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JONES POLYNOMIAL OF A KNOT WITH SMALL SPAN

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Abstract

We decide the Jones polynomials of knots with span up to four and enumerate the potential Jones polynomials of knots with span five and six.

1. Introduction

The *span* of a Laurent polynomial $f(t) \in \mathbb{Z}[t^{\pm 1}]$ is the difference between the highest and lowest degrees of $f(t)$. The span of the Jones polynomial of a knot K is less than or equal to the least crossing number of K. Moreover, the span of the Jones polynomial of a nonsplit alternating link equals its crossing number [7, 13, 14].

The Jones polynomial of a knot with span zero is 1 [5, Corollary 3] and that with span one or two does not exist [4, Lemma 3.9]. In this note we show the following.

Theorem 1.1. *The Jones polynomial of a knot with span at most six is one of the polynomials $f_k(t)$ or $f_k(t^{-1})$, $k = 1, 2, \ldots, 19$, listed in Table 1. Note that $f_k(t) = f_k(t^{-1})$ for $k = 1, 3, 11, 13, 16$.*

Therefore, we can decide the Jones polynomials of knots with span up to four.

Corollary 1.2. *If the span of the Jones polynomial $V(t)$ of a knot is at most four, then it is the Jones polynomial of the unknot, trefoil knot, or figure-eight knot, that is, $V(t) = 1$, $t + t^3 - t^4$, $-t^{-4} + t^{-3} + t^{-1}$, or $t^{-2} - t^{-1} + 1 - t + t^2$.*

In Table 1 the notation $(r)[c_0 c_1 c_2 \ldots c_n]$ denotes the polynomial $t^r(c_0 + c_1 t + c_2 t^2 + \cdots + c_n t^n)$. In column "Knots" we list the knots with up to 13 crossings whose Jones polynomial is $f_k(t)$. For $f_8(t)$ and $f_{19}(t)$ we cannot find such knots with up to 18 crossings; see Remark 3.1 and Question 3.2. We denote the mirror image of a knot K by $K!$. Then $V(K!;t) = V(K;t^{-1})$. We use the knot names in [9] for a knot with up to 13 crossings and those in [2] for a knot with 14–18 crossings.

k	Polynomial	Knots
1	$(0)[+1]$	U
2	$(1)[+1+0+1-1]$	3_1
3	$(-2)[+1-1+1-1+1]$	4_1, 11n_19
4	$(2)[+1+0+1-1+1-1]$	5_1, $10_{132}!$
5	$(1)[+1-1+2-1+1-1]$	5_2, 11n_57, 12n_475, 13n_3082
6	$(3)[+1+0+1+0+0-1]$	8_{19}
7	$(4)[+2-1+2-2+1-1]$	12n_200!
8	$(8)[+2+0+2-2+1-2]$	
9	$(-2)[+1-1+2-2+1-1+1]$	6_1
10	$(-1)[+1-1+2-2+2-2+1]$	6_2, 12n_25, 13n_1169, 13n_4304
11	$(-3)[-1+2-2+3-2+2-1]$	6_3, 13n_2922
12	$(2)[+1+0+2-2+1-2+1]$	$3_1\#3_1$
13	$(-3)[-1+1-1+3-1+1-1]$	$3_1!\#3_1$
14	$(-5)[-1+1-1+2-1+2-1]$	8_{20}
15	$(1)[+2-2+3-3+2-2+1]$	8_{21}
16	$(-3)[+1-1+1-1+1-1+1]$	9_{42}
17	$(0)[+2-1+1-2+1-1+1]$	$9_{46}!$
18	$(4)[+1+0+1+0+0+0-1]$	10_{124}
19	$(6)[+1+1+0+1-1+0-1]$	

Table 1. Potential knot Jones polynomials $f_k(t)$ with span up to six.

This note is organized as follows. In Sect. 2 we review the Jones polynomial and give some restrictions a knot Jones polynomial satisfies. Using them we prove Theorem 1.1 in Sect. 3. In Sect. 4 we consider potential knot Jones polynomials with span ≥ 7.

2. Jones polynomial

The Jones polynomial $V(L;t) \in \mathbb{Z}[t^{\pm 1/2}]$ [6] is an invariant of the isotopy type of an oriented link L, which are defined by the following formulas:

$$V(U;t) = 1, \tag{1}$$

$$t^{-1}V(L_+;t) - tV(L_-;t) = (t^{1/2} - t^{-1/2})V(L_0;t), \tag{2}$$

where U is the unknot and (L_+, L_-, L_0) is a skein triple, which is an ordered set of three oriented links that are identical except near one point where they are as in Fig. 1.

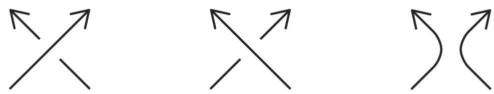

FIGURE 1. A skein triple.

