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ON A CLASSIFICATION OF IRREDUCIBLE PERIODIC DIFFEOMORPHISMS ON SURFACES WHICH COMMUTE WITH CERTAIN INVOLUTION

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Abstract

Ishizaka classified up to conjugacy hyperelliptic periodic automorphisms of a surface. Here, an involution I on a surface Σ_g is hyperelliptic if and only if $\Sigma_g/\langle I \rangle$ is homeomorphic to S^2 . In this article, we give a classification, up to conjugacy, of irreducible periodic automorphisms of a surface Σ_g which commute with an involution ι such that $\Sigma_g/\langle \iota \rangle$ is homeomorphic to T^2 .

1. Introduction

Let Σ_g of genus $g > 1$ be a connected oriented closed surface. The *mapping class group* $\text{Mod}(\Sigma_g)$ of Σ_g is the group of isotopy classes of orientation-preserving diffeomorphisms of Σ_g . We call elements of $\text{Mod}(\Sigma_g)$ *automorphisms* of Σ_g . An involution on Σ_g is called *hyperelliptic* if it fixes $2g + 2$ points. It is well known that such an involution is unique up to conjugacy and has the maximum number of fixed points among involutions on Σ_g . An automorphism of Σ_g which commutes with a hyperelliptic involution is called *hyperelliptic*. Ishizaka [12, 13] classified hyperelliptic periodic automorphisms up to conjugacy and gave right-handed Dehn twist presentations based on resolutions of singularities of families of Riemann surfaces. An involution I of Σ_g is hyperelliptic if and only if the quotient space $\Sigma_g/\langle I \rangle$ is homeomorphic to a sphere.

In this paper, we will study periodic automorphisms of Σ_g which commute with an involution ι_g whose quotient space $\Sigma_g/\langle \iota_g \rangle$ is homeomorphic to a torus (see Figure 1). There is a significant difference in the classification from the

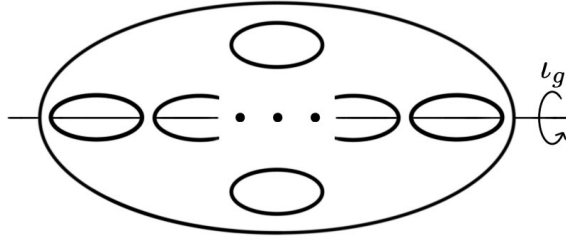


FIGURE 1. An involution ι_g such that $\Sigma_g/\langle\iota_g\rangle$ is homeomorphic to T^2

hyperelliptic case considered in [13]. Recall that an automorphism φ of Σ_g is called *reducible* if there is a nonempty set $\{C_1, C_2, \dots, C_n\}$ of isotopy classes of essential simple closed curves in Σ_g so that the geometrical intersection number of C_i and C_j is equal to 0 for all $i, j \in \{1, 2, \dots, n\}$ and so that $\{\varphi(C_i) \mid i = 1, 2, \dots, n\} = \{C_1, C_2, \dots, C_n\}$, and *irreducible* if otherwise [5, Chapter 13, Section 2]. Such $\{C_1, C_2, \dots, C_n\}$ is called an invariant reduction system.

By the classification in [13], every hyperelliptic periodic automorphism of Σ_g is the power of an irreducible periodic automorphism. In turn, as we will see below, this is not the case for periodic automorphisms which commute with an involution ι_g such that $\Sigma_g/\langle\iota_g\rangle$ is homeomorphic to T^2 (see Examples 1.1, 1.2 and Proposition 5.1). In this article, we focus on the classification in the irreducible case. Irreducible periodic automorphisms of Σ_g have been studied from various viewpoints (see, e.g., [9, 11, 7, 4]).

By Kerckhoff's theorem [16, Theorem 5], it is known that any finite subgroup of $\text{Mod}(\Sigma_g)$ can be lifted to the group of orientation-preserving diffeomorphisms of Σ_g . Therefore, when considering periodic automorphism of Σ_g from now on, we will fix one representative diffeomorphism for discussion. In particular, note that thereafter, we will only consider the case where periodic diffeomorphisms are orientation-preserving.

To state the result, let us present an example of an irreducible periodic diffeomorphism $h_{n,p}$ of an oriented closed surface parametrized by a positive integer $n \geq 3$ and $p \in \{1, \dots, n-1\}$, which was considered also in [19, Section 2], [3, Chapter 2], [20] and [21]. Consider a regular $2n$ -gon. Let α_i and β_i ($i = 0, \dots, n-1$) be the edges of the $2n$ -gon as shown in Figure 2. For each $i = 0, 1, \dots, n-1$, equip α_i the clockwise orientation and β_i the counter-clockwise orientation, respectively. By identifying α_i with β_j so that the orientations are compatible for

every pair of i and $j \in \{0, \dots, n-1\}$ with $j-i \equiv p \pmod n$, we obtain a surface Σ of genus $g = \frac{n - \gcd(n, p) - \gcd(n, p+1) + 1}{2}$ (see Proposition 3.1). Let $h_{n,p}$ denote the diffeomorphism of Σ induced from the clockwise $\frac{2\pi}{n}$ -rotation of the $2n$ -gon. To simplify the figures, both α_0 and β_p will be denoted by α , and we omit the symbols of other edges in the rest of the paper.

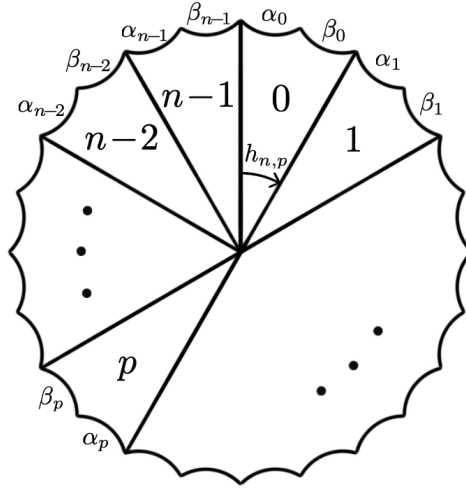


FIGURE 2. A diffeomorphism $h_{n,p}$

Example 1.1([20, Section 1]). We can construct an example of a reducible periodic diffeomorphism on Σ_{2g+1} which commutes with an involution ι_{2g+1} from $h_{4g,2g}$ and $h_{4g,2g}^{4g-1}$ by a connected sum (Figure 5): Remove small disk neighborhoods of the fixed points of $h_{4g,2g}$ and $h_{4g,2g}^{4g-1}$. Let us denote the circle boundaries $\partial_{1,1}, \partial_{1,2}, \partial_{2,1}$ and $\partial_{2,2}$ as shown in Figures 3 and 4. For $i = 1$ and 2, we glue $\partial_{1,i}$ to $\partial_{2,i}$ (see Figure 5). Then, we get a periodic diffeomorphism F_1 on Σ_{2g+1} of period $4g$. The restriction of F_1^{2g} to each of two surfaces coincides with $h_{4g,2g}^{2g}$ and $h_{4g,2g}^{2g(4g-1)}$, respectively. By Theorem 1 in [21], the diffeomorphism $h_{4g,2g}$ is hyperelliptic periodic diffeomorphism and we have $\langle h_{4g,2g}, I \rangle = \langle h_{4g,2g} \rangle$ (This group is exactly the group G_2 of Theorem 1 in [21]). Since both $h_{4g,2g}^{2g}$ and $h_{4g,2g}^{2g(4g-1)}$ are hyperelliptic involutions, we can see that F_1^{2g} is an involution whose quotient space is a torus. Since $F_1^{2g} \circ F_1 = F_1^{2g+1} = F_1 \circ F_1^{2g}$, the map F_1^{2g} commutes with F_1 .

