mu\nu} + (\Box\phi - \gamma u^\mu \partial_\mu \phi - V'(\phi)) \varphi_a + \dots \right. \\ \left. + \frac{i}{2} A \varphi_a^2 + \frac{i}{4} W_{\mu\alpha, \nu\beta} \mathcal{G}_a^{\mu\nu} \mathcal{G}_a^{\alpha\beta} + \dots \right], \quad (3.101)$$

where the second line is the newly added $\mathcal{O}(a^2)$ terms. Here we have added a Gaussian noise term constructed from $\mathcal{G}_{a\mu\nu}$ parameterized by the coefficient $W_{\mu\alpha, \nu\beta}$, which respects the physical and noise diffeomorphism symmetries and is also at the same order in derivatives as the noise term of the scalar. As we will discuss in the next section, this describes noise effects of gravitational waves. Besides, the dots in the second line stand for $\mathcal{O}(a^2)$ terms that include a -variables other than $\phi_a, g_{a\mu\nu}, X_a^\mu$, as well as non-Gaussian noise terms coming from higher orders.

Hydro model. Now we model the environment by HydroEFT. For illustration, we take into account the first-order corrections to ideal fluids, which are at the same order in derivatives as the noise and dissipation effects in the scalar sector. More explicitly, the SK action for first-order fluids in the Landau frame takes the form [33, 34],

$$S_{\text{hydro}} = \int d^4x \sqrt{-g} \left[\frac{1}{2} T_{\mu\nu}^{(\text{hydro})} \mathcal{G}_a^{\mu\nu} + \frac{i}{4} W_{\mu\alpha, \nu\beta} \mathcal{G}_a^{\mu\nu} \mathcal{G}_a^{\alpha\beta} \right], \quad (3.102)$$

where the energy-momentum tensor $T_{\mu\nu}^{(\text{hydro})}$ is corrected from the ideal fluid one (3.94) as

$$T_{\mu\nu}^{(\text{hydro})} = T_{\mu\nu}^{(0)} + T_{\mu\nu}^{(1)} \quad \text{with} \quad T_{\mu\nu}^{(1)} \equiv -2\eta \left(K_{\mu\nu} - \frac{1}{3} g_{\mu\nu} K \right) - \zeta \Delta_{\mu\nu} K. \quad (3.103)$$

Here $K_{\mu\nu}$ is the extrinsic curvature constructed from the fluid velocity vector u^μ and K is its trace part:

$$K_{\mu\nu} = \Delta_\mu{}^\rho \nabla_\rho u_\nu, \quad K = \Delta^{\mu\nu} K_{\mu\nu}. \quad (3.104)$$

The shear viscosity η and the bulk viscosity ζ are functions of β . On the other hand, the second term representing noise effects has the same form as the one in the general argument (3.101). To elaborate on it, let us recall that noise terms can be correlated to dissipation terms, once the environment state is specified. For example, when the fluids are locally in thermal equilibrium, the dynamical KMS symmetry implies that $W_{\mu\alpha,\nu\beta}$ is of the form (see Appendix B)¹⁰,

$$W_{\mu\alpha,\nu\beta} = \frac{2\eta}{\beta} \left(\Delta_{\alpha(\mu} \Delta_{\nu)\beta} - \frac{1}{3} \Delta_{\mu\nu} \Delta_{\alpha\beta} \right) + \frac{\zeta}{\beta} \Delta_{\mu\nu} \Delta_{\alpha\beta}, \quad (3.105)$$

with β identified with the local inverse temperature. Note that the dynamical KMS symmetry gives a first-law type constraint $\rho_0 + p_0 = -\beta \frac{\partial p_0}{\partial \beta}$ on the ideal fluid part as well.

Combining it with the general discussion, we obtain the following SK action for a dissipative scalar coupled to dynamical gravity and the environment modeled by HydroEFT:

$$S = \int d^4x \sqrt{-g} \left[\frac{1}{2} \left(-M_{\text{Pl}}^2 G_{\mu\nu} + T_{\mu\nu}^{(\phi)} + T_{\mu\nu}^{(\text{hydro})} \right) \mathcal{G}_a^{\mu\nu} + (\Box\phi - \gamma u^\mu \partial_\mu \phi - V'(\phi)) \varphi_a \right. \\ \left. + i \frac{\gamma}{\beta} \varphi_a^2 + \frac{i}{4} W_{\mu\alpha,\nu\beta} \mathcal{G}_a^{\mu\nu} \mathcal{G}_a^{\alpha\beta} \right], \quad (3.106)$$

where $T_{\mu\nu}^{(\text{hydro})}$ and $W_{\mu\alpha,\nu\beta}$ are defined in Eq. (3.103) and Eq. (3.105), and γ is now a function of the local inverse temperature β . For simplicity, we have assumed that the environment fluids are locally in thermal equilibrium by imposing the dynamical KMS symmetry, which fixes the noise term of the scalar as $A = 2\gamma/\beta$. Relaxing this assumption is straightforward, but there appear some more new terms in the EFT that are not presented here.

3.3.5 Decoupling regime

As we discussed, it is necessary to take into account the energy-momentum tensor of the environment sector appropriately, in order to describe dissipative systems in the presence of dynamical gravity. An advantage of the EFT constructed in this manner is that one may study impacts of fluctuations in the environment sector on the system sector of interest, even though it requires modeling of the environment sector. For example, the hydro model of the environment can be used to quantify how the dynamics of the system sector (the scalar ϕ in the present model) is affected, e.g., by temperature fluctuations.

¹⁰The round (square) brackets imply that the indices are symmetrized (anti-symmetrized), e.g., as $A_{(\mu\nu)} = \frac{1}{2}(A_{\mu\nu} + A_{\nu\mu})$ ($A_{[\mu\nu]} = \frac{1}{2}(A_{\mu\nu} - A_{\nu\mu})$).

On the other hand, there are certain situations where fluctuations in the environment sector are irrelevant to the dynamics of the system sector. For example, in the harmonic oscillator model of Sec. 3.1, Green's functions of the environment oscillators Y_n were approximated by the free theory propagator at a fixed finite temperature (see footnote 3) and the resultant effective theory (3.32) was nothing but the one for Brownian motion used in the context of open systems as the simplest example. Intuitively, this Markovian type approximation is justified for Brownian motion because energy of the Brownian particle is too small to change properties (e.g., temperature) of the environment. At the cost of limitation of applicability, this type of approximation has an advantage in universality, i.e., the effective theory of the system sector does not depend on details of the environment. In the following, we call the regime where such approximation works the decoupling regime.

Decoupling regime in hydro model. Having said this, let us use the hydro model for the environment to identify the decoupling regime, where fluctuations in the environment sector are irrelevant. For illustration, let us consider the $\mathcal{O}(a)$ action for a dissipative scalar ϕ coupled to gravity and the environment sector described by an ideal fluid:

$$S = \int d^4x \sqrt{-g} \left[\frac{1}{2} (-M_{\text{Pl}}^2 G_{\mu\nu} + T_{\mu\nu}^{(\phi)} + \rho_0 u_\mu u_\nu + p_0 \Delta_{\mu\nu}) \mathcal{G}_a^{\mu\nu} + (\Box\phi - \gamma u^\mu \partial_\mu \phi - V'(\phi)) \varphi_a \right]. \quad (3.107)$$

Also let us denote the background configurations of β and u^μ by $\bar{\beta}$ and $\bar{u}^\mu$, and their fluctuations by $\delta\beta$ and δu^μ , respectively. To identify the decoupling regime, let us estimate impacts of $\delta\beta$, δu^μ , and X_a^μ on the ϕ dynamics. For example, the dissipation term can be expanded around the background as¹¹

$$-(\gamma(\bar{\beta}) + \gamma'(\bar{\beta})\delta\beta)(\bar{u}^\mu + \delta u^\mu) \partial_\mu \phi (\phi_a + X_a^\nu \partial_\nu \phi), \quad (3.108)$$

so that $\delta\beta$, δu^μ , and X_a^μ are negligible as long as the following conditions are satisfied:

$$|\gamma'(\bar{\beta})\delta\beta| \ll |\gamma(\bar{\beta})|, \quad |\delta u| \ll 1, \quad |X_a \partial \phi| \ll |\phi_a|. \quad (3.109)$$

Here the first condition says that temperature fluctuations are small enough to not change the dissipation coefficient. The second condition says that the time-direction of dissipation is fixed approximately. The last condition says that noise in the fluid configuration is small enough compared to the typical modulation scale of the scalar. Similar conditions can be derived in the same manner for other terms in Eq. (3.107) and also for higher order corrections in derivatives and a -fields.

