



Tunneling in de Sitter spacetime as a stochastic process

宮地, 大河

(Degree)

博士 (理学)

(Date of Degree)

2025-03-25

(Date of Publication)

2026-03-01

(Resource Type)

doctoral thesis

(Report Number)

甲第9143号

(URL)

<https://hdl.handle.net/20.500.14094/0100496424>

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Doctoral Dissertation

Tunneling in de Sitter spacetime as a stochastic process

(確率過程としてみたド・ジッター時空中のトンネリング)

January 2025

Graduate School of Science, Kobe University

Taiga Miyachi

(宮地 大河)

Abstract

Tunneling processes in de Sitter spacetime are studied by using the stochastic approach. We evaluate the Martin–Siggia–Rose–Janssen–de Dominicis (MSRJD) functional integral by using the saddle-point approximation to obtain the tunneling rate. The applicability conditions of this method are clarified using the Schwinger-Keldysh formalism. In the case of a shallow potential barrier, we reproduce the Hawking-Moss (HM) tunneling rate. Remarkably, in contrast to the HM picture, the configuration derived from the MSRJD functional integral satisfies physically natural boundary conditions. We also discuss the case of a steep potential barrier and find an interesting Coleman-de Luccia (CDL) bubble-like configuration. Since the starting point of our analysis is the Schwinger-Keldysh path integral, which can be formulated in a more general setup and incorporates quantum effects, our formalism sheds light on further studies of tunneling phenomena from a real-time perspective.

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Chapter 1

Introduction

In the thermal history of the universe, it is believed that several phase transitions occurred in the early universe, where high-energy particle physics played a central role. In the Standard Model, the electroweak phase transition (100 GeV) and the quark-hadron phase transition (100 MeV) are well-known examples. At even higher energy scales, such as those predicted by grand unified theories (10^{15} GeV) and string theories (10^{17} GeV), various additional phase transitions are hypothesized.

The phase transition is typically described by the finite-temperature one-loop effective potential, as given below[1, 2, 3]¹.

$$V_{\text{eff}}(\varphi) = V(\varphi) + V_1(\varphi) + \frac{T^4}{2\pi^2} \left[\sum_b n_b J_B(m_b^2(\varphi)/T^2) - \sum_f n_f J_F(m_f^2(\varphi)/T^2) \right], \quad (1.1)$$

where φ is the vacuum expectation value of the scalar field, which serves as the order parameter. $V(\varphi)$ and $V_1(\varphi)$ represent the tree-level potential and the one-loop potential, respectively. The third and fourth terms correspond to the finite-temperature one-loop effects from bosonic fields and fermionic fields, respectively. The summations account for the different species of fields. $m_{b,f}^2(\varphi)$ are the effective masses of φ , and $n_{b,f}$ represent the degrees of freedom

¹We neglect the gravity.

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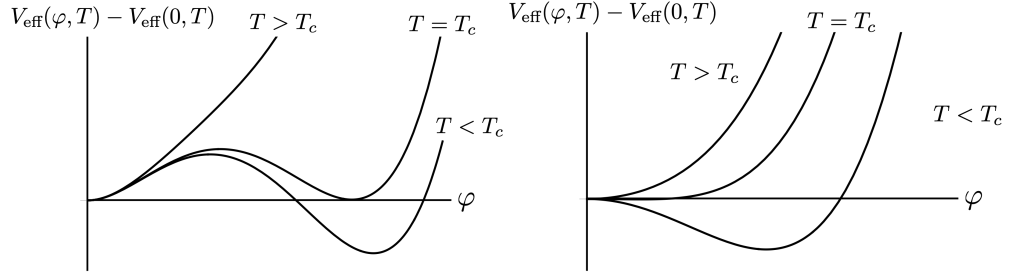


Figure 1.1: The effective potentials for the first-order phase transition (left) and second-order phase transition (right).

for each field. The functions $J_{B,F}(x)$ are defined by

$$J_B(x) := \int_0^\infty dp \, p^2 \log \left(1 - e^{-\sqrt{p^2+x}} \right), \quad (1.2)$$

$$J_F(x) := \int_0^\infty dp \, p^2 \log \left(1 + e^{-\sqrt{p^2+x}} \right). \quad (1.3)$$

Using the effective potential (1.1), we can perturbatively analyze the order of the phase transition. In the case of a first-order phase transition, the minimum point of the effective potential changes discontinuously at the critical temperature $T = T_c$ (FIG. 1.1, left). On the other hand, in the case of a second-order phase transition, the minimum point of the effective potential changes continuously, but its derivative with respect to temperature is discontinuous (FIG. 1.1, right). If the minimum point is continuous for all factorial derivatives with respect to temperature, there is no phase transition; this scenario is referred to as a crossover.

The first-order phase transition is particularly attractive because it gives a characteristic object known as a *bubble*. When the temperature drops below the critical temperature (see FIG. 1.1, left), the original lowest point of the effective potential becomes unstable. To transition from the metastable minimum to the true minimum, the potential barrier must be overcome through the quantum effect known as *tunneling*. Then, the phase transition occurs locally, and bubbles form, within which the scalar field takes on a value corresponding to the true minimum, similar to the behavior of supercooled water.

Bubbles give rise to rich physical phenomena in the early universe. In electroweak baryogenesis [4, 5, 6], bubbles play a significant role in creating the

non-equilibrium conditions required for baryon number generation, fulfilling one of the three Sakharov conditions [7]. The formation of primordial black holes during a first-order phase transition is also an intriguing phenomenon [8, 9]. During bubble nucleation, metastable regions surrounded by bubbles appear and collapse as the bubbles expand. The latent heat contained within these metastable regions is then gravitationally collapses, leading to the formation of primordial black holes. Additionally, gravitational waves produced by bubble collisions [10, 11, 12, 13, 14] are particularly noteworthy, especially in light of the recent detection of gravitational waves by LIGO in 2015 [15] and the Pulsar Timing Array in 2023 [16, 17, 18, 19]. In particular, the gravitational waves detected by the Pulsar Timing Array might originate from a first-order phase transition [20]. In the future, space-based interferometers such as LISA [21], BBO [22], and DECIGO [23] are expected to usher in a new era of gravitational wave cosmology.

In the Standard Model, the electroweak phase transition is a crossover when the mass of the Higgs boson exceeds approximately 80 GeV [24, 25, 26, 27]. In reality, the Higgs boson mass is measured to be 125.20 ± 0.11 GeV [28, 29, 30]. Lattice simulations have shown that the quark-hadron phase transition is also a crossover under realistic conditions, such as the physical up, down, and strange quark masses and zero baryon chemical potential [31, 32, 33]. On the other hand, many well-motivated particle physics models predict first-order phase transitions in the early universe. For detailed reviews, see [34, 35] and references therein. The detection of gravitational waves from a first-order phase transition would provide a powerful tool to probe physics beyond the Standard Model.

The tunneling rate (or the false vacuum decay rate) is a fundamental physical quantity that represents the probability of transitioning from the metastable state (the false vacuum) to the true vacuum, which corresponds to the lowest minimum point of the potential. Since the tunneling rate determines the timing of phase transition completion, it also characterizes the peak density and frequency of the resulting gravitational waves today [14]. Therefore, understanding the accurate calculation of the tunneling rate is crucial in early universe physics.

Since our universe is not flat spacetime, understanding tunneling processes in curved spacetime becomes crucial, particularly when the tunneling involves mass scales close to the Planck scale or when the nucleated bubble is large enough to be influenced by spacetime curvature. In this context, tunneling in de Sitter spacetime is especially worth detailed study, as it rep-

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resents the simplest example of a non-trivial curved spacetime. Moreover, insights gained from de Sitter spacetime can be directly applied to tunneling phenomena that occur during inflation.

While both Minkowski and de Sitter spacetimes possess maximal symmetry, leading to some similarities, there are key differences. For instance, the Euclidean instanton method used to evaluate tunneling rates in Minkowski spacetime [36, 37] can also be applied to tunneling processes in de Sitter spacetime [38]. However, extending the Euclidean method to de Sitter spacetime presents interpretational challenges. In de Sitter spacetime, there exist two saddle point solutions: the Coleman-De Luccia (CDL) instanton [38] and the Hawking-Moss (HM) instanton [39]. The HM instanton, in particular, is a homogeneous solution, making it less straightforward to interpret the tunneling rate as the probability of transitioning from a false vacuum to a true vacuum [40, 41, 42].

To circumvent the difficulty, the real time formalism might be useful. In the case of de Sitter spacetime in the flat chart, the stochastic formalism has been used for investigating time evolution of fluctuations during inflation [43, 44]. In this formalism, Langevin equations describe the dynamics of long wavelength modes. Stochastic noises stem from short wavelength quantum modes. The stochastic formalism has been also used to analyze the tunneling processes in de Sitter spacetime [45, 46, 47, 48, 49, 50, 51].

However, most of the previous works treated only the HM transition by utilizing the Fokker-Plank (FP) equation for a homogeneous scalar field, which can be derived from the Langevin equation. Obviously, the HM transition cannot describe the bubble nucleation in contrast to CDL tunneling. To describe the bubble, we have to include the spatial gradient term and the FP equation becomes the functional differential equation instead of the partial differential equation. Hence, at first glance, it seems difficult to treat the CDL bubble by the stochastic approach. However, we make an observation that there exists a path integral representation to solutions of the FP equation. In condensed matter physics, it is called Martin-Siggia-Rose-Janssen-de Dominicis (MSRJD) functional integral [52, 53, 54, 55].

In this paper, we employ the MSRJD path integral formalism to study tunneling processes on a fixed de Sitter background. The applicability conditions for this method are clarified using the Schwinger-Keldysh formalism. We consider a real scalar field theory² with a potential $V(\phi)$, which has a

²We implicitly assume the stage below the critical temperature (FIG. 1.1, left).

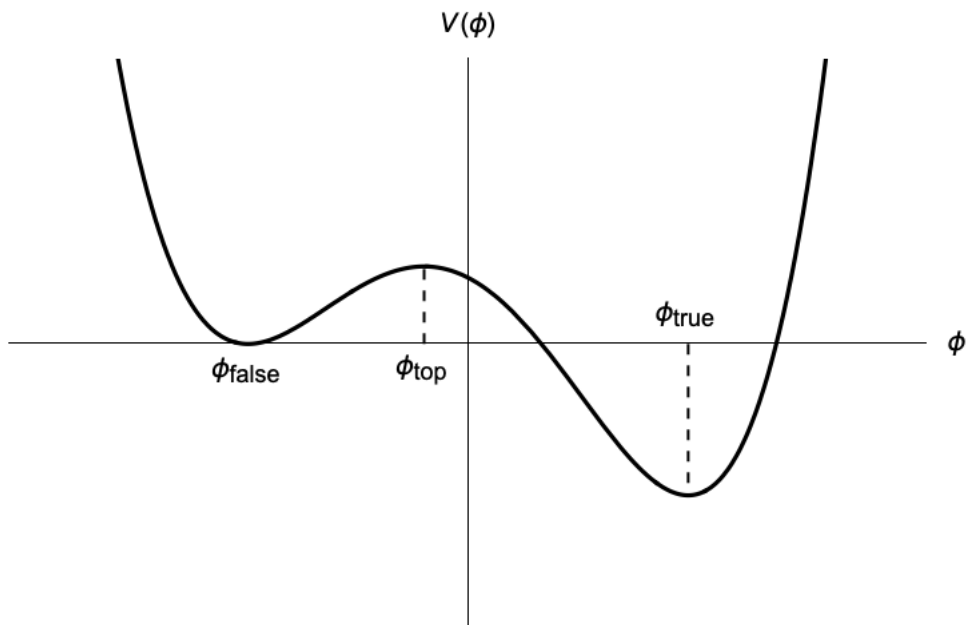


Figure 1.2: The schematic picture of the potential $V(\phi)$ which has a false vacuum at $\phi = \phi_{\text{false}}$, true vacuum at $\phi = \phi_{\text{true}}$ and the top of the potential at $\phi = \phi_{\text{top}}$.

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local minimum at $\phi = \phi_{\text{false}}$ (the false vacuum), a true minimum at $\phi = \phi_{\text{true}}$ (the true vacuum), and a local maximum at $\phi = \phi_{\text{top}}$, as shown in FIG. 1.2. We use the saddle point method and evaluate the tunneling rate by computing the action for the tunneling configurations [56]. We investigate both HM and CDL tunneling processes.

In the case of a shallow potential barrier, the conventional HM tunneling rate is reproduced. Notably, the tunneling configuration derived from the MSRJD functional integral method naturally incorporates physically intuitive boundary conditions, in contrast to the Euclidean approach. It means that the problem of Hawking-Moss tunneling, as proposed by Weinberg [42], does not arise in our formalism. We clarify that the HM tunneling rate corresponds to the tunneling probability of a (roughly Hubble-sized) coarse-grained patch transitioning from ϕ_{false} to ϕ_{top} . For a steep potential barrier, we discover an intriguing CDL bubble-like configuration. This configuration is obtained by solving the scalar field equation in Euclidean anti de Sitter space, even though we do not work in imaginary time. Our results demonstrate how the bubble nucleation process can be effectively described within the stochastic formalism.

The organization of this thesis is structured as follows:

- Chapter 2 is a review of the usual formalism for the tunneling process named Euclidean method. It begins by presenting the definition of the tunneling rate in the quantum mechanics and how to calculate it by Euclidean method. Following this, the formula of the tunneling rate is extended to the quantum field theory in flat spacetime and de Sitter spacetime. Especially in de Sitter spacetime, we briefly review the two famous tunneling process, CDL tunneling and HM tunneling.
- Chapter 3 is a review of the stochastic approach and the MSRJD functional integral which is constructed from the Langevin equations. We also outline the derivation of MSRJD functional integral starting from the Schwinger-Keldysh formalism.
- Chapter 4 is the main part of this thesis, which is based on the author's paper [57]. We apply the MSRJD functional integral to the case of the shallow potential barrier and study the HM tunneling process. Also we consider the case of the steep potential barrier and find the CDL bubble-like configuration. We derive the tunneling rate using the configuration and compare it with the one obtained in the case of. We

also compare our results with those in the Euclidean method reviewed in Chapter 2.

- Chapter 5 is devoted to the conclusion and future prospects.

Conventions

In this thesis, unless otherwise noted, the following natural unit system is adopted

$$c = \hbar = G = 1,$$

where c is the speed of light, $\hbar$ is the reduced Planck constant and G is the Newton constant. The signature of the four-dimensional metric is $(-, +, +, +)$.

Chapter 2

Tunneling formalism in Euclidean method

In this chapter, we review the usual formalism to describe the tunneling process in the quantum field theory, Euclidean method. In section 2.1, we consider the quantum mechanical case to see the usual discussion about the tunneling process in Euclidean method. In section 2.2, we extend the formulation of tunneling rate to the quantum field theory. Finally, we review the tunneling process in de Sitter background in section 2.3.

This review is grounded in existing literature, with the reference including [36, 37, 38, 40, 42, 58].

2.1 Quantum mechanics

2.1.1 Definition of the tunneling rate

Firstly, let us consider a quantum mechanical particle in one dimension with the Hamiltonian

$$H = \frac{p^2}{2m} + V(q), \tag{2.1}$$

where the potential $V(q)$ has a metastable minimum at $x = a$ (FIG. 2.1). For convenience, we take the value of the potential at this minimum to be zero.

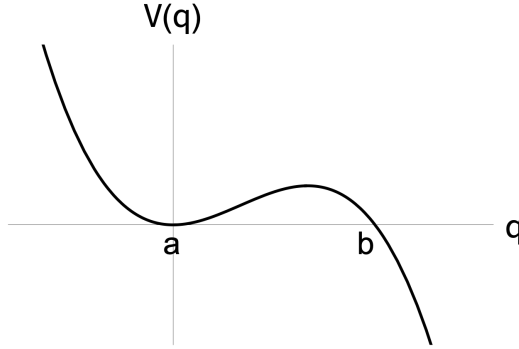


Figure 2.1: Potential $V(q)$ with a metastable minimum.

To start, let us consider the matrix element

$$\langle a | e^{-\mathcal{T}H} | a \rangle = \sum_n e^{-E_n \mathcal{T}} |\langle a | n \rangle|^2 = \int \mathcal{D}q(\tau) e^{-S_E[q]}, \quad (2.2)$$

where $\mathcal{T}$ and τ is Euclidean time and S_E is Euclidean action. At large $\mathcal{T}$, this matrix element is approximated as

$$\lim_{\mathcal{T} \rightarrow \infty} \langle a | e^{-\mathcal{T}H} | a \rangle = e^{-E_0 \mathcal{T}} |\langle a | 0 \rangle|^2. \quad (2.3)$$

Then if we return to the real time T by Wick rotation $T = -i\mathcal{T}$, the probability of the particle staying at the minimum is

$$\lim_{T \rightarrow \infty} |\langle a | e^{iTH} | a \rangle|^2 = e^{-2\text{Im}E_0 T} |\langle a | 0 \rangle|^2. \quad (2.4)$$

This probability exponentially decay with the tunneling (decay) rate

$$\Gamma = -2\text{Im}E_0, \quad (2.5)$$

and this quantity is what we want to obtain.

2.1.2 Saddle point approximation

To obtain the imaginary part of the energy E , we evaluate the path integral in (2.2) by the saddle point approximation. Given a classical Euclidean solution $\bar{q}(\tau)$ we write

$$q(\tau) = \bar{q}(\tau) + \sum_n c_n \psi_n(\tau), \quad (2.6)$$

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where $\psi(\tau)$ is an eigenmode with eigenvalue λ of

$$\left. \frac{\delta^2 S_E}{\delta q(\tau) \delta q(\tau')} \right|_{q=\bar{q}\tau} = -\frac{d^2}{d\tau^2} + V''(\bar{q}(\tau)) =: S_E''[\bar{q}]. \quad (2.7)$$

The boundary condition is $\psi_n(-\mathcal{T}/2) = \psi_n(\mathcal{T}/2) = 0$. We change integration variables from $q(\tau)$ to the mode coefficients and write

$$\mathcal{D}q(\tau) = \prod_n \frac{dc_n}{\sqrt{2\pi}}. \quad (2.8)$$

The contribution to the path integral from this stationary point then becomes

$$I = \int \prod_n \frac{dc_n}{\sqrt{2\pi}} e^{-(S[\bar{q}] + \frac{1}{2} \sum_k \lambda_k c_k^2 + \dots)}, \quad (2.9)$$

where the ellipsis denotes terms that are cubic and higher order in the deviation from $\bar{q}$. Neglecting the ellipsis and integrating the Gauss integral, we obtain

$$\begin{aligned} I &= e^{-S_E[\bar{q}]} \prod_n \lambda_n^{-1/2} [1 + \dots] \\ &= e^{-S_E[\bar{q}]} (\det S_E''[\bar{q}])^{-1/2} [1 + \dots]. \end{aligned} \quad (2.10)$$

From below, we omit the ellipsis.

Let us now enumerate the relevant stationary points. For $\langle a | e^{-\mathcal{T}H} | a \rangle$ we have the trivial constant solution $q_0(\tau) = a$. With our convention that $V(q)$ vanishes at the minimum, $S[q_0] = 0$ and the contribution to the path integral is simply

$$I_0 = (\det S_E''[q_0])^{-1/2}. \quad (2.11)$$

Also we have the nontrivial solution named *bounce* which can be noticed from the transmittance obtained from the WKB approximation [36]. The solution q_b runs from a at $\tau = -\mathcal{T}/2$, reaches at the turning point b and back to a at $\tau = \mathcal{T}/2$ (FIG. 2.2). The existence of such solution is obvious because the potential is flipped in the Euclidean dynamics. According to our formula, this solution should give a contribution

$$e^{-S_E[q_b]} (\det S_E''[q_b])^{-1/2}. \quad (2.12)$$

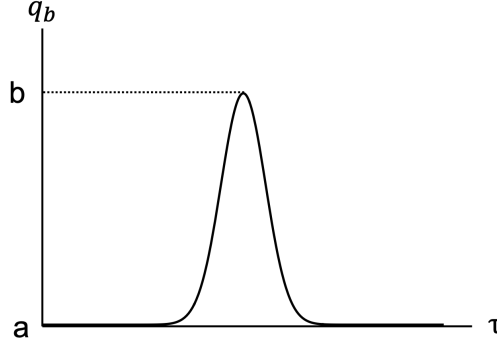


Figure 2.2: Bounce solution in quantum mechanics

However, this is problematic because there is a zero mode of S''

$$\psi_0(\tau) = N^{-1/2} \frac{dq_b}{d\tau}, \quad (2.13)$$

where $N = S_E[q_b]$ is the normalization constant¹. This zero mode reflects the broken τ -translation symmetry. Because of the zero eigenvalue, $\det S''$ vanishes and the pre-exponential factor diverges.

The solution is to replace the coefficient of the zero mode by a collective coordinate z that specifies the location of the center of the bounce, and to include only the nonzero modes in the determinant. Instead of a Gaussian integral over the zero-mode coefficient, we must integrate over the full range of z from $-\mathcal{T}/2$ to $\mathcal{T}/2$. From (2.6)

$$\psi_0 dc_0 = \frac{dq}{d\tau} dz, \quad (2.16)$$

¹This normalization condition is defined as

$$1 = \int_{-\mathcal{T}/2}^{\mathcal{T}/2} \psi_0(\tau)^2 d\tau = \int_{-\mathcal{T}/2}^{\mathcal{T}/2} N^{-1} \left(\frac{dq_b}{d\tau} \right)^2 d\tau. \quad (2.14)$$

By using Euclidean energy conservation law

$$\frac{1}{2} \left(\frac{dq_b}{d\tau} \right)^2 - V(q_b) = 0, \quad (2.15)$$

we obtain $N = S_E[q_b]$.

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then we have

$$\frac{dc_0}{\sqrt{2\pi}} = \left(\frac{N}{2\pi}\right)^{1/2} dz. \quad (2.17)$$

Therefore, the net contribution of the bounce solution is

$$I_b = e^{-S_E[q_b]} \left(\frac{N}{2\pi}\right)^{1/2} T(\det' S_E''[q_b])^{-1/2}. \quad (2.18)$$

with the prime on the determinant indicating that only the nonzero modes are to be included. Finally, let us define

$$K = \left(\frac{N}{2\pi}\right)^{1/2} \left(\frac{\det' S_E''[q_b]}{\det S_E''[q_0]}\right)^{-1/2} \quad (2.19)$$

so that

$$I_b = e^{-S_E[q_b]} (\det S_E''[q_0])^{-1/2} K T. \quad (2.20)$$

In addition to these saddle solutions, we must include multi-bounce solutions which reach to b several times. For the configuration q_n with a n -bounces, its action is just n times that of a single instanton $S[q_n] = nS[q_b]$. It has n approximate zero modes corresponding to independent τ -translations of its component bounces. Then a collective coordinate z_j must be introduced for each of these, leading to a total Jacobian factor $(N/2\pi)^n/2$. Integrating over the bounce locations, which satisfy the constraint $z_1 < z_2 < \dots < z_n$ gives a factor of

$$\int_{-\mathcal{T}/2}^{\mathcal{T}/2} dz_1 \int_{z_1}^{\mathcal{T}/2} dz_2 \dots \int_{z_{n-1}}^{\mathcal{T}/2} dz_n = \frac{\mathcal{T}^n}{n!}. \quad (2.21)$$

The final factor will be included in the functional determinant of $S''[q_n]$ (with zero modes excluded). The effect of single bounce on the spectrum of the fluctuations is localized around the center of the bounce i.e. the ratio of determinants that in eq (2.19) is independent of $\mathcal{T}$ for Large $\mathcal{T}$. Hence, the determinant factor for a contribution with n well-separated bounces can be written as

$$(\det S_E''[q_0])^{-1/2} \left(\frac{\det' S_E''[q_n]}{\det S_E''[q_0]}\right)^{-1/2} = (\det S_E''[q_0])^{-1/2} \left(\frac{\det' S_E''[q_b]}{\det S_E''[q_0]}\right)^{-n/2}. \quad (2.22)$$

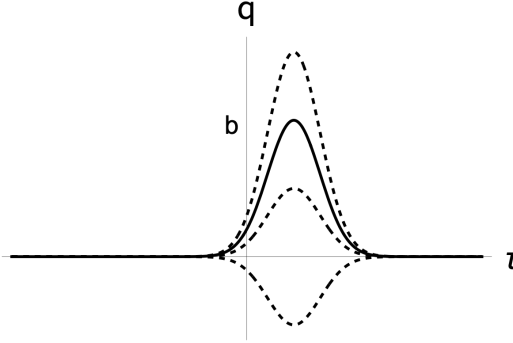


Figure 2.3: A series of the configurations. The solid line with maximal value at b corresponds to the bounce. The dashed curve above this is a configuration that extends to the classically allowed region beyond the potential barrier, while the dashed curves below it are configurations that remain within the potential barrier.