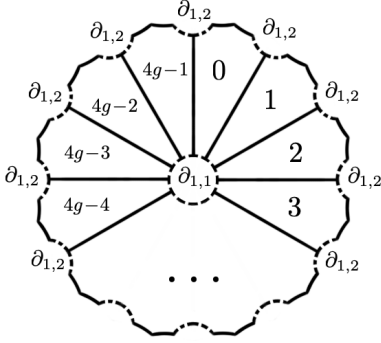


FIGURE 3. Boundaries $\partial_{1,1}$ and $\partial_{1,2}$

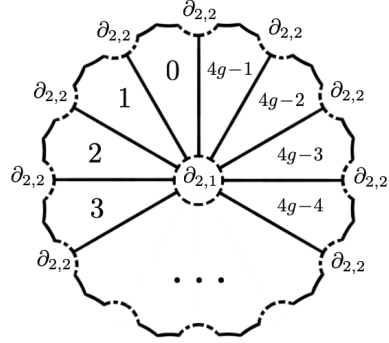


FIGURE 4. Boundaries $\partial_{2,1}$ and $\partial_{2,2}$

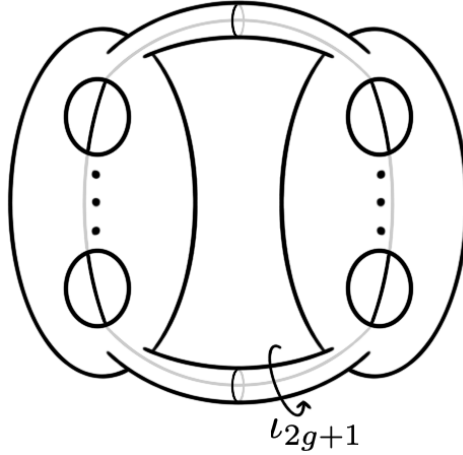


FIGURE 5. Glue the boundaries of the complements of disk neighborhoods of the fixed points of $(\Sigma_g, h_{4g,2g})$ and $(\Sigma_g, h_{4g,2g}^{4g-1})$ to obtain a diffeomorphism that commutes with an involution ι_{2g+1}

The genus of the surface is odd in the previous example. In the next, we present an example with an even genus.

Example 1.2. Remove small disk neighborhoods of the fixed points of $h_{2g+1,1}$ and $h_{2g+1,1}^{2g}$. We call the circle boundaries $\partial_{1,1}, \partial_{1,2}, \partial_{1,3}, \partial_{2,1}, \partial_{2,2}$ and $\partial_{2,3}$ as shown in Figures 6 and 7. Then, glue the circle boundary $\partial_{1,i}$ to $\partial_{2,i}$ for $i = 1, 2$, and 3 (see Figure 8). Then we get a periodic diffeomorphism F_2 on Σ_{2g+2} of

period $2g + 1$ (see Lemma A.1). Since the diffeomorphisms $h_{2g+1,1}$ and $h_{2g+1,1}^{2g}$ are hyperelliptic (see Appendix A) and the intersection of $\text{Fix}(h_{2g+1,1})$ and $\text{Fix}(I)$ consists of a single point (see, Proposition A.2), we can see that hyperelliptic involutions on two surfaces induce an involution on the connected sum whose quotient space is T^2 . The periodic diffeomorphism F_2 commutes with the involution as shown in Figure 8.

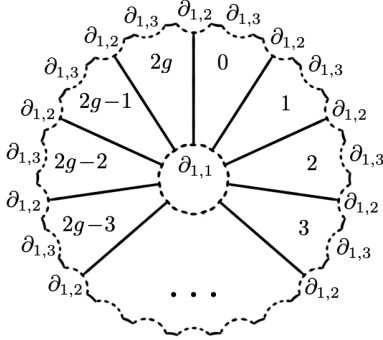


FIGURE 6. Boundaries $\partial_{1,1}$, $\partial_{1,2}$ and $\partial_{1,3}$

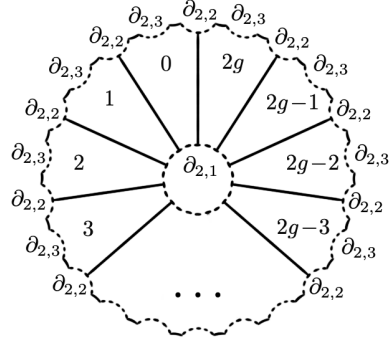


FIGURE 7. Boundaries $\partial_{2,1}$, $\partial_{2,2}$ and $\partial_{2,3}$

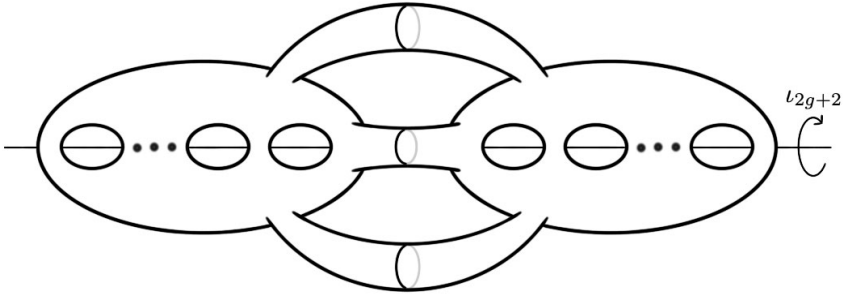


FIGURE 8. As in Example 1.2, glue the boundaries of the complements of disk neighborhoods of the fixed points of $(\Sigma_g, h_{2g+1,1})$ and $(\Sigma_g, h_{2g+1,1}^{2g})$

Now, let $\text{Diff}_+(\Sigma_g)$ denote the orientation-preserving diffeomorphism group of Σ_g . Let us recall the notion of reducibility and irreducibility of a subgroup of the diffeomorphism group of surfaces [6] in the case where the surface is closed.

Recall that a partition of a closed surface is a set of mutually disjoint simple closed curves such that no two are freely homotopic and none of which is homotopic to a point. A subgroup G in $\text{Diff}_+(\Sigma_g)$ is defined to be reducible if it has an invariant partition and is irreducible otherwise.

Let us explain the relation between this reducibility and the reducibility of periodic automorphism used above. The reducibility of a subgroup of $\text{Mod}(\Sigma_g)$ is defined by the existence of an invariant reduction system on Σ_g as in the case of periodic automorphisms.(see, [14]) This definition of the reducibility of subgroups of $\text{Mod}(\Sigma_g)$ naturally generalizes that of periodic automorphisms. Indeed, by definition, a periodic automorphism φ is reducible if and only if the finite group generated by φ is reducible. The definition of reducibility of finite group actions on Σ_g is compatible with that of finite subgroups of $\text{Mod}(\Sigma_g)$. See Remark 2.9 for a short argument that shows the equivalence of reducibility of a finite subgroup G of $\text{Diff}_+(\Sigma_g)$ and the reducibility of the image of G under the projection $\text{Diff}_+(\Sigma_g) \rightarrow \text{Mod}(\Sigma_g)$. In the following, we will consider irreducible finite subgroups of $\text{Diff}_+(\Sigma_g)$.

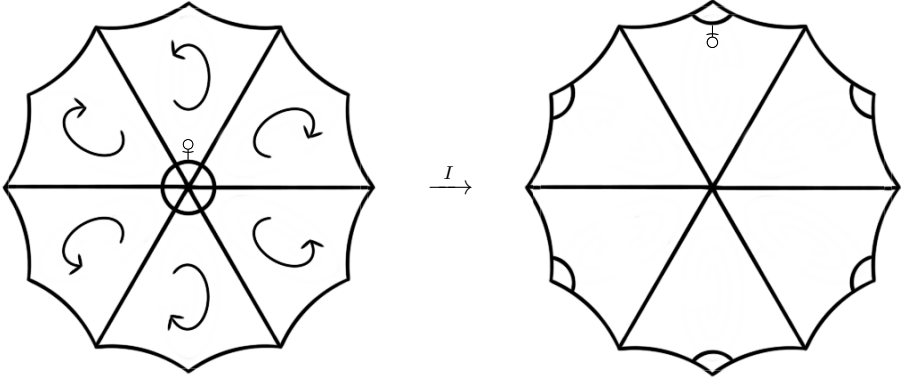
Let us state the main result of this article. In this result, we classify irreducible periodic diffeomorphisms which commute with ι_g .

Theorem 1.3. *Let I and ι_g be a hyperelliptic involution and an involution of Σ_g such that $\Sigma_g/\langle\iota_g\rangle$ is homeomorphic to a torus (Figure 1), respectively. For any periodic diffeomorphism f of Σ_g which commutes with an involution ι_g , if the subgroup $G = \langle f, \iota_g \rangle$ of $\text{Diff}_+(\Sigma_g)$ is irreducible, then G is conjugate in $\text{Diff}_+(\Sigma_g)$ to a subgroup of one of the followings:*

- (1) $\langle h_{6,1}, I \rangle$ ($g = 2$ and see Figures 9 and 10),
- (2) $\langle h_{8,1} \rangle$ ($g = 3$ and see Figure 10),
- (3) $\langle h_{8,5} \rangle$ ($g = 3$ and see Figure 10),
- (4) $\langle h_{12,2} \rangle$ ($g = 4$ and see Figure 10),
- (5) $\langle h_{12,3} \rangle$ ($g = 3$ and see Figure 10).