¹¹We assumed that the scalar ϕ vanishes at the background level.

Finally, let us write down the conditions (3.109) in terms of the path-integral variables. As we explained in Sec. 3.3.3, path-integral variables of HydroEFT are $\sigma^A(x)$ and X_a^μ . If we write fluctuations of σ as π , the kinetic term of (π, X_a) is schematically of the form,

$$(\rho_0 + p_0) \partial\pi \partial X_a, \quad (3.110)$$

where for simplicity we assumed that β -dependence of ρ_0, p_0 is small enough. Then, canonically normalized fields π_c and X_{ac} can be introduced schematically as

$$\pi_c \sim \sqrt{\rho_0 + p_0} \pi, \quad X_{ac} \sim \sqrt{\rho_0 + p_0} X_a. \quad (3.111)$$

Now the conditions (3.109) determining the decoupling regime are rephrased as

$$|\partial X_{ac}|, |\partial \pi_c| \ll \sqrt{\rho_0 + p_0}, \quad (3.112)$$

where we used $\delta u \sim \partial\pi$ and $\delta\beta \sim \bar{\beta} \partial\pi$, and also assumed for simplicity that the β -dependence of γ is not so strong $|\frac{\ln \gamma}{\ln \beta}| \lesssim 1$. By dimensional analysis, we conclude that the decoupling regime is given in terms of the typical energy scale E of fluctuations as

$$E^4 \ll \rho_0 + p_0, \quad (3.113)$$

which means that energy of fluctuations is small enough compared to the background energy of the environment. This is nothing but the condition intuitively explained earlier using Brownian motion. In this manner, the decoupling regime of environment fluctuations can be quantified by explicitly modeling the environment, which is an advantage of the current formulation.

Chapter 4

Generalized second law in the Schwinger-Keldysh formalism

For stationary black holes, the event horizon, apparent horizon, and Killing horizon all coincide, hence there is no ambiguity in the area law. While it is still an ongoing issue on how to extend the notion of the area law to dynamical black holes, a plausible extension is to use the area of the apparent horizon [20, 23–31], which can be defined without following the full time-evolution of the black hole, in contrast to the event horizon. In this context, the temperature of the black hole is defined using the dynamical surface gravity determined by the Kodama vector [24, 66, 67]. Meanwhile, the fluid surrounding the black hole has a locally defined temperature, leading to a situation where two distinct temperatures are present. In the following, we show that the generalized entropy is generated by the temperature gradient between the fluid and the black hole, as in standard thermodynamics.

This chapter begins by extending the results from the previous chapter to discuss the dissipation of gravity itself in Sec. 4.1. In doing so, it is confirmed that the fluid, modeled as an environmental sector, captures the dissipation of gravity based on discussions of gravitational waves in flat spacetime. Furthermore, in Sec. 4.2, the case of a fluid surrounding a black hole is considered as an application. By introducing the thermodynamics of dynamic black holes, the generalized entropy is defined by combining the entropy of the fluid and the black hole. It is shown that the increase or decrease of the generalized entropy is characterized by the temperature gradient between the local temperature of the fluid, introduced from the perspective of dynamical KMS symmetry, and the temperature of the black hole.

4.1 Dissipative gravity

In the previous chapter, we constructed the SKEFT for a dissipative scalar coupled to dynamical gravity. There, it was crucial to introduce the energy-momentum tensor of the environment sector, essentially because gravity couples to all degrees of freedom universally. In particular,

we used HydroEFT to model the environment. In this section, we apply the same formalism to dissipative gravity, simply by truncating the scalar field. As an application, we discuss dissipative gravitational waves in Sec. 4.1.1.

4.1.1 Dissipative gravitational wave

First we construct a SKEFT for dissipative gravitational waves whose linear EOM around a flat spacetime is schematically of the form,

$$\ddot{h}_{ij}^{\text{TT}} + \gamma \dot{h}_{ij}^{\text{TT}} - \partial_k^2 h_{ij}^{\text{TT}} = 0. \quad (4.1)$$

Here h_{ij}^{TT} is the transverse-traceless (TT) component of metric fluctuations with i, j, k as the spatial indices and γ is the dissipation coefficient. We begin by a naive attempt analogous to the naive model of a dissipative scalar studied in Sec. 3.3.2 and then incorporate the environment sector to which the energy of gravitational waves can escape.

A naive model

To specify the time-direction of dissipation, let us introduce a time-like unit vector u^μ ($u_\mu u^\mu = -1$). Using this, we also define a projector $\Delta_{\mu\nu}$ onto the three-dimensional spatial surface orthogonal to u^μ and the associated extrinsic curvature $K_{\mu\nu}$ by

$$\Delta_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu, \quad K_{\mu\nu} = \Delta_\mu{}^\rho \nabla_\rho u_\nu. \quad (4.2)$$

At the linear order in perturbations around a flat spacetime with $u^\mu = \delta_0^\mu$, the TT components h_{ij}^{TT} of metric fluctuations are embedded into the Einstein tensor $G_{\mu\nu}$ and the extrinsic curvature $K_{\mu\nu}$ as

$$G_{ij} \ni \frac{1}{2} \left(\ddot{h}_{ij}^{\text{TT}} - \partial_k^2 h_{ij}^{\text{TT}} \right), \quad K_{ij} \ni \frac{1}{2} \dot{h}_{ij}^{\text{TT}}. \quad (4.3)$$

Therefore, a naive candidate for non-linear EOMs of dissipative gravitational waves is

$$G_{\mu\nu} + \gamma K_{\mu\nu} = 0, \quad (4.4)$$

whose TT components reproduce Eq. (4.1) at the linear level. The corresponding SK action is

$$S_{\text{pre}} = -\frac{M_{\text{Pl}}^2}{2} \int d^4x \sqrt{-g} (G_{\mu\nu} + \gamma K_{\mu\nu}) g_a^{\mu\nu}. \quad (4.5)$$

The γ term breaks the noise diffeomorphism symmetry, but it can be recovered by introducing the Stückelberg fields X_a^μ as

$$\begin{aligned} S_{\text{pre}} &= -\frac{M_{\text{Pl}}^2}{2} \int d^4x \sqrt{-g} (G_{\mu\nu} + \gamma K_{\mu\nu}) \mathcal{G}_a^{\mu\nu} \\ &= -\frac{M_{\text{Pl}}^2}{2} \int d^4x \sqrt{-g} (G_{\mu\nu} g_a^{\mu\nu} + \gamma K_{\mu\nu} \mathcal{G}_a^{\mu\nu}), \end{aligned} \quad (4.6)$$

where $\mathcal{G}_{a\mu\nu}$ is defined in Eq. (3.84).