Combining all factors, we find that the contributions of n -bounce solution is given as

$$I_n = e^{-nS_E[q_b]} (\det S''_E[q_0])^{-1/2} K^n \frac{T^n}{n!}. \quad (2.23)$$

Adding together all contributions from saddle point solutions, we have

$$\begin{aligned} \langle a | e^{-\mathcal{T}H} | a \rangle &= \sum_{n=0}^{\infty} I_n \\ &= (\det S''_E[q_0])^{-1/2} \exp(K\mathcal{T}e^{-S_E[q_b]}). \end{aligned} \quad (2.24)$$

Actually, we have another problem about the factor K . It can be seen that the zero mode eigenfunction (2.13) has one node at the center of the bounce i.e. there is a negative mode. Then the Gaussian integral of the negative mode is divergent.

The origin of this negative mode can be understood from the configuration space along the direction of negative mode. FIG. 2.3 show a series of the configurations labeled by c , the maximal value of $q(\tau)$. Then $c = b$ corresponds to the bounce solution and the trivial solution $q(\tau) = 0$ corresponds to the stational solution. The Euclidean action of these configurations as a function of c is illustrated in FIG. 2.4. The local minimum and minimum

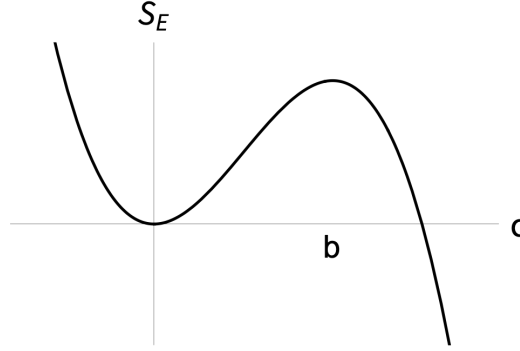


Figure 2.4: The Euclidean action for a family of configurations such as shown in FIG. 2.3. The maximum at $c = b$ corresponds to the bounce.

correspond to the stationary solution and bounce respectively. We can notice that the bounce is a saddle point.

Our path integral is an infinite-dimensional integral, but the problematic thing is only the integration over the coefficient of the one negative mode. We can see how to handle this integration by considering the analogous integral

$$J = \int_{-\infty}^{\infty} \frac{dc}{\sqrt{2\pi}} e^{-S(c)}, \quad (2.25)$$

where $S(c)$ is the function plotted in FIG. 2.4. As $c \rightarrow -\infty$ the action increase and the integral converges. On the other hand, as $c \rightarrow \infty$ the action tends toward infinity and the integral diverges. However, we can relate it to a finite integral by deforming the contour of integration from the real axis into the upper half plane (FIG. 2.5). The integration from $-\infty$ to b is clearly real. The imaginary part which comes from the remainder of the contour and the steepest descent approximation is given by

$$\begin{aligned} \text{Im} J &= \text{Im} \int_b^{b+i\infty} \frac{dc}{\sqrt{2\pi}} e^{-S(b)} e^{-\frac{1}{2}S''(b)(c-b)^2} \\ &= \frac{1}{2} e^{-S(b)} |S''(b)|^{-1/2} \end{aligned} \quad (2.26)$$

with the factor of $1/2$ being due to the fact that the contour only goes through half of the Gaussian peak.

With this factor taken into account, the imaginary part of (2.19) is given

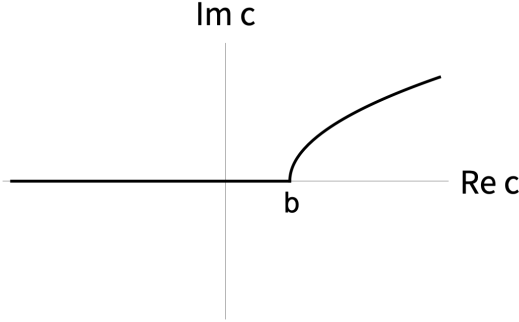


Figure 2.5: The contour of integration that makes the integral in (2.25) well-defined.

by

$$\text{Im}K = \frac{1}{2} \left(\frac{N}{2\pi} \right)^{1/2} \left| \frac{\det' S_E''[q_b]}{\det S_E''[q_0]} \right|^{-1/2}. \quad (2.27)$$

Finally, substituting the value of $N = S_E[q_b]$, we obtain the below decay rate

$$\begin{aligned} \Gamma &= -2\text{Im}E_0 = 2\text{Im}K e^{-S_E[q_b]} \\ &= \left(\frac{S_E[q_b]}{2\pi} \right)^{1/2} \left| \frac{\det' S_E''[q_b]}{\det S_E''[q_0]} \right|^{-1/2} e^{-S_E[q_b]}. \end{aligned} \quad (2.28)$$

This result is was obtained under the assumption that $V(a) = 0$. If $V(a) \neq 0$, we obtain instead

$$\Gamma = \left(\frac{B}{2\pi} \right)^{1/2} \left| \frac{\det' S_E''[q_b]}{\det S_E''[q_0]} \right|^{-1/2} e^{-B}, \quad (2.29)$$

where B is the difference between the action of the bounce and that of the trivial solution. This exponent B is the leading contribution to the decay rate and a main quantity in this dissertation.

2.2 Quantum field theory

In this section, we extend the formulation of the decay rate (2.28) or (2.29) to the quantum field theory in flat spacetime. We consider a scalar field theory

$$S[\phi] = \int d^4x \left(-\frac{1}{2} \eta^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right), \quad (2.30)$$

where the scalar field potential has two non-degenerate local minima as in FIG. 1.2. The lower of the two corresponds to the vacuum state of the theory called the true vacuum. The higher minimum is called the false vacuum.

Since the actual potential energy is

$$U[\phi[\mathbf{x}]] = \int d^3x \left[\frac{1}{2} (\nabla \phi)^2 + V(\phi) \right], \quad (2.31)$$

there is an infinite potential between the spatially homogeneous false and true vacuum configurations. Then, the tunneling amplitude for this vanishes. Instead, the false vacuum decays by a tunneling process that takes a spatially homogeneous state to one with a region of approximate true vacuum, a *bubble*, embedded in a false vacuum background. Because the bubble can be nucleated anywhere, the decay rate is proportional to the volume of space, and formally infinite. The finite physically measurable quantity is the bubble nucleation rate per unit volume, $\Gamma/\mathcal{V}$.

We will denote the bounce solution by $\phi_b(t, \mathbf{x})$. The nucleation rate is then of the form

$$\frac{\Gamma}{\mathcal{V}} = A e^{-B}, \quad (2.32)$$

where

$$B = S[\phi_b] - S[\phi_{\text{false}}] \quad (2.33)$$

like as (2.29). Here $S[\phi_b]$ is the Euclidean action of the bounce solution and

$$S[\phi_{\text{false}}] = \int d\tau d^3x V(\phi_{\text{false}}) \quad (2.34)$$

is that of the homogeneous false vacuum.

In the quantum mechanics, the tunneling process is described by the bounce solution in the Euclidean equation of motion, which in the present case is the field equation

$$\frac{d^2\phi}{d\tau^2} + \nabla^2\phi - \frac{dV}{d\phi} = 0 \quad (2.35)$$

that follows from the Euclidean action

$$S_E[\phi] = \int d\tau d^3x \left[\frac{1}{2} \left(\frac{\partial\phi}{\partial\tau} \right)^2 + \frac{1}{2} (\nabla\phi)^2 + V(\phi) \right]. \quad (2.36)$$

We also impose boundary conditions as

$$\phi(\pm\infty, \mathbf{x}) = \phi_{\text{false}}, \quad (2.37)$$

$$\phi(\tau, |\mathbf{x}| = \infty) = \phi_{\text{false}}. \quad (2.38)$$

The first boundary condition is the same as in quantum mechanics. The second is needed to obtain a finite action.

Since both of equation and boundary conditions displays an $O(4)$ invariance, we use an $O(4)$ -invariance ansatz when seeking the bounce. Indeed, it is proved that such a symmetric solution gives the lowest action for the case of a theory with a single scalar field [59]. Thus let us define $s = \sqrt{\tau^2 + \mathbf{x}^2}$ and seek a bounce solution that depends only on s . The Euclidean field equation then becomes

$$\frac{d^2\phi}{ds^2} + \frac{3}{s} \frac{d\phi}{ds} = \frac{dV}{d\phi}. \quad (2.39)$$

The boundary conditions on the bounce reduce to

$$\phi(\infty) = \phi_{\text{false}} \quad (2.40)$$

while requiring that the solution be nonsingular at the origin gives

$$\left. \frac{d\phi}{ds} \right|_{s=0} = 0. \quad (2.41)$$

The existence of a solution to these equation can be demonstrated by an "overshoot-undershoot" argument due to Coleman [36].

An expression for the pre-exponential factor (2.29) in quantum mechanics can be generalized to the quantum field theory. Since the bounce solution

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is not spatially homogeneous, there are three zero modes in addition to the zero mode corresponding to the translation in Euclidean time. Then we have to introduce three collective coordinates, which represent the location of the center of the bounce.

Making the appropriate modification to (2.29), we obtain

$$\Gamma = \mathcal{V} \left[\prod_{a=1}^4 \left(\frac{N_a}{2\pi} \right)^{1/2} \right] \left| \frac{\det' S''[\phi_b]}{\det S''[\phi_{\text{false}}]} \right|^{-1/2} e^{-B}. \quad (2.42)$$

Here N_a is the normalization factor for the zero mode. The factor $\mathcal{V}$, the volume of space, comes from integrating over the location of the bounce.

For an $O(4)$ -symmetric bounce, the N_a are all equal and given by

$$N_a = \int d^4x (\partial_a \phi_b)^2 = B. \quad (2.43)$$

Then we obtain ²

$$\frac{\Gamma}{\mathcal{V}} = \left(\frac{B}{2\pi} \right)^2 \left| \frac{\det' S''[\phi_b]}{\det S''[\phi_{\text{false}}]} \right|^{-1/2} e^{-B}. \quad (2.44)$$

2.3 Including gravity

The gravitational effects on the bubble nucleation is often neglected. However, there are situations where the effects of gravity are significant. The most obvious case is a transition involves mass scale close to the Plank scale, but gravitational effects can also come into play at lower mass scales if the bubble nucleates with a size large enough to be sensitive to the curvature of spacetime. Even when gravitational effects are negligible for bubble nucleation itself, they can be quite significant for the evolution of the bubble after nucleation.

The extension of the argument in section 2.2 to include gravity is a non-trivial manner, mainly because the globally defining energy in a curved space-time is difficult, and there is no complete derivation of an expression for the

²Until now we have been working implicitly with the action expressed in terms of unrenormalized fields, masses and coupling constants. If we want take into account the renormalization, we only have to add the leading counterterm to the calculation of the Euclidean actions [37].

tunneling rate. Nevertheless, one can argue from analogy, as first done by Coleman and De Luccia [38].

In section 2.2, we discuss the decay rate in flat spacetime and obtain the form $\Gamma = Ae^{-B}$. In this section, we use this form in de Sitter spacetime. We only discuss the exponent B and set aside the prefactor A because the one-loop calculation with gravity becomes so difficult. We will argue the two types of saddle point solutions, Coleman-de Ruccia instanton [38] and Hawking-Moss instanton [39]. We also discuss the case where the back-reaction to the geometry is negligible.

2.3.1 Coleman-de Ruccia instanton

Let us consider the following Euclidean action.

$$S_E[\phi, g_{\mu\nu}] = \int d^4x \sqrt{g} \left[-\frac{1}{2\kappa} R + \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + V(\phi) \right] + S_{GHY} \quad (2.45)$$

that includes a Euclideanized Einstein–Hilbert term with the metric $g_{\mu\nu}$. S_{GHY} is the Euclidean version of the Gibbons–Hawking–York boundary action [60, 61] that must be added to remove the second derivative terms from the action. Also $\kappa := 8\pi G = 8\pi/M_{\text{pl}}^2$.

In flat spacetime, at least for a theory with a single scalar field, the bounce solution with the minimum action exhibits $O(4)$ symmetry [59]. Although the analogous result in curved spacetime has not been fully proven, it is widely believed to hold true³. Assuming this to be the case, any $O(4)$ -symmetric Euclidean metric can be expressed in the form

$$ds^2 = d\xi^2 + \rho(\xi)^2 d\Omega_3^2. \quad (2.46)$$

The Ricci scalar is then

$$R = -\frac{6}{\rho^2}(\rho\rho'' + \rho'^2 - 1) \quad (2.47)$$

³There is an attempt to prove this by AdS/CFT correspondence [62].

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with primes denoting differentiation with respect to ξ , and the action becomes

$$\begin{aligned}
S &= 2\pi^2 \int_{\xi_{\min}}^{\xi_{\max}} d\xi \left[\rho^3 \left\{ \frac{1}{2} \phi'^2 + V(\phi) \right\} + \frac{3}{\kappa} (\rho^2 \rho'' + \rho \rho'^2 - \rho) \right] + S_{GHY} \\
&= 2\pi^2 \int_{\xi_{\min}}^{\xi_{\max}} d\xi \left[\rho^3 \left\{ \frac{1}{2} \phi'^2 + V(\phi) \right\} - \frac{3}{\kappa} (\rho \rho'^2 + \rho) \right] \\
&\quad + \frac{6\pi^2}{\kappa} (\rho^2 \rho') \Big|_{\xi=\xi_{\min}}^{\xi=\xi_{\max}} + S_{GHY}.
\end{aligned} \tag{2.48}$$

The boundary term from the integration by part is exactly canceled by the Gibbons-Hawking-York term, and then we have

$$S_E = 2\pi^2 \int_{\xi_{\min}}^{\xi_{\max}} d\xi \left[\rho^3 \left\{ \frac{1}{2} \phi'^2 + V(\phi) \right\} - \frac{3}{\kappa} (\rho \rho'^2 + \rho) \right]. \tag{2.49}$$

The scalar field equation is

$$\phi'' + \frac{3\rho'}{\rho} \phi' = \frac{dV}{d\phi} \tag{2.50}$$

while the $\xi\xi$ -component of the Euclidean Einstein equation gives

$$\rho'^2 = 1 + \frac{\kappa}{3} \rho^2 \left(\frac{1}{2} \phi'^2 - V \right). \tag{2.51}$$

Differentiating (2.50) and using (2.51), we obtain another useful equation.

$$\rho'' = -\frac{\kappa}{3} (\phi'^2 + V) \rho. \tag{2.52}$$

Combining (2.49) and (2.51), the action for the solution of the field equation can be expressed as

$$S_E = 4\pi^2 \int_{\xi_{\min}}^{\xi_{\max}} d\xi \left[\rho^3 V - \frac{3}{\kappa} \rho \right]. \tag{2.53}$$

The boundary conditions for the field equations depend on the topology of the solution. Subtracting (2.51) and (2.52) shows that $\rho'^2 - \rho''\rho \geq 1$, from which we can conclude that ρ cannot have a nonzero minimum and must have at least one zero. The invariance of the metric under translation of ξ

allows us to set $\xi = \xi_{\min} = 0$. If this is the only zero point of ρ , ξ ranges from 0 to ∞ , and the solution has the topology of $\mathbb{R}^4$. The boundary conditions for (2.50) are $\phi'(0) = 0$ to avoid a singularity at the origin, and $\phi(\infty) = \phi_{\text{false}}$, as in flat spacetime.

However, it is possible that ρ has a second zero point at some finite $\xi_{\max}$. This is always the case if $V(\phi) \geq 0$ everywhere and is not identically zero [63]. In this case, the solution has the topology of a four-sphere, and the boundary conditions for (2.50) become $\phi'(0) = \phi'(\xi_{\max}) = 0$. Here, there is no requirement for any point where $\phi = \phi_{\text{false}}$.

Let us now study the Euclidean solutions for the case of a de Sitter false vacuum. We require both the bounce solution and the Euclidean version of the false vacuum. The latter is easily obtained. With ϕ constant and $V(\phi_{\text{false}})$ positive, (2.51) gives

$$\rho = \Lambda_{\text{fv}} \sin(\xi/\Lambda_{\text{fv}}), \quad (2.54)$$

where Λ is related to the vacuum energy V by $\Lambda = \sqrt{3/\kappa V}$ and $\xi_{\max} = \pi\Lambda_{\text{fv}}$. With this form for ρ , the metric gives the standard round metric on a four-sphere of radius Λ_{fv} which is the Euclidean counterpart of the Lorentzian hyperboloid. We find that its action is

$$S[\phi_{\text{false}}] = -\frac{24\pi^2}{\kappa^2 V(\phi_{\text{false}})} = -\frac{8\pi^2}{\kappa} \Lambda_{\text{fv}}^2. \quad (2.55)$$

Let us now turn to the bounce itself, beginning with the limit where gravitational effects are expected to be small. Suppose that $V(\phi)$ satisfies the following conditions:

1. The variation in V between ϕ_{false} and ϕ_{true} is much smaller than M_{pl}^4 .
2. The radius of the flat-spacetime bounce (i.e., the region where ϕ is appreciably different from ϕ_{false}) is much less than Λ_{fv} .

Using (2.51), the first condition implies $\rho' \approx 1$, and thus $\rho \approx \xi$ in the region $0 \leq \xi \ll \Lambda_{\text{fv}}$. (2.50) then reduces to the flat spacetime (2.39), showing that the bounce is essentially identical to that in flat spacetime in this region. The second condition ensures that ϕ becomes exponentially close to ϕ_{false} before exiting this region. Outside this region, ρ is well approximated by (2.54), with a small shift $\xi \rightarrow \xi - \Delta$, where Δ is of the order of M/M_{pl}^2 . Here, M characterizes the mass scale of the barrier in $V(\phi)$ that separates the two

vacua. The Euclidean action is finite and negative, but when subtracting the false vacuum action, it yields a positive value for B . This value differs from the flat-spacetime result only by a fractional amount on the order of M^2/M_{pl}^2 .

The above discussion includes quantum gravity; however, it is also reasonable to neglect quantum gravity effects and focus solely on field theory in curved spacetime. In other word, we fix the background geometry and consider the classical gravity.

Let us consider the real scalar field theory in 3+1 dimensional de Sitter background.

$$\begin{aligned} ds^2 &= -dt^2 + a^2 d\mathbf{x}^2, \quad a(t) = e^{-Ht} \\ &= \frac{1}{H^2 \eta^2} (-d\eta^2 + d\mathbf{x}^2), \quad \eta := -\frac{1}{H\eta}, \end{aligned} \quad (2.56)$$

$$S[\phi] = \int d^4x \sqrt{-g} \left(-g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right). \quad (2.57)$$

where H is Hubble parameter and the potential $V(\phi)$ has the false vacuum like as FIG. 1.2. In Euclidean method, we Euclideanize the action by taking the Wick rotation. Generally, the ways of Euclideanization for de Sitter spacetime depend on the coordinates. In this paper, we choose the second coordinate in (2.56)⁴.

By taking the wick rotation as $\eta \rightarrow -i\tau$, the coordinate becomes

$$-ds^2 = \frac{1}{H^2 \tau^2} (d\tau^2 + d\mathbf{x}^2). \quad (2.58)$$

This is the Euclidean AdS spacetime and we can take the global coordinate;

$$-ds^2 = H^{-2} (d\rho^2 + \sinh^2 \rho d\Omega^2). \quad (2.59)$$

We also impose that the field ϕ only depends on ρ . Therefore, the Euclidean action S_E and the equation of motion becomes

$$S_E = 2\pi^2 \int_0^\infty d\rho \sinh^3 \rho \left[\frac{1}{2H^2} (\partial_\rho \phi)^2 + \frac{V(\phi)}{H^4} \right], \quad (2.60)$$

$$\frac{d^2 \phi}{d\rho^2} + \frac{3}{\tanh \rho} \frac{d\phi}{d\rho} = \frac{V'(\phi)}{H^2}. \quad (2.61)$$

⁴The analytic continuations of this coordinate is examined in [40]

The second equation is the same as (4.43) and we can obtain the same bounce solutions discussed in section 4.2.

We take the following non-dimensionalizations;

$$\phi = \alpha \tilde{\phi}, \quad (2.62)$$

$$V(\phi) = g^2 \alpha^4 \tilde{V}(\tilde{\phi}). \quad (2.63)$$

Therefore, the Euclidean action (2.60) becomes

$$\begin{aligned} S_E &= 2\pi^2 \frac{\alpha^2}{H^2} \int_0^\infty d\rho \sinh^3 \rho \left[\frac{1}{2} (\partial_\rho \tilde{\phi})^2 + \frac{g^2 \alpha^2}{H^2} \tilde{V}(\tilde{\phi}) \right] \\ &:= 2\pi^2 \frac{\alpha^2}{H^2} \tilde{I}^E. \end{aligned} \quad (2.64)$$

2.3.2 Hawking-Moss instanton

For a generic potential, the situation is more complex than in the corresponding flat spacetime case. It has already been noted that, with an everywhere positive potential, the bounce is topologically a four-sphere. One consequence is that the second term in (2.50), which corresponds to a damping term in the flat spacetime case, changes sign and becomes an anti-damping term for part of the interval. This invalidates the overshoot–undershoot argument that guaranteed the existence of a bounce in the absence of gravity. In fact, there are situations where there is no ‘Coleman-De Luccia’ bounce, that is, no symmetric $O(4)$ bounce with non-constant ϕ crossing the top of the potential barrier.

To understand this, note that the width of the bubble wall is largely determined by the shape of the barrier in $V(\phi)$, with a natural scale for this width being set by the curvature of V at $\phi = \phi_{\text{top}}$, the top of the barrier. A very flat potential barrier corresponds to a low mass scale, or a large distance scale, and thus to a thick wall. In flat spacetime, this is not a problem, and one can even have a “thin-wall” bounce with a thick wall by arranging for the bubble radius to be large enough. However, it is definitely a problem with gravity included, because the compact Euclidean de Sitter space places an upper limit on the size of the bounce. The critical test is whether $|V''(\phi_{\text{top}})|$ is greater or less than $4/\Lambda_{\text{top}}$ [64, 65, 66, 67]. If $V''(\phi_{\text{top}})$ is greater, a Coleman–De Luccia bounce is guaranteed to exist. If not, there may be no Coleman–De Luccia bounce. Even if one does exist, it may be

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quite different from what one might naively expect. We have already seen that, in flat spacetime, there are bounces in which ϕ never gets near its true vacuum value. In curved spacetime, one finds bounces in which ϕ is never close to its false vacuum value either.