It is remarkable that there are only finitely many cases in the classification of Theorem 1.3. It contrasts with the classification of hyperelliptic periodic classes, where we have three infinite families of conjugacy classes.

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FIGURE 9. The action of I on $(\Sigma_2, h_{6,1})$

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2. Periodic diffeomorphisms and their total valencies

We first recall the classification of the conjugacy classes of periodic diffeomorphisms on Σ_g in terms of the total valency introduced in [1]. Let $f \in \text{Diff}_+(\Sigma_g)$ be a periodic diffeomorphism of order n and C the cyclic subgroup of $\text{Diff}_+(\Sigma_g)$ generated by f . A C -orbit $Cx = \{f^i(x) \mid 0 \leq i \leq n-1\}$ is called *multiple* if $|Cx| < n$ and is called *free* if $|Cx| = n$. The points in multiple orbits are *multiple points* of f . Let M_f be the set of all multiple points of f and $p : \Sigma_g \rightarrow \Sigma_g/C$ the canonical projection. We call $p(M_f)$ the *branch locus* of p and its elements the *branched points* of p . We say that p is *branched* at the points in $p(M_f)$. For a subgroup G of $\text{Diff}_+(\Sigma_g)$ and $x \in \Sigma_g$, $\text{Stab}_G(x) = \{f \in G \mid f(x) = x\}$ is called the *stabilizer subgroup* of G with respect to x . For $x \in \Sigma_g$, the minimum positive integer ℓ such that $f^\ell(x) = x$ is called the *period* of Cx . Then, we have $\text{Stab}_C(x) = \langle f^\ell \rangle$. For a multiple point $x \in M_f$ with period ℓ , the integer $\lambda = \frac{n}{\ell}$ is called the *branching index* of the branched point $p(x)$. Then, there uniquely exists $\mu \in \{1, 2, \dots, \lambda-1\}$ such that the restriction of f^ℓ to a small disk neighborhood of x is the clockwise $\frac{2\pi\mu}{\lambda}$ -rotation. Since the group $\langle f^\ell \rangle$ of order λ acts effectively on a disk neighborhood of x and a generator f^ℓ is $\frac{2\pi\mu}{\lambda}$ -rotation, μ and λ are coprime. Since μ and λ are coprime, there uniquely exists an integer $\theta \in \{1, 2, \dots, \lambda-1\}$ such that $\mu\theta \equiv 1 \pmod{\lambda}$. The *valency* of Cx is defined by $\frac{\theta}{\lambda}$. Let $\frac{\theta_1}{\lambda_1}, \frac{\theta_2}{\lambda_2}, \dots, \frac{\theta_s}{\lambda_s}$

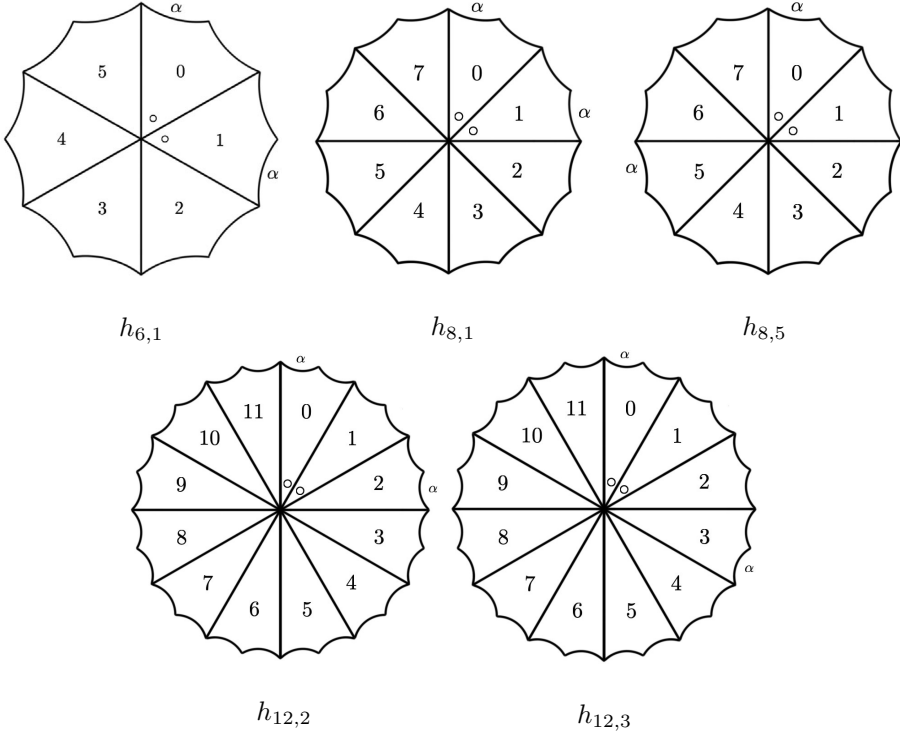


FIGURE 10. The diffeomorphisms appearing in Theorem 1.3
Each barycenter is a fixed point, and each small circle represents
how the angles move

be the valencies of all multiple orbits of f . The data $\left[g, n; \frac{\theta_1}{\lambda_1} + \frac{\theta_2}{\lambda_2} + \cdots + \frac{\theta_s}{\lambda_s} \right]$ is called the *total valency* of f .

By the following theorem, total valencies determine periodic diffeomorphisms of Σ_g up to conjugacy.

Theorem 2.1(Nielsen [17, Section 11], **see also** [1, Section 1.3], [10, Theorem 2.1]). *Let f, f' be periodic diffeomorphisms of Σ_g with total valencies $\left[g, n; \frac{\theta_1}{\lambda_1} + \frac{\theta_2}{\lambda_2} + \cdots + \frac{\theta_s}{\lambda_s} \right]$ and $\left[g, n'; \frac{\theta'_1}{\lambda'_1} + \frac{\theta'_2}{\lambda'_2} + \cdots + \frac{\theta'_{s'}}{\lambda'_{s'}} \right]$, respectively. f is conjugate to f' if and only if the following are satisfied:*

- (i) $s = s'$,
- (ii) $n = n'$,
- (iii) *after relabeling the indices, we have $\frac{\theta_i}{\lambda_i} = \frac{\theta'_i}{\lambda'_i}$ for $i = 1, 2, \dots, s$.*

By Theorem 2.1, we can identify the total valency of a periodic diffeomorphism with its conjugacy class. We will use the following well-known observations of Nielsen.

Proposition 2.2(Nielsen [17, Equation (4.6)]). *For any periodic diffeomorphism, the sum of the valencies of all multiple orbits is an integer.*

Remark 2.3. Every multiple orbit of an involution is a fixed point whose valency is $\frac{1}{2}$. Thus, by Proposition 2.2, the number of fixed points of an involution is even.

In general, the composition of two periodic diffeomorphisms may not be periodic, and hence, the product of total valencies does not make sense. But one can define powers of the total valency of a periodic diffeomorphism f by the total valency of powers of f .

Let us recall the well-known Riemann-Hurwitz formula for the convenience of the reader.

Proposition 2.4(Riemann-Hurwitz formula). *Consider an n -fold branched covering from Σ_g to $\Sigma_{g'}$ with branching indices $\lambda_1, \dots, \lambda_s$. Then we have*

$$2g - 2 = n \left(2g' - 2 + \sum_{1 \leq i \leq s} \left(1 - \frac{1}{\lambda_i} \right) \right).$$

We will use the following result due to Harvey, which gives a necessary condition for the existence of finite covering maps that have given branching indices.

Proposition 2.5(Harvey [8, Theorem 4]). *Assume $g > 1$ and let $n \in \mathbb{Z}_{\geq 2}$ and $\lambda_1, \dots, \lambda_s \in \mathbb{Z}_{\geq 2}$. Set $M = \text{lcm}(\lambda_1, \lambda_2, \dots, \lambda_s)$. If there is an n -fold cyclic covering from Σ_g to $\Sigma_{g'}$ with branching indices $\lambda_1, \dots, \lambda_s$, then the following conditions are satisfied:*

- (i) $\text{lcm}(\lambda_1, \lambda_2, \dots, \hat{\lambda}_i, \dots, \lambda_s) = M$ for all $i \in \{1, 2, \dots, s\}$, where $\hat{\lambda}_i$ denotes the omission of λ_i .
- (ii) M divides n , and if $g' = 0$, then $M = n$.
- (iii) $s \neq 1$, and, if $g' = 0$, then $s \geq 3$.