Necessity of the environment sector. Similarly to the naive model (3.83) of a dissipative scalar, the current model (4.6) cannot describe dissipative gravitational waves due to absence of the environment energy-momentum tensor. To see this, consider the EOM for X_a^μ , or equivalently the divergence of the EOM (4.4):

$$\nabla_\mu K^{\mu\nu} = 0. \quad (4.7)$$

If we assume that the time-like unit vector u^μ has no acceleration $u^\nu \nabla_\nu u^\mu = 0$,¹ the EOM (4.7) is reduced to

$$\Delta^{\mu\rho} \nabla_\rho K_{\mu\nu} = 0. \quad (4.8)$$

Essentially because the derivative acting on $K_{\mu\nu}$ is projected onto the spatial directions, the left hand side does not contain second-time-derivatives of propagating modes, i.e., $\ddot{h}_{ij}^{\text{TT}}$, and time-derivatives of auxiliary fields (lapse and shift), hence Eq. (4.8) gives a constraint equation due to which the initial conditions cannot be chosen arbitrarily. Indeed, by explicitly calculating the energy-momentum tensor of linear gravitational waves, one can show that this constraint equation prohibits energy loss of gravitational waves and the current naive model (4.6) cannot describe dissipative gravitational waves, similarly to the naive model of Sec. 3.3.2.

Introduction of the environment sector

Generality. Now we take into account the environment sector to which the energy of gravitational waves can escape. For generality, consider a SKEFT that contains $g_{\mu\nu}$, $g_{a\mu\nu}$ and X_a^μ as well as other r - and a -variables collectively denoted by Φ and Φ_a . Also suppose that the SK action up to the first order in a -variables is schematically of the form,

$$S = \frac{1}{2} \int d^4x \sqrt{-g} [-M_{\text{Pl}}^2 (G_{\mu\nu} + \gamma K_{\mu\nu}) + T_{\mu\nu}^{(\text{env})}] \mathcal{G}_a^{\mu\nu} + \dots, \quad (4.9)$$

where $T_{\mu\nu}^{(\text{env})}$ is the energy-momentum tensor of the environment that is composed of $g_{\mu\nu}$ and Φ . The time-like vector u^μ defining the extrinsic curvature $K_{\mu\nu}$ and the dissipation coefficient γ may have Φ -dependence. The dots stand for terms linear in Φ_a , if any. Then, the EOM for X_a^μ reads

$$\nabla^\mu (\gamma K_{\mu\nu}) = \nabla^\mu T_{\mu\nu}^{(\text{env})}, \quad (4.10)$$

which is no more a constraint equation as long as the right hand side $\nabla^\mu T_{\mu\nu}^{(\text{env})}$ contains second-time-derivatives of Φ or time-derivatives of auxiliary fields. Physically, this equates the energy loss of gravitational waves (the left hand side) and the energy increase in the environment sector (the right hand side). One may check this statement explicitly by expanding Eq. (4.10) around a given background.

¹This is a covariant generalization of the assumption that u^μ is a constant vector.

Hydro model. HydroEFT again offers a simple and natural setup that realizes the above scenario. More concretely, consider the following SK action up to the first order in a -variables:

$$S = \frac{1}{2} \int d^4x \sqrt{-g} \left(-M_{\text{Pl}}^2 G_{\mu\nu} + T_{\mu\nu}^{(\text{hydro})} \right) \mathcal{G}_a^{\mu\nu}. \quad (4.11)$$

Here $T_{\mu\nu}^{(\text{hydro})}$ is the energy-momentum tensor of fluids that is composed of σ^A as well as $g_{\mu\nu}$. Its explicit form up to the first order in the Landau frame is

$$T_{\mu\nu}^{(\text{hydro})} = \rho_0 u_\mu u_\nu + p_0 \Delta_{\mu\nu} - \left[2\eta \left(K_{\mu\nu} - \frac{1}{3} g_{\mu\nu} K \right) + \zeta \Delta_{\mu\nu} K \right], \quad (4.12)$$

where ρ_0 , p_0 , η , and ζ are functions of the local inverse temperature β . Notably, the shear viscosity η term contains the extrinsic curvature, which sources dissipation of gravitational waves. For example, if the background fluid is homogeneous and the corresponding energy-momentum tensor is small enough, the background spacetime is approximated as a flat spacetime and $u^\mu = \delta_0^\mu$ in the rest frame. Then, the linear EOM for the TT component of metric fluctuations reproduces Eq. (4.1) with the dissipation coefficient $\gamma = 2\eta/M_{\text{Pl}}^2$. Note that fluctuations of the hydro variables X^μ do not appear in the linear EOM (4.1) simply because there are no spin-2 modes in the hydro sector. However, they appear from the second order of fluctuations, which are responsible for describing the energy flow from the gravitational wave sector to the environment hydro sector and contribute to the non-linear dynamics.

Noise effects. To close this subsection, we discuss the noise effects of dissipative gravitational waves, which appear through the effective operators at the second order in a -variables. Up to the second order in a -variables, the SK action of first-order fluids coupled to dynamical gravity takes the form,

$$S = \int d^4x \sqrt{-g} \left[\frac{1}{2} \left(-M_{\text{Pl}}^2 G_{\mu\nu} + T_{\mu\nu}^{(\text{hydro})} \right) \mathcal{G}_a^{\mu\nu} + \frac{i}{2} W_{\mu\alpha,\nu\beta} \mathcal{G}_a^{\mu\nu} \mathcal{G}_a^{\alpha\beta} \right]. \quad (4.13)$$

In the Landau frame, $T_{\mu\nu}^{(\text{hydro})}$ is given by (4.12), whereas $W_{\mu\alpha,\nu\beta}$ is

$$W_{\mu\alpha,\nu\beta} = \frac{2\eta}{\beta} \left(\Delta_{\alpha(\mu} \Delta_{\nu)\beta} - \frac{1}{3} \Delta_{\mu\nu} \Delta_{\alpha\beta} \right) + \frac{\zeta}{\beta} \Delta_{\mu\nu} \Delta_{\alpha\beta}. \quad (4.14)$$

Note that the dynamical KMS condition specific to thermal setups was used to relate the shear viscosity η and the bulk viscosity ζ with the coefficients in $W_{\mu\alpha,\nu\beta}$, but it can be relaxed if necessary. While the $W_{\mu\alpha,\nu\beta}$ term describes noise effects of fluids, it also captures the noise effects of gravitational waves as well. For example, if the background fluid is homogeneous and the spacetime curvature is negligible, the second order SK action for the TT component of metric fluctuations in the fluid rest frame reads

$$S = \frac{M_{\text{Pl}}^2}{4} \int d^4x \left[- \left(\ddot{h}_{ij}^{\text{TT}} + \gamma \dot{h}_{ij}^{\text{TT}} - \partial_k^2 h_{ij}^{\text{TT}} \right) h_{aij}^{\text{TT}} + i \frac{\gamma}{\beta} h_{aij}^{\text{TT}} h_{aij}^{\text{TT}} \right] \quad (4.15)$$

with the dissipation coefficient $\gamma = 2\eta/M_{\text{Pl}}^2$. We confirm that the dynamical KMS symmetry implies the fluctuation-dissipation relation of dissipative gravitational waves.

4.2 Generalized second law of black hole thermodynamics

In this section, we discuss the generalized second law [3, 6, 68] in the context of HydroEFT coupled to dynamical gravity introduced in the previous subsection. For simplicity, we focus on first-order fluids coupled to the Einstein gravity (4.13), which is free from higher derivative terms with Riemann tensors and follows the area law of black hole entropy. We consider a scenario where a dynamical black hole is surrounded by a fluid and evaluate the change of generalized entropy within the setup (4.13).

4.2.1 Dynamical black hole thermodynamics

First, we review the thermodynamics of dynamical black holes based on Refs. [24, 66].

Metric ansatz. We start by considering a spherically symmetric spacetime, which takes the following general form parameterized by two functions $F(v, r)$ and $h(v, r)$:

$$ds^2 = -e^{2h(v,r)} F(v, r) dv^2 + 2e^{h(v,r)} dv dr + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) , \quad (4.16)$$

where θ and ϕ are the angular coordinates, r is the radial coordinate, and v is the ingoing null coordinate. To describe a dynamical black hole in an asymptotically flat spacetime, we further parameterize $F(v, r)$ as

$$F(v, r) = 1 - \frac{2GM(v, r)}{r} , \quad M(v, r) > 0 , \quad (4.17)$$

and assume that $\lim_{r \rightarrow \infty} F(v, r) = 1$. Here we introduced the Newton constant $G = \frac{1}{8\pi M_{\text{Pl}}^2}$. We also assume that $h(v, r)$ is finite in the spacetime region of our interest. Note that the Vaidya spacetime is its special case with $M(v, r) = M(v)$ and $h(v, r) = 0$, but our setup is not limited to this.