When a potential is so flat that no Coleman–De Luccia bounce exists, there is always a second type of solution, first pointed out by Hawking and Moss [39]. This is a homogeneous solution with $\phi = \phi_{\text{top}}$ everywhere and

$$\rho = \Lambda_{\text{top}} \sin(\xi/\Lambda_{\text{top}}). \quad (2.65)$$

Its Euclidean action is given by

$$S_{HM} = -\frac{24\pi^2}{\kappa^2 V_{\text{top}}} \quad (2.66)$$

leading to a tunneling exponent

$$B_{HM} = S_{HM} - S[\phi_{\text{false}}] = \frac{24\pi^2}{\kappa^2} \left(\frac{1}{V(\phi_{\text{false}})} - \frac{1}{V(\phi_{\text{top}})} \right). \quad (2.67)$$

In the case where $(V(\phi_{\text{top}}) - V(\phi_{\text{false}}))/V(\phi_{\text{false}}) \ll 1$, this can be approximated as [42]

$$B_{HM} \sim \frac{\frac{4\pi}{3} H_{\text{fv}}^{-3} (V(\phi_{\text{top}}) - V(\phi_{\text{false}}))}{T_{\text{fv}}}, \quad (2.68)$$

where $H_{\text{fv}} = (8\pi V(\phi_{\text{false}})/3M_{\text{pl}})^{1/2}$ and $T_{\text{fv}} = H_{\text{fv}}/2\pi$ is the de Sitter temperature of the false vacuum. This expression suggest an interpretation of the Hawking Moss instanton as a thermal fluctuation of a horizon-sized region up to the top of the barrier. Also the back-reaction to the geometry is negligible because the energy difference between the false vacuum and the top of the potential is small.

For later discussion in chapter 4, we define the ratio of the Coleman-de Luccia Euclidean action (2.64) to the Hawking-Moss action (2.68) in fixed de Sitter background.

$$\gamma_E = \frac{S_{\text{bounce}}^E}{S_{\text{HM}}^E} = \frac{H^2}{g^2 \alpha^2} \frac{9\tilde{I}_{\text{bounce}}^E}{(1-\beta)^3(\beta+3)}, \quad (2.69)$$

where we take $V(\phi_{\text{false}}) = 0$.

Chapter 3

Stochastic approach

This chapter provides a review of the stochastic approach and the MSRJD functional integral. In Section 3.1, we introduce the scalar field theory in fixed de Sitter background. In Section 3.2, we derive the Langevin equation and provide its details. Finally, Section 3.3 presents MSRJD functional integral from the Langevin equation. We also outline the derivation of this formalism starting from the Schwinger-Keldysh formalism. The more detailed discussion is showed in Appendix A.

Section 3.1 and 3.2 are dedicated to the review of fundamental concept of the stochastic approach, based on the literature [43, 44]. The review of MSRJD functional integral in Section 3.3 is based on [68, 69] and its derivation from Schwinger-Keldysh formalism uses the formalism in [70, 71, 72].

3.1 Setup

We consider a real scalar field ϕ in (3+1) dimensional de Sitter background. The background geometry is described by the metric

$$ds^2 = -dt^2 + a(t)^2 d\mathbf{x}^2 = a(\eta^2)(-d\eta^2 + d\mathbf{x}^2), \quad (3.1)$$

with the scale factor

$$a(t) = e^{Ht} = -\frac{1}{H\eta}, \quad (3.2)$$

where H is the Hubble constant. The action is given by

$$S = \int dt d^3x a^3 \left[\frac{1}{2} \dot{\phi}^2 - \frac{1}{2} a^{-2} (\nabla \phi)^2 - V(\phi) \right], \quad (3.3)$$

where a dot denotes a derivative respect to t and $V(\phi)$ is a potential function. From this action, the canonical conjugate momentum Π is defined as $\Pi := a^3 \dot{\phi}$ and the Hamilton's equations of motion are given by

$$\dot{\phi} = a^{-3} \Pi, \quad \dot{\Pi} = a \nabla^2 \phi - a^3 V'(\phi). \quad (3.4)$$

3.2 Langevin equation

3.2.1 Derivation of Langevin equation

First, we divide the quantum fields ϕ and Π into an infrared (IR) part and an ultraviolet (UV) part respectively as

$$\phi = \phi_{\text{IR}} + \phi_{\text{UV}}, \quad \Pi = \Pi_{\text{IR}} + \Pi_{\text{UV}}. \quad (3.5)$$

The UV part is defined as

$$\begin{aligned} \phi_{\text{UV}} &:= \int \frac{d^3 k}{(2\pi)^3} \theta(k - k_c(t)) \left[a_{\mathbf{k}} u_{\mathbf{k}}(t) e^{i\mathbf{k} \cdot \mathbf{x}} + a_{\mathbf{k}}^\dagger u_{\mathbf{k}}^*(t) e^{-i\mathbf{k} \cdot \mathbf{x}} \right], \\ \Pi_{\text{UV}} &:= \int \frac{d^3 k}{(2\pi)^3} \theta(k - k_c(t)) a(t)^3 \left[a_{\mathbf{k}} \dot{u}_{\mathbf{k}}(t) e^{i\mathbf{k} \cdot \mathbf{x}} + a_{\mathbf{k}}^\dagger \dot{u}_{\mathbf{k}}^*(t) e^{-i\mathbf{k} \cdot \mathbf{x}} \right], \end{aligned} \quad (3.6)$$

where $u_{\mathbf{k}}$ is a mode function that will be specified later, and $a_{\mathbf{k}}$ and $a_{\mathbf{k}}^\dagger$ are annihilation and creation operators, respectively. Here, $\theta(k - k_c(t))$ is a step function and $k_c(t)$ is the cutoff scale defined as

$$k_c(t) := \varepsilon a(t) H, \quad (3.7)$$

where ε is a constant. The value of ε will be discussed later. Substituting Eqs.(3.6) into Hamilton's equations (3.4), we obtain the following equations;

$$\dot{\phi}_{\text{IR}} = a^{-3} \Pi_{\text{IR}} + \xi^\phi, \quad \dot{\Pi}_{\text{IR}} = a \nabla^2 \phi_{\text{IR}} - a^3 V'(\phi_{\text{IR}}) + \xi^\Pi, \quad (3.8)$$

where ξ^ϕ and ξ^Π are defined as

$$\begin{aligned} \xi^\phi &:= \dot{k}_c(t) \int \frac{d^3 k}{(2\pi)^3} \delta(k - k_c(t)) \left[a_{\mathbf{k}} u_{\mathbf{k}}(t) e^{i\mathbf{k} \cdot \mathbf{x}} + a_{\mathbf{k}}^\dagger u_{\mathbf{k}}^*(t) e^{-i\mathbf{k} \cdot \mathbf{x}} \right], \\ \xi^\Pi &:= \dot{k}_c(t) \int \frac{d^3 k}{(2\pi)^3} \delta(k - k_c(t)) \left[a_{\mathbf{k}} a(t)^3 \dot{u}_{\mathbf{k}}(t) e^{i\mathbf{k} \cdot \mathbf{x}} + a_{\mathbf{k}}^\dagger a(t)^3 \dot{u}_{\mathbf{k}}^*(t) e^{-i\mathbf{k} \cdot \mathbf{x}} \right]. \end{aligned} \quad (3.9)$$

To derive Eqs.(3.8) and (3.9), we assumed that the time evolution of UV modes can be well approximated by the free theory that is defined around the false vacuum. The mode function $u_{\mathbf{k}}(t)$ introduced in (3.9) then satisfies the following equation

$$\ddot{u}_{\mathbf{k}} + 3H\dot{u}_{\mathbf{k}} + a^{-2}k^2u_{\mathbf{k}} + V''(\phi_{\text{IR}})u_{\mathbf{k}} = 0. \quad (3.10)$$

From this equation, we see the mode function only depends on $k := |\mathbf{k}|$. Hence, we simply denote the mode function as u_k . Note that this mode function is used for UV modes $k > k_c(t)$.

Let us consider statistical properties of ξ^ϕ and ξ^Π . The annihilation and creation operators $a_{\mathbf{k}}$ and $a_{\mathbf{k}}^\dagger$ satisfy following commutation relations:

$$[a_{\mathbf{k}}, a_{\mathbf{k}'}^\dagger] = (2\pi)^3 \delta^{(3)}(\mathbf{k} - \mathbf{k}') , \quad [a_{\mathbf{k}}, a_{\mathbf{k}'}] = [a_{\mathbf{k}}^\dagger, a_{\mathbf{k}'}^\dagger] = 0. \quad (3.11)$$

Defining the vacuum state $|0\rangle$ as

$$a_{\mathbf{k}}|0\rangle = 0, \quad \forall \mathbf{k}, \quad (3.12)$$

we can calculate the one point and two point correlation functions of ξ^ϕ and ξ^Π as

$$\begin{aligned} \langle 0 | \xi^\alpha(t, \mathbf{x}) | 0 \rangle &= 0, \quad (\alpha, \beta = \phi, \Pi), \\ \langle 0 | \xi^\alpha(t, \mathbf{x}) \xi^\beta(t', \mathbf{x}') | 0 \rangle &= \frac{1}{2\pi^2} \dot{k}_c(t) k_c(t)^2 \frac{\sin(k_c(t)r)}{k_c(t)r} g^{\alpha\beta}(t) \delta(t - t'), \end{aligned} \quad (3.13)$$

where $r := |\mathbf{x} - \mathbf{x}'|$ and

$$\begin{cases} g^{\phi\phi}(t) &:= |u_{k_c}(t)|^2 \\ g^{\Pi\Pi}(t) &:= a(t)^6 |\dot{u}_{k_c}(t)|^2 \\ g^{\phi\Pi}(t) &= (g^{\Pi\phi})^* := a(t)^3 u_{k_c}(t) \dot{u}_{k_c}^*(t). \end{cases} \quad (3.14)$$

Because of the Wick theorem, all higher correlation functions can be decomposed into two point functions. In the stochastic approach, we replace the operator ξ^α by the real random field which has the following statistical properties

$$\begin{aligned} \langle \xi^\alpha(t, \mathbf{x}) \rangle_\xi &= 0, \quad (\alpha, \beta = \phi, \Pi), \\ \langle \xi^\alpha(t, \mathbf{x}) \xi^\beta(t', \mathbf{x}') \rangle_\xi &= \frac{1}{2\pi^2} \dot{k}_c(t) k_c(t)^2 \frac{\sin(k_c(t)r)}{k_c(t)r} \text{Re}[g^{\alpha\beta}(t)] \delta(t - t'), \end{aligned} \quad (3.15)$$

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where $\langle \cdots \rangle_\xi$ represents an expectation value with the distribution function of ξ^α . With this prescription, Eqs.(3.8) can be interpreted as Langevin equations with the noise ξ^α stemming from quantum fluctuations. Since ξ^α is now regarded as a classical variable, we ignore the imaginary parts of $g^{\phi\Pi}$ and $g^{\Pi\phi}$, which arise from non-commutativity. Accordingly, the IR variables ϕ_{IR} and Π_{IR} are also regarded as classical stochastic variables under this prescription. The validity of this classical approximation will be discussed in the next section.

3.2.2 Bunch-Davies vacuum

Now we define the vacuum state $|0\rangle$ by specifying the mode function u_k . Eq.(3.10) is the equation for the mode $k > k_c$. Let us assume that we can ignore the term V'' in solving (3.10). This can be justified when $|V''|$ is sufficiently small to satisfy $\max[(k_c/a)^2, H^2] \gg |V''|$. Under this approximation, (3.10) can be solved as

$$u_k = (-k\eta)^{\frac{3}{2}} \left[C_1 H_{\frac{3}{2}}^{(1)}(-k\eta) + C_2 H_{\frac{3}{2}}^{(2)}(-k\eta) \right], \quad (3.16)$$

where C_1, C_2 are constants of integration and $H_\nu^{(1)}(z), H_\nu^{(2)}(z)$ are Hankel functions. We choose the Bunch-Davies vacuum, i.e., the vacuum coincides with Minkowski vacuum at past infinity ($\eta \rightarrow -\infty$) so that

$$u_k \rightarrow \frac{-H\eta}{\sqrt{2k}} e^{-ik\eta}, \quad (-k\eta \rightarrow \infty). \quad (3.17)$$

This determines C_1, C_2 as

$$C_1 = -\sqrt{\frac{\pi}{4k}} \frac{H}{k}, \quad C_2 = 0. \quad (3.18)$$

Thus, we obtain the mode function

$$u_k = \frac{H\eta}{\sqrt{2k}} \left(\frac{1}{-ik\eta} - 1 \right) e^{-ik\eta}. \quad (3.19)$$

To check (3.17), we used the following formulae

$$H_{\frac{3}{2}}^{(1)}(z) = \sqrt{\frac{2}{\pi z}} \left(\frac{1}{iz} - 1 \right) e^{iz}, \quad H_{\frac{3}{2}}^{(2)}(z) = \sqrt{\frac{2}{\pi z}} \left(\frac{1}{-iz} - 1 \right) e^{-iz}. \quad (3.20)$$

Substituting (3.19) into (3.14) and using (3.7), we obtain

$$\begin{cases} g^{\phi\phi} &= \frac{H^2\eta^2}{2k_c} \left(\frac{1}{k_c^2\eta^2} + 1 \right) = \frac{1}{2Ha^3} \left(\varepsilon^{-3} + \varepsilon^{-1} \right) \\ g^{\text{III}} &= \frac{k_c}{2H^2\eta^2} = \frac{Ha^3}{2}\varepsilon \\ g^{\phi\Pi} &= g^{\Pi\phi*} = \frac{1}{2} \left(\frac{1}{k_c\eta} + i \right) = -\frac{1}{2}\varepsilon^{-1} + \frac{i}{2} \end{cases}. \quad (3.21)$$

From (3.8), we have to scale $g^{\alpha\beta}$ like as $g^{\phi\phi} \rightarrow g^{\phi\phi}$, $g^{\text{III}} \rightarrow g^{\text{III}}/(H^2a^6)$ and $g^{\phi\Pi} \rightarrow g^{\phi\Pi}/(Ha^3)$ in order to compare these quantities. Then, $g^{\phi\phi}$ and g^{III} become dominant for the case $\varepsilon \ll 1$ and $\varepsilon \gg 1$, respectively.

3.2.3 Reduced Langevin equation

Let us focus on physics at the super-horizon scales by taking $\varepsilon \ll 1$ and assume a shallow potential so that we can use the massless approximation to evaluate the noise correlations. In this setup, we can use (3.21) from which we find that $g^{\phi\phi}$ becomes dominant. We then ignore the noise for momentum ξ^Π . We also ignore the gradient term in (3.8) because we focus on the physics at super-horizon scales. Since the potential is assumed to be shallow, the time variation of $\phi_{\text{IR}}(t)$ will be then very small. Therefore, we may integrate the second equation of (3.8) as

$$\Pi_{\text{IR}} \simeq -\frac{a^3}{3H} V'(\phi_{\text{IR}}). \quad (3.22)$$

Substituting this relation into the first equation of (3.8), we obtain

$$\dot{\phi}_{\text{IR}} = -\frac{V'(\phi_{\text{IR}})}{3H} + \xi^\phi, \quad (3.23)$$

where the two point correlation function of ξ^ϕ is given at the leading order by

$$\langle \xi^\phi(t, \mathbf{x}) \xi^\phi(t', \mathbf{x}') \rangle_\xi = \frac{H^3}{4\pi^2} j_0(k_c r) \delta(t - t'), \quad j_0(k_c r) := \frac{\sin(k_c r)}{k_c r}. \quad (3.24)$$

Eq.(3.23) is the Langevin equation with the noise correlation (3.24) derived in [44].

3.3 Path integral representation

In this section, we introduce path integral approach for the Langevin equation following [68]. From now on, we replace the IR field and its conjugate momentum as $(\phi_{\text{IR}}, \Pi_{\text{IR}}) \rightarrow (\phi_c, \Pi_c)$. As we see in section 3.3.2, this notation corresponds to the Keldysh basis. In section 3.3.2, we provide a first-principle derivation of the stochastic approach based on the Schwinger-Keldysh formalism following [71, 72].¹ We briefly show the sketch of the derivation in the main text, and details of derivations are given in appendix A.

3.3.1 MSRJD functional integral representation

0+1 dimensional theories

Eq.(3.23) can be regarded as the equation determining the position of a particle $\phi(t)$ with the white noise ξ^ϕ defined by Eq.(3.24). For $k_c r \ll 1$, we have $\sin(k_c r)/(k_c r) \simeq 1$ in a good approximation. Let us discretize the field and noise as $\phi_c(t) \rightarrow \phi_i$, $\xi^\phi(t) \rightarrow \xi_i^\phi$, ($i \in \mathbf{Z}$). Then, Eq.(3.23) reads

$$X_i := \phi_i - \phi_{i-1} + \Delta t \left(\frac{V'(\phi_i)}{3H} - \xi_i^\phi \right) = 0, \quad (3.25)$$

where Δt is the temporal discretization interval. Note that the discretization (3.25) is called Ito discretization.² Denoting a solution of (3.23) by $\phi_c[\xi]$, the expectation value of an observable $\langle \mathcal{O} \rangle_\xi$ can be formally expressed as

$$\langle \mathcal{O}[\phi_c(t)] \rangle_\xi = \int \mathcal{D}\phi_c \mathcal{O}[\phi_c] \langle \delta(\phi_c - \phi_c[\xi]) \rangle_\xi = \int \mathcal{D}\phi_c \mathcal{O}[\phi_c] \left\langle \left| \frac{\delta X}{\delta \phi_c} \right| \delta(X) \right\rangle_\xi. \quad (3.26)$$

where $\mathcal{D}\phi_c := \prod_i d\phi_i$ is the functional measure, $\delta(\phi_c - \phi_c[\xi]) := \prod_i \delta(\phi_i - \phi_i[\xi])$, and $|\delta X/\delta \phi_c|$ is the determinant of $\{\delta X_i/\delta \phi_j\}$. When we choose the above discretization (3.25), $\delta X/\delta \phi_c$ becomes triangular matrix with a unit

¹For earlier discussions of stochastic approach based on the Schwinger-Keldysh formalism, see e.g., [70, 47].

²Generally, we can choose other discretization like as Stratonovich discretization. This ambiguity does not affect the result since the amplitudes of the noise do not depend on fields within the range of our approximation.

in the diagonal components, i.e., the functional determinant is unity. Thus, substituting (3.25) into (3.26), we obtain

$$\langle \mathcal{O}[\phi_c(t)] \rangle_\xi = \int \mathcal{D}\phi_c \mathcal{O}[\phi_c] \left\langle \prod_i \delta \left(\phi_i - \phi_{i-1} + \Delta t \left(\frac{V'(\phi_i)}{3H} - \xi_i^\phi \right) \right) \right\rangle_\xi. \quad (3.27)$$

Representing the delta functions in terms of a Fourier integral and taking the continuum limit, we obtain the following path integral representation

$$\langle \mathcal{O}[\phi_c(t)] \rangle_\xi = \int \mathcal{D}(\phi_c, \tilde{\phi}) \mathcal{O}[\phi_c] \left\langle \exp \left[i \int dt \tilde{\phi} \left(\dot{\phi}_c + \frac{V'(\phi_c)}{3H} - \xi^\phi \right) \right] \right\rangle_\xi. \quad (3.28)$$

After averaging over the noise assuming Gaussian statistics, we obtain the Martin–Siggia–Rose–Janssen–de Dominicis (MSRJD) functional integral

$$\langle \mathcal{O}[\phi_c(t)] \rangle_\xi = \int \mathcal{D}(\phi_c, \tilde{\phi}) \mathcal{O}[\phi_c] \exp \left[\int dt \left(i \tilde{\phi} \left(\dot{\phi}_c + \frac{V'(\phi_c)}{3H} \right) - \frac{H^3}{8\pi^2} \tilde{\phi}^2 \right) \right]. \quad (3.29)$$

Putting $\mathcal{O}[\phi_c] = \delta(\phi_c - \phi_c(t))$ and taking the boundary conditions $\phi_c(t') = \phi'_c$ and $\phi_c(t) = \phi_c$, we obtain the transition probability $p(\phi_c, t | \phi'_c, t')$

$$p(\phi_c, t | \phi'_c, t') = \int_{\phi_c(t')=\phi'_c}^{\phi_c(t)=\phi_c} \mathcal{D}(\phi_c, \tilde{\phi}) \exp \left[\int dt \left(i \tilde{\phi} \left(\dot{\phi}_c + \frac{V'(\phi_c)}{3H} \right) - \frac{H^3}{8\pi^2} \tilde{\phi}^2 \right) \right]. \quad (3.30)$$

Finally, introducing new variables $\Pi_\Delta := i\tilde{\phi}$, we obtain the "phase space" path integral

$$p(\phi_c, t | \phi'_c, t') = \int_{\phi_c(t')=\phi'_c}^{\phi_c(t)=\phi_c} \mathcal{D}(\phi_c, \Pi_\Delta) \exp \left[\int dt (\Pi_\Delta \dot{\phi}_c - H(\phi_c, \Pi_\Delta)) \right], \quad (3.31)$$

where we defined the Hamiltonian as

$$H(\phi_c, \Pi_\Delta) := -\frac{V'(\phi_c)}{3H} \Pi_\Delta - \frac{H^3}{8\pi^2} \Pi_\Delta^2. \quad (3.32)$$

We emphasize that this is just a change of integration variable, not the Wick rotation. A set of saddle point equations is defined as the Hamilton's equations for the Hamiltonian defined above.

3+1 dimensional theories

Let us consider the case of the full Langevin Eqs. (3.8), (3.13) and (3.21). When the space is discretized, (3.8) can be seen as the dynamics of many body particles. Thus, the procedure in the previous subsection can be utilized. Taking the continuum limit, we obtain the following expression which is analogous to (3.30),

$$p(\phi_c(\mathbf{x}), t | \phi'_c(\mathbf{x}), t') = \int_{\phi_c(t', \mathbf{x}) = \phi'_c(\mathbf{x})}^{\phi_c(t, \mathbf{x}) = \phi_c(\mathbf{x})} \mathcal{D}(\phi_c, \Pi_c, \phi_\Delta, \Pi_\Delta) \times \exp \left[i \int d^4x \left(\Pi_\Delta \dot{\phi}_c - \phi_\Delta \dot{\Pi}_c - H(\phi_c, \Pi_c, \phi_\Delta, \Pi_\Delta) \right) \right]. \quad (3.33)$$

We defined the Hamiltonian $H(\phi_c, \Pi_c, \phi_\Delta, \Pi_\Delta)$ as

$$H(\phi_c, \Pi_c, \phi_\Delta, \Pi_\Delta) = \frac{\Pi_c \Pi_\Delta}{a^3} - (a \nabla^2 \phi_c - a^3 V'(\phi_c)) \phi_\Delta - \frac{i}{2} \sum_{\alpha, \beta} \int d^4x' X_\alpha(x) G^{\alpha\beta}(x, x') X_\beta(x'), \quad (3.34)$$

where the Greek indices $\alpha, \beta = (\phi, \Pi)$ label the Δ fields as $(X_\phi, X_\Pi) = (\Pi_\Delta, -\phi_\Delta)$ and $G^{\alpha\beta}$ denotes the correlations of noises:

$$G^{\alpha\beta}(x, x') := \langle \xi^\alpha(x) \xi^\beta(x') \rangle_\xi, \quad (3.35)$$

where the RHS is given in (3.15). We use the formula (3.33) in chapter 4. Note that we have not changed the integration variables as $(\phi_\Delta, \Pi_\Delta) \rightarrow (-i\phi_\Delta, -i\Pi_\Delta)$.

In chapter 4, we take the initial state and the final state to be the false vacuum and the field configurations after tunneling (true vacuum or bubble), respectively.