We will use the following characterizations of reducibility of periodic diffeomorphisms.

Theorem 2.6(Kasahara [15, Theorem 4.1]). *Let $f \in \text{Diff}_+(\Sigma_g)$ be a periodic diffeomorphism of order n . If f is irreducible, then $n \geq 2g + 1$.*

Theorem 2.7(Wiman, see [11, Theorem 5.1]). *Let $f \in \text{Diff}_+(\Sigma_g)$ be a periodic diffeomorphism of order n . Then, $n \leq 4g + 2$.*

Moreover, for finite group actions, we will also use the following characterization due to Gilman.

Theorem 2.8(Gilman [6, Lemma 3.9]). *Let G be a finite subgroup of $\text{Diff}_+(\Sigma_g)$ and M_G be the union of all the multiple orbits of G . Then G is irreducible if and only if $(\Sigma_g \setminus M_G)/G$ is homeomorphic to $S^2 \setminus \{3\text{-points}\}$.*

Remark 2.9. Let G be a finite subgroup of $\text{Diff}_+(\Sigma_g)$. Let us prove that G is reducible if and only if so is the image of G under the projection $\text{Diff}_+(\Sigma_g) \rightarrow \text{Mod}(\Sigma_g)$. We can show a stronger result: By Kerckhoff's theorem [16], every finite subgroup H of $\text{Mod}(\Sigma_g)$ can be lifted to $\text{Diff}_+(\Sigma_g)$, namely, there exists a subgroup $\tilde{H}$ of $\text{Diff}_+(\Sigma_g)$ such that the restriction of the projection $\text{Diff}_+(\Sigma_g) \rightarrow \text{Mod}(\Sigma_g)$ to $\tilde{H}$ is an isomorphism from $\tilde{H}$ to H . Moreover, $\tilde{H}$ preserves a hyperbolic metric on Σ_g . We will prove that H is reducible if and only if $\tilde{H}$ is reducible.

If $\tilde{H}$ is reducible, then there is an $\tilde{H}$ -invariant partition $\mathcal{T}$ on Σ_g . The isotopy classes of $\mathcal{T}$ give a reduction system on Σ_g for H , which implies the reducibility of H . Conversely, assume that H is reducible. Then there exists an H -invariant reduction system $\mathcal{C} = \{C_1, \dots, C_n\}$. Since Σ_g is hyperbolic, C_i is represented by a unique simple closed geodesic D_i on Σ_g for $i = 1, \dots, n$ (see, for example, [18, Lemmas 3.3 and 3.4]). By a classical result (for example, see [18, Théorème 15]), a pair of simple closed geodesics on a closed hyperbolic surface intersect with each other at the minimal number of points in their isotopy classes. Since the geometric intersection number of C_i and C_j is 0 for $i \neq j$, the geodesics $D_1, \dots, D_n$ are mutually disjoint. Since $\tilde{H}$ preserves the hyperbolic metric on Σ_g and D_i is the unique geodesic that represents C_i up to free homotopy, it follows that $\{D_1, \dots, D_n\}$ is $\tilde{H}$ -invariant. Therefore, having an $\tilde{H}$ -invariant partition $\{D_1, \dots, D_n\}$, $\tilde{H}$ is reducible.

3. Preliminaries

Let us compute the total valency of $h_{n,p}$ defined in the introduction.

Proposition 3.1([19, Section 2], [3, Theorem 2.2.4] and [20, Proposition 4.1]).
The total valency of $h_{n,p}$ is $\left[g, n; \frac{1}{n} + \frac{p}{n} + \frac{n-p-1}{n}\right]$, where

$$g = \frac{n - \gcd(n, p) - \gcd(n, p+1) + 1}{2}.$$

Proof. Let P be the regular $2n$ -gon shown in Figure 2. As shown below, $h_{n,p}$ has 3 multiple orbits: The barycenter x of P , the middle points $\{y_0, y_1, \dots, y_{n-1}\}$ of the edges of P and the vertices $\{z_0, z_1, \dots, z_{n-1}\}$ of P (see Figure 11). Thus, the total valency of $h_{n,p}$ is of the form $\left[g, n; \frac{\theta_1}{\lambda_1} + \frac{\theta_2}{\lambda_2} + \frac{\theta_3}{\lambda_3}\right]$ for some mutually coprime integers λ_i and θ_i such that λ_i divides n for $i = 1, 2, 3$.

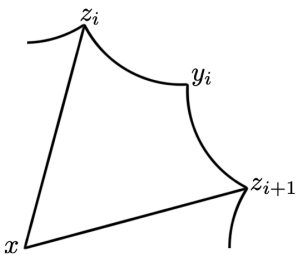


FIGURE 11. The barycenter x , the middle points y_i of edges and the vertices z_i of P

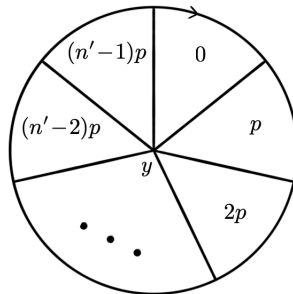


FIGURE 12. The cyclic order of fundamental domains of $h_{n,p}$ around y

Since the action is a rotation by $\frac{2\pi}{n}$ on the $2n$ -gon, the valency of the fixed point x is $\frac{1}{n}$. Hence, we can assume $\frac{\theta_1}{\lambda_1} = \frac{1}{n}$.

Let us compute the valency $\frac{\theta_2}{\lambda_2}$ of the multiple orbit $\{y_0, y_1, \dots, y_{n-1}\}$. Set $k = \gcd(n, p)$. Let n' and p' be the integers such that $n = kn'$ and $p = kp'$, respectively. The period of the multiple orbits is k , and hence, $\lambda_2 = n'$. Take a point y in the orbit and consider a loop that goes around the boundary of a small disk centered at y as shown in Figure 12. The fundamental domains appear along this loop in the order

$$0, p, 2p, \dots, (n' - 2)p, (n' - 1)p$$

modulo n . Then, there uniquely exists $\nu \in \{1, 2, \dots, n'\}$ such that $\nu p \equiv k \pmod{n}$. Hence, we have $\nu p' \equiv 1 \pmod{n'}$. By the definition of the valency, $\theta_2 = p'$, and hence, $\frac{\theta_2}{\lambda_2} = \frac{p'}{n'} = \frac{p}{n}$.

By Nielsen's result (Proposition 2.2), we have

$$\frac{1}{n} + \frac{p}{n} + \frac{\theta_3}{\lambda_3} \in \mathbb{Z}.$$

Thus, we have $\frac{\theta_3}{\lambda_3} = \frac{n-p-1}{n}$.

Lastly, we compute the genus g of the surface. The branching indices of $h_{n,p}$ are n , $\frac{n}{\gcd(n,p)}$ and $\frac{n}{\gcd(n,n-p-1)}$. It follows from the definition of the map $h_{n,p}$ that the quotient space of $h_{n,p}$ is a sphere. Then, by the Riemann-Hurwitz formula (Proposition 2.4), we have

$$\begin{aligned} 2g - 2 &= n \left(-2 + 3 - \left(\frac{1}{n} + \frac{\gcd(n,p)}{n} + \frac{\gcd(n,n-p-1)}{n} \right) \right) \\ &= n - \gcd(n,p) - \gcd(n,p+1) - 1, \end{aligned}$$

and hence, we have

$$g = \frac{n - \gcd(n,p) - \gcd(n,p+1) + 1}{2}.$$

□

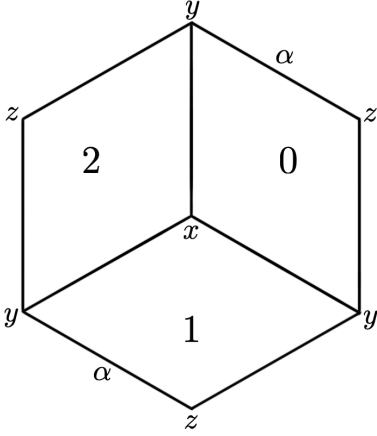
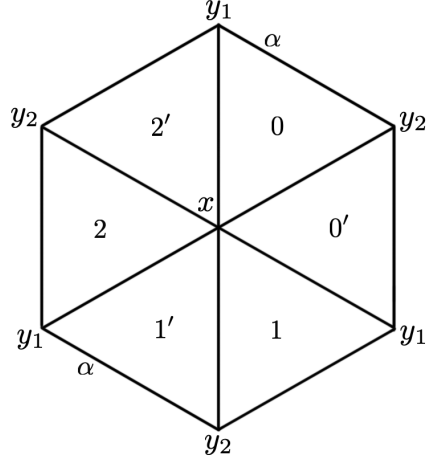
Applying the Riemann-Hurwitz formula (Proposition 2.4), we can obtain the following well-known classification of periodic diffeomorphisms on tori.