Apparent horizon. The location of the apparent horizon is determined by solving $F(v, r) = 0$. We introduce its radius $r_{\text{H}}(v)$ as a function of v such that

$$r_{\text{H}}(v) = 2GM_{\text{BH}}(v) \quad \text{with} \quad M_{\text{BH}}(v) = M(v, r_{\text{H}}(v)) , \quad (4.18)$$

where $M_{\text{BH}}(v)$ is the Misner-Sharp mass [69] inside the horizon that we will call it the black hole mass in the following. Note that normalization of the energy is set by the Kodama vector [70]²

$$k^\mu(v, r) \partial_\mu = e^{-h(v,r)} \partial_v . \quad (4.19)$$

²The Kodama vector is defined by $k^\mu \equiv \epsilon^{\mu\nu} \nabla_\nu r$, where $\epsilon^{\mu\nu}$ is the Levi-Civita tensor on the 2-dimensional v - r spacetime, which is normalized as $\epsilon^{vr} = -\epsilon^{rv} = 1/\sqrt{-\det q_{\mu\nu}}$, with $q_{\mu\nu}$ being the induced metric on the 2 dimensional spacetime [70].

With the above definition of the black hole mass, its time-dependence reads

$$\dot{M}_{\text{BH}}(v) = \dot{M}(v, r_{\text{H}}(v)) + M'(v, r_{\text{H}}(v))\dot{r}_{\text{H}}(v). \quad (4.20)$$

Here and in what follows, we use dot and prime to denote derivatives in v and r , respectively. To see the physical meaning of each term, it is convenient to introduce the black hole volume $V_{\text{BH}}(v)$ as

$$V_{\text{BH}}(v) \equiv \frac{4\pi r_{\text{H}}(v)^3}{3}, \quad (4.21)$$

and one can obtain from Eq. (4.20) that

$$\dot{M}_{\text{BH}}(v) = \dot{M}(v, r_{\text{H}}(v)) + \frac{M'(v, r_{\text{H}}(v))}{4\pi r_{\text{H}}(v)^2} \dot{V}_{\text{BH}}(v). \quad (4.22)$$

We interpret the terms $\dot{M}_{\text{BH}}(v)$, $\dot{M}(v, r_{\text{H}}(v))$ and $\frac{M'(v, r_{\text{H}}(v))}{4\pi r_{\text{H}}(v)^2} \dot{V}_{\text{BH}}(v)$ as the changes of internal energy, heat, and work, respectively. Using Eq. (4.18), the time-dependence of the horizon radius reads

$$\dot{r}_{\text{H}}(v) = 2G\dot{M}_{\text{BH}}(v) = \frac{2G\dot{M}(v, r_{\text{H}}(v))}{1 - 2GM'(v, r_{\text{H}}(v))}. \quad (4.23)$$

We also introduce the entropy $S_{\text{BH}}(v)$ of the time-evolving black hole proportional to the area of the apparent horizon as

$$S_{\text{BH}}(v) = \frac{4\pi r_{\text{H}}(v)^2}{4G}. \quad (4.24)$$

Then, its time-dependence reads

$$\dot{S}_{\text{BH}}(v) = \frac{4\pi r_{\text{H}}(v)}{1 - 2GM'(v, r_{\text{H}}(v))} \dot{M}(v, r_{\text{H}}(v)). \quad (4.25)$$

Black hole temperature. The relation (4.25) motivates us to introduce the inverse temperature $\beta_{\text{BH}}(v)$ of the time-evolving black hole by

$$\beta_{\text{BH}}(v) = \frac{4\pi r_{\text{H}}(v)}{1 - 2GM'(v, r_{\text{H}}(v))}, \quad (4.26)$$

in terms of which Eq. (4.25) enjoys the following first-law type relation:

$$\dot{S}_{\text{BH}}(v) = \beta_{\text{BH}}(v) \dot{M}(v, r_{\text{H}}(v)). \quad (4.27)$$

Indeed, it is related to the so-called dynamical surface gravity $\kappa(v)$ as³

$$\beta_{\text{BH}}(v) = \frac{2\pi}{\kappa(v)} = \frac{4\pi r_{\text{H}}(v)}{1 - 2GM'(v, r_{\text{H}}(v))}, \quad (4.28)$$

³We followed the definition (4.29) proposed in [24] and further studied in [66, 67, 71–75]. See however Refs. [67, 76–79] for other approaches to surface gravity in dynamical spacetimes.

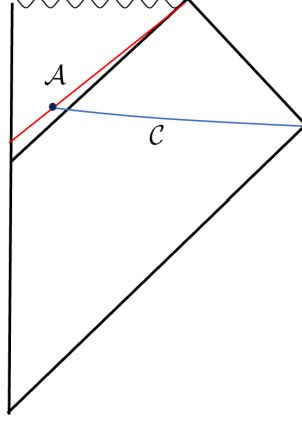


Figure 4.1: This figure shows the Penrose diagram of a non-stationary black hole: The black line corresponds to the event horizon and the red line corresponds to the apparent horizon. Here we consider the area of the apparent horizon $\mathcal{A}$ and the entropy of matter on the time slice $\mathcal{C}$.

where the general definition of $\kappa(v)$ is given in terms of the Kodama vector k^μ by

$$k^\mu \nabla_{[\mu} k_{\nu]} \big|_{r=r_H(v)} = \kappa(v) k_\nu \big|_{r=r_H(v)} , \quad (4.29)$$

and its concrete form in our setup (4.16) reads

$$\kappa(v) = \frac{1 - 2GM'(v, r_H(v))}{4GM(v, r_H(v))} = \frac{1 - 2GM'(v, r_H(v))}{2r_H(v)} . \quad (4.30)$$

4.2.2 Generalized second law

Setup. Now we suppose that time-evolution of the black hole is induced by a flow of fluid matter and discuss the generalized second law in this context. We parameterize Cauchy surfaces by the coordinate v at its intersection with the apparent horizon. See Fig. 4.1.

Energy flow. On the apparent horizon, $\dot{M}$ and M' are related to the energy-momentum tensor $T_{\mu\nu}$ of the fluid through the Einstein equation as

$$\dot{M}(v, r_H(v)) = 4\pi r_H(v)^2 e^{-h} T_{vv} , \quad M'(v, r_H(v)) = -4\pi r_H(v)^2 e^{-h} T_{vr} . \quad (4.31)$$

Here and in what follows we suppress the superscript “hydro” of $T_{\mu\nu}^{(\text{hydro})}$ for visual clarity. Also we refer to $h(v, r_H(v))$ and $T_{\mu\nu}(v, r_H(v))$ as h and $T_{\mu\nu}$, as long as there is no risk of confusion. Then, time-dependence of the black hole mass M_{BH} from (4.22) reads

$$\dot{M}_{\text{BH}}(v) = 4\pi r_H(v)^2 T_v{}^\mu \partial_\mu (r - r_H(v)) ,$$

which is nothing but the rate of the total energy flowing across the apparent horizon.