3.3.2 Transition probability from the Schwinger-Keldysh formalism

In the above prescription, we treat IR fields $(\phi_{\text{IR}}, \Pi_{\text{IR}})$ as classical stochastic variables. To quantify the validity of this approximation, we derive the stochastic approach from the first principles based on the Schwinger-Keldysh

formalism. Note that we recover the label “IR” for IR fields in this subsection following the notation adopted in appendix A.

Suppose that the system is in the false vacuum $|\Psi_{\text{false}}\rangle$ at the past infinity $t = -\infty$ and it evolves into a certain quantum state $|\{\phi^{\text{IR}}(T, \mathbf{x})\}_{\mathbf{x} \in \mathcal{D}}\rangle$ at the final time $t = T$, where $|\{\phi^{\text{IR}}(T, \mathbf{x})\}_{\mathbf{x} \in \mathcal{D}}\rangle$ is an eigenstate of an IR field $\hat{\phi}^{\text{IR}}(\mathbf{x})$ at spatial points $\mathbf{x}$ in the domain $\mathcal{D}$ with an eigenvalue $\phi^{\text{IR}}(T, \mathbf{x})$. Here, $\hat{\phi}^{\text{IR}}(\mathbf{x})$ is an IR field in the Schrödinger picture.³ The transition probability p for this process is given by

$$p(\{\phi^{\text{IR}}(T, \mathbf{x})\}_{\mathbf{x} \in \mathcal{D}}) = \left| \langle \{\phi^{\text{IR}}(T, \mathbf{x})\}_{\mathbf{x} \in \mathcal{D}} | \hat{U}(t, -\infty) | \Psi_{\text{false}} \rangle \right|^2, \quad (3.36)$$

where $\hat{U}(t, t')$ describes the unitary time evolution from t' to t . The RHS is the Fourier component of the generating functional $Z[J^{\text{IR}}]$ for IR fields, and we have

$$p(\{\phi^{\text{IR}}(T, \mathbf{x})\}_{\mathbf{x} \in \mathcal{D}}) = \prod_{\mathbf{x} \in \mathcal{D}} \int dJ^{\text{IR}}(T, \mathbf{x}) Z[J^{\text{IR}}(T)] e^{-iJ^{\text{IR}}(T, \mathbf{x})\phi^{\text{IR}}(T, \mathbf{x})} \Bigg|_{\substack{\{J^{\text{IR}}(T, \mathbf{y})=0\}_{\mathbf{y} \notin \mathcal{D}}}}, \quad (3.37)$$

where the generating functional $Z[J^{\text{IR}}(T)]$ is defined by

$$Z[J^{\text{IR}}(T)] := \text{Tr} \left[\hat{U}(T, -\infty) | \Psi_{\text{false}} \rangle \langle \Psi_{\text{false}} | \hat{U}^\dagger(T, -\infty) \prod_{\mathbf{x}} e^{iJ^{\text{IR}}(T, \mathbf{x})\phi^{\text{IR}}(T, \mathbf{x})} \right]. \quad (3.38)$$

We can evaluate $Z[J^{\text{IR}}(T)]$ non-perturbatively in the IR sector based on the method developed in [71, 72]. Our strategy is that we integrate out UV sector $k > k_c(t)$ perturbatively in nonlinear couplings to obtain $Z[J^{\text{IR}}(T)]$. This is analogous to tracing out environmental degrees of freedom to evaluate the reduced density matrix for the system under considerations. Each step we take can be summarized as follows:

1. First, we derive a path integral representation of $Z[J^{\text{IR}}(T)]$. We then split the integration variables into UV fields and IR fields such that

³Precisely, $\hat{\phi}^{\text{IR}}(\mathbf{x})$ is defined in terms of the the Schrödinger picture field $\hat{\phi}(\mathbf{k})$ in momentum space as $\hat{\phi}^{\text{IR}}(\mathbf{x}) := \int \frac{d^3k}{(2\pi)^3} \hat{\phi}(\mathbf{k}) e^{i\mathbf{k} \cdot \mathbf{x}} \theta(k_c(t) - k)$. We suppressed the trivial time dependence stemming from the step function.

38 Stochastic approach

the integration contour of UV variables of ϕ is closed: see discussions around Eq.(A.4) for more details.⁴

2. We perform the integration over UV variables and evaluate an IR effective action, the so-called Feynman-Vernon influence functional [73] perturbatively.
3. We substitute the obtained expression of $Z[J^{\text{IR}}(T)]$ into (3.37), giving rise to the path integral expression for the transition probability p .

Technical details are shown in appendix A. At the step 2, we assumed that the quantum state for non-zero modes is given by the Bunch-Davies vacuum state for a free field defined around the false vacuum, and that the exact zero mode provides a non-fluctuating classical background field configuration $\phi = \phi_{\text{false}}$.

In this way, we can obtain the path integral expression for the probability p as

$$p(\{\phi^{\text{IR}}(T, \mathbf{x})\}_{\mathbf{x} \in \mathcal{D}}) = \int \mathcal{D}(\phi_c^{\text{IR}}, \phi_\Delta^{\text{IR}}, \Pi_c^{\text{IR}}, \Pi_\Delta^{\text{IR}}) e^{i\mathcal{S}} \delta[\mathcal{C}_{\text{fin}}] \delta[\mathcal{C}_{\text{ini}}] . \quad (3.39)$$

Here, $\delta[\mathcal{C}_{\text{fin}}]$ and $\delta[\mathcal{C}_{\text{ini}}]$ fix the boundary conditions of path integral at the final time and the initial time, respectively:

$$\delta[\mathcal{C}_{\text{fin}}] := \prod_{\mathbf{x} \in \mathcal{D}} \delta(\phi_c^{\text{IR}}(T, \mathbf{x}) - \phi^{\text{IR}}(T, \mathbf{x})) \prod_{\mathbf{y}} \delta(\phi_\Delta^{\text{IR}}(T, \mathbf{y})) , \quad (3.40a)$$

$$\delta[\mathcal{C}_{\text{ini}}] := \prod_{\mathbf{x}} \delta(\phi_c^{\text{IR}}(-\infty, \mathbf{x}) - \phi_{\text{false}}) \delta(\Pi_c^{\text{IR}}(-\infty, \mathbf{x})) \delta[\mathcal{C}_{\text{detail}}] , \quad (3.40b)$$

where $\delta[\mathcal{C}_{\text{detail}}]$ is not relevant for the main points here, and we define it in appendix A. A term $\mathcal{S}$ may be understood as an IR effective action,

$$\mathcal{S} := \int d^4x \left[\Pi_\Delta^{\text{IR}} \dot{\phi}_c^{\text{IR}} - \phi_\Delta^{\text{IR}} \dot{\Pi}_c^{\text{IR}} - H(\phi_c^{\text{IR}}, \Pi_c^{\text{IR}}, \phi_\Delta^{\text{IR}}, \Pi_\Delta^{\text{IR}}) - \delta H \right] , \quad (3.41a)$$

$$\delta H := a^3(t) \left[V(\phi_c^{\text{IR}} + (\phi_\Delta^{\text{IR}}/2)) - V(\phi_c^{\text{IR}} - (\phi_\Delta^{\text{IR}}/2)) - V'(\phi_c^{\text{IR}}) \phi_\Delta^{\text{IR}} \right] + \delta H_{\text{higher}} , \quad (3.41b)$$

⁴There is a subtlety that modes satisfying $k \geq k_c(T)$ are initially regarded as the “UV” degrees of freedom (DoF) while they become “IR” DoFs due to the accelerating expansion of the spacetime. However, by adopting this splitting procedure, we can use the Schwinger-Keldysh (or *closed-time-path*) formalism to evaluate the integration over UV variables first as usual.

where H is given in (3.34). δH_{higher} denotes the higher-order terms in the coupling constant. Note that eq (3.39) allows us to understand the transition probability p in terms of the stochastic dynamics: this point is discussed in appendix A.3.

Validity of Eq.(3.33). Now we find that $\mathcal{S}$ coincides with the term in the exponent of (3.33) up to the term δH . Hence, (3.39) shows that the result (3.33) in the previous section can be justified from the first-principles when the term δH is negligible. Suppose that the potential $V(\phi)$ is sufficiently flat in the regime of our interest such that perturbation theory works when integrating out UV modes. In this case, we can systematically calculate δH_{higher} perturbatively and we have $\delta H_{\text{higher}} = 0$ at the leading order. We set $\delta H_{\text{higher}} = 0$ in the present analysis. We should keep in mind however that the non-perturbative physics at UV scales $k > k_c(t) = \varepsilon a(t)H$ is lost by setting $\delta H_{\text{higher}} = 0$. From this viewpoint, we should choose ε as large as possible.

Even after setting $\delta H_{\text{higher}} = 0$, we have nonzero δH . In the classical approximation, we may regard the Δ -variables $(\phi_{\Delta}^{\text{IR}}, \Pi_{\Delta}^{\text{IR}})$ as tiny quantities and keep only the terms which are linear in the Δ -variables in $\mathcal{S}$, leading to $\delta H \simeq 0$. In the case $\varepsilon \ll 1$, this approximation would work thanks to the squeezing of quantum fluctuations at super-horizon scales. However, it is more subtle if we can set $\delta H = 0$ in the case $\varepsilon \gtrsim 1$. We will revisit this issue in chapter 4.

Chapter 4

Tunneling as a stochastic process

In this chapter, we apply the MSRJD path integral to the tunneling process in de Sitter spacetime, based on the author's paper [57]. In Section 4.1, we firstly investigate the case of the HM tunneling and obtain the same tunneling rate appeared in Section 2.3. In Section 4.2, we construct a CDL-like process in our formalism and calculate the tunneling rate. We also discuss its validity and compare with that of the Euclidean method.

4.1 Hawking-Moss tunneling

In this section, we discuss the Hawking-Moss (HM) tunneling from the perspective of MSRJD functional integral. We calculate the tunneling rate defined by (3.31) and (3.33) in sections 4.1.1 and 4.1.2, respectively. Hereafter, for the numerical calculations, we consider the potential [58] in FIG. 4.1;

$$V(\phi) = \frac{g^2}{4} \left[(\phi^2 - \alpha^2)^2 + \frac{4\beta}{3} (\alpha\phi^3 - 3\alpha^3\phi - 2\alpha^4) \right], \quad (4.1)$$

for which the derivative with respect to ϕ is given by a simple form

$$V'(\phi) = g^2(\phi - \alpha)(\phi + \alpha)(\phi + \alpha\beta). \quad (4.2)$$

For this potential, we have $\phi_{\text{false}} = -\alpha$, $\phi_{\text{true}} = \alpha$ and $\phi_{\text{top}} = -\beta\alpha$. The dimensionless parameter β must be $0 < \beta < 1$, otherwise $\phi = -\alpha$ is not a

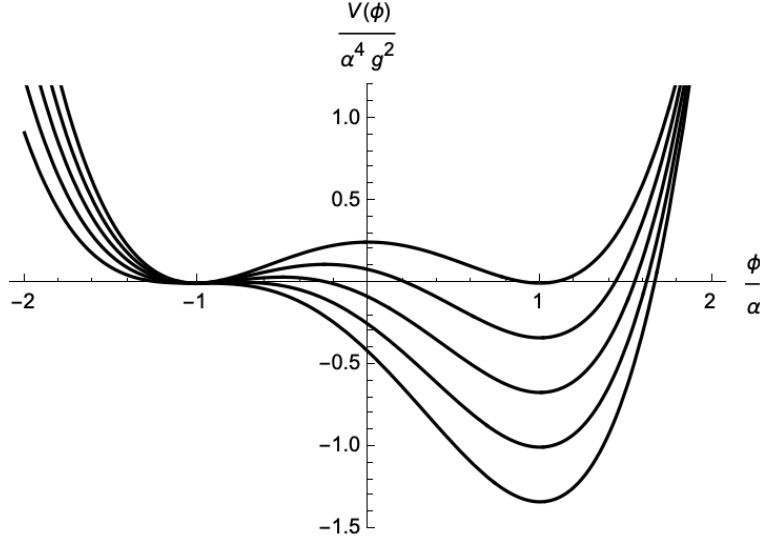


Figure 4.1: The potential of (4.1) for various β . From top to bottom, the curves correspond to $\beta = 0, 0.25, 0.5, 0.75, 1$.

false vacuum. The potential at each point takes the following values:

$$V(-\alpha) = 0, \quad V(\alpha) = -\frac{4\beta g^2 \alpha^4}{3}, \quad V(-\beta\alpha) = \frac{g^2 \alpha^4}{12} (1 - \beta)^3 (3 + \beta). \quad (4.3)$$

The height of potential barrier ΔV between the false vacuum and the true vacuum is

$$\Delta V := V(-\beta\alpha) - V(-\alpha) = \frac{g^2 \alpha^4}{12} (1 - \beta)^3 (3 + \beta). \quad (4.4)$$

The mass of fluctuations around a given point ϕ is

$$V''(\phi) = \alpha^2 g^2 [3(\phi/\alpha)^2 + 2\beta(\phi/\alpha) - 1]. \quad (4.5)$$

In our analysis, we work in the quantum field theory on the fixed de Sitter background. This assumption will be valid when the condition

$$\max[|V(-\alpha)|, |V(-\beta\alpha)|, |V(\alpha)|] \ll 3M_{\text{pl}}^2 H^2 \quad (4.6)$$

is satisfied. This is always satisfied when the following condition is imposed,

$$H \ll \frac{3gM_{\text{pl}}}{2} \left(\frac{H}{g\alpha} \right)^2 \approx 2.9 \times 10^{12} \text{ GeV} \left(\frac{g}{1.12 \times 10^{-4}} \right) \left(\frac{\sqrt{140}}{g\alpha/H} \right)^2, \quad (4.7)$$

where we use $0 < \beta < 1$ and choose parameters $(g, g\alpha/H) = (1.12 \times 10^{-4}, \sqrt{140})$ as a benchmark point for later convenience. This shows that we can ignore the backreaction consistently with the weak coupling $g \ll 1$ provided that H is well below the Planck scale $M_{\text{pl}} \approx 2.44 \times 10^{18}$ GeV. Note that under this condition, our discussion below is also applicable in the case of a quasi de Sitter background. However, it is non-trivial to generalize our analysis to the case when a non-negligible backreaction on H exists. This is because it is necessary to extend the first-principles derivation of the stochastic approach to account for the backreaction. This is in itself an interesting issue which we leave for future work.

4.1.1 The case for the reduced Langevin equation

Let us start from the simpler formula (3.31). The Hamilton's equations for the Hamiltonian $H(\phi_c, \Pi_\Delta)$ defined in (3.32) are given as

$$\begin{aligned}\dot{\phi}_c &= -\frac{V'(\phi_c)}{3H} - \frac{H^3}{4\pi^2}\Pi_\Delta, \\ \dot{\Pi}_\Delta &= \frac{V''(\phi_c)}{3H}\Pi_\Delta.\end{aligned}\tag{4.8}$$

There exist many Hamiltonian flow lines in phase space (ϕ_c, Π_Δ) that represent the solution of Hamilton's equations as shown in FIG.4.2. Each flow line can be specified by the value of Hamiltonian since it is conserved on the given flow line. In the FIG.4.2, there are two important flow lines corresponding to $H(\phi_c, \Pi_\Delta) = 0$. Since the Hamiltonian is the quadratic polynomial of Π_Δ , there are two Π_Δ satisfying $H(\phi_c, \Pi_\Delta) = 0$:

$$\Pi_\Delta = 0, -\frac{8\pi^2}{3H^4}V' .\tag{4.9}$$

In the case of $\Pi_\Delta = 0$, the equations (4.8) reduce to

$$\dot{\phi}_c = -\frac{V'(\phi_c)}{3H} .\tag{4.10}$$

Similarly, in the case of $\Pi_\Delta = -\frac{8\pi^2}{3H^4}V'(\phi_c)$, the equations (4.8) reduce to

$$\dot{\phi}_c = \frac{V'(\phi_c)}{3H} .\tag{4.11}$$

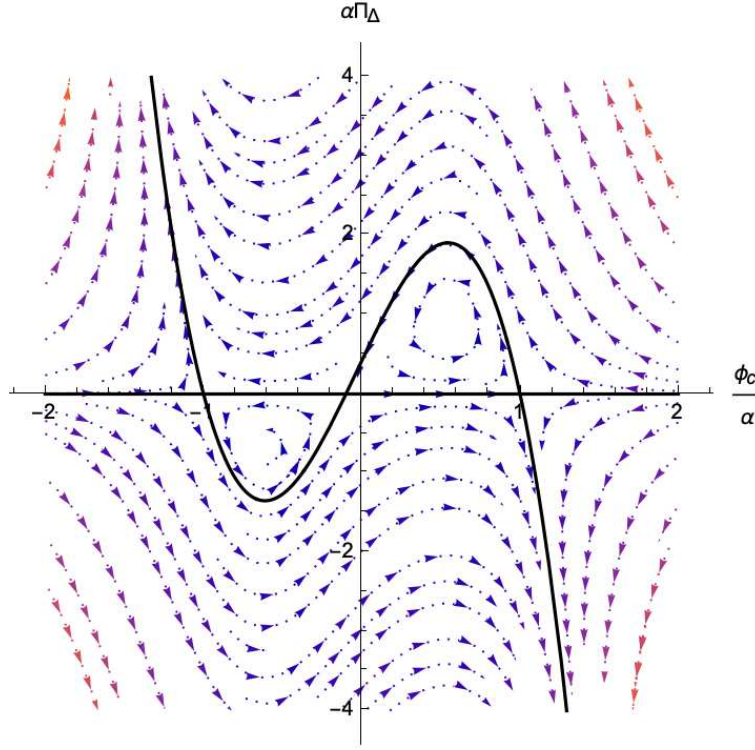


Figure 4.2: The Hamilton flow for (4.8) where we take $\alpha = H$, $g = 0.4$ and $\beta = 0.1$. The solid lines correspond to $H(\phi_c, \Pi_\Delta) = 0$. From left to right, the intersections of the solid lines correspond to the false vacuum, the top of potential hill and the true vacuum.

These two flow lines have three intersections which correspond to the false vacuum, the top of potential hill and the true vacuum from left to right. On the flow line with $\Pi_\Delta = 0$, the false and true vacuum are stable and the top of potential is unstable. While on the flow line with $\Pi_\Delta \neq 0$, the stability is reversed. The different signs of the potential forces account for the reverse of the stability at the intersections. We see that the tunneling configuration exists; starting from the false vacuum $(\phi_c, \Pi_\Delta) = (-\alpha, 0)$, going to the top of the potential $(\phi_c, \Pi_\Delta) = (-\beta\alpha, 0)$ through the flow line with $\Pi_\Delta \neq 0$ and finally going to the true vacuum through the flow line with $\Pi_\Delta = 0$. We numerically solve the second equation with the initial conditions $\phi_c(t') = -\alpha$ and switch to the first equation when the field reaches the top of potential $\phi_c(t) = -\beta\alpha$. Thus, we obtain the tunneling configuration in FIG.4.3.

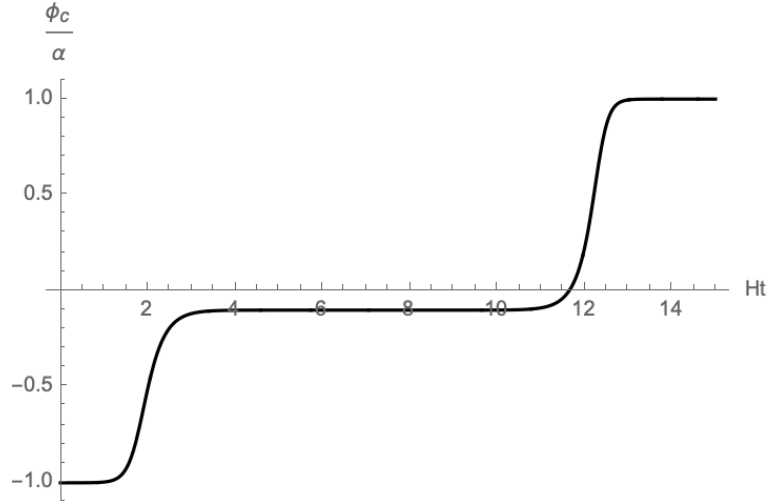


Figure 4.3: The tunneling configuration for the homogeneous fields is plotted for $\beta = 0.1$ and $\alpha g/H = \sqrt{10}$. We choose the initial condition as $\phi_c(10^{-15})/\alpha = -1 + 10^{-5}$ and solve the equation of (4.11). When ϕ_c/α gets close to the top of potential $\phi_c/\alpha = -\beta$, we switch to the equation of (4.10) with the initial condition $\phi_c/\alpha = -\beta + 10^{-5}$.

Now, we evaluate the transition probability (3.31) by substituting the above tunneling configurations. The action in (3.31) can be calculated without the concrete solutions for ϕ_c ;

$$I = \int_{t'}^t dt \left[\Pi_\Delta \dot{\phi}_c - H(\phi_c, \Pi_\Delta) \right] = -\frac{8\pi^2}{3H^4} \int_{t'}^{t_*} dt \dot{\phi}_c V'(\phi_c) = -\frac{8\pi^2}{3H^4} \Delta V, \quad (4.12)$$

where t_* is the time that ϕ_c reaches the top of potential and ΔV is the difference of energy density between the top of the potential and the false vacuum. The trajectory from the top of the potential to the true vacuum does not contribute to the transition probability because $H(\phi_c, \Pi_\Delta) = 0$ and $\Pi_\Delta = 0$ on its trajectory. Thus, the transition probability of the tunneling can be evaluated as

$$p(\alpha, t | -\alpha, t') \sim \exp \left(-\frac{8\pi^2}{3H^4} \Delta V \right). \quad (4.13)$$

This shows a complete agreement with the result of the HM instanton on the fixed background (2.68).

Comments on this result are in order. It is pointed out in [42] that when the potential $V(\phi)$ has several degenerate local maxima, all of them yield the same tunneling action in the Euclidean method, despite differing distances from the false vacuum. In our formalism, however, a transition to a distant maximum is described by a saddle-point solution that passes through all the intermediate maxima, and the transition probability is given by the product of the factors on the RHS of (4.13) for each maximum. The aforementioned subtlety is absent in our formalism.

Since the analysis in section 4.1.1 is based on (3.23), which describes the stochastic dynamics of the IR field ϕ_c at a single spatial point, eq.(4.13) calculates the tunneling probability at a single spatial point, not for the whole universe.¹ Hence, physically speaking, (4.13) provides the tunneling probability of a coarse-grained patch with a physical radius $(\varepsilon H)^{-1}$. This point is also made clear in sec. 4.1.2.

4.1.2 The case for the full Langevin equation

Next, we evaluate the tunneling rate based on the formula (3.33). An advantage of (3.33) is that one can discuss various non-trivial dynamics in the whole universe covered by the flat chart in principle.

In the previous section, we focused on the dynamics at a single spatial point and found the HM tunneling process as a non-trivial solution of the Hamilton's equations. In this section, we discuss the corresponding process in the global picture based on (3.33) by identifying a non-trivial configuration that satisfies the Hamilton's equations in a good approximation. We then reproduce the result (4.12). We also estimate the characteristic time scale of the tunneling. In sec. 4.1.2, we remark some technical complications that arise in the global picture.