Proposition 3.2. *Let f be a nontrivial orientation-preserving periodic diffeomorphism on a torus T^2 . Then $T^2/\langle f \rangle$ is homeomorphic to either T^2 or S^2 .*

If $T^2/\langle f \rangle$ is homeomorphic to T^2 , then f has no multiple orbit. If $T^2/\langle f \rangle$ is homeomorphic to S^2 , then f is conjugate to one of the following:

- (1) $h_{4,1}^2$ or $h_{6,3}^3$ whose total valency is $\left[1, 2; \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} \right]$.
- (2) $h_{4,1}$ whose total valency is $\left[1, 4; \frac{1}{4} + \frac{1}{4} + \frac{1}{2} \right]$.
- (3) $h_{4,1}^3$ whose total valency is $\left[1, 4; \frac{3}{4} + \frac{3}{4} + \frac{1}{2} \right]$.
- (4) $h_{6,3}$ whose total valency is $\left[1, 6; \frac{1}{6} + \frac{1}{3} + \frac{1}{2} \right]$.
- (5) $h_{3,1}$ or $h_{6,3}^2$ whose total valency is $\left[1, 3; \frac{1}{3} + \frac{1}{3} + \frac{1}{3} \right]$.
- (6) $h_{3,1}^2$ or $h_{6,3}^4$ whose total valency is $\left[1, 3; \frac{2}{3} + \frac{2}{3} + \frac{2}{3} \right]$.
- (7) $h_{6,3}^5$ whose total valency is $\left[1, 6; \frac{5}{6} + \frac{2}{3} + \frac{1}{2} \right]$.

The diffeomorphism $h_{3,1}$ is conjugate to the diffeomorphism $h_{6,3}^2$. Indeed, both are represented by a $\frac{\pi}{3}$ -rotation of a regular hexagon (see Figures 13 and 14).


 FIGURE 13. $h_{3,1}$

 FIGURE 14. $h_{6,3}^2$

Since any multiple-point free periodic diffeomorphism on a torus is reducible, we get the following consequence.

Proposition 3.3. *Every irreducible periodic diffeomorphism on a torus is conjugate to one of the following:*

- (1) $h_{3,1}$ or its inverse,
- (2) $h_{4,1}$ or its inverse,
- (3) $h_{6,3}$ or its inverse.

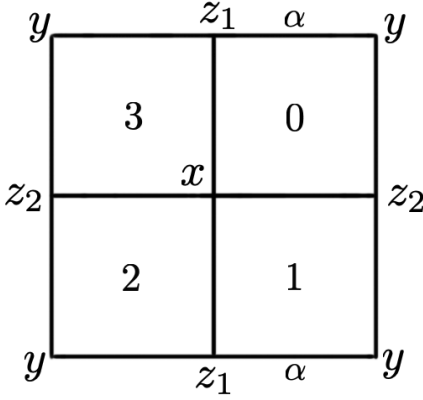
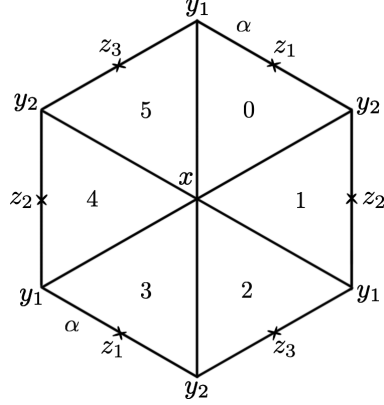
Here $h_{3,1}$, $h_{4,1}$ and $h_{6,3}$ are shown in Figures 13, 15 and 16, respectively.

4. Cyclic branched coverings of tori

In this section, we will classify irreducible periodic diffeomorphisms that commute with an involution ι_g whose quotient $\Sigma_g/\langle\iota_g\rangle$ is a torus to prove Theorem 1.3.

Take an involution ι_g of Σ_g whose quotient space is homeomorphic to a torus. Let f be a periodic diffeomorphism of Σ_g of order n , which commutes with ι_g . Let G be the subgroup of $\text{Diff}_+(\Sigma_g)$ generated by f and ι_g .

Let $\bar{f}$ be a periodic diffeomorphism of $\Sigma_g/\langle\iota_g\rangle$ induced by f . Let n and $\bar{n}$ be the order of f and $\bar{f}$, respectively. We have the following simple observation.

FIGURE 15. $h_{4,1}$ FIGURE 16. $h_{6,3}$

Lemma 4.1. (1) *We have $n = 2\bar{n}$ if and only if $f^{\bar{n}} = \iota_g$.*
 (2) *We have $n = \bar{n}$ if and only if $f^{\bar{n}} \neq \iota_g$.*

Proof. We have $\bar{f}^{\bar{n}} = \text{id}_{\Sigma_g/\langle \iota_g \rangle}$. Since the lifts of $\text{id}_{\Sigma_g/\langle \iota_g \rangle}$ to Σ_g are ι_g and id_{Σ_g} , we have either $f^{\bar{n}} = \iota_g$ or $f^{\bar{n}} = \text{id}_{\Sigma_g}$. The latter is equivalent to $n = \bar{n}$, and the former implies $n = 2\bar{n}$. The proof is concluded. $\square$

Corollary 4.2. *n is even and G is cyclic if and only if $n = 2\bar{n}$.*

Proof. By Lemma 4.1, the necessity holds. Assume that n is even and G is cyclic. Then we have $f^{\frac{n}{2}} = \iota_g$, which implies that $G/\langle \iota_g \rangle$ is of order $\frac{n}{2}$. Since $\bar{f}$ is a generator of $G/\langle \iota_g \rangle$ acting on $\Sigma_g/\langle \iota_g \rangle$, it follows that $\bar{n} = \frac{n}{2}$. $\square$

We will use the following result.

Lemma 4.3. *For any $n \in \mathbb{Z}_{\geq 2}$, there exists no n -fold cyclic branched covering $p : S^2 \rightarrow S^2$ which has more than two branched points such that for the branched points $\{x_1, x_2, x_3\}$, $p^{-1}(x_i)$ is a one-point set for $i = 1, 2, 3$. In particular, there exists no periodic diffeomorphism of S^2 which fixes more than two points except the identity.*

Proof. Assume that there exists an n -fold branched covering $p : S^2 \rightarrow S^2$ which has more than two branched points $x_1, x_2, \dots, x_s \in S^2$ such that $p^{-1}(x_i)$ is a one-point set for $i = 1, 2, 3$. By Riemann-Hurwitz formula (Proposition 2.4),

we have

$$-2 = n \left(-2 + 3 \left(1 - \frac{1}{n} \right) + \sum_{i=4}^s \left(1 - \frac{1}{\lambda_i} \right) \right),$$

where λ_i is the branching index of x_i for $i = 4, \dots, s$. Then we have

$$1 - n = \sum_{i=4}^s \left(1 - \frac{1}{\lambda_i} \right) \geq 0,$$

which implies $n \leq 1$, and hence a contradiction. $\square$

The following simple observation is fundamental.

Lemma 4.4. (1) *We have $f(\text{Fix}(\iota_g)) = \text{Fix}(\iota_g)$, where $\text{Fix}(\iota_g)$ is the fixed-point set of ι_g .*

(2) *The involution ι_g maps every f -orbit to an f -orbit of the same valency.*

Proof. For any $x \in \text{Fix}(\iota_g)$, we have $\iota_g(f(x)) = f(\iota_g(x)) = f(x)$, which implies $f(x) \in \text{Fix}(\iota_g)$. We can prove $f^{-1}(x) \in \text{Fix}(\iota_g)$ in a similar way. Therefore we have the first assertion $f(\text{Fix}(\iota_g)) = \text{Fix}(\iota_g)$.