Fluid entropy. We next consider entropy of fluids outside the apparent horizon:

$$S_{\text{mat}}(\bar{v}) = \int d^4x \sqrt{-g} \delta(v - \bar{v}) \theta(r - r_{\text{H}}(v)) s^v, \quad (4.32)$$

where $s^\mu \equiv su^\mu = (\rho_0 + p_0)\beta u^\mu$ is the entropy current of the fluid (see, e.g., Refs. [32–34] and Appendix B). Its time-dependence follows from the Stokes' theorem that

$$\begin{aligned} \frac{dS_{\text{mat}}(\bar{v})}{d\bar{v}} &= - \int d^4x \sqrt{-g} \delta'(v - \bar{v}) \theta(r - r_{\text{H}}(v)) s^v \\ &= - \int d^4x \sqrt{-g} \partial_\mu \delta(v - \bar{v}) \theta(r - r_{\text{H}}(v)) s^\mu \\ &= \int d^4x \sqrt{-g} \delta(v - \bar{v}) \left[\theta(r - r_{\text{H}}(v)) \nabla_\mu s^\mu + \delta(r - r_{\text{H}}(v)) s^\mu \partial_\mu (r - r_{\text{H}}(v)) \right], \end{aligned} \quad (4.33)$$

where the local second law $\nabla_\mu s^\mu \geq 0$ (that follows from unitarity and the dynamical KMS symmetry [33]) implies that the first term in the square brackets is non-negative. Then, the second term gives a lower bound on the entropy change:

$$\frac{dS_{\text{mat}}(v)}{dv} \geq 4\pi r_{\text{H}}(v)^2 e^h s^\mu \partial_\mu (r - r_{\text{H}}(v)), \quad (4.34)$$

where $h = h(v, r_{\text{H}}(v))$ and $s^\mu = s^\mu(v, r_{\text{H}}(v))$ as before.

Generalized second law. With the entropy change of the black hole and fluid matter introduced in the above, we can now discuss the generalized entropy S_{gen} given by the sum,

$$S_{\text{gen}}(v) = S_{\text{BH}}(v) + S_{\text{mat}}(v). \quad (4.35)$$

Using the results so far, its time-dependence follows as

$$\begin{aligned} \frac{dS_{\text{gen}}(v)}{dv} &\geq \beta_{\text{BH}}(v) \left(\dot{M}_{\text{BH}}(v) - \frac{M'(v, r_{\text{H}}(v))}{4\pi r_{\text{H}}(v)^2} \dot{V}_{\text{BH}}(v) \right) + 4\pi r_{\text{H}}(v)^2 e^h s^\mu \partial_\mu (r - r_{\text{H}}(v)) \\ &= 4\pi r_{\text{H}}^2 e^h (\beta_{\text{BH}} - \beta_{\text{eff}}) T_{\mu\nu} k^\mu k^\nu + 4\pi r_{\text{H}}^2 e^h \beta_{\text{eff}} (T_{\mu\nu}^{(1)} k^\mu k^\nu - T_{\text{eff}} s^\mu \partial_\mu r_{\text{H}}(v)), \end{aligned} \quad (4.36)$$

where we have defined the effective temperature of the fluid as

$$\beta_{\text{eff}} = \frac{1}{T_{\text{eff}}} \equiv - \frac{1}{(k^\mu u_\mu) T}. \quad (4.37)$$

Note that the effective temperature does not in general coincide with the local temperature T but differs by a factor $-(k^\mu u_\mu)$, where k^μ is defined in (4.19). This correction factor plays a similar role as the blue-shift factor in the definition of Tolman temperature [67]. The blue-shift factor and hence the effective temperature vanishes when k^μ and u^μ are parallel $k^\mu u_\mu = 0$, whereas it becomes larger as the fluid is boosted to the inward direction⁴.

⁴It is worth noticing that for the Schwarzschild spacetime ($h(v, r) = 0$ and $M(v, r) = M$ is a constant), $k^\mu u_\mu = -1$ when taking u^μ as a velocity of a free-fall particle emitted from infinity, indicating that there is no blue/red-shift. This is consistent with the equivalence principle.

Quasi-static approximation. While it is not easy to provide a full interpretation of the general result (4.36), a certain simplification occurs if we focus on quasi-static processes and neglect terms proportional to $\dot{r}_H(v)$:

$$\frac{dS_{\text{gen}}(v)}{dv} \geq 4\pi r_H^2 e^h (\beta_{\text{BH}} - \beta_{\text{eff}}) T_{\mu\nu} k^\mu k^\nu + 4\pi r_H^2 e^h \beta_{\text{eff}} T_{\mu\nu}^{(1)} k^\mu k^\nu \quad (\text{quasi-static}), \quad (4.38)$$

where the second term is from the first order corrections to ideal fluids, hence it is subdominant at least when $\beta_{\text{BH}} \neq \beta_{\text{eff}}$. Interestingly, the sign of the dominant first term is controlled by the temperature gradient between the black hole and fluids, and also the null energy. Below, we discuss its property for the $T_{kk} > 0$ case and the $T_{kk} < 0$ case separately.

Case 1: $T_{vv} > 0$. When the matter flow satisfies the null energy condition, the generalized entropy increases when $\beta_{\text{BH}} > \beta_{\text{eff}}$. This is consistent with the general thermodynamic expectation that the total entropy of a closed system increases when heat flows from a warmer region to a colder region.

Case 2: $T_{vv} < 0$. Conversely, when the null energy condition is violated, the generalized entropy increases when $\beta_{\text{BH}} < \beta_{\text{eff}}$. A typical example for this is particle emission through the Hawking radiation [1]. Again, it is consistent with thermodynamic expectation that heat flow is controlled by the temperature gradient. Furthermore, for Hawking particles in quasi-static processes, we expect that u^μ is proportional to k^μ , in which limit the inverse effective temperature goes as $\beta_{\text{eff}} \rightarrow \infty$ and $\beta_{\text{BH}} < \beta_{\text{eff}}$ is trivially satisfied.

Chapter 5

Conclusion

In this thesis, motivated by the description of black hole thermodynamics in dynamical contexts, we discuss the effective field theory (EFT) of dynamical gravity within the Schwinger-Keldysh (SK) formalism as a framework for describing non-equilibrium gravitational theories. In the dynamical gravitational theory under the SK formalism, the time evolution is described by the closed-time-path, leading to a doubling of diffeomorphism symmetry. Both the diagonal parts, referred to as the physical diffeomorphism, and the off-diagonal parts, referred to as the noise diffeomorphism, must be preserved. However, the breaking of the noise diffeomorphism is related to the non-equilibrium nature of the system. For the theory to remain consistent, the noise diffeomorphism symmetry can be restored by introducing Stückelberg fields. By dynamically promoting these Stückelberg fields, an environment sector is naturally introduced, replacing the role of capturing dissipation from the breaking of the noise diffeomorphism to the dynamics of the Stückelberg fields. Furthermore, by selecting a fluid as the model for the EFT of the environment sector, a dissipative gravitational theory can be constructed in which dissipation is captured through the dynamics of the fluid.

This framework is constructed as an EFT based on considerations of symmetry and the quantum unitarity. As a further application to gravitational theories, this thesis employs results from the SKEFT of hydrodynamics to explore the generalized second law in scenarios where a black hole evolves dynamically while surrounded by a fluid. For dynamical black holes, the entropy is typically associated with the apparent horizon. It is demonstrated that generalized entropy production arises from the temperature gradient between the local fluid temperature, defined based on local equilibrium principles linked to dynamical Kubo-Martin-Schwinger (KMS) symmetry, and the black hole temperature, determined by spacetime symmetries. Additionally, this temperature gradient is affected by the blue-shift effect, which depends on the direction of the fluid flow. The analysis confirms that entropy production is enhanced in both of the following cases: (i) when fluid flows into the black hole under typical conditions, and (ii) when exotic matter, such as Hawking radiation, flows out of the black hole. In both scenarios, the alignment of the matter flow direction and the temperature gradient plays a critical role in driving the

increase in entropy.