We start with simplifying the expression (3.33). As discussed in chapter 3, $G^{\phi\phi}$ becomes dominant and the others can be neglected for $\varepsilon \ll 1$ in (3.34). Under this approximation, the exponent of the integrand of (3.33) is linear in ϕ_Δ . We can then perform the path integral $\int \mathcal{D}\phi_\Delta$ in (3.33), yielding the product of delta functions $\prod_{\mathbf{x}} \delta(\dot{\Pi}_c - a\nabla^2\phi_c + a^3V'(\phi_c))$. These delta functions give an equation of motion for Π_c and eliminate the path integral $\int \mathcal{D}\Pi_c$. We solve this equation as $\Pi_c \simeq -a^3V'(\phi_c)/3H$, ignoring the spatial

¹This is consistent with the observation made in [41] that the HM solution corresponds to the transition over a region of a Hubble horizon volume.

gradient and the time variation of ϕ_c . Furthermore, we change the integration variable as $\Pi_\Delta \rightarrow -i\Pi_\Delta$ in (3.33). Consequently, (3.33) reduces to

$$p(\phi_c(\mathbf{x}), t | \phi'_c(\mathbf{x}), t') \simeq \int_{\phi_c(t', \mathbf{x}) = \phi'_c(\mathbf{x})}^{\phi_c(t, \mathbf{x}) = \phi_c(\mathbf{x})} \mathcal{D}(\phi_c, \Pi_\Delta) \exp \left[\int d^4x \left(\Pi_\Delta \dot{\phi}_c - H_{\text{HM}}(\phi_c, \Pi_\Delta) \right) \right], \quad (4.14)$$

where the Hamiltonian H_{HM} is defined as

$$H_{\text{HM}}(\phi_c, \Pi_\Delta) := -\frac{1}{3H} V'(\phi_c) \Pi_\Delta - \frac{H^3}{8\pi^2} \Pi_\Delta \bar{\Pi}_\Delta, \quad (4.15)$$

and $\bar{\Pi}_\Delta$ is given by

$$\bar{\Pi}_\Delta(t, \mathbf{x}) := \int d^3x' j_0(k_c(t) |\mathbf{x} - \mathbf{x}'|) \Pi_\Delta(t, \mathbf{x}') = \int \frac{d^3k}{(2\pi)^3} \frac{4\pi^2}{H^3} \frac{H^2}{2k^3} \delta(t - t_k) \Pi_\Delta(t, \mathbf{k}) e^{i\mathbf{k} \cdot \mathbf{x}}. \quad (4.16)$$

Here we define t_k by the condition $k_c(t_k) = k$. The Hamilton's equations are

$$\dot{\phi}_c = -\frac{1}{3H} V'(\phi_c) - \frac{H^3}{4\pi^2} \bar{\Pi}_\Delta, \quad (4.17a)$$

$$\dot{\Pi}_\Delta = \frac{1}{3H} V''(\phi_c) \Pi_\Delta. \quad (4.17b)$$

Here, the term $\dot{\Pi}_\Delta(t, \mathbf{x})$ on the LHS of (4.17b) is defined as the Fourier transform of $\dot{\Pi}_\Delta(t, \mathbf{k})$ i.e., $\dot{\Pi}_\Delta(t, \mathbf{x}) := (2\pi)^{-3} \int d^3k \dot{\Pi}_\Delta(t, \mathbf{k}) e^{i\mathbf{k} \cdot \mathbf{x}}$. The same rule applies to other IR variables in the real space as in eq. (A.25). Intuitively, this is because the stochastic formalism is formulated in the momentum space and its Fourier transform provides the stochastic formalism in the real-space: see appendix A for more details.

Eqs. (4.17) admit a trivial configuration $\Pi_\Delta(t, \mathbf{k}) = 0$ under which we have $H_{\text{HM}} = 0$ and (4.17a) becomes $\dot{\phi}_c = -V'(\phi_c)/(3H)$ describing the standard classical time evolution. Now we are interested in the non-trivial configuration which satisfies (4.17). In the previous section, we focus on the non-trivial dynamics at the single spatial point $\mathbf{x} = \mathbf{x}_0$ and it is found that the tunneling configuration is obtained by $\Pi_\Delta(t, \mathbf{x}_0) = -\frac{8\pi^2}{3H^4} V'(\phi_c(t, \mathbf{x}_0))$. This configuration can be naturally extended to the one in the global picture by considering the following configuration

$$\Pi_\Delta(t, \mathbf{k}) = -\frac{8\pi^2}{3H^4} V'(\phi_c(t, \mathbf{x}_0)) e^{-i\mathbf{k} \cdot \mathbf{x}_0}. \quad (4.18)$$

From now on, we show that this configuration satisfies a set of Hamilton equations (4.17) in a good approximation. For this configuration, we have

$$\Pi_{\Delta}(t, \mathbf{x}) = -\frac{8\pi^2}{3H^4} V'(\phi_c(t, \mathbf{x}_0)) f(r; t), \quad (4.19a)$$

$$\dot{\Pi}_{\Delta}(t, \mathbf{x}) = -\frac{8\pi^2}{3H^4} V''(\phi_c(t, \mathbf{x}_0)) \dot{\phi}_c(t, \mathbf{x}_0) f(r; t), \quad (4.19b)$$

$$\bar{\Pi}_{\Delta}(t, \mathbf{x}) = -\frac{8\pi^2}{3H^4} V'(\phi_c(t, \mathbf{x}_0)) j_0(k_c(t)r), \quad (4.19c)$$

where $r := |\mathbf{x} - \mathbf{x}_0|$ and $f(r; t) := (2\pi)^{-3} \int d^3k \theta(k_c(t) - k) e^{i\mathbf{k} \cdot (\mathbf{x} - \mathbf{x}_0)}$. The functions $j_0(k_c(t)r)$ and $f(r; t)$ rapidly oscillate for $r \gg k_c^{-1}(t)$. Hence, they exponentially decay to zero after they are smeared over the Hubble time: for instance, j_0 and f becomes the smooth function after the suitable coarse-graining as ²

$$\begin{aligned} j_0(k_c(t)r) &\xrightarrow{\text{time c.g.}} W(r; t) := \int dt' w(t', t) j_0(k_c(t')r) \approx \begin{cases} 1 & (r \ll k_c^{-1}(t)) \\ 0 & (r \gg k_c^{-1}(t)) \end{cases}, \\ f(r; t) &\xrightarrow{\text{time c.g.}} W_f(r; t) := \int dt' w(t', t) f(r; t') \approx \begin{cases} k_c^3(t)/(6\pi^2) & (r \ll k_c^{-1}(t)) \\ 0 & (r \gg k_c^{-1}(t)) \end{cases}. \end{aligned} \quad (4.21)$$

Here, we perform the smearing by using a window function $w(t', t)$ which is approximately constant in the domain $|t - t'| \ll H^{-1}$ while it exponentially decays to zero at $|t - t'| \gg H^{-1}$. We also impose the normalization condition $\int dt' w(t', t) = 1$. We can choose $w(t', t) = H\pi^{-1/2} \exp[-H^2(t - t')^2]$ for example. Note that the property (4.21) is based on the behavior $j_0(k_c r) \simeq 1$ and $f(r; t) \simeq k_c^3/(6\pi^2)$ for $r \ll k_c^{-1}$ and the rapid oscillations of $j_0(k_c r)$ and $f_0(r; t)$ for $r \gg k_c^{-1}$. Hence, the property (4.21) will be robust against the choice of $w(t', t)$.

Therefore, substituting Eqs. (4.19c) and (4.21) into Eq. (4.17a) and per-

²Note that this behavior may motivate one to replace the term $j_0(k_c(t)r)$ in (4.15) by the step function

$$j_0(k_c(t)r) \rightarrow \theta(1 - k_c(t)r). \quad (4.20)$$

This is the approximation adopted in [44].

forming the smearing in time, we obtain

$$\dot{\phi}_c(t, \mathbf{x}) = -\frac{1}{3H} [V'(\phi_c(t, \mathbf{x})) - 2V'(\phi_c(t, \mathbf{x}_0))j_0(k_c(t)r)] \quad (4.22a)$$

$$\xrightarrow{\text{time c.g.}} \begin{cases} -V'(\phi_c(t, \mathbf{x}))/ (3H) & (r \gg k_c^{-1}(t)) \\ V'(\phi_c(t, \mathbf{x}_0))/ (3H) & (r \ll k_c^{-1}(t)) \end{cases} . \quad (4.22b)$$

This exhibits the non-trivial dynamics only in the domain $r \lesssim k_c^{-1}$, i.e., a coarse-grained patch centered at a point $\mathbf{x}_0$ with a physical radius $(\varepsilon H)^{-1}$. Here, we used $\phi_c(t, \mathbf{x}) \simeq \phi_c(t, \mathbf{x}_0)$ for $r \ll k_c^{-1}(t)$. In the second line, it is assumed that $V'(\phi_c(t, \mathbf{x}_0))$ does not cancel the rapid oscillations of $j_0(k_c(t)r)$ so that their product $V'j_0$ decays exponentially at $r \gg k_c^{-1}$ after the suitable smearing. This assumption will be valid thanks to the approximate constancy of super-Horizon fluctuations $\phi_c(t, \mathbf{k})$ over the Hubble time: $\phi_c(t, \mathbf{k}) \simeq \phi_c(t', \mathbf{k})$ for $|t - t'| \lesssim H^{-1}$. Adopting the same assumption, from (4.19a) and (4.19b) we also obtain the smeared expressions of Π_Δ and $\tilde{\Pi}_\Delta$ which turn out to satisfy (4.17b) under the condition (4.17a). Hence, we conclude that (4.19) describes a non-trivial saddle-point solution in a good approximation. Furthermore, after the smearing we can find $H_{\text{HM}} \approx 0$ for this configuration.

Now we use the configuration (4.18) or (4.19) to obtain the tunneling configuration shown in FIG.4.3 again. We can calculate the action for this tunneling configuration similarly to (4.12). By substituting the exact expres-

sions (4.19) and (4.22a), we obtain ³

$$\begin{aligned}
 I &= -\frac{H^3}{8\pi^2} \int_{t'}^{t_*} dt \int d^3x \Pi_\Delta(t, \mathbf{x}) \bar{\Pi}_\Delta(t, \mathbf{x}) \\
 &= -\frac{8\pi^2}{3H^4} \int_{t'}^{t_*} dt V'(\phi_c(t, \mathbf{x}_0)) \frac{V'(\phi_c(t, \mathbf{x}_0))}{3H} \int d^3x j_0(k_c(t)r) f(r; t) \\
 &= -\frac{8\pi^2}{3H^4} \int_{t'}^{t_*} dt \dot{\phi}_c(t, \mathbf{x}_0) V'(\phi_c(t, \mathbf{x}_0)) = -\frac{8\pi^2}{3H^4} \Delta V. \tag{4.23}
 \end{aligned}$$

In the first line, we used (4.17a) to eliminate $\dot{\phi}_c$ from the integrand. In the second line, we substituted the concrete configurations (4.19). In the third line, we used (4.22a) and performed the spatial integral as $\int d^3x j_0(k_c(t)r) f(r; t) = 1$. Thus, we reproduce the previous result (4.12). Our analysis confirms that (4.23) gives the tunneling probability of a coarse-grained patch with a physical radius $(\varepsilon H)^{-1}$.

It is worth mentioning the time scale of the HM tunneling. We may naively calculate the typical time scale of the HM tunneling as $t_* - t' = \int_{\phi_{\text{false}}}^{\phi_{\text{top}}} d\phi_c \frac{3H}{V'(\phi_c)}$. Here, the space-time argument of the field is suppressed. However, this integral is in general divergent since $V' = 0$ at the both ends of the integral. In the vicinity of the domain where $V' = 0$, we expect that the quantum fluctuations of the field would play an important role. Since the typical size of quantum fluctuations accumulated over the Hubble time is $H/(2\pi)$, we may define the regulated quantity t_{HM} as

$$t_{\text{HM}} := \int_{\phi_{\text{false}} + \frac{H}{2\pi}}^{\phi_{\text{top}} - \frac{H}{2\pi}} d\phi_c \frac{3H}{V'(\phi_c)}, \tag{4.24}$$

and we expect that t_{HM} would correctly characterize the typical time scale

³Note that we can also estimate the action I by performing the coarse-graining in time suitably and substituting the smeared expression of Π_Δ into the action:

$$I \simeq -\frac{8\pi^2}{3H^4} \int_{t'}^{t_*} dt V'(\phi_c(t, \mathbf{x}_0)) \int d^3x \dot{\phi}_c(t, \mathbf{x}) W_f(r; t) \simeq -\frac{8\pi^2}{3H^4} \int_{t'}^{t_*} dt \dot{\phi}_c(t, \mathbf{x}_0) V'(\phi_c(t, \mathbf{x}_0)) = -\frac{8\pi^2}{3H^4} \Delta V,$$

reproducing the previous result (4.12). In the first approximate equality, we used $H_{\text{HM}} = 0$ which is well satisfied after the smearing. In the second one, we use $\phi_c(t, \mathbf{x}) \simeq \phi_c(t, \mathbf{x}_0)$ for $r \ll k_c^{-1}(t)$ and the condition $\int d^3x W_f(r; t) = 1$ which follows from the definition of W_f . This estimate may be more analogous to the calculation of the action in sec. 4.1.1 since the approximate locality of the dynamics is restored after the smearing.

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of the HM tunneling. In particular, for our potential (4.1) we have

$$t_{\text{HM}} = \frac{3H}{2g^2\alpha^2(1+\beta)} \left[\left(\frac{3+\beta}{1-\beta} \right) \log \left| 1 - \frac{1-\beta}{H/2\pi\alpha} \right| - \log \left| \frac{1 - \frac{2}{H/2\pi\alpha}}{1 + \frac{1+\beta}{H/2\pi\alpha}} \right| \right] \sim \frac{1}{H} \cdot \mathcal{O}((H/g\alpha)^2) . \quad (4.25)$$

Since we have $(H/g\alpha)^2 \sim (H^2/V'')_{\phi_{\text{false}} \lesssim \phi_c \lesssim \phi_{\text{top}}} \gg 1$ for the shallow potential, we conclude that the time scale of HM tunneling is much longer than the Hubble time.

Remark

So far it is found that, starting from (3.33), the configuration (4.18) approximately solves the Hamilton's equations (4.17) and reproduce the previous result (4.12) when it is substituted into the action. There is actually an important reason behind why we should choose the configuration (4.18) to evaluate the HM tunneling process.

To see this, it is important to realize an important difference between the current Hamilton's equations (4.17) and the previous one (4.8); the equation for ϕ_c now contains $\bar{\Pi}_\Delta$ rather than Π_Δ . Due to this difference, the structures of Hamilton's equations (4.17) are understood as follows;

- the equation for ϕ_c (4.17a) is determined once $\bar{\Pi}_\Delta$ is specified,
- $\bar{\Pi}_\Delta(t, \mathbf{x})$ is understood as initial conditions for (4.17b) since $\bar{\Pi}_\Delta(t, \mathbf{x})$ contains only the boundary modes satisfying $k = k_c(t)$. Hence, $\Pi_\Delta(t, \mathbf{x})$ is obtained by solving (4.17b) for given $\bar{\Pi}_\Delta$ under the condition (4.17a).

On top of that, substituting the Hamilton's equation for ϕ_c into the action, we have the first line of (4.23) which can be rewritten as

$$I = -\frac{1}{2} \int_{t'}^{t_*} dt \int \frac{d^3k}{(2\pi)^3} \Pi_\Delta(t, \mathbf{k}) \Pi_\Delta(t, -\mathbf{k}) \frac{H^2}{2k^3} \delta(t - t_k) . \quad (4.26)$$

The RHS depends only on the value of $\Pi_\Delta(t, \mathbf{k})|_{k=k_c(t)}$. Hence, we can evaluate the tunneling action once $\bar{\Pi}_\Delta$ is specified, provided that (4.26) or equivalently (3.33) is reliable.

We claim that (3.33) will be valid when $\bar{\Pi}_\Delta$ is chosen so that $\Pi_\Delta(t > t_k, \mathbf{k})$ becomes a smooth function in time. Otherwise, it will not be justified to ignore the higher-order corrections δH_{higher} which are neglected in (3.33). Intuitively, this states that perturbation theory tend to be broken down around

exotic configurations. When Π_Δ is a smooth function, we can relate $\bar{\Pi}_\Delta$ to Π_Δ after the suitable smearing in time:

$$\bar{\Pi}_\Delta(t, \mathbf{x}) \xrightarrow{\text{time c.g.}} \int d^3x' \Pi_\Delta(t, \mathbf{x}') W(|\mathbf{x} - \mathbf{x}'|; t) \sim \mathcal{V} \Pi_\Delta(t, \mathbf{x}). \quad (4.27)$$

Here, we also used $\Pi_\Delta(t, \mathbf{x}) \sim \Pi_\Delta(t, \mathbf{x}')$ for $|\mathbf{x} - \mathbf{x}'| \lesssim k_c^{-1}$ and defined $\mathcal{V} := (4\pi/3)k_c^{-3}(t) \sim \int d^3x' W(|\mathbf{x} - \mathbf{x}'|; t)$. We then require that the Hamilton's equation for Π_Δ (4.17b) is satisfied in a good approximation to ensure the validity of (3.33). This constrains the choice of $\bar{\Pi}_\Delta$. Indeed, the configuration (4.18) approximately solves (4.17b) and satisfies the smoothness and henceforth (4.27) after the smearing. This would be the reason why our analysis with (4.18) based on (3.33) can reproduce the correct result.

4.1.3 On the ε -independence of the Hawking-Moss tunneling

In sections 4.1.1 and 4.1.2, the tunneling probability of a coarse-grained patch with a physical radius $(\varepsilon H)^{-1}$ is calculated. Our results coincide with the HM tunneling and are independent of ε .

The ε -independence of the results would be a consequence of the scale-independence of the dynamics of light scalar fields at super-horizon scales. To see this, it is useful to notice that a coarse-grained patch of physical radius $(\varepsilon_1 H)^{-1}$ at a time $t = T$ expands to a patch of larger physical radius $(\varepsilon_2 H)^{-1}$ at a later time $t = T + \delta t > T$ with $\varepsilon_2 = \varepsilon_1 \exp[-H\delta t] < \varepsilon_1 \ll 1$. The value of IR field does not evolve from $t = T$ to $T + \delta t$ because the fluctuations of light scalar field at super-horizon scales are approximately time-independent,

$$\int \frac{d^3k}{(2\pi)^3} \phi_c(t, \mathbf{k}) e^{i\mathbf{k} \cdot \mathbf{x}} \theta(k_c(t) - k) \simeq \int \frac{d^3k}{(2\pi)^3} \phi_c(t + \delta t, \mathbf{k}) e^{i\mathbf{k} \cdot \mathbf{x}} \theta(k_c(t) - k), \quad (4.28)$$

unless δt is much longer than H^{-1} . Hence, we can relate the coarse-grained dynamics of different $\varepsilon \ll 1$ by considering the time shift $t \rightarrow t + \delta t$ with the value of IR field being kept fixed. This means that the tunneling probability from $\phi = \phi_{\text{false}}$ to ϕ_{true} of a patch with physical radius $(\varepsilon H)^{-1}$ should be independent of ε .

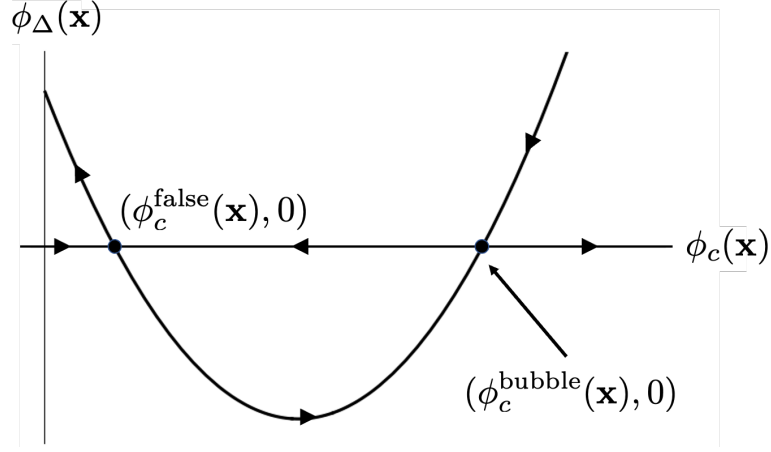


Figure 4.4: A schematic picture of Hamiltonian flows in the "field" phase space. We expect that there is a flow line which starts from the false vacuum (left intersection) and reaches the bubble configuration (right intersection) through a non-zero $\phi_\Delta(\mathbf{x})$ path.

4.2 Coleman-de Luccia tunneling

In the previous section, the result of the HM instanton was reproduced using (3.33). We expect that the formula (3.33) can describe not only HM tunneling but also CDL tunneling. In other words, we anticipate that there will be a flow line starting from the false vacuum to the bubble configurations, as illustrated in FIG. 4.4. In this section, we explicitly present an interesting configuration by following the prescription outlined in the previous section.

4.2.1 Coleman-de Luccia bubble solution

In Euclidean method, the CDL instanton is dominant rather than the HM instanton when the potential barrier is steep [64, 65, 66, 67]. By taking $\varepsilon \gg 1$, we can study nonperturbative physics at sub-horizon scales such as the formation of bubble. In this case, G^{III} becomes dominant and the other noises can be ignored in (3.34). Under this approximation, the exponent of the integrand of (3.33) is linear in Π_Δ . We can then perform the path integral $\int \mathcal{D}\Pi_\Delta$ in (3.33), yielding the product of delta functions $\prod_{\mathbf{x}} \delta(\dot{\phi}_c - a^{-3}\Pi_c)$. These delta functions eliminate the path integral $\int \mathcal{D}\Pi_c$.⁴ Furthermore, we

⁴This is the reason why we illustrate flow lines in the (ϕ_c, ϕ_Δ) -plane in FIG. 4.4.

change the integration variable as $\phi_\Delta \rightarrow -i\phi_\Delta$ in (3.33), leading to

$$p \simeq \int \mathcal{D}(\phi_c, \phi_\Delta) \exp \left[\int d^4x \left(a^3 \dot{\phi}_\Delta \dot{\phi}_c - H_{\text{CDL}}(\phi_c, \phi_\Delta) \right) \right] \quad (4.29)$$

with appropriate boundary conditions; we consider bubble configurations as boundary conditions later. Here, the Hamiltonian H_{CDL} is defined by

$$H_{\text{CDL}}(\phi_c, \phi_\Delta) := - \left(a \nabla^2 \phi_c - a^3 V'(\phi_c) \right) \phi_\Delta - \frac{a^3}{2} \phi_\Delta(x) \bar{\phi}_\Delta(x), \quad (4.30)$$

with

$$\bar{\phi}_\Delta(t, \mathbf{x}) := \frac{H k_c^4(t)}{4\pi^2 a(t)} \int d^3x' j_0(k_c(t)|\mathbf{x} - \mathbf{x}'|) \phi_\Delta(t, \mathbf{x}'). \quad (4.31)$$

We would like to find the appropriate configuration $\bar{\phi}_\Delta$ which describes the tunneling process. Following the choice made in the last section, let us *suppose* that the appropriate choice is given by the configuration for which the Hamiltonian (4.30) vanishes:

$$\bar{\phi}_\Delta(x) = 0, \quad -2(a^{-2} \nabla^2 \phi_c - V'(\phi_c)). \quad (4.32)$$

Note that this condition is imposed up to the suitable smearing in time because $\bar{\phi}_\Delta(t, \mathbf{x})$ includes only the boundary Fourier mode $k = k_c(t)$. As discussed in sec. 4.1.2, ϕ_Δ becomes sufficiently smooth for the appropriate $\bar{\phi}_\Delta$, and hence we also suppose

$$\bar{\phi}_\Delta \sim \frac{H^2 \varepsilon}{3\pi} \phi_\Delta(x) \quad (4.33)$$

after the suitable averaging over the Hubble time, similarly to (4.27). We use the relation (4.33) to evaluate the action later. This would work at least for the purpose of estimating the action.