Let $x \in \Sigma_g$ be a multiple point of period $\ell < n$. Take a disk neighborhood U of x such that f^ℓ acts on U by the clockwise $\frac{2\pi\mu}{\lambda}$ -rotation for some $1 \leq \mu < \lambda$, where $\lambda = \frac{n}{\ell}$. Since we have $\iota_g \circ f^\ell \circ (\iota_g)^{-1} = f^\ell$, the map f^ℓ acts on $\iota_g(U)$ by the clockwise $\frac{2\pi\mu}{\lambda}$ -rotation. Thus, by the definition, the valency of $\{x, f(x), \dots, f^{\ell-1}(x)\}$ is equal to that of $\{\iota_g(x), f(\iota_g(x)), \dots, f^{\ell-1}(\iota_g(x))\}$. $\square$

In the remainder of this section, we will classify the G -action up to conjugacy in the case where G is cyclic. Let $\bar{f}$ be the diffeomorphism on $\Sigma_g/\langle\iota_g\rangle$ induced by f . Let p denote the double covering $\Sigma_g \rightarrow \Sigma_g/\langle\iota_g\rangle$. In the sequel, we will use the following criterion.

Lemma 4.5. *Let $x \in \Sigma_g$ be a multiple point of f and $\text{Stab}_G(x)$ the stabilizer subgroup of G with respect to x . If ι_g fixes x and $\text{Stab}_G(x)$ is of even order, then G is cyclic.*

Proof. Let k be the period of x . By assumption, the stabilizer subgroup $\text{Stab}_G(x)$ of G with respect to x is generated by f^k and ι_g . Since $\frac{n}{k}$ is even, the subgroup of $\text{Stab}_G(x)$ generated by f^k is of even order. Thus, it contains $f^{\frac{n}{2}}$, which is a π -rotation around x . Since ι_g is also a π -rotation around x , it follows that $\iota_g \circ (f^{\frac{n}{2}})^{-1}$ acts on a neighborhood V of x by the identity. Let $h = \iota_g \circ (f^{\frac{n}{2}})^{-1}$ and H be the subgroup of $\text{Diff}_+(\Sigma_g)$ generated by h . Since $\Sigma_g/\langle h \rangle$ is connected,

by [2, Proposition 1.2.5], there exists an open dense subset U of Σ_g such that the isotropy group of the H -action is the same at all points in U . Since $U \cap V \neq \emptyset$, it follows that h fixes a point in U , and hence h fixes all points in U , which implies that $h = \text{id}_{\Sigma_g}$. Therefore, it follows that $f^{\frac{n}{2}} = \iota_g$, which implies that G is cyclic. $\square$

Remark 4.6. If $\bar{f}$ is reducible, then we can take a partition $\mathcal{C}$ preserved by $\bar{f}$ such that $\mathcal{C} \cap \text{Fix}(\iota_g) = \emptyset$. Then G preserves $p^{-1}(\mathcal{C})$. Therefore, if $\bar{f}$ is reducible, then G is reducible. Hence, to classify the irreducible group G , it is sufficient to consider the case where $\bar{f}$ is irreducible.

Lemma 4.7. Assume that $\bar{f}$ is conjugate to a power of $h_{n,p}$. If the canonical projection $p : \Sigma_g \rightarrow \Sigma_g / \langle \iota_g \rangle$ is branched at a free orbit of $\bar{f}$, then G is reducible.

Proof. Assume that p is branched at a point y , which is not a multiple point of $\bar{f}$. Let $\{y_0, y_1, \dots, y_{\bar{n}-1}\}$ be the $\bar{f}$ -orbit of y . For $i = 0, 1, \dots, \bar{n} - 1$, take an embedded arc γ_i that connects y_i and the barycenter x of the $2n$ -gon so that $\gamma_i \cap \gamma_j = \{x\}$ for $i \neq j$. Let U be a simply-connected small open neighborhood of $\bigcup_{i=0}^{\bar{n}-1} \gamma_i$. Let $\mathcal{C}$ be the path that goes around the boundary of U once, as shown in Figure 17. Then each component of the preimage $p^{-1}(\mathcal{C})$ is essential and preserved by f and ι_g , hence G is reducible. $\square$

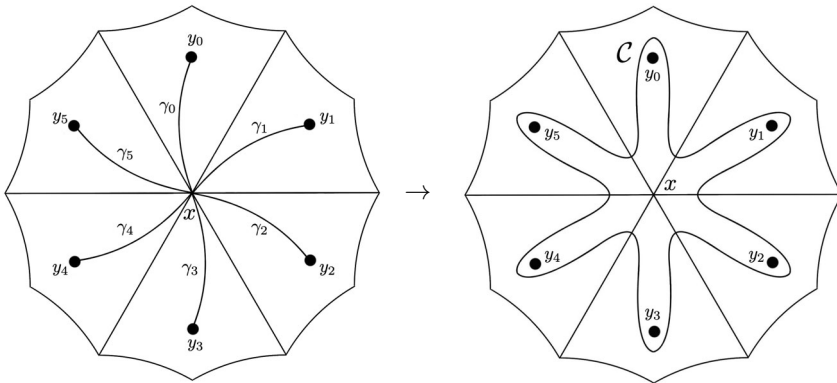


FIGURE 17. A construction of a simple closed curve $\mathcal{C}$ ($n = 6$)

Now, we will determine irreducible f that induces $h_{4,1}$ on the torus $\Sigma_g / \langle \iota_g \rangle$.

Since ι_g has $2g - 2$ fixed points, p is branched at $2g - 2$ points. By Lemma 4.4, $\bar{f}$ maps the branch locus of p to itself. By classifying nonempty $\bar{f}$ -invariant sets with an even number of points in the union of the multiple orbits of $\bar{f}$, we obtain the following lemma.

Lemma 4.8. *If $\bar{f} = h_{4,1}$, then the branch locus B_p of p is one of the following:*

- (1) $\{x, y\}$,
- (2) $\{z_1, z_2\}$,
- (3) $\{x, y, z_1, z_2\}$,

where $x, y, z_1, z_2 \in \Sigma_g / \langle \iota_g \rangle$ are as shown in Figure 15.

Proposition 4.9. *If $\bar{f} = h_{4,1}$, then G is conjugate to either of $\langle h_{8,1} \rangle$ or $\langle h_{8,5} \rangle$.*

Proof. Let q be the projection $\Sigma_g \rightarrow \Sigma_g / G$. We can see that G must be cyclic in all three cases in Lemma 4.8. Indeed, since the order of $\bar{f}$ is equal to 4, and the period of multiple points of $\bar{f}$ is either 1 or 2, we can see that the branching index of every branched point of q is even. Then, by Lemma 4.5, G is cyclic. By Corollary 4.2, we have that $n = 8$.

Let us consider the case 1 in Lemma 4.8. We can see that q has three branched points of index 8, 8 and 2, respectively. Then the total valency of f is of the form $[2, 8; \frac{\theta_1}{8} + \frac{\theta_2}{8} + \frac{1}{2}]$, where $\theta_1, \theta_2 \in \{1, 3, 5, 7\}$ with $\theta_1 \leq \theta_2$. Here $\frac{\theta_1}{8} + \frac{\theta_2}{8} + \frac{1}{2}$ is an integer by Nielsen's condition (Proposition 2.2). Thus $\theta_1 + \theta_2$ is equal to either 4 or 12, and hence, we have either $(\theta_1, \theta_2) = (1, 3)$ or $(5, 7)$. The involution f^4 fixes two fixed points of f . In addition, since the f -orbit of valency $\frac{1}{2}$ consists of four points, f^4 fixes each point in the f -orbit. Thus, having 6 fixed points, f^4 is a hyperelliptic involution. Therefore, despite the value of (θ_1, θ_2) , f^4 is a hyperelliptic involution, which contradicts Lemma 4.1(i). Thus, this case does not occur.

In the case 2 of Lemma 4.8, we can see that q has three branched points of index 4. Thus, the total valency of f is of the form $[2, 8; \frac{\theta_1}{4} + \frac{\theta_2}{4} + \frac{\theta_3}{4}]$, where $\theta_1, \theta_2, \theta_3 \in \{1, 3\}$. Then, since $\theta_1 + \theta_2 + \theta_3$ is odd, there are no $\theta_1, \theta_2, \theta_3$ such that $\frac{\theta_1}{4} + \frac{\theta_2}{4} + \frac{\theta_3}{4}$ is an integer. By Nielsen's condition (Proposition 2.2), this case does not occur.