We finish this thesis by outlining several potential future directions. To begin with, our proposed framework is highly versatile. It is interesting to apply the dissipative scalar model of Sec. 3.3 to cosmology. As we discussed there, interactions between the scalar sector and the environment sector are present in general. Impacts of these new interactions on dissipative inflation will be worth studying. Second, our EFT recipe can directly be applied to the open EFT of inflation [42, 52]. Even though the current studies in the literature are limited to fixed background spacetime, our present work offers a framework to incorporate dynamical gravity and go beyond the decoupling limit of gravity, by which dynamics of gravitational waves in the open EFT of inflation can be studied for example. Another important issue is the model-dependence of the environment sector. Although it is natural and simple to use HydroEFT as a fluid model for the dynamical Stückelberg field associated with the physical diffeomorphism symmetry, we have implicitly imposed a set of internal symmetries which defines a fluid. However, the HydroEFT formalism is not limited to this and one can modify the set of symmetries imposed on the Stückelberg fields to describe other phases of matter such as solid and superfluid [46, 80]. It is interesting to utilize the flexibility of the HydroEFT formalism to study various environment sector scenarios and explore a possibility to identify the phase of the environment matter from the dynamics of dissipative open systems. Besides, more fundamentally, the SK formalism is a framework useful to study real-time dynamics at finite temperature, so it will offer a proper arena to study thermodynamic properties of dynamical gravity and spacetime. We hope that our present work on the SKEFT with dynamical gravity gives useful insights for such studies as well.

Appendix A

Raychaudhuri's equation

A.1 The null version of Raychaudhuri's equation

In this appendix, we derive the null version of Raychaudhuri's equation. First, let us consider a generator of null hypersurfaces k^μ affinely parameterized by λ which satisfies the following:

$$k^\mu k_\mu = 0, \quad k^\mu \nabla_\mu k^\nu = 0. \quad (\text{A.1})$$

Then, to construct the transverse metric $h_{\mu\nu}$ for the null hypersurface, we select an auxiliary null vector N^μ and choose its normalization as $k^\mu N_\mu = -1$. By using this, the transverse metric is given as

$$h_{\mu\nu} = g_{\mu\nu} + k_\mu N_\nu + N_\mu k_\nu. \quad (\text{A.2})$$

Now we introduce the tensor

$$B_{\mu\nu} \equiv \nabla_\mu k_\nu, \quad (\text{A.3})$$

and define

$$\tilde{B}_{\mu\nu} \equiv h^\alpha{}_\mu h^\beta{}_\nu B_{\alpha\beta}. \quad (\text{A.4})$$

This tensor describes the force experienced by the congruence of k^μ as

$$\tilde{B}_{\mu\nu} = \frac{1}{2}\theta h_{\mu\nu} + \sigma_{\mu\nu} + \omega_{\mu\nu}, \quad (\text{A.5})$$

where $\theta = \tilde{B}^\mu{}_\mu$ is the expansion, $\sigma_{\mu\nu} = \frac{1}{2}(\tilde{B}_{\mu\nu} + \tilde{B}_{\nu\mu}) - \frac{1}{2}\theta h_{\mu\nu}$ is the shear tensor, and $\omega_{\mu\nu} = \frac{1}{2}(\tilde{B}_{\mu\nu} - \tilde{B}_{\nu\mu})$ is the rotation tensor. From this $B_{\mu\nu}$, one can obtain

$$\begin{aligned} k^\mu \nabla_\mu \tilde{B}^\nu{}_\nu &= -\tilde{B}_{\mu\nu} \tilde{B}^{\mu\nu} - R_{\mu\nu} k^\mu k^\nu, \\ \Rightarrow \frac{d\theta}{d\lambda} &= -\frac{1}{2}\theta - \sigma^{\mu\nu} \sigma_{\mu\nu} + \omega^{\mu\nu} \omega_{\mu\nu} - R_{\mu\nu} k^\mu k^\nu. \end{aligned} \quad (\text{A.6})$$

This is Raychaudhuri's equation for a congruence of null geodesics.

Finally, let us mention an important property of the expansion. For a coordinate $x^\mu = (\lambda, r, x^i)$, the cross-sectional area element of the null hypersurface generated by k^μ is defined by $\sqrt{a}d^2x$. Here $a = \det(a_{ij})$ and it is given as

$$a_{ij} = g_{\mu\nu}e_i^\mu e_j^\nu, \quad k_\mu e_i^\mu = 0 = N_\mu e_i^\mu. \quad (\text{A.7})$$

This a_{ij} is inverse of $h_{\mu\nu}$ on the geodesic:

$$h^{\mu\nu} = a^{ij}e_i^\mu e_j^\nu. \quad (\text{A.8})$$

Now one can obtain the relation

$$\frac{d\theta}{d\lambda} = \left(\tilde{B}_{\mu\nu} + \tilde{B}_{\nu\mu} \right) e_i^\mu e_j^\nu, \quad (\text{A.9})$$

and taking its trace give

$$\theta = \frac{1}{\sqrt{a}} \frac{d}{d\lambda} \sqrt{a}. \quad (\text{A.10})$$

Appendix B

Schwinger-Keldysh EFT of hydrodynamics

This appendix reviews and explains all the essential features and ideas of HydroEFT used in this paper. We refer the readers to [32–35] for proofs and derivations.

B.1 Stückelberg fields of HydroEFT

In this subsection, we explain the physical origin of the Stückelberg fields that arise in HydroEFT. Recall that in Eq. (3.102), we presented a hydro action

$$S_{\text{hydro}} = \int d^4x \sqrt{-g} \left[\frac{1}{2} T_{\mu\nu}^{(\text{hydro})} \mathcal{G}_a^{\mu\nu} + \frac{i}{4} W_{\mu\alpha, \nu\beta} \mathcal{G}_a^{\mu\nu} \mathcal{G}_a^{\alpha\beta} \right], \quad (\text{B.1})$$

which implicitly contains the dynamical fields X^μ and X_a^μ in the r - a basis or equivalently X_1^μ and X_2^μ in the 1-2 basis. These fields were introduced in Eq. (3.84) through the Stückelberg trick to recover the diffeomorphism symmetries at the action level on the closed-time path. To make a connection to conventional hydrodynamics and discuss the physical meaning of these fields, it is best done by going back to the original form of the action constructed in [32, 33]. The exact form of the action is not important here and is schematically written as

$$S_{\text{hydro}} = \int d^4\sigma \mathcal{L}_{\text{eff}}[X_1^\mu(\sigma), X_2^\mu(\sigma); g_{1\mu\nu}, g_{2\mu\nu}]. \quad (\text{B.2})$$

The effective Lagrangian, $\mathcal{L}_{\text{eff}}$, is written in terms of coordinates σ^A . In this form, this hydro action is expressed in a “fluid spacetime” parameterized by the coordinates σ^A ($A = \bar{0}, \bar{1}, \bar{2}, \bar{3}$). The dynamical fields $X_1^\mu(\sigma)/X_2^\mu(\sigma)$ describes a map from the fluid spacetime to a physical spacetime parameterized by the coordinates x_1^μ/x_2^μ (see Figure B.1). This is equivalent to the Lagrangian frame of conventional hydrodynamics. Note that we have two copies of the physical spacetime. $g_{1\mu\nu}$ and $g_{2\mu\nu}$ in Eq. (B.2) are the two physical spacetime metrics and act as the (non-dynamical) source of the stress tensor living on each physical spacetime, respectively. In this language, the induced metric $\mathbf{h}_{sAB}$ on the fluid spacetime in the 1-2 basis reads

$$\mathbf{h}_{sAB}(\sigma) = g_{s\mu\nu}(X_s) \frac{\partial X_s^\mu}{\partial \sigma^A} \frac{\partial X_s^\nu}{\partial \sigma^B}, \quad (s = 1, 2). \quad (\text{B.3})$$

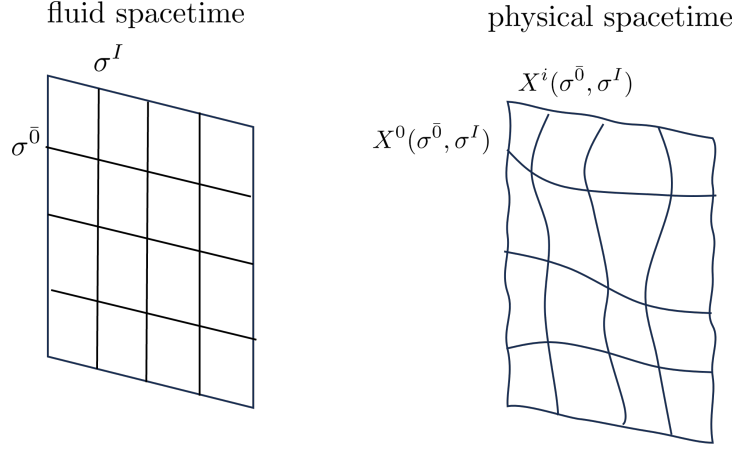


Figure B.1: The left figure shows the fluid spacetime where $\mathbf{h}_{AB}$ lives: Fluid elements are labeled by $\sigma^{\bar{0}}$ and σ^I ($I = \bar{1}, \bar{2}, \bar{3}$). The right figure shows the physical spacetime where $g_{\mu\nu}$ lives: $X^\mu(\sigma)$ tells us the position of the fluids in the spacetime. Note that there are two copies of physical spacetime but only one is shown here.