In principle, we do not need to use the estimate (4.33) in evaluating the action since it is determined once $\bar{\phi}_\Delta$ is specified based on the analogous logic discussed around (4.26). For this purpose, we need to specify the configurations $\{\phi_\Delta(t, \mathbf{k})|_{k=k_c(t)}\}$ in momentum space which result in the sufficiently smooth ϕ_Δ that solves the Hamilton equations. It is not easy to find such appropriate configurations precisely, however. Hence, we leave more careful

analysis on the choice of $\bar{\phi}_\Delta$ for future work. In this study, instead, we focus on how we can proceed the analysis for given configuration (4.32) with adopting the estimate (4.33) and how the bubble nucleation process could be described in the stochastic approach.

Using the relation (4.33), the Hamiltonian (4.30) and the configuration (4.32) become

$$H_{\text{CDL}} \sim -\left(a\nabla^2\phi_c - a^3V'(\phi_c)\right)\phi_\Delta - \frac{H^2\varepsilon}{6\pi}a^3\phi_\Delta^2, \quad (4.34)$$

$$\phi_\Delta(x) = 0, \quad -\frac{6\pi}{H^2\varepsilon}(a^{-2}\nabla^2\phi_c - V'(\phi_c)). \quad (4.35)$$

The Hamilton's equations under the constraints (4.35) give the following equation of motions for ϕ_c :

$$\begin{aligned} \ddot{\phi}_c + 3H\dot{\phi}_c &= -(V'(\phi_c) - a^{-2}\nabla^2\phi_c), \quad (\phi_\Delta = 0), \\ \ddot{\phi}_c + 3H\dot{\phi}_c &= V'(\phi_c) - a^{-2}\nabla^2\phi_c, \quad (\phi_\Delta = -\frac{6\pi}{H^2\varepsilon}(a^{-2}\nabla^2\phi_c - V'(\phi_c))). \end{aligned} \quad (4.36)$$

Similar to the previous case (4.10), the signs of the potential term and the gradient term are flipped in the equation of motion with the non-trivial $\phi_\Delta \neq 0$. Hence, we can expect that the tunneling process is realized. From (4.29), we find that only the solutions satisfying $\phi_\Delta \neq 0$ contribute to the action. Then we focus on the second equation in (4.36).

Interestingly, this equation is the classical equation of motion in the Euclidean anti-de Sitter (AdS) space which is defined by the embedding equation

$$-X_0^2 + X_1^2 + X_2^2 + X_3^2 + X_4^2 = -H^{-2} \quad (4.37)$$

in five-dimensional Minkowski spacetime

$$ds^2 = -dX_0^2 + dX_1^2 + dX_2^2 + dX_3^2 + dX_4^2. \quad (4.38)$$

In fact, the d'Alembert operator $\partial_t^2 + 3H\partial_t + a^{-2}\nabla^2$ can be obtained from the following induced metrics;

$$ds^2 = dt^2 + e^{2Ht}d\mathbf{x}^2 = \frac{1}{H^2\eta^2}(d\eta^2 + d\mathbf{x}^2) = H^{-2}(d\rho^2 + \sinh^2\rho d\Omega^2), \quad (4.39)$$

where $d\Omega^2 := d\theta_1^2 + \sin^2 \theta_1 d\theta_2^2 + \sin^2 \theta_1 \sin^2 \theta_2 d\theta_3^2$ and $0 < \rho < \infty$, $0 \leq \theta_1 \leq \pi$, $0 \leq \theta_2 \leq \pi$ and $0 \leq \theta_3 \leq 2\pi$. The second metric is that in the Poincaré coordinates and the third one is that in the global coordinates. The explicit coordinate transformations are given by

$$\begin{aligned} X_0 &= \frac{1}{2} \frac{\eta^2 + \mathbf{x}^2 + H^{-2}}{-\eta} = H^{-1} \cosh \rho, \\ X_1 &= \frac{x_1}{-H\eta} = H^{-1} \sinh \rho \sin \theta_1 \cos \theta_2, \\ X_2 &= \frac{x_2}{-H\eta} = H^{-1} \sinh \rho \sin \theta_1 \sin \theta_2 \cos \theta_3, \\ X_3 &= \frac{x_3}{-H\eta} = H^{-1} \sinh \rho \sin \theta_1 \sin \theta_2 \sin \theta_3, \\ X_4 &= \frac{1}{2} \frac{\eta^2 + \mathbf{x}^2 - H^{-2}}{-\eta} = H^{-1} \sinh \rho \cos \theta_1. \end{aligned} \quad (4.40)$$

For later convenience, we derive the relation between $(\eta, r := |\mathbf{x}|)$ and (ρ, σ_1) as

$$-H\eta = \frac{1}{\cosh \rho - \sinh \rho \cos \theta_1}, \quad r = H^{-1} \frac{\sinh \rho \sin \theta_1}{\cosh \rho - \sinh \rho \cos \theta_1}. \quad (4.41)$$

From this, we can check that a point $(\eta, r) = (-H^{-1}, 0)$ is mapped to $\rho = 0$ for any $\theta_1 \in [0, \pi]$. Except for this, there is one-to-one correspondence between points in the (η, r) -plane with $\eta \leq 0$ and $r \geq 0$ and points in the (ρ, θ_1) -plane with $\rho > 0$ and $0 \leq \theta_1 \leq \pi$. On a constant- η slice in the (η, r) -plane, the value of ρ for given spatial point r is given by

$$\rho = \operatorname{arccosh} \left(\frac{1}{2} \frac{H^2(\eta^2 + r^2) + 1}{-H\eta} \right). \quad (4.42)$$

This function increases monotonically in r for given η in the region $r > 0$.

From the global coordinate (4.39), we see the Euclidean AdS spacetime has $O(4)$ symmetry. Hence, we assume that the field ϕ_c depends only on ρ . Thus the second equation of (4.36) reads

$$\frac{d^2 \phi_c}{d\rho^2} + \frac{3}{\tanh \rho} \frac{d\phi_c}{d\rho} = \frac{V'(\phi_c)}{H^2}. \quad (4.43)$$

Note that this is the same equation utilized in the Euclidean method [38, 40]. Imposing the boundary conditions [36, 38]

$$\lim_{\rho \rightarrow \infty} \phi_c = -\alpha, \quad \left. \frac{d\phi_c}{d\rho} \right|_{\rho=0} = 0, \quad (4.44)$$

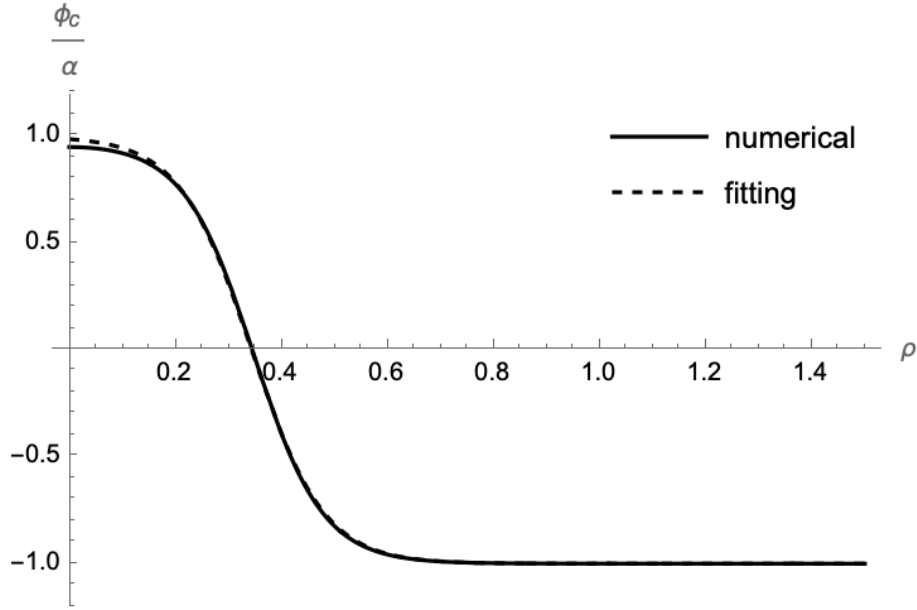


Figure 4.5: The tunneling configuration for the inhomogeneous field is plotted. The solid curve is the numerical solution for $\beta = 0.5$ and $\alpha g/H = \sqrt{140}$. We take the initial conditions as $\phi_c(10^{-15})/\alpha = 1 - 4.98284 \times 10^{-2}$ and $\phi'_c(10^{-15})/\alpha = 0$. The dashed curve is the fitting function (4.45) where $\mu = 7.32515$ and $\bar{\rho} = 0.34397$.

we obtain the bubble solutions as is shown in FIG. 4.5. Note that the latter condition ensures the continuity of $\partial_\eta \phi_c(\eta, r)$ and $\partial_\eta^2 \phi_c(\eta, r)$ at $(H\eta, Hr) = (-1, 0)$.

It is useful to fit the bubble configurations by the following fitting function (FIG. 4.5);

$$\phi_c(\rho) = -\alpha \tanh \mu(\rho - \bar{\rho}), \quad (4.45)$$

where μ and $\bar{\rho}$ represent the thickness and the position of the bubble wall, respectively. Note that, for $\beta \gtrsim 0.6$, the deviation from the true vacuum at the $\rho = 0$ becomes significant [58] and the fitting (4.45) becomes bad.

We also have to examine the boundary conditions in the Hamilton flow. As in the case of HM tunneling, we expect that there are several intersections where $\phi_\Delta = 0$ (FIG. 4.4). Since we have the concrete ϕ_Δ (4.35) and the bubble configurations (4.45), it is possible to obtain the curves in (r, η) -plane

on which $\phi_\Delta = 0$. Using (4.35) and (4.36), such curves are given as

$$F(\eta, r) := \partial_\eta^2 \phi_c - \frac{2}{\eta} \partial_\eta \phi_c = 0. \quad (4.46)$$

Note that this is equivalent to solving $a^{-2} \nabla^2 \phi_c - V'(\phi_c) = 0$. Substituting (4.35) into (4.46) and using the fitting formula (4.45), we find four solutions which are expressed as four lines in the (η, r) -plane; two of them are placed at $\eta = -\infty$ with fixed r and $r = \infty$ with fixed η . These correspond to $\rho = \infty$ where $\phi = -\alpha$ (FIG. 4.5). Also, we numerically find other two non-trivial hypersurfaces (FIG. 4.6(a)). It can be seen that one of the curves is totally spacelike but another is partially timelike. Thus, it seems natural to take $\eta = -\infty$ curve as an initial time slice on which ϕ is false vacuum and totally spacelike non-trivial curve as a final time slice on which the bubble is nucleated. We denote the region between the two spacelike curves as Σ .

As a matter of fact, the configurations on the final time slice has a bubble. It can be checked as follows; the value of ϕ_c at the origin exceeds the top of potential hill. In the case of FIG. 4.6(a) where we set $(\beta, \alpha g/H) = (0.5, \sqrt{140})$, we have $\phi_c/\alpha \sim -0.1 > -0.5 = -\beta$ at $Hr = 0$, $H\eta \sim -0.7$.

Now, we can evaluate the action for the bubble configurations for Σ (FIG. 4.6(b) or (d)). We use the fitting function (4.45) for the numerical integration.

$$\begin{aligned} I &\simeq \int_\Sigma d^4x \left[-a^3 \phi_\Delta \left(\ddot{\phi}_c + 3H\dot{\phi}_c \right) - H_{\text{CDL}} \right] \\ &= -\frac{6\pi}{H^6 \varepsilon} \int_\Sigma d\rho d\theta_1 d\theta_2 d\theta_3 \frac{\sin^2 \theta_1 \sin \theta_2 \sinh^3 \rho}{(\cosh \rho - \sinh \rho \cos \theta_1)^4} \left(\partial_\eta^2 \phi_c - \frac{2}{\eta} \partial_\eta \phi_c \right)^2 \\ &= -\frac{24\pi^2}{H^2 \varepsilon} \int_\Sigma d\rho d\theta_1 \sin^2 \theta_1 \sinh^3 \rho \left[\frac{(\sinh \rho - \cosh \rho \cos \theta_1)^2}{(\cosh \rho - \sinh \rho \cos \theta_1)^2} \frac{\partial^2 \phi_c}{\partial \rho^2} \right. \\ &\quad \left. + \left\{ 3 \frac{\sinh \rho - \cosh \rho \cos \theta_1}{\cosh \rho - \sinh \rho \cos \theta_1} + \frac{\sin^2 \theta_1}{\tanh \rho (\cosh \rho - \sinh \rho \cos \theta_1)^2} \right\} \frac{\partial \phi_c}{\partial \rho} \right]^2 \\ &:= -\frac{24\pi^2 \alpha^2}{H^2 \varepsilon} \tilde{I}(\mu, \bar{\rho}). \end{aligned} \quad (4.47)$$

In the second line, we used $H_{\text{CDL}} = 0$, (4.35) and (4.36). Also we performed a coordinate transformation $(t, r) \rightarrow (\rho, \theta_1)$. In the third line, the following relations are used

$$-\frac{2}{\eta} \frac{\partial \phi_c(\rho)}{\partial \eta} = -\frac{2}{\eta} \frac{\partial \rho}{\partial \eta} \frac{\partial \phi_c(\rho)}{\partial \rho} = 2H^2 (\cosh \rho - \sinh \rho \cos \theta_1) (\sinh \rho - \cosh \rho \cos \theta_1) \frac{\partial \phi_c(\rho)}{\partial \rho},$$

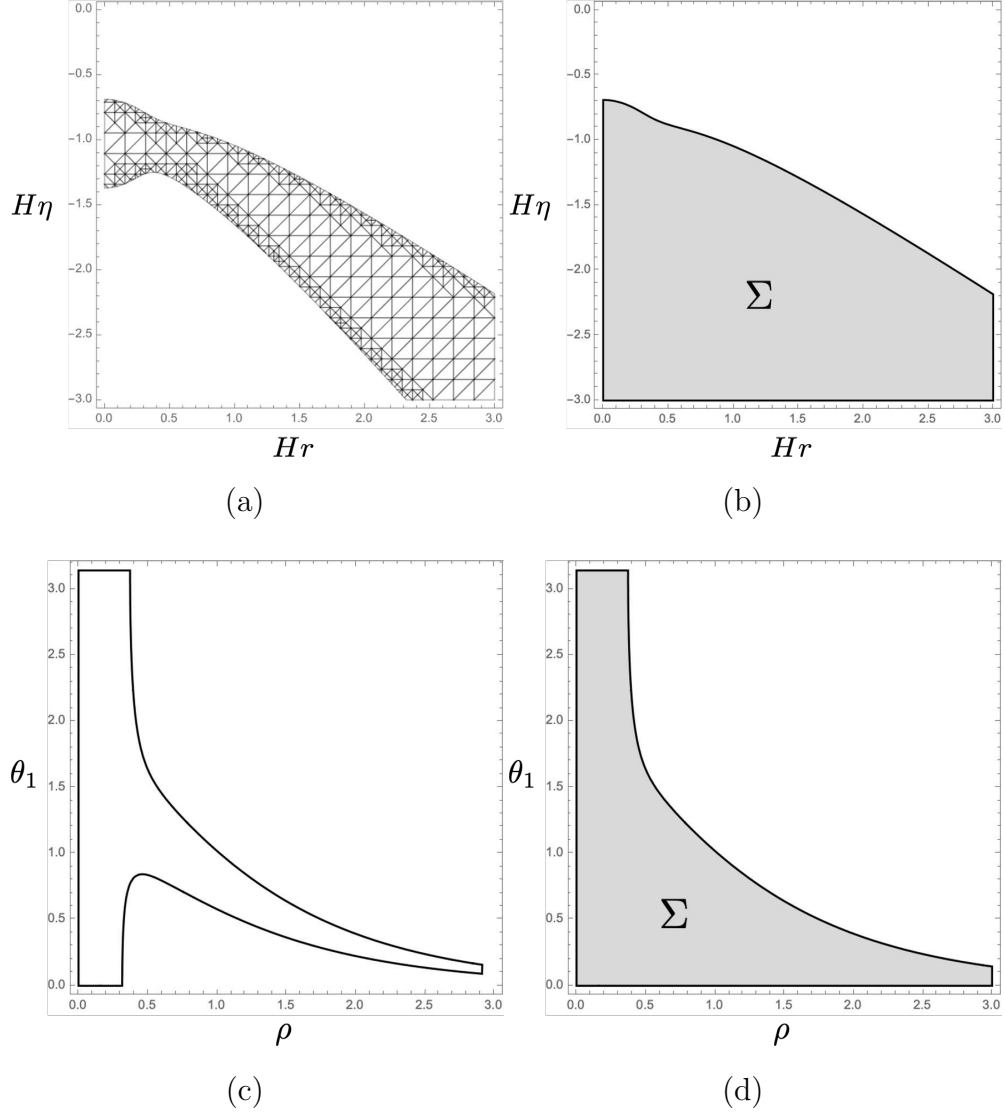


Figure 4.6: (a): The two non-trivial curves on which (4.46) is satisfied. We insert the mesh to confirm whether the curves are spacelike or timelike. It can be seen that the upper line is spacelike but the lower one is partially timelike. (b): The integration region for (4.47). There are another spacelike curve at $\eta = -\infty$. (c): The two non-trivial curves in (ρ, θ_1) plane is plotted. (d): The integration region for (4.47) in (ρ, θ_1) plane. All plots are the case for $\beta = 0.5$ and $\alpha g/H = \sqrt{140}$.

and

$$\begin{aligned} \frac{\partial^2 \phi_c(\rho)}{\partial \eta^2} &= H^2 \left[(\sinh \rho - \cosh \rho \cos \theta_1)^2 \frac{\partial^2}{\partial \rho^2} \right. \\ &\quad \left. + \left\{ (\sinh \rho - \cosh \rho \cos \theta_1)(\cosh \rho - \sinh \rho \cos \theta_1) + \frac{\sin^2 \theta_1}{\tanh \rho} \right\} \frac{\partial}{\partial \rho} \right] \phi_c(\rho). \end{aligned}$$

Here, the partial differentiation with respect to η is taken while keeping r constant.

4.2.2 Discussions

Appropriate choice of ε

Since our results (4.47) depend on ε , we need to choose some specific value of ε to predict the tunneling rate. We emphasize that the ε dependence does not imply the pathology of our result. Rather our results should depend on ε because the non-perturbative effects from UV modes are missed in our formalism as discussed in section 3.3.2. We then choose ε as large as possible to evaluate the tunneling rate.

Our analysis will be valid only when the value of ε lies in a certain range. Below, we briefly discuss necessary conditions for ε to justify our analysis.

Lower bound on ε . We need to choose sufficiently large ε so that IR fields can describe the bubble configuration. This imposes that $(\varepsilon H)^{-1}$ should be smaller than the typical physical size of the bubble or the bubble wall. This condition will be satisfied by taking $\varepsilon \gtrsim \mu$ for our parameter choices.

Upper bound on ε . Our analysis is based on Eq.(3.33) which will be invalid when quantum fluctuations of IR fields are too large. This imposes an upper bound on ε because the size of fluctuations of ϕ_c at the scale $k \sim k_c(t)$ is proportional to ε :

$$\delta \phi_c|_{k \sim k_c} := \left[\int \frac{d^3 k}{(2\pi)^3} \delta(\ln(k/k_c(t))) \langle |\hat{\phi}(t, \mathbf{k})|^2 \rangle \right]^{1/2} \approx \frac{k_c/a}{2\pi} = \frac{\varepsilon H}{2\pi}, \quad (4.48)$$

where we assumed $(k_c/a)^2 \gg V''(\phi_c)$ ⁵ and approximated ϕ as a massless scalar field.

⁵This is compatible with the condition $\varepsilon < \alpha/H$ when the weak coupling $g \ll 1$ is considered.

As discussed in section 3.3.2, (3.33) will be justified when the term δH is negligible. Then, we define the ratio $\mathcal{Q}$ of the term δH to the final term on the RHS of (4.34) as ⁶

$$\mathcal{Q} := \left| \frac{\delta H}{H^2 \varepsilon a^3 \phi_\Delta^2 / (6\pi)} \right|_{\phi_\Delta = -\frac{6\pi}{H^2 \varepsilon} (a^{-2} \nabla^2 \phi_c - V'(\phi_c))} = \frac{3\pi^2}{2\varepsilon^2 H^4} |V'''(\phi_c) (a^{-2} \nabla^2 \phi_c - V'(\phi_c))| ,$$

and we impose $\mathcal{Q} < 1$. In the second equality, we used $\delta H = (a^3/24)V'''(\phi_c)\phi_\Delta^3$. Similarly to (4.48), we can estimate the size of quantum fluctuations of $a^{-2}\nabla^2\phi$. Then, we estimate $\mathcal{Q}$ at large ε as $\mathcal{Q} \sim g^2\alpha\varepsilon/H$ where we also used $|V'''(\phi_c)| \sim g^2\alpha$ for $|\phi_c| \lesssim \alpha$. The condition $\varepsilon < H/(g^2\alpha)$ follows from $\mathcal{Q} < 1$.

Summary. In summary, it is necessary to choose ε satisfying the following condition to justify our analysis,

$$\mu \lesssim \varepsilon \lesssim \varepsilon_{\max} := H/(g^2\alpha) \approx 755 \times \left(\frac{1.12 \times 10^{-4}}{g} \right) \left(\frac{\sqrt{140}}{g\alpha/H} \right) . \quad (4.49)$$

This suggests that, at least when g is sufficiently tiny, we can reliably choose large ε . Note that we cannot make the action I given in (4.47) arbitrarily small even if we take $\varepsilon_{\max} \rightarrow \infty$ in the free-theory limit $g \rightarrow 0$; for $\varepsilon = \varepsilon_{\max}$, we have $I|_{\varepsilon=\varepsilon_{\max}} \sim -g^{-1}(g\alpha/H)^3 \tilde{I}(\mu, \bar{\rho})$. Roughly speaking, the small $g\alpha/H$ corresponds to the bubble radius being larger than the Hubble radius. Since we are interested in the bubble smaller than the Hubble radius, $g\alpha/H$ cannot be taken small arbitrarily.

A few more comments on the ε -dependence of our results are in order. Our result (4.47) shows that the value of I becomes larger for smaller choice of ε . This is because the size of quantum fluctuations of modes $k \sim \varepsilon a(t)H$ is proportional to ε , while the nonperturbative physics caused by UV modes $k > \varepsilon a(t)H$ is truncated in our formalism. This interpretation however implicitly assumes that the modes $k \sim \varepsilon_{\max} a(t)H$ are relevant for the tunneling process. We expect that this is the case because a kinetic energy of fluctuations of modes $k \sim \varepsilon_{\max} a(t)H$ is estimated as $(\varepsilon_{\max} H)^4$ which is much smaller than the height of potential barrier $\Delta V \sim g^2\alpha^4$ for our parameter choices.

⁶We can also compare δH with the second and the third term on the RHS of (4.34). Such considerations do not change our estimate of $\varepsilon_{\max}$ in (4.49).

By contrast, very short-scale physics which is insensitive to the detailed structure of $V(\phi)$ near its origin might be irrelevant for the tunneling process. Hence, we expect that the tunneling rate would converge to some finite value in the limit $\varepsilon \rightarrow \infty$. It would be interesting to see whether the tunneling rate converges to a finite value as ε increases, and if so, from which value of ε this convergence begins.