Finally, let us consider the case 3 of Lemma 4.8. We can see that q has three branched points of index 8, 8 and 4, respectively. Let $[3, 8; \frac{\theta_1}{8} + \frac{\theta_2}{8} + \frac{\theta_3}{4}]$ be the total valency of f , where $\theta_1, \theta_2 \in \{1, 3, 5, 7\}$ with $\theta_1 \leq \theta_2$ and $\theta_3 \in \{1, 3\}$. Now

$\frac{\theta_1}{8} + \frac{\theta_2}{8} + \frac{\theta_3}{4}$ is an integer by Nielsen's condition (Proposition 2.2). First, consider the case where $\theta_3 = 1$. In this case, $\theta_1 + \theta_2$ is equal to either 6 or 14, and hence, we have either $(\theta_1, \theta_2, \theta_3) = (1, 5, 1), (3, 3, 1), (7, 7, 1)$. In the case where $\theta_3 = 3$, similarly, we have either $(\theta_1, \theta_2, \theta_3) = (1, 1, 3), (5, 5, 3)$ or $(3, 7, 3)$. By Proposition 3.1, the total valency of $h_{8,1}$ and $h_{8,5}$ are $[3, 8; \frac{1}{8} + \frac{1}{8} + \frac{3}{4}]$ and $[3, 8; \frac{1}{8} + \frac{5}{8} + \frac{1}{4}]$, respectively. Thus f is conjugate to either $h_{8,1}^k$ for some $k \in \{1, 3, 5, 7\}$ or $h_{8,5}^\ell$ for some $\ell \in \{1, 3\}$, which implies that $G = \langle f \rangle$ is conjugate to either $\langle h_{8,1} \rangle$ or $\langle h_{8,5} \rangle$. $\square$

Let us consider the case where $\bar{f} = h_{6,3}$. We obtain the following lemma by an argument similar to the case where $\bar{f} = h_{4,1}$.

Lemma 4.10. *If $\bar{f} = h_{6,3}$, then the branch locus B_p of p is one of the following:*

- (1) $\{y_1, y_2\}$,
- (2) $\{x, z_1, z_2, z_3\}$,
- (3) $\{x, y_1, y_2, z_1, z_2, z_3\}$,

where $x, y_1, y_2, z_1, z_2, z_3 \in \Sigma_g / \langle \iota_g \rangle$ are as shown in Figure 16.

Proposition 4.11. *If $\bar{f} = h_{6,3}$, then G is conjugate to a subgroup of either of $\langle h_{6,1}, I \rangle$, $\langle h_{12,2} \rangle$, or $\langle h_{12,3} \rangle$.*

Proof. Let us consider the case 2 of Lemma 4.10. Since p is branched at a fixed point x of $\bar{f}$ and the order of f is even, by Lemma 4.5, G is cyclic. Then, by Corollary 4.2, we have $n = 12$ and by the assumption on the branched locus, the branching indices of the branched points of $\bar{f}$ are 12, 4 and 3. Let $[3, 12; \frac{\theta_1}{12} + \frac{\theta_2}{4} + \frac{\theta_3}{3}]$ be the total valency of f , where $\theta_1 \in \{1, 5, 7, 11\}, \theta_2 \in \{1, 3\}$ and $\theta_3 \in \{1, 2\}$. By Nielsen's condition (Proposition 2.2), $\frac{\theta_1}{12} + \frac{\theta_2}{4} + \frac{\theta_3}{3}$ is an integer. By $0 < \theta_1 \leq 11, 0 < \theta_2 \leq 3$ and $0 < \theta_3 \leq 2$, we have either $\frac{\theta_1}{12} + \frac{\theta_2}{4} + \frac{\theta_3}{3} = 1$ or 2. Since $\frac{\theta_2}{4} + \frac{\theta_3}{3}$ is equal to either $\frac{7}{12}, \frac{11}{12}, \frac{13}{12}$ or $\frac{17}{12}$, we can determine all $(\theta_1, \theta_2, \theta_3)$ that satisfy the condition: $(\theta_1, \theta_2, \theta_3)$ is equal to either $(5, 1, 1), (1, 1, 2), (11, 3, 1)$ or $(7, 3, 2)$. By Proposition 3.1, the total valency of $h_{12,3}$ is $[3, 12; \frac{1}{12} + \frac{1}{4} + \frac{2}{3}]$. Thus f is conjugate to $h_{12,3}^k$ for some $k \in \{1, 5, 7, 11\}$, which implies that $G = \langle f \rangle$ is conjugate to $\langle h_{12,3} \rangle$.

Let us consider the case 3 of Lemma 4.10. Since p is branched at a fixed point x of $\bar{f}$ and the order of f is even, by Lemma 4.5, G is cyclic. Then, by Corollary 4.2, we have $n = 12$, and by the assumption on the branched locus, the branching indices of the branched points of $\bar{f}$ are 12, 6 and 4. Let $[3, 12; \frac{\theta_1}{12} + \frac{\theta_2}{6} + \frac{\theta_3}{4}]$

be the total valency of f , where $\theta_1 \in \{1, 5, 7, 11\}$, $\theta_2 \in \{1, 5\}$ and $\theta_3 \in \{1, 3\}$. By Nielsen's condition (Proposition 2.2), $\frac{\theta_1}{12} + \frac{\theta_2}{6} + \frac{\theta_3}{4}$ is an integer, and it is equal to either 1 or 2 by the same argument as the last paragraph. Since $\frac{\theta_2}{6} + \frac{\theta_3}{4}$ is equal to either $\frac{5}{12}$, $\frac{11}{12}$, $\frac{13}{12}$ or $\frac{19}{12}$, we can determine all $(\theta_1, \theta_2, \theta_3)$ that satisfy the condition: $(\theta_1, \theta_2, \theta_3)$ is equal to either $(7, 1, 1)$, $(11, 5, 1)$, $(1, 1, 3)$ or $(5, 5, 3)$. By Proposition 3.1, the total valency of $h_{12,2}$ is $[3, 12; \frac{1}{12} + \frac{1}{6} + \frac{3}{4}]$. Thus f is conjugate to $h_{12,2}^k$ for some $k \in \{1, 5, 7, 11\}$, which implies that $G = \langle f \rangle$ is conjugate to $\langle h_{12,2} \rangle$.

Finally, let us consider the case 1 in Lemma 4.10. Since p is branched at 2 points, we have $g = 2$. Since $\bar{f}$ fixes x , it follows that $p^{-1}(x)$ is either a multiple orbit of period two of f , or f fixes $p^{-1}(x)$ pointwise. In the former case, $\iota_g \circ f$ is a lift of $\bar{f}$ which fixes $p^{-1}(x)$ pointwise. Thus, to classify the conjugacy class of G , we can assume that f fixes $p^{-1}(x)$ pointwise without loss of generality. In this case, by Wiman's result (Theorem 2.7), we have $n \leq 10$, and hence $n = 6$. Since the valency of $\{y_1, y_2\}$ as an $\bar{f}$ -orbit is $\frac{1}{3}$, by a local argument, we can see that the valency of $p^{-1}(\{y_1, y_2\})$ is either $\frac{2}{3}$ or $\frac{1}{6}$. But $n = 6$ yields that only the former case occurs. Similarly, $p^{-1}(\{z_1, z_2, z_3\})$ is either a free orbit or the union of two multiple orbits of valency $\frac{1}{2}$. The Riemann-Hurwitz formula (Proposition 2.4) yields that only the former case occurs here. Indeed, if we substitute $g = 2$ and $(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5) = (6, 6, 3, 2, 2)$ to the formula, then we get $g' = -\frac{1}{2}$, which contradicts the fact that g' is the genus of $\Sigma_2/\langle f \rangle$. Thus, it follows that the total valency of f is $[2, 6; \frac{1}{6} + \frac{1}{6} + \frac{2}{3}]$. Thus f is conjugate to $h_{6,1}$. Now $f^3 \circ \iota_2$ is a hyperelliptic involution which fixes all points in $p^{-1}(\{z_1, z_2, z_3\})$. By [21, Lemma 13], G is conjugate to $\langle h_{6,1}, I \rangle$, where I is a hyperelliptic involution on Σ_2 . $\square$

Finally, let us consider the case where $\bar{f} = h_{3,1}$.