To convert the hydro action to a more familiar form, i.e., the effective Lagrangian and all the fields are functions of the physical spacetime, formally we can introduce an inverse map of the dynamical variables and express the fluid spacetime coordinates in terms of the physical spacetime coordinates. Such inverse map was presented and discussed in [34].

B.2 Symmetries and local temperature of HydroEFT

Having discussed the physical meaning of the dynamical fields of HydroEFT, the next step is to understand the symmetries of a fluid. After that, we also explain how we can introduce a local temperature to the system and how the concept of local thermal equilibrium is connected to the KMS symmetry in this subsection.

Fluid symmetries. In the Lagrangian frame of hydrodynamics, we label each fluid element and follows their trajectory (c.f. the Eulerian frame where we focus on a fixed spacetime point and observe how fluid elements pass through it.). Such role is played by the spatial coordinates σ^I ($I = \bar{1}, \bar{2}, \bar{3}$) in the fluid spacetime. The worldline (trajectory) of each fluid element is given by a constant- σ^I curve and parameterized by $\sigma^{\bar{0}}$. Although the existence of a fluid breaks part of the diffeomorphism symmetry, the following symmetries remains:

1. Time-independent reparameterizations of σ^A

$$\sigma^{\bar{0}} \rightarrow \sigma^{\bar{0}}, \quad \sigma^I \rightarrow \sigma'^I(\sigma^I), \quad (I = \bar{1}, \bar{2}, \bar{3}). \quad (\text{B.4})$$

2. Time-diffeomorphisms of $\sigma^{\bar{0}}$

$$\sigma^{\bar{0}} \rightarrow \sigma'^{\bar{0}}(\sigma^{\bar{0}}, \sigma^I), \quad \sigma^I \rightarrow \sigma^I, \quad (I = \bar{1}, \bar{2}, \bar{3}). \quad (\text{B.5})$$

The first condition (B.4) implies the invariance under a relabeling of fluid elements, while (B.5) implies the time-reparametrization invariance of the fluid element's worldline. These symmetries define a fluid.

Local temperature. In formulating the HydroEFT, we have assumed an initial thermal density matrix $\rho = \frac{1}{Z} e^{-\beta_0 H}$ with an inverse temperature β_0 . Under a perturbation with a timescale much larger than the local thermalization (relaxation) time, thermodynamics quantities including the temperature can fluctuate locally. To describe the local temperature of a fluid, it is convenient to introduce a scalar field $\tau(\sigma)$ which defines the local proper temperature $T(\sigma)$ as

$$T(\sigma) \equiv T_0 e^{-\tau(\sigma)} \quad (\text{B.6})$$

with a reference temperature set by the initial state $T_0 = \beta_0^{-1}$. The local temperature $T_{\text{local}}(\sigma)$ normalized by the temporal coordinate $\sigma^{\bar{0}}$ is defined as

$$T_{\text{local}}(\sigma) = E_r T(\sigma) = E_r T_0 e^{-\tau}, \quad E_r \equiv \frac{1}{2} \left(\sqrt{-h_{1\bar{0}\bar{0}}} + \sqrt{-h_{2\bar{0}\bar{0}}} \right). \quad (\text{B.7})$$

From here and onward, we fix $E_r = e^\tau$ by using the time-diffeomorphisms (B.5) to obtain

$$T_{\text{local}}(\sigma) = T_0, \quad (\text{B.8})$$

and the local proper temperature becomes

$$T(\sigma) = \frac{T_0}{E_r} = \frac{1}{\beta(\sigma)}. \quad (\text{B.9})$$

These procedures correspond to defining the local temperature by imposing the imaginary fluid time periodicity $\sigma^{\bar{0}} \sim \sigma^{\bar{0}} + i\beta_0^{-1}$. This gauge choice does not completely fix (B.5) and the following gauge transformation remains:

$$\sigma^{\bar{0}} \rightarrow \sigma^{\bar{0}} + \lambda(\sigma^I). \quad (\text{B.10})$$

Dynamical KMS symmetry. If the microscopic system has a Z_2 symmetry which includes time reversal, the generating function $e^{iW[g_1, g_2]}$ of a system with an initial state in thermal equilibrium (with an inverse temperature β_0) satisfies the Kubo-Martin-Schwinger (KMS) condition:

$$W[g_1(x), g_2(x)] = W[\tilde{g}_1(x), \tilde{g}_2(x)] \quad (\text{B.11})$$

with

$$\tilde{g}_{1\mu\nu}(x) = \Theta g_{1\mu\nu}(t, \vec{x}), \quad \tilde{g}_{2\mu\nu}(x) = \Theta g_{2\mu\nu}(t + i\beta_0, \vec{x}), \quad (\text{B.12})$$

where Θ is one of the following Z_2 transformations $T/PT/CPT$, where C , P and T represents the charge conjugation, parity and time reversal, respectively. The explicit choice of this transformation depends on the microscopic system of interest but at least contains T .

One can generalize the KMS condition to accommodate systems that are out-of-equilibrium but in local thermal equilibrium. It was proposed in [34] (and also in [37] in a different context) that the local equilibrium condition can be implemented as a symmetry at the level of the effective action:

$$S_{\text{eff}}[X_1(\sigma), X_2(\sigma); \mathbf{h}_1, \mathbf{h}_2] = S_{\text{eff}}[\tilde{X}_1(\sigma), \tilde{X}_2(\sigma); \tilde{\mathbf{h}}_1, \tilde{\mathbf{h}}_2] \quad (\text{B.13})$$

with

$$\tilde{\mathbf{h}}_{1AB}(\sigma) = \Theta \mathbf{h}_{1AB}(\sigma^{\bar{0}}, \sigma^I), \quad \tilde{\mathbf{h}}_{2AB}(\sigma) = \Theta \mathbf{h}_{2AB}(\sigma^{\bar{0}} + i\beta_0, \sigma^I), \quad (\text{B.14})$$

and

$$\tilde{X}_1^\mu(\sigma) = \Theta X_1^\mu(\sigma^{\bar{0}}, \sigma^I), \quad \tilde{X}_2^\mu(\sigma) = \Theta X_2^\mu(\sigma^{\bar{0}} + i\beta_0, \sigma^I) - i\beta_0 \delta_0^\mu. \quad (\text{B.15})$$

(B.12) is implied by (B.14) and (B.15). The symmetry associated with the transformations (B.14) and (B.15) is called the dynamical KMS symmetry. Note that the KMS symmetry (B.11) is a symmetry based on the temperature of the initial state in equilibrium, whereas the dynamical KMS symmetry (B.13) is a symmetry based on the periodicity of $\sigma^{\bar{0}}$. Therefore, theories with the dynamical KMS symmetry can take any local equilibrium state as the initial state.

B.3 Ingredients of HydroEFT and symmetries in the classical limit

To construct the HydroEFT perturbatively, we need to identify a suitable set of variables with nice transformation properties under the symmetries discussed above. In this subsection, we discuss this procedure in more details. We will only consider the classical limit which is adequate for this thesis.