Hawking-Moss vs. Coleman-de Luccia

From the formula (3.33), we derived the two configurations which describe the tunneling from the false vacuum to the true vacuum. One is HM configuration and the other is CDL-like configuration. Using these configurations, we evaluate the probability of the tunneling. Here, we compare these probabilities. Denoting the action (4.47) and (4.12) as I_{bubble} and I_{HM} , the ratio of the two actions is given by

$$\gamma := \frac{I_{\text{bubble}}}{I_{\text{HM}}} = \frac{9\alpha^2 H^2}{\Delta V \varepsilon} \tilde{I}(\mu, \bar{\rho}) = \frac{1}{\varepsilon} \frac{108}{(1-\beta)^3(\beta+3)} \frac{H^2}{\alpha^2 g^2} \tilde{I}(\mu, \bar{\rho}). \quad (4.50)$$

For example, if we choose $\beta = 0.5$ and $\alpha g/H = \sqrt{140}$ (see FIG. 4.5), the fitting parameters become $\mu = 7.32515$ and $\bar{\rho} = 0.34397$ and the ratio becomes $\gamma = 12.9840/\varepsilon$. For these parameters, the upper bound of ϵ is given as (4.49) and then γ can be smaller than one for appropriately small g . This means that the CDL bubble configuration is dominant rather than HM configuration. Results for other parameters are also shown in Table 4.1.

Comparison with the Euclidean method

We also compute the CDL action in Euclidean method (Chapter 2) and compare the ratios of the CDL action to the HM action (Table 4.1). For $\varepsilon > \varepsilon_*$, our tunneling process becomes more probable than the one predicted by the Euclidean method. As discussed in section 4.2.2, we can consistently choose $\varepsilon > \varepsilon_*$ for certain choices of parameters (g, α, β) . However, the implications of this choice should be analyzed with caution, as the quantitative properties of ε are not yet fully understood.

Bubble nucleation hypersurface and the subsequent evolution

We find the non-trivial spacelike hypersurface on which the bubble is nucleated (FIG. 4.6(a)), where the hypersurface is given by the condition $F(\eta, r) =$

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β	$g^2\alpha^2/H^2$	$1 - \phi_c(10^{-15})/\alpha$	μ	$\bar{\rho}$	$\varepsilon\gamma$	γ^E	ϵ_*
0.3	80	4.87210×10^{-5}	6.22563	1.05197	355.089	0.247004	1437.58
0.4	80	5.20273×10^{-3}	5.99645	0.65257	64.1185	0.0909796	704.758
0.5	80	3.89106×10^{-2}	5.63444	0.475247	33.9526	0.0599470	566.376
0.3	140	3.50945×10^{-4}	8.13990	0.673305	52.2515	0.0486678	1073.64
0.4	140	8.96003×10^{-3}	7.82419	0.459659	19.2912	0.0239903	804.123
0.5	140	4.98284×10^{-2}	7.32515	0.34397	12.9840	0.0172063	754.606

Table 4.1: The values of γ for several parameter sets. First and second ones are potential parameters. Third one is the initial condition for the equation of motion (4.43). Forth and fifth ones are fitting parameters for (4.45). Sixth and seventh ones are the numerical results of (4.50) and (2.69). The last one is the cut-off parameter ϵ for which (4.50) coincides with (2.69).

0 or equivalently $a^{-2}\nabla^2\phi_c - V'(\phi_c) = 0$. The subsequent evolution is then described by the one shown in the first line of (4.36) which is the standard classical equation of motion. The configuration on the hypersurface gives the initial data for the classical dynamics after the bubble nucleation.

The field value on the spacelike hypersurface does not reach the true vacuum but rather it lies between the true vacuum and the top of the potential as mentioned in the subsection 4.2.1. This is possible because the location of hypersurface is defined by the condition $a^{-2}\nabla^2\phi_c - V'(\phi_c) = 0$ where the gradient force is balanced with the potential force. Though it is non-trivial to solve the evolution starting from the general spacelike hypersurfaces, it is desirable to solve it to fully understand the formation of the true-vacuum bubble in our scenario.

We conclude this section by pointing out that our formalism naturally predicts that the condition $\bar{\phi}_\Delta = 0$ (or $\phi_\Delta = 0$ via (4.33)) defines the hypersurface on which the quantum dynamics is switched to the classical dynamics and the non-trivial field configuration is nucleated. This would be the generic prediction of our formalism that holds true once the appropriate configuration $\bar{\phi}_\Delta$ is specified.

Chapter 5

Conclusion and prospects

We studied tunneling processes in a de Sitter background using the stochastic approach. A key novelty of our work is the application of the MSRJD functional integral to this problem. In this formalism, the tunneling rate is obtained by evaluating the functional integral via the saddle-point approximation. Using this method, we analyzed both the HM and CDL tunneling processes.

For the HM case, we first analyzed (3.31), the MSRJD functional integral of the stochastic equation (3.23) at a single spatial point. We successfully derived the tunneling solution in the "phase space" (FIG.4.2), which describes the tunneling process from the false vacuum, through the top of the potential hill, and finally to the true vacuum (FIG.4.3). This solution naturally satisfies the tunneling boundary conditions, which cannot be obtained using the Euclidean method. The tunneling rate for this configuration coincides with that of the HM instanton at leading order. In section 4.1.2, we re-derived this result starting from (3.33), which describes the stochastic dynamics in the global region covered by the flat chart. We also estimated the time scale of the HM tunneling as (4.24), which is much longer than the Hubble time scale. Our analysis clarifies the physical picture of the HM instanton: the HM transition probability represents the transition probability of a coarse-grained patch with a physical radius $(\varepsilon H)^{-1}$. Some technical complications that arise in dealing with (3.33) and the methods for addressing them are discussed in section 4.1.2. Physically, these complications stem from the spatial correlations of stochastic noises.

For the CDL case, we found configurations that describe the bubble nucleation process. The tunneling rates for these bubble configurations depend

on the cutoff scale dividing IR and UV fields. We argued that this dependence arises from the truncation of non-perturbative effects from UV modes. We also discussed the appropriate choice of the cutoff scale for which the MSRJD functional integral remains reliable. This consideration is based on the first-principles derivation of the stochastic approach from the Schwinger-Keldysh formalism. With an appropriate cutoff scale, we found that the CDL tunneling rate is larger than the HM rate for a steep potential barrier. However, as mentioned above (4.34), it has not yet been fully investigated whether our configuration can be reliably used for estimating the tunneling action. This aspect, which is crucial for evaluating the tunneling action, is left for future work. Nonetheless, we believe that our study clarifies how the analysis can be extended and how the bubble nucleation process can be described within the stochastic approach. For instance, our formalism naturally defines the hypersurface on which the quantum dynamics transitions to classical dynamics and the non-trivial field configuration is nucleated.

It is interesting to understand the relation between the stochastic approach and the Euclidean method. We compared the CDL tunneling rate in our method with that in the Euclidean method and found that the former becomes larger than the latter for the potential considered with a certain cut-off scale. Investigating the meaning of this result would be worthwhile. In particular, we need to clarify the precise relation between the CDL instanton and our configuration. Intriguingly, our method is also related to that discussed in [74, 75, 76, 77, 78], where the bubble emerges spontaneously from initial quantum fluctuations. In our method, however, we assume that non-perturbative effects from UV modes, the quantum component of the potential in the form $V'''(\phi_c)\phi_\Delta^3$, can be neglected, and that the configuration (4.35) can be used to estimate the tunneling rate. The first two are summarized into the term δH . We also assume a fixed background spacetime. Relaxing these assumptions would be important to explore the relations among the three methods. It would also be useful to apply our path-integral method to tunneling in flat space and compare it with the Euclidean method.

Once this relaxation is achieved, it would also be interesting to apply our method to tunneling phenomena where the back-reaction to the background geometry is non-negligible and to make comparisons with recently discussed methods. For instance, the formalism of the Wheeler-DeWitt equation [79, 80, 81] and the tunneling potential [82] must be related to ours because the HM exponent is reproduced. Tunneling in black hole spacetimes [83, 84] is also of interest. In particular, [84] discussed the HM transition with a black

hole from the viewpoint of the stochastic approach.

Our method, based on the saddle-point approximation, effectively amounts to solving real-time quantum dynamics, which reduces to Starobinsky's stochastic approach [43, 44] at leading order when the super-horizon dynamics in de Sitter space is considered. However, since our starting point is the Schwinger-Keldysh path integral, which can be formulated in a more generic setup and, in principle, incorporates all quantum effects, we believe that our formalism sheds light on further studies of tunneling phenomena from a real-time perspective.

One of the advantages of our method is that it is applicable to a dynamical setup. The application of our method to inflationary models, such as chain inflation [85] and warm inflation [86], is also intriguing. Using our method, we can study the tunneling process in an inflationary background under dissipation and fluctuations coming from the circumference. We leave these issues for future work.

Acknowledgments

I express my gratitude to my supervisor, Jiro Soda, for his unwavering encouragement and fruitful discussions throughout this research endeavor. Additionally, I extend my thanks to Junsei Tokuda for his valuable suggestions. I am sincerely appreciative of all the individuals I have encountered during my time at Kobe University.

The author acknowledges support from JST SPRING, Grant Number JPMJSP2148 and JSPS KAKENHI Grant Number JP23KJ1543.

Appendix A

Stochastic approach from the first principle

In this appendix, we evaluate the generating functional for IR fields $Z[J^{\text{IR}}(T)]$ defined in Eq.(3.38) to obtain (3.39). Note that each step we need to take for evaluating $Z[J^{\text{IR}}(T)]$ is summarized in section 3.3.2.

A.1 Path integral representation

A path integral representation of $Z[J^{\text{IR}}(T)]$ takes the following standard form,

$$Z[J^{\text{IR}}(T)] = \int \mathcal{D}(\phi_+, \phi_-, \Pi_+, \Pi_-) e^{i\mathbf{J}^{\text{IR}} \cdot (\phi_+^{\text{IR}} + \phi_-^{\text{IR}})/2} \\ \times e^{i(S_H^+ - S_H^-)} \prod_{\mathbf{x}} \delta(\phi_+(T, \mathbf{x}) - \phi_-(T, \mathbf{x})) \Psi_0[\phi_+] \Psi_0^*[\phi_-], \quad (\text{A.1})$$

where $\Psi_0[\phi]$ is an initial wave functional. We specify it more concretely in section A.2.1. $S_H^\pm = \int^{t=T} d^4x (\Pi^\pm \dot{\phi}^\pm - H[\phi^\pm, \Pi^\pm])$ is a Hamiltonian action. After the rotation of the basis $(X_+, X_-) \rightarrow (X_c, X_\Delta) := (\frac{X_+ + X_-}{2}, X_+ - X_-)$ with $X = \phi, \Pi$, (A.1) is written as

$$Z[J^{\text{IR}}(T)] = \int \mathcal{D}(\phi_c, \phi_\Delta, \Pi_c, \Pi_\Delta) e^{i\mathbf{J}^{\text{IR}} \cdot \phi_c^{\text{IR}}} e^{iS_H[\phi_c, \phi_\Delta, \Pi_c, \Pi_\Delta]} \prod_{\mathbf{x}} \delta(\phi_\Delta(T, \mathbf{x})) \rho_0[\phi_c, \phi_\Delta]. \quad (\text{A.2})$$

Here, we defined $S_H[\phi_c, \phi_\Delta, \Pi_c, \Pi_\Delta] := (S_H^+ - S_H^-)|_{(X_+, X_-) \rightarrow (X_c, X_\Delta)}$ with $X = \phi, \Pi$. We also defined $\rho_0[\phi_c, \phi_\Delta] := \Psi_0[\phi_c + (\phi_\Delta/2)]\Psi_0^*[\phi_c - (\phi_\Delta/2)]$.

A.2 Nonperturbative generating functional for IR sector

Now we explain how to calculate $Z[J^{\text{IR}}(T)]$ non-perturbatively in the IR sector. This allows us to capture the non-perturbative physics of the sector $k \leq k_c(t)$. Our strategy is to split the integration variables into UV modes $k > k_c(t)$ and IR modes $k \leq k_c(t)$ for each time step t , and perform the integration over UV variables to get $Z[J^{\text{IR}}(T)]$. One may simply split the integration variables $X_a(t, \mathbf{k})$,¹ where $X = (\phi, \Pi)$ and $a = (c, \Delta)$, by the following replacement in (A.2):

$$X_a(t, \mathbf{k}) \rightarrow \begin{cases} X_a^{\text{IR}}(t, \mathbf{k}) & (t \geq t_k) \\ X_a^{\text{UV}}(t, \mathbf{k}) & (t < t_k), \end{cases} \quad (\text{A.3})$$

where t_k is defined by the condition $k_c(t_k) = k$. It is then tempted to perform the integration over UV variables X_a^{UV} by using the Schwinger-Keldysh (or *closed-time-path*) formalism. However, the expression obtained from (A.2) after the replacement (A.3) does not have the product of delta functions $\prod_{\mathbf{k}} \delta(\phi_\Delta^{\text{UV}}(t_k - 0^+, \mathbf{k}))$ at the final time $t = t_k - 0^+$.² That is, the time contour for each UV ϕ -variable with modes $0 < k \leq k_c(T)$ is not closed at the final time. Hence, the usual Schwinger-Keldysh formalism does not apply for evaluating the integration over UV variables. To resolve this issue, Refs.[71, 72] proposed a new way of splitting integration variables. The splitting is given by (A.3) for $X = \Pi$, while for ϕ -variable it is defined by the following rule rather than (A.3):

$$\phi_c(t, \mathbf{k}) \rightarrow \begin{cases} \phi_c^{\text{IR}}(t, \mathbf{k}) & (t > t_k) \\ \phi_c^{\text{UV}}(t, \mathbf{k}) & (t \leq t_k), \end{cases} \quad \phi_\Delta(t, \mathbf{k}) \rightarrow \begin{cases} \phi_\Delta^{\text{IR}}(t, \mathbf{k}) & (t \geq t_k) \\ \phi_\Delta^{\text{UV}}(t, \mathbf{k}) & (t < t_k). \end{cases} \quad (\text{A.4})$$

¹In the main text, we use the notation $X_{\mathbf{k}}(t)$ to write variables in momentum space. We however adopt the notation $X(t, \mathbf{k})$ in this appendix since we have many subscripts such as c and Δ .

²Here, 0^+ is the infinitesimal time step which is introduced to obtain path integral representation of unitary time evolution as usual.

In this replacement (A.4), the variables $\phi_\Delta^{\text{UV}}(t_k, \mathbf{k})$ are absent while the variables $\phi_c^{\text{UV}}(t_k, \mathbf{k})$ are present. Therefore, in the expression obtained from (A.2) after the new replacement, the time contour for every UV ϕ -variable including those for modes $\mathbf{k}$ with $0 < k \leq k_c(T)$ is closed. This allows us to perform the integration over UV variables based on the Schwinger-Keldysh formalism as we see below. Note that there is no subtlety in the replacement of integration variables $X_a(t, \mathbf{k})$ for a zero mode $k = 0$ and those for deep UV modes $k > k_c(T)$: in both replacement rules mentioned above, they are simply replaced by IR variables $X_a^{\text{IR}}(t, \mathbf{k})$ and UV variables $X_a^{\text{UV}}(t, \mathbf{k})$, respectively. The time contours for them are closed thanks to the product of delta functions in the original expression (A.2).

After the new replacement, the term $S_H[\phi_c, \phi_\Delta, \Pi_c, \Pi_\Delta]$ in the exponent of (A.2) is decomposed into three pieces: purely IR terms $S_H^{\text{IR}} := S_H[\phi_c^{\text{IR}}, \phi_\Delta^{\text{IR}}, \Pi_c^{\text{IR}}, \Pi_\Delta^{\text{IR}}]$, purely UV terms $S_H^{\text{UV}} := S_H[\phi_c^{\text{UV}}, \phi_\Delta^{\text{UV}}, \Pi_c^{\text{UV}}, \Pi_\Delta^{\text{UV}}]$, and the remaining IR-UV mixing terms S_{mixed} :

$$S_H[\phi_c, \phi_\Delta, \Pi_c, \Pi_\Delta] \rightarrow S_H^{\text{IR}} + S_H^{\text{UV}} + S_{\text{mixed}}. \quad (\text{A.5})$$

In terms of these quantities, $Z[J^{\text{IR}}(T)]$ can be formally written as [71, 72]

$$\begin{aligned} Z[J^{\text{IR}}(T)] &= \int \mathcal{D}(\phi_c^{\text{IR}}, \phi_\Delta^{\text{IR}}, \Pi_c^{\text{IR}}, \Pi_\Delta^{\text{IR}}) e^{i\mathbf{J}^{\text{IR}} \cdot \phi_c^{\text{IR}}} e^{iS_H^{\text{IR}}} e^{i\Gamma} \\ &\quad \times \prod_{k \leq k_c(T)} \delta(\phi_\Delta^{\text{IR}}(T, \mathbf{k})) \prod_{0 < k \leq k_c(T)} \delta(\phi_c^{\text{IR}}(t_k, \mathbf{k})), \end{aligned} \quad (\text{A.6})$$

where $i\Gamma$ is the effective action for IR fields,

$$e^{i\Gamma} := \int \mathcal{D}(\phi_c^{\text{UV}}, \phi_\Delta^{\text{UV}}, \Pi_c^{\text{UV}}, \Pi_\Delta^{\text{UV}}) e^{i(S_H^{\text{UV}} + S_{\text{mixed}})} \prod_{k > 0} \delta(\phi_\Delta^{\text{IR}}(t_f(k), \mathbf{k})) \rho_0[\phi_c^{\text{UV}}, \phi_\Delta^{\text{UV}}; \phi_c^{\text{IR}}, \phi_\Delta^{\text{IR}}], \quad (\text{A.7})$$

with $t_f(k) := \min[T, t_k]$. Here, $\rho_0[\phi_c, \phi_\Delta] \rightarrow \rho_0[\phi_c^{\text{UV}}, \phi_\Delta^{\text{UV}}; \phi_c^{\text{IR}}, \phi_\Delta^{\text{IR}}]$ due to the replacement of variables. The IR variables $(\phi_c^{\text{IR}}, \phi_\Delta^{\text{IR}})$ in the argument of ρ_0 contain only a zero mode. Since the time path for each UV ϕ -variable is closed in (A.7), we can calculate $i\Gamma$ perturbatively as usual by writing the connected diagrams with n external IR fields that are connected by UV propagators. The diagrammatic rules such as vertex factors and symmetry factors follow the standard rules of Scheinger-Keldysh formalism for given vertexes in $S_H^{\text{UV}} + S_{\text{mixed}}$.

Interestingly, due to the non-trivial replacement rule (A.4), the $\Pi\dot{\phi}$ terms in S_H also contribute to S_{mixed} [70, 47, 71, 72]. Referring to such contributions and the remaining terms in S_{mixed} as S_{tr} and $S_{\text{mix-int}}$, respectively, we have $S_{\text{mixed}} = S_{\text{tr}} + S_{\text{mix-int}}$ with ³

$$S_{\text{tr}} := \int_{-\infty}^T dt \int \frac{d^3k}{(2\pi)^3} \delta(t - t_k) [\phi_{\Delta}^{\text{IR}}(t, -\mathbf{k}) \Pi_c^{\text{UV}}(t, \mathbf{k}) - \Pi_{\Delta}^{\text{IR}}(-\mathbf{k}, t) \phi_c^{\text{UV}}(t, \mathbf{k})] . \quad (\text{A.9})$$

The bi-linear vertexes exist only at the UV $\rightarrow$ IR transition time t_k for given modes with $0 < k \leq k_c(T)$. Physically, these vertexes set the initial fluctuations for such modes in (A.6).

A.2.1 Integrate out short-wavelength modes

Now we calculate $i\Gamma$ perturbatively by specifying the setup more concretely. We organize the perturbation theory around the false vacuum state ρ_0 for which the expectation value of the zero mode is ϕ_{false} . We assume that the fluctuations of zero modes are tiny enough so that the couplings between zero-mode fluctuations and those of non-zero modes $k > 0$ are switched off in the past infinity $t = -\infty$. We then take the initial state of non-zero modes to be the Bunch-Davies vacuum state for a free field, where the time evolution of a free field is defined by the quadratic action expanded around the homogeneous background ϕ_{false} . These assumptions can be implemented by writing the initial quantum state as

$$\rho_0[\phi_c^{\text{UV}}, \phi_{\Delta}^{\text{UV}}; \phi_c^{\text{IR}}, \phi_{\Delta}^{\text{IR}}] = \rho_{\text{BD}}[\phi_c^{\text{UV}}, \phi_{\Delta}^{\text{UV}}; \phi_{\text{false}}] \rho_0[\phi_c^{\text{IR}} - \phi_{\text{false}}, \phi_{\Delta}^{\text{IR}}] \quad (\text{A.10})$$

³Precisely speaking, we need to perform the UV-IR splitting in the path integral with infinitesimal discrete time step for deriving S_{tr} correctly. After taking the continuum limit, we obtain (A.9); see also [72]. Furthermore, the form of $S_{\text{mix-int}}$ depends on the model. In our case, we have

$$S_{\text{mix-int}} = \int d^4x [V(\phi_+^{\text{UV}} + \phi_+^{\text{IR}}) - V(\phi_+^{\text{UV}}) - V(\phi_+^{\text{IR}}) - V(\phi_-^{\text{UV}} + \phi_-^{\text{IR}}) + V(\phi_-^{\text{UV}}) + V(\phi_-^{\text{IR}})]_{\phi_{\pm}^{\text{UV}} = \phi_c^{\text{UV}} \pm (\phi_{\Delta}^{\text{UV}}/2), \phi_{\pm}^{\text{IR}} = \phi_c^{\text{IR}} \pm (\phi_{\Delta}^{\text{IR}}/2)} . \quad (\text{A.8})$$

which is properly normalized as

$$\begin{aligned}
 1 &= \prod_{k>0} \int d\phi_c^{\text{UV}}(-\infty, \mathbf{k}) \int d\phi_\Delta^{\text{UV}}(-\infty, \mathbf{k}) \rho_{\text{BD}}[\phi_c^{\text{UV}}, \phi_\Delta^{\text{UV}}; \phi_{\text{false}}] \\
 &= \int d\phi_c^{\text{IR}}(-\infty, \mathbf{0}) \int d\phi_\Delta^{\text{IR}}(-\infty, \mathbf{0}) \rho_0[\phi_c^{\text{IR}} - \phi_{\text{false}}, \phi_\Delta^{\text{IR}}]. \quad (\text{A.11})
 \end{aligned}$$

$\rho_{\text{BD}}[\phi_c^{\text{UV}}, \phi_\Delta^{\text{UV}}; \phi_{\text{false}}]$ is the density matrix for non-zero modes. An initial state for the zero mode is given by $\rho_0[\phi_c^{\text{IR}} - \phi_{\text{false}}, \phi_\Delta^{\text{IR}}]$ for which we have

$$\langle \Psi(-\infty) | \hat{\phi}(\mathbf{k} = \mathbf{0}) | \Psi(-\infty) \rangle = \int d\phi_c^{\text{IR}} \rho_0[\phi_c^{\text{IR}} - \phi_{\text{false}}, \phi_\Delta^{\text{IR}}] \phi_c^{\text{IR}} = \phi_{\text{false}} (2\pi)^3 \delta(\mathbf{0}). \quad (\text{A.12})$$

Combining this with $\langle \Psi(-\infty) | \hat{\phi}(\mathbf{k}) | \Psi(-\infty) \rangle = 0$ for non-zero modes $k > 0$, we have

$$\langle \Psi(-\infty) | \hat{\phi}(\mathbf{k}) | \Psi(-\infty) \rangle = \phi_{\text{false}} (2\pi)^3 \delta(\mathbf{k}). \quad (\text{A.13})$$

Hence, the VEV of the IR field $\hat{\phi}^{\text{IR}}(t, \mathbf{x})$ in the past infinity $t = -\infty$ is given by ϕ_{false} .