Proposition 4.12. *If $\bar{f} = h_{3,1}$, then G is conjugate to a subgroup of $\langle h_{6,1}, I \rangle$.*

Proof. The multiple orbits of $h_{3,1}$ are three fixed points. A nonempty $\bar{f}$ -invariant subset with an even number of points in the set of these three fixed points is a two-point set. All of them are mutually conjugate, and hence we can assume that p is branched at y and z in Figure 13. Thus, the branch locus of p is the same as the case 1 in Lemma 4.10. By an argument parallel to that in the final paragraph of the proof of Proposition 4.11, we can see that G is conjugate to a subgroup of $\langle h_{6,1}, I \rangle$. $\square$

Theorem 1.3 follows from Propositions 4.9, 4.11 and 4.12.

5. Nonexistence of irreducible roots of F_1 and F_2

In this section, we will show the nonexistence of irreducible roots of F_1 and F_2 constructed in Examples 1.1 and 1.2. The argument of this section is independent of the other sections.

Proposition 5.1. *Neither F_1 nor F_2 is conjugate to a power of any irreducible periodic diffeomorphism.*

Proof. By Proposition 3.1, the total valency of $h_{4g,2g}$ and $h_{4g,2g}^{4g-1}$ is equal to $\left[g, 4g; \frac{1}{4g} + \frac{2g-1}{4g} + \frac{1}{2}\right]$ and $\left[g, 4g; \frac{4g-1}{4g} + \frac{2g+1}{4g} + \frac{1}{2}\right]$, respectively.

By the construction of F_1 and F_2 , the total valency of F_1 and F_2 is equal to $\left[2g+1, 4g; \frac{1}{2} + \frac{1}{2}\right]$ and $\left[2g+2, 2g+1; \emptyset\right]$, respectively.

In the case of F_1 , by the Riemann-Hurwitz formula, we have

$$2(2g+1) - 2 = 4g \left(2g'_1 - 2 + \frac{1}{2} + \frac{1}{2} \right),$$

therefore, we have $g'_1 = 1$, where g'_1 is the genus of $\Sigma_{2g+1}/\langle F_1 \rangle$. Assume that there exists an irreducible periodic diffeomorphism h of period n such that $h^k = F_1$ for some $k \geq 2$. Then, we have $n = 4 \gcd(n, k)g$. By Kasahara's Theorem 2.6 and Wiman's Theorem 2.7, we have $4g + 3 \leq 4 \gcd(n, k)g \leq 8g + 6$, which implies that $\gcd(n, k) = 2$. However, by Remark 4.6, h induces an irreducible periodic diffeomorphism of order 2 on T^2 . This contradicts Proposition 3.3.

In the case of F_2 , by Riemann-Hurwitz formula, we have

$$2(2g+2) - 2 = (2g+1)(2g'_2 - 2),$$

therefore, we have $g'_2 = 2$, where g'_2 is the genus of $\Sigma_{2g+2}/\langle F_2 \rangle$. Assume that there exists an irreducible periodic diffeomorphism h of period n such that $h^k = F_2$ for some $k \geq 2$. Then, we have $n = (2g+1) \gcd(n, k)$. By Wiman's Theorem 2.7, we have $(2g+1) \gcd(n, k) \leq 8g + 10$, which implies that $\gcd(n, k) \leq 5$. However, by Remark 4.6 and the general theory of covering spaces, h induces an irreducible periodic diffeomorphism of order $\gcd(n, k)$ on Σ_2 . If $\gcd(n, k) \leq 4$, it contradicts Kasahara's result (Theorem 2.6). If $\gcd(n, k) = 5$, we have $g = 2$ and $n = 25$. Then, since F_2 acts freely, so does h , which contradicts Gilman's theorem (Theorem 2.8). $\square$

Appendix A.

In this appendix, we provide a more detailed explanation of $h_{2g+1,1}$ that is considered in Example 1.2. The map $h_{4g+2,2g+1}$ is a hyperelliptic periodic diffeomorphism on Σ_g . Indeed, let $I = h_{4g+2,2g+1}^{2g+1}$, which is an involution that commutes with $h_{4g+2,2g+1}$. This I fixes the $(2g+2)$ -points that are the center of the $(8g+4)$ -gon and the middle point of every edge of the $(8g+4)$ -gon, which means that I is hyperelliptic.

Lemma A.1. *The diffeomorphism $h_{2g+1,1}$ is hyperelliptic and is conjugate to $h_{4g+2,2g+1}^{2g}$.*

Proof. By Proposition 3.1, the total valency of $h_{4g+2,2g+1}$ is equal to $\left[g, 4g+2; \frac{1}{4g+2} + \frac{g}{2g+1} + \frac{1}{2} \right]$. Since $\frac{4g+2}{\gcd(4g+2, 4g+2)} = 1$ and $1 \cdot 1 \equiv 1 \pmod{4g+2}$, the multiple orbit of valency $\frac{1}{4g+2}$ is a fixed point and $h_{4g+2,2g+1}$ acts on a small neighborhood of the point by $\frac{2\pi}{4g+2}$ rotation. Since $\frac{4g+2}{\gcd(2g+1, 4g+2)} = 2$ and $g(2g-1) \equiv 1 \pmod{2g+1}$, the multiple orbit of valency $\frac{g}{2g+1}$ consists of two points and $h_{4g+2,2g+1}^2$ acts on a small neighborhood of each point by $\frac{2(2g-1)\pi}{2g+1}$ rotation. Since $\frac{4g+2}{\gcd(2, 4g+2)} = 2g+1$ and $1 \cdot 1 \equiv 1 \pmod{2}$, the multiple orbit of valency $\frac{1}{2}$ consists of $2g+1$ points and $h_{4g+2,2g+1}^{2g+1}$ acts on a small neighborhood of each point by $\frac{2\pi}{2}$ rotation.

Thus, the $h_{4g+2,2g+1}$ -orbit of the rotation angle $\frac{2\pi}{4g+2}$ is a fixed point of $h_{4g+2,2g+1}^{2g}$ of rotation angle $\frac{2g\pi}{2g+1}$, and the $h_{4g+2,2g+1}$ -orbit of the rotation angle $\frac{2\pi}{2}$ is a free $h_{4g+2,2g+1}^{2g}$ -orbit.

On the other hand, since the $h_{4g+2,2g+1}$ -orbit of the rotation angle $\frac{2(2g-1)\pi}{2g+1}$ consists of two points and $2g$ is even, the orbit splits into two fixed points of $h_{4g+2,2g+1}^{2g}$ with rotation angle $\frac{2\pi}{2g+1}$. Therefore, the total valency of $h_{4g+2,2g+1}^{2g}$ is equal to $\left[g, 2g+1; \frac{1}{2g+1} + \frac{1}{2g+1} + \frac{2g-1}{2g+1} \right]$. Now, by Proposition 3.1, the total valency of $h_{2g+1,1}$ is equal to the one of $h_{4g+2,2g+1}^{2g}$, and hence, by Theorem 2.1, we obtain the conclusion.

Finally, by the equation

$$h_{2g+1,1} \circ I = h_{4g+2,2g+1}^{2g} \circ h_{4g+2,2g+1}^{2g+1} = h_{4g+2,2g+1}^{2g+1} \circ h_{4g+2,2g+1}^{2g} = I \circ h_{2g+1,1},$$

the diffeomorphism $h_{2g+1,1}$ is hyperelliptic. $\square$

Proposition A.2. *The intersection of $\text{Fix}(h_{2g+1,1})$ and $\text{Fix}(I)$ consists of a single point.*

Proof. By Lemma A.1, we can identify $h_{2g+1,1}$ with $h_{4g+2,2g+1}^{2g}$. Moreover, we already have $h_{4g+2,2g+1}^{2g+1} = I$. Therefore, $\text{Fix}(h_{2g+1,1}) \cap \text{Fix}(I)$ contains at least the fixed point of $h_{4g+2,2g+1}$.

Thus, if we let the projection $\Sigma_g \rightarrow \Sigma_g / \langle h_{2g+1,1} \rangle$ be p , since I induces an involution on $\Sigma_g / \langle h_{2g+1,1} \rangle$, which is a sphere with three cone points, I exchanges two multiple orbits whose valencies are $\frac{1}{2g+1}$ and preserves the remaining multiple orbit. $\square$

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