***r*-a basis.** It is convenient to use the *r*-a basis for considering the classical limit:

$$g_{\mu\nu} = \frac{1}{2} (g_{1\mu\nu} + g_{2\mu\nu}), \quad \hbar g_{a\mu\nu} = g_{1\mu\nu} - g_{2\mu\nu}, \quad (\text{B.16})$$

$$X^\mu = \frac{1}{2} (X_1^\mu + X_2^\mu), \quad \hbar X_a^\mu = X_1^\mu - X_2^\mu, \quad (\text{B.17})$$

$$\mathbf{h}_{AB} = \frac{1}{2} (\mathbf{h}_{1AB} + \mathbf{h}_{2AB}), \quad \hbar \mathbf{h}_{aAB} = \mathbf{h}_{1AB} - \mathbf{h}_{2AB}. \quad (\text{B.18})$$

As explained in Sec. 3.2.3, the spacetime diffeomorphisms in the classical limit are given by (3.70c) and (3.70d). In what follows, we introduce the transformation properties of the fluid diffeomorphisms in the classical limit, along with the corresponding ingredients for constructing the hydro EFT. Furthermore, we present the transformation properties of the dynamical KMS symmetries in the same limit.

Fluid spacetime metric. Next, we associate the fluid spacetime metric h_{AB} and the physical spacetime metric $g_{\mu\nu}$ in the classical limit:

$$h_{AB}(\sigma) \equiv \frac{\partial X^\mu}{\partial \sigma^A} \frac{\partial X^\nu}{\partial \sigma^B} g_{\mu\nu}(X). \quad (\text{B.19})$$

Similarly, the “noise” metrics h_{aAB} on the fluid spacetime reads

$$h_{aAB}(\sigma) = \frac{\partial X^\mu}{\partial \sigma^A} \frac{\partial X^\nu}{\partial \sigma^B} \mathcal{G}_{a\mu\nu}(X), \quad \mathcal{G}_{a\mu\nu}(X) \equiv g_{a\mu\nu} + \nabla_\mu X_{a\nu} + \nabla_\nu X_{a\mu}. \quad (\text{B.20})$$

Notice that $\mathcal{G}_{a\mu\nu}$ is invariant under the noise diffeomorphism (3.70d), hence it is a suitable quantity for constructing the EFT.

Ingredients for hydro EFT. Now we identify ingredients of the hydro effective action. First, we define the canonically normalized four-velocity u^μ by

$$u^\mu = \frac{1}{\sqrt{-h_{00}}} \frac{\partial X^\mu}{\partial \sigma^0}, \quad (u_\mu u^\mu = -1), \quad (\text{B.21})$$

which is invariant under the fluid diffeomorphisms (B.4) and (B.5). The transverse projector orthogonal to u^μ is defined by

$$\Delta^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu, \quad (\text{B.22})$$

which satisfies $u_\mu \Delta^{\mu\nu} = 0$ and $\Delta^{\mu\nu} \Delta_{\nu\gamma} = \delta_\gamma^\mu$. Also it is invariant under (B.4) and (B.5). In the classical limit, we have $E_r = \sqrt{-h_{00}}$, so that the local proper temperature reads

$$T(\sigma) = \frac{T_0}{\sqrt{-h_{00}}} = \frac{1}{\beta(\sigma)}. \quad (\text{B.23})$$

Note that $\beta(\sigma)$ is invariant under (B.10). Thus, the variables that transform nicely under the symmetries and are convenient for constructing the action are

$$\mathcal{G}_{a\mu\nu}, \quad u^\mu, \quad \Delta^{\mu\nu}, \quad \beta, \quad (\text{B.24})$$

and their derivatives.

Dynamical KMS transformations. Let us also look at the dynamical KMS transformations for the r - a variables in the semiclassical limit. If we recover $\hbar$ explicitly, β_0 becomes $\hbar\beta_0$, so that we obtain

$$\tilde{g}_{\mu\nu}(x) = \Theta g_{\mu\nu}(x), \quad \tilde{g}_{a\mu\nu}(x) = \Theta g_{a\mu\nu}(x) + i\Theta \mathcal{L}_{\beta_0} g_{\mu\nu}(x), \quad (\text{B.25})$$

$$\tilde{X}^\mu(\sigma) = \Theta X^\mu(\sigma), \quad \tilde{X}_a^\mu(\sigma) = \Theta X_a^\mu(\sigma) - i\Theta \beta^\mu(\sigma) + i\beta_0^\mu, \quad (\text{B.26})$$

$$\tilde{h}_{AB}(\sigma) = \Theta h_{AB}(\sigma), \quad \tilde{h}_{aAB}(\sigma) = \Theta h_{aAB}(\sigma) + i\Theta \mathcal{L}_\beta h_{AB}(\sigma), \quad (\text{B.27})$$

where $\beta^\mu(x) \equiv \beta(x)u^\mu$, $\beta_0^\mu \equiv \beta_0 \delta_0^\mu$, and the physical variables transform as

$$\tilde{u}^\mu(x) = \Theta u^\mu(x), \quad \tilde{\beta}(x) = \Theta \beta(x), \quad \tilde{\mathcal{G}}_{a\mu\nu}(x) = \Theta \mathcal{G}_{a\mu\nu}(x) + i\Theta \mathcal{L}_\beta g_{\mu\nu}(x). \quad (\text{B.28})$$

B.4 Entropy current associated with the dynamical KMS invariance

If dynamical KMS symmetry is a symmetry of the action, we can construct a quantity similar to a Noether current by using the Lagrangian formulated in the previous subsection. For HydroEFT, this quantity coincides with the usual entropy current defined in thermodynamics. We will review the construction in this subsection. For simplicity, we only consider the action (B.1) which includes derivatives up to first order.

In general, dynamical KMS invariant theories satisfy

$$\mathcal{L}_{\text{eff}} = \tilde{\mathcal{L}}_{\text{eff}} - i\nabla_\mu V^\mu, \quad (\text{B.29})$$

where $\tilde{\mathcal{L}}_{\text{eff}}$ is obtained by taking the KMS transformation and $x \rightarrow x' = \Theta x$, with $\Theta = T, PT, CPT$, of the Lagrangian density. Applying the KMS transformation to (B.1) and keeping only upto first order in derivatives, V^μ is given by

$$V^\mu = p_0 \beta^\mu, \quad (\text{B.30})$$

where we have used the thermodynamics identity $\rho_0 + p_0 = -\beta \partial_\beta p_0$. On the other hand, using the EOM, the KMS transformation of the Lagrangian becomes

$$\tilde{\mathcal{L}}_{\text{eff}} - \mathcal{L}_{\text{eff}} = i\nabla_\mu (T^{\mu\nu} \beta_\nu). \quad (\text{B.31})$$

where $T_{\mu\nu} \equiv T_{\mu\nu}^{(0)} + T_{\mu\nu}^{(1)}$. From these calculations, we can define something akin to a Noether current based on the dynamical KMS invariance:

$$\begin{aligned} s^\mu &\equiv V^\mu - T^{\mu\nu} \beta_\nu \\ &= p_0 \beta^\mu - T^{\mu\nu} \beta_\nu. \end{aligned} \quad (\text{B.32})$$

This s^μ coincides with the usual entropy current in hydrodynamics. Considering the divergence of this entropy current s^μ , we have

$$\nabla_\mu s^\mu = \frac{\beta}{2} \eta \sigma^{\mu\nu} \sigma_{\mu\nu} + \beta \zeta \vartheta^2 \geq 0, \quad (\text{B.33})$$

where we have used the constitution relation of the stress tensor upto first order in derivative and the semi-positive definite properties of η and ζ , i.e. $\zeta \geq 0$, $\eta \geq 0$, from quantum unitarity condition (3.47). For a perfect fluid, the entropy current is conserved. Here, we only presented a proof of the second law upto first order correction. A proof to all orders can be found in [33].

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