The interaction vertexes for calculating $i\Gamma$ are the bi-linear vertexes (A.9) and non-linear vertexes coming from interacting potential $V(\phi)$. We assume that the potential is sufficiently flat under the region we are interested in. It is then natural to separate $i\Gamma$ into the leading order pieces and higher-order terms in the coupling constant as follows,

$$\begin{aligned}
 e^{i\Gamma} &= \rho_0[\phi_c^{\text{IR}} - \phi_{\text{false}}, \phi_\Delta^{\text{IR}}] e^{i\Gamma_{\text{LO}}^{k>0}} e^{-i \int d^4x \delta H_{\text{higher}}}, \\
 e^{i\Gamma_{\text{LO}}^{k>0}} &:= \int \mathcal{D}(\phi_c^{\text{UV}}, \phi_\Delta^{\text{UV}}, \Pi_c^{\text{UV}}, \Pi_\Delta^{\text{UV}}) e^{iS_{H,\text{free}}^{\text{UV}} + S_{\text{tr}}} \prod_{k>0} \delta(\phi_\Delta^{\text{IR}}(t_f(k), \mathbf{k})) \rho_{\text{BD}}[\phi_c^{\text{UV}}, \phi_\Delta^{\text{UV}}; \phi_{\text{false}}]. \quad (\text{A.14})
 \end{aligned}$$

Here, $S_{H,\text{free}}^{\text{UV}}[\phi_c^{\text{UV}}, \phi_\Delta^{\text{UV}}, \Pi_c^{\text{UV}}, \Pi_\Delta^{\text{UV}}; \phi_{\text{false}}]$ is the free part of the Hamiltonian action defined around the false vacuum as explained above. Higher-order corrections to Γ in the coupling constant are represented by the term δH_{higher} which can be evaluated perturbatively. We have $\delta H_{\text{higher}} = 0$ at the leading order.

To calculate $i\Gamma_{\text{LO}}^{k>0}$ by writing down the connected diagrams with external IR fields, we only need to consider the bi-linear vertexes S_{tr} given in (A.9).

72 Stochastic approach from the first principle

The number of diagrams are only four, and $i\Gamma_{\text{LO}}^{k>0}$ can be calculated as [71, 72]

$$\begin{aligned} i\Gamma_{\text{LO}}^{k>0} &= \frac{-1}{2} \int dt \int dt' \int \frac{d^3k}{(2\pi)^3} X_\alpha(t, \mathbf{k}) g^{\alpha\beta}(t, t', k) X_\beta(t', -\mathbf{k}) \\ &= \frac{-1}{2} \int d^4x \int d^4x' X_\alpha(x) G^{\alpha\beta}(x, x') X_\beta(x'). \end{aligned} \quad (\text{A.15})$$

Here, the Greek indices $\alpha, \beta = (\phi, \Pi)$ label the IR- Δ fields: $(X_\phi, X_\Pi) = (\Pi_\Delta^{\text{IR}}, -\phi_\Delta^{\text{IR}})$. Substituting Eq.(A.14) into (A.6), we find

$$\begin{aligned} Z[J^{\text{IR}}(T)] &= \int \mathcal{D}(\phi_c^{\text{IR}}, \phi_\Delta^{\text{IR}}, \Pi_c^{\text{IR}}, \Pi_\Delta^{\text{IR}}) e^{i\mathbf{J}^{\text{IR}} \cdot \phi_c^{\text{IR}}} \rho_0[\phi_c^{\text{IR}} - \phi_{\text{false}}, \phi_\Delta^{\text{IR}}] e^{i(S_H^{\text{IR}} + \Gamma_{\text{LO}}^{k>0})} e^{-i \int d^4x \delta H_{\text{higher}}} \\ &\quad \times \prod_{k \leq k_c(T)} \delta(\phi_\Delta^{\text{IR}}(T, \mathbf{k})) \prod_{0 < k \leq k_c(T)} \delta(\phi_c^{\text{IR}}(t_k, \mathbf{k})). \end{aligned} \quad (\text{A.16})$$

Now we introduce $i\Gamma_{k=0}$ as

$$e^{i\Gamma_{k=0}[\phi_\Delta^{\text{IR}}, \Pi_\Delta^{\text{IR}}]} := \int d\phi_c^{\text{IR}} \rho_0[\phi_c^{\text{IR}} - \phi_{\text{false}}, \phi_\Delta^{\text{IR}}] e^{-i\Pi_\Delta^{\text{IR}}(\phi_c^{\text{IR}} - \phi_{\text{false}})(2\pi)^3\delta(\mathbf{0})}. \quad (\text{A.17})$$

By definition, we have

$$\begin{aligned} &\int d\phi_c^{\text{IR}} \rho_0[\phi_c^{\text{IR}} - \phi_{\text{false}}, \phi_\Delta^{\text{IR}}] e^{-i\Pi_\Delta^{\text{IR}}(\phi_c^{\text{IR}} - \phi_{\text{false}})(2\pi)^3\delta(\mathbf{0})} \\ &= \int d\phi_c^{\text{IR}} e^{-i\Pi_\Delta^{\text{IR}}\phi_c^{\text{IR}}} e^{i\Gamma_{k=0}[\phi_\Delta^{\text{IR}}, \Pi_\Delta^{\text{IR}}]} \delta(\phi_c^{\text{IR}} - \phi_{\text{false}}(2\pi)^3\delta(\mathbf{0})). \end{aligned} \quad (\text{A.18})$$

Substituting (A.18) into (A.16), we find

$$\begin{aligned} Z[J^{\text{IR}}(T)] &= \int \mathcal{D}(\phi_c^{\text{IR}}, \phi_\Delta^{\text{IR}}, \Pi_c^{\text{IR}}, \Pi_\Delta^{\text{IR}}) e^{i\mathbf{J}^{\text{IR}} \cdot \phi_c^{\text{IR}}} e^{i(S_H^{\text{IR}} + \Gamma_{\text{LO}}^{k>0} + \Gamma_{k=0})} e^{-i \int d^4x \delta H_{\text{higher}}} \\ &\quad \times \prod_{k \leq k_c(T)} \delta(\phi_\Delta^{\text{IR}}(T, \mathbf{k})) \delta(\phi_c^{\text{IR}}(t_k, \mathbf{k}) - \phi_{\text{false}}(2\pi)^3\delta(\mathbf{k})), \end{aligned} \quad (\text{A.19})$$

with defining $t_k|_{k=0} = -\infty$.

A.2.2 Long-wavelength sector

For later convenience, let us focus on the term $\int dt \Pi_c^{\text{IR}}(t, \mathbf{k}) \dot{\phi}_\Delta^{\text{IR}}(t, -\mathbf{k})$ in S_H^{IR} . We use the following trick to perform the integration by parts:

$$\begin{aligned}
 & \delta(\phi_\Delta^{\text{IR}}(T, \mathbf{k})) e^{i \int_{t_k}^T dt \Pi_c^{\text{IR}}(t, \mathbf{k}) \dot{\phi}_\Delta^{\text{IR}}(t, -\mathbf{k})} \\
 &= \delta(\phi_\Delta^{\text{IR}}(T, \mathbf{k})) \int d\Pi_c^{\text{IR}}(t_k - 0^+, \mathbf{k}) e^{i \int_{t_k - 0^+}^T dt \Pi_c^{\text{IR}}(t, \mathbf{k}) \dot{\phi}_\Delta^{\text{IR}}(t, -\mathbf{k})} \delta(\Pi_c^{\text{IR}}(t_k - 0^+, \mathbf{k})) \\
 &= \delta(\phi_\Delta^{\text{IR}}(T, \mathbf{k})) \int d\Pi_c^{\text{IR}}(t_k - 0^+, \mathbf{k}) e^{-i \int_{t_k - 0^+}^T dt \dot{\Pi}_c^{\text{IR}}(t, \mathbf{k}) \phi_\Delta^{\text{IR}}(t, -\mathbf{k})} \delta(\Pi_c^{\text{IR}}(t_k - 0^+, \mathbf{k})).
 \end{aligned} \tag{A.20}$$

Thanks to this trick, (A.19) can be written as

$$\begin{aligned}
 Z[J^{\text{IR}}(T)] &= \int \mathcal{D}(\phi_c^{\text{IR}}, \phi_\Delta^{\text{IR}}, \Pi_c^{\text{IR}}, \Pi_\Delta^{\text{IR}}) \int \tilde{\mathcal{D}}\Pi_c^{\text{IR}} e^{i \mathbf{J}^{\text{IR}} \cdot \phi_c^{\text{IR}}} e^{i(S_{H,(d)}^{\text{IR}} + S_{H,(s)}^{\text{IR}} + \Gamma_{\text{LO}}^{k>0} + \Gamma_{k=0})} e^{-i \int d^4x \delta H_{\text{higher}}} \\
 &\quad \times \prod_{k \leq k_c(T)} \delta(\phi_\Delta^{\text{IR}}(T, \mathbf{k})) \delta(\phi_c^{\text{IR}}(t_k, \mathbf{k}) - \phi_{\text{false}}(2\pi)^3 \delta(\mathbf{k})) \delta(\Pi_c^{\text{IR}}(t_k - 0^+, \mathbf{k})),
 \end{aligned} \tag{A.21}$$

where, $\tilde{\mathcal{D}}\Pi_c^{\text{IR}} := \prod_{k \leq k_c(T)} d\Pi_c^{\text{IR}}(t_k - 0^+, \mathbf{k})$. Below, we simply omit this integration measure just for simplifying the notation. We also decomposed S_H^{IR} as

$S_H^{\text{IR}} = S_{H,(d)}^{\text{IR}} + S_{H,(s)}^{\text{IR}}$ up to boundary terms which vanish thanks to the trick (A.20) as

$$S_{H,(d)}^{\text{IR}} = \int d^4x \left[\Pi_\Delta^{\text{IR}} \left(\dot{\phi}_c^{\text{IR}} - a^{-3} \Pi_c^{\text{IR}} \right) - \phi_\Delta^{\text{IR}} \left(\dot{\Pi}_c^{\text{IR}} - a(t) \nabla^2 \phi_c^{\text{IR}} + a^3(t) V'(\phi_c^{\text{IR}}) \right) \right], \tag{A.22}$$

$$S_{H,(s)}^{\text{IR}} = - \int d^4x a^3(t) \left[V(\phi_c^{\text{IR}} + (\phi_\Delta^{\text{IR}}/2)) - V(\phi_c^{\text{IR}} - (\phi_\Delta^{\text{IR}}/2)) - V'(\phi_c^{\text{IR}}) \phi_\Delta^{\text{IR}} \right]. \tag{A.23}$$

Here, the space-time argument of IR variables are omitted. We introduced the IR variables and their time derivatives as

$$X^{\text{IR}}(t, \mathbf{x}) := \int \frac{d^3k}{(2\pi)^3} \theta(k_c(t) - k) X^{\text{IR}}(t, \mathbf{k}) e^{i\mathbf{k} \cdot \mathbf{x}}, \tag{A.24}$$

$$\dot{X}^{\text{IR}}(t, \mathbf{x}) := \int \frac{d^3k}{(2\pi)^3} \theta(k_c(t) - k) \dot{X}^{\text{IR}}(t, \mathbf{k}) e^{i\mathbf{k} \cdot \mathbf{x}}, \tag{A.25}$$

where X^{IR} represents the IR variables: $X^{\text{IR}} = (\phi_c^{\text{IR}}, \phi_\Delta^{\text{IR}}, \Pi_c^{\text{IR}}, \Pi_\Delta^{\text{IR}})$. Precisely speaking, for $X^{\text{IR}} = \Pi_c^{\text{IR}}$, we should replace $k_c(t)$ by $k_c(t + 0^+)$ so that we have $k_c(t) = k$ at $t = t_k - 0^+$. One may concern about the subtlety of the time derivatives $\dot{\phi}_c^{\text{IR}}(t, \mathbf{x})$ and $\dot{\Pi}_c^{\text{IR}}(t, \mathbf{x})$ in (A.22): We in general have $\dot{X}^{\text{IR}}(t, \mathbf{x}) \neq \frac{\partial}{\partial t}(X^{\text{IR}}(t, \mathbf{x}))$, where the LHS is defined by (A.25), due to the presence of time-dependent step function. However, there is no subtlety thanks to the initial conditions for all IR modes $k \leq k_c(T)$ of the form

$$\phi_c^{\text{IR}}(t_k, \mathbf{k}) = \phi_{\text{false}}(2\pi)^3 \delta(\mathbf{k}), \quad \Pi_c^{\text{IR}}(t_k - 0^+, \mathbf{k}) = 0, \quad (\text{A.26})$$

which are imposed by the delta functions in the final line in (A.21). The corresponding conditions in the real space for all $\mathbf{x}$ are

$$\dot{X}^{\text{IR}}(t, \mathbf{x}) = \partial_t(X^{\text{IR}}(t, \mathbf{x})) \quad \text{for} \quad X^{\text{IR}} = (\phi_c^{\text{IR}}, \Pi_c^{\text{IR}}), \quad (\text{A.27a})$$

$$\phi_c^{\text{IR}}(-\infty, \mathbf{x}) = \phi_{\text{false}}, \quad \Pi_c^{\text{IR}}(-\infty, \mathbf{x}) = 0. \quad (\text{A.27b})$$

To implement these conditions in the path integral, we define $\delta[\mathcal{C}_{\text{ini}}]$ by (3.40b) with

$$\delta[\mathcal{C}_{\text{detail}}] := \prod_{-\infty \leq t \leq T} \prod_{X^{\text{IR}} = (\phi_c^{\text{IR}}, \Pi_c^{\text{IR}})} \prod_{\mathbf{x}} \delta\left(\dot{X}^{\text{IR}}(t, \mathbf{x}) - \partial_t(X^{\text{IR}}(t, \mathbf{x}))\right). \quad (\text{A.28})$$

We find from eqs. (3.37) and (A.21) that the transition probability can be written in terms of path integral as

$$p(\{\phi^{\text{IR}}(T, \mathbf{x})\}_{T, \mathbf{x} \in \mathcal{D}}) = \int \mathcal{D}(\phi_c^{\text{IR}}, \phi_\Delta^{\text{IR}}, \Pi_c^{\text{IR}}, \Pi_\Delta^{\text{IR}}) e^{i\mathcal{S}} e^{i\Gamma_{k=0}} \delta[\mathcal{C}_{\text{fin}}] \delta[\mathcal{C}_{\text{ini}}], \quad (\text{A.29})$$

where $\delta[\mathcal{C}_{\text{fin}}]$ is defined in (3.40a), and the term $\mathcal{S}$ in the exponent is given by

$$\mathcal{S} = (S_{H,(d)}^{\text{IR}} + S_{H,(s)}^{\text{IR}} + \Gamma_{\text{LO}}^{k>0}) - \int d^4x \delta H_{\text{higher}} \quad (\text{A.30})$$

which agrees with the definition of $\mathcal{S}$ given in (3.41). When fluctuations of zero mode are negligible, we can set $\Gamma_{k=0} = 0$. This is the setup adopted in the main text. Then, (A.29) coincides with (3.39).

A.3 Stochastic interpretation of the tunneling

It is useful to see how the stochastic equations emerge as a consequence of performing the path integral over IR variables non-perturbatively. This leads to the stochastic interpretation of the tunneling probability. A final result of this section is (A.40).

For simplicity, let us ignore higher-order corrections δH_{higher} for a while. We start with introducing the auxiliary fields (ξ_ϕ, ξ_Π) as

$$e^{i(\Gamma_{\text{LO}}^{k>0} + \Gamma_{k=0} + S_{H,(s)}^{\text{IR}})} = \int \mathcal{D}(\xi_\phi, \xi_\Pi) P[\xi_\phi, \xi_\Pi] e^{i \int d^4x (\xi_\Pi(x) \phi_\Delta^{\text{IR}}(x) - \xi_\phi(x) \Pi_\Delta^{\text{IR}}(x))}. \quad (\text{A.31})$$

We substitute (A.31) into (A.21). Then the exponent becomes linear in $\phi_\Delta^{\text{IR}}(t, \mathbf{k})$ or $\Pi_\Delta^{\text{IR}}(t, \mathbf{k})$ with $t_0 \leq t < T$:

$$\begin{aligned} Z[J^{\text{IR}}(T)] &\simeq \int \mathcal{D}(\phi_c^{\text{IR}}, \phi_\Delta^{\text{IR}}, \Pi_c^{\text{IR}}, \Pi_\Delta^{\text{IR}}) \int \tilde{\mathcal{D}}\Pi_c^{\text{IR}} e^{i\mathbf{J}^{\text{IR}} \cdot \phi_c^{\text{IR}}} \int \mathcal{D}(\xi_\phi, \xi_\Pi) P[\xi_\phi, \xi_\Pi] \\ &\times \exp \left\{ i \int_{-\infty}^T dt \int \frac{d^3k}{(2\pi)^3} [\Pi_\Delta^{\text{IR}}(t, -\mathbf{k}) \mathcal{L}_\phi(t, \mathbf{k}) - \phi_\Delta^{\text{IR}}(t, -\mathbf{k}) \mathcal{L}_\Pi(t, \mathbf{k})] \Theta(k_c(t) - k) \right\} \\ &\times \prod_{k \leq k_c(T)} \delta(\phi_\Delta^{\text{IR}}(T, \mathbf{k})) \delta(\phi_c^{\text{IR}}(t_k, \mathbf{k}) - \phi_{\text{false}}(2\pi)^3 \delta(\mathbf{k})) \delta(\Pi_c^{\text{IR}}(t_k - 0^+, \mathbf{k})), \end{aligned} \quad (\text{A.32})$$

where

$$\mathcal{L}_\phi(t, \mathbf{k}) := \dot{\phi}_c^{\text{IR}}(t, \mathbf{k}) - a^{-3}(t) \Pi_c^{\text{IR}}(t, \mathbf{k}) - \xi_\phi(t, \mathbf{k}), \quad (\text{A.33a})$$

$$\mathcal{L}_\Pi(t, \mathbf{k}) := \dot{\Pi}_c^{\text{IR}}(t, \mathbf{k}) + a(t) k^2 \phi_c^{\text{IR}}(t, \mathbf{k}) + a^3(t) V'(\phi_c^{\text{IR}})(t, \mathbf{k}) - \xi_\Pi(t, \mathbf{k}). \quad (\text{A.33b})$$

Consequently, we can perform the path integral $\int \mathcal{D}(\phi_\Delta^{\text{IR}}, \Pi_\Delta^{\text{IR}})$, leading to

$$Z[J^{\text{IR}}(T)] \simeq \prod_{k \leq k_c(T)} \int d\phi_c^{\text{IR}}(T, \mathbf{k}) e^{i\mathbf{J}^{\text{IR}} \cdot \phi_c^{\text{IR}}} \int \mathcal{D}(\xi_\phi, \xi_\Pi) P[\xi_\phi, \xi_\Pi] \Big|_{\mathcal{L}_\phi = \mathcal{L}_\Pi = 0 \text{ \& (A.26)}}. \quad (\text{A.34})$$

Now we regard auxiliary fields (ξ_ϕ, ξ_Π) as noise variables whose correlations are given by

$$\begin{aligned} \langle \xi_\alpha(t, \mathbf{k}) \xi_\beta(t', \mathbf{k}') \rangle &:= \int \mathcal{D}(\xi_\phi, \xi_\Pi) P[\xi_\phi, \xi_\Pi] \xi_\alpha(t, \mathbf{k}) \xi_\beta(t', \mathbf{k}') \\ &= \left[\frac{i\delta}{\delta X_\alpha(t, \mathbf{k})} \frac{i\delta}{\delta X_\beta(t', \mathbf{k}')} \right] e^{i(\Gamma_{\text{LO}}^{k>0} + \Gamma_{k=0} + S_{H,(s)}^{\text{IR}})} \Big|_{\phi_\Delta^{\text{IR}} = \Pi_\Delta^{\text{IR}} = 0}, \end{aligned} \quad (\text{A.35})$$

where $\alpha, \beta = (\phi, \Pi)$ and $(X_\phi, X_\Pi) = (\Pi_\Delta^{\text{IR}}, -\phi_\Delta^{\text{IR}})$ following the notation in (A.15). We also have the corresponding expression in the momentum space. If we ignore $S_{H,(s)}$ and $i\Gamma_{k=0}$, we have

$$\langle \xi_\alpha(t, \mathbf{k}) \xi_\beta(t', \mathbf{k}') \rangle = (2\pi)^3 \delta(\mathbf{k} + \mathbf{k}') \text{Re} [g^{\alpha\beta}(t)] \delta(t - t') \delta(t - t_k). \quad (\text{A.36})$$

Eq.(A.34) shows that the effective dynamics of IR fields are described by the set of Langevin equations $\mathcal{L}_\phi = \mathcal{L}_\Pi = 0$ with the initial condition (A.26) and the noise correlations (A.36).

In the real space, the dynamics is described by the set of Langevin equations,

$$\dot{\phi}_c^{\text{IR}}(x) = a^{-3}(t) \Pi_c^{\text{IR}}(x) + \xi_\phi(x), \quad (\text{A.37a})$$

$$\dot{\Pi}_c^{\text{IR}}(x) = a(t) \nabla^2 \phi_c^{\text{IR}}(x) - a^3 V'(\phi_c^{\text{IR}}(x)) + \xi_\Pi(x), \quad (\text{A.37b})$$

with the initial conditions (A.27) and the noise correlations

$$\langle \xi_\alpha(x) \xi_\beta(y) \rangle = G^{\alpha\beta}(x, y). \quad (\text{A.38})$$

Eq.(A.34) takes the following form in the real space:

$$Z[J^{\text{IR}}(T)] \simeq \prod_{\mathbf{x}} \int d\phi_c^{\text{IR}}(T, \mathbf{x}) e^{iJ^{\text{IR}} \cdot \phi_c^{\text{IR}}} \int \mathcal{D}(\xi_\phi, \xi_\Pi) P[\xi_\phi, \xi_\Pi] \Big|_{(\text{A.27}) \ \& \ (\text{A.37})}. \quad (\text{A.39})$$

Substituting (A.39) into (3.37), we have

$$\begin{aligned} p(\{\phi^{\text{IR}}(T, \mathbf{x})\}_{T, \mathbf{x} \in \mathcal{D}}) &\simeq \prod_{\mathbf{y}} \int d\phi_c^{\text{IR}}(T, \mathbf{y}) \prod_{\mathbf{x} \in \mathcal{D}} \delta(\phi_c^{\text{IR}}(T, \mathbf{x}) - \phi^{\text{IR}}(T, \mathbf{x})) \\ &\times \int \mathcal{D}(\xi_\phi, \xi_\Pi) P[\xi_\phi, \xi_\Pi] \Big|_{(\text{A.27}) \ \& \ (\text{A.37})}. \end{aligned} \quad (\text{A.40})$$

This expression provides a stochastic interpretation of the transition probability; the dynamics of ϕ_c^{IR} is described by the set of Langevin equations and the transition probability is simply understood as the probability to realize the field configuration $\phi_c^{\text{IR}}(T, \mathbf{x}) = \phi^{\text{IR}}(T, \mathbf{x})$ in the domain $\mathbf{x} \in \mathcal{D}$ at the final time T starting from the initial configuration $\phi_c^{\text{IR}}(-\infty, \mathbf{x}) = \phi_{\text{false}}$. In particular, the set of Langevin equations is given by (3.8) with the noise correlations (3.15) when we ignore the terms δH_{higher} , $S_{H,(s)}^{\text{IR}}$, and $\Gamma_{k=0}$. Note that such terms simply correct the Langevin equations⁴ and do not invalidate the above stochastic interpretation.

⁴For instance, leading-order corrections to the stochastic dynamics at the super-horizon scales are calculated in [71].

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