Let $V(t)$ be the Jones polynomial of a knot K. Then we have the following evaluations:

$$V(1) = 1, \tag{3}$$

$$V'(1) = 0, \tag{4}$$

$$V(e^{2\pi i/3}) = 1, \tag{5}$$

$$V(i) = \pm 1, \tag{6}$$

$$V(e^{\pi i/3}) = \pm(i\sqrt{3})^d, \tag{7}$$

$$V(-1) \equiv 0 \pmod{3^d}, \tag{8}$$

where $V'(1)$ is the value of the first derivative at $t = 1$, and $d = \dim H_1(\Sigma_2(K); \mathbb{Z}_3)$ with $\Sigma_2(K)$ the double covering space of S^3 branched over K. Equations (3)–(5) follow from the fact that $V(t) - 1$ is divisible by $(1-t)(1-t^3)$ [6, Proposition 12.5]; cf. [5, Theorem 1]. Equation (6) follows from $V(i) = (-1)^{\mathrm{Arf}(K)}$, where $\mathrm{Arf}(K)$ is the Arf (or Robertello) invariant of K [8, 12], cf.[6, (12.6)]. Note that $\mathrm{Arf}(K) \equiv a_2(K) \pmod 2$ and that $a_2(K) = -V''(1)/6$ [11], where $a_2(K)$ is the second coefficient of the Conway polynomial of K. Equation (7) is proved in [8]. Since $|V(-1)|$ is the determinant of K, that is, $|H_1(\Sigma_2(K); \mathbb{Z})|$, we obtain Eq. (8).

3. Proof of Theorem 1.1

First, we prove for span ≤ 4 (Corollary 1.2). Let $V(t) = t^r(c_0 + c_1 t + c_2 t^2 + c_3 t^3 + c_4 t^4)$, where r, c_0, c_1, c_2, c_3, $c_4 \in \mathbb{Z}$. Then using Eq. (3), we obtain

$$c_0 + c_1 + c_2 + c_3 + c_4 = 1. \tag{9}$$

Using Eq. (4), we obtain $r(c_0 + c_1 + c_2 + c_3 + c_4) + (c_1 + 2c_2 + 3c_3 + 4c_4) = 0$, and so

$$r + c_1 + 2c_2 + 3c_3 + 4c_4 = 0. \tag{10}$$

Let $\xi = e^{2\pi i/3}$. Then we have

$$V(\xi) = \begin{cases} (c_0 - c_2 + c_3) + (c_1 - c_2 + c_4)\xi & \text{if } r \equiv 0 \pmod 3, \\ (-c_1 + c_2 - c_4) + (c_0 - c_1 + c_3 - c_4)\xi & \text{if } r \equiv 1 \pmod 3, \\ (-c_0 + c_1 - c_3 + c_4) + (-c_0 + c_2 - c_3)\xi & \text{if } r \equiv 2 \pmod 3. \end{cases} \tag{11}$$

Using Eq. (5), we have

$$c_0 - c_2 + c_3 = 1, \qquad c_1 - c_2 + c_4 = 0 \quad \text{if } r \equiv 0 \pmod 3, \tag{12}$$

$$-c_1 + c_2 - c_4 = 1, \qquad c_0 - c_1 + c_3 - c_4 = 0 \quad \text{if } r \equiv 1 \pmod 3, \tag{13}$$

$$-c_0 + c_1 - c_3 + c_4 = 1, \qquad -c_0 + c_2 - c_3 = 0 \quad \text{if } r \equiv 2 \pmod 3. \tag{14}$$

Since

$$V(i) = i^r((c_0 - c_2 + c_4) + (c_1 - c_3)i), \tag{15}$$

using Eq. (6), we have

$$c_1 - c_3 = 0, \qquad c_0 - c_2 + c_4 = \pm 1 \quad \text{if } r \equiv 0 \pmod 2, \tag{16}$$

$$c_0 - c_2 + c_4 = 0, \qquad -c_1 + c_3 = \pm 1 \quad \text{if } r \equiv 1 \pmod 2. \tag{17}$$

Since $V(t) \neq 0$ by Eq. (3), we may assume $c_0 \neq 0$. Then, we obtain

$$(r, c_0, c_1, c_2, c_3, c_4) = \begin{cases} (0,1,0,0,0,0) & \text{if } r \equiv 0 \pmod 6, \\ (1,1,0,1,-1,0) & \text{if } r \equiv 1 \pmod 6, \\ (-4,-1,1,0,1,0) & \text{if } r \equiv 2 \pmod 6, \\ (3,1,1,0,0,-1) & \text{if } r \equiv 3 \pmod 6, \\ (-2,1,-1,1,-1,1) & \text{if } r \equiv 4 \pmod 6, \\ (-7,-1,0,0,1,1) & \text{if } r \equiv 5 \pmod 6. \end{cases} \tag{18}$$

The solutions $(3,1,1,0,0,-1)$ and $(-7,-1,0,0,1,1)$ give the polynomials $\varphi(t) = t^3 + t^4 - t^7$ and $\varphi(t^{-1}) = -t^{-7} + t^{-4} + t^{-3}$, respectively. However, they cannot be the Jones polynomials of knots. In fact, $\varphi(e^{\pi i/3}) = -2 - i\sqrt{3}$, which contradicts Eq. (7). The other solutions yield $f_1(t)$, $f_2(t)$, $f_2(t^{-1})$, $f_3(t)$.

Next, we prove for span 5. Let $V(t)$ be the Jones polynomial of a knot with span 5. By using Eqs. (3)–(7), $V(t)$ is one of the following polynomials: $f_k(t)$, $f_k(t^{-1})$ $(k = 5, 6, 7)$, $\psi(t, a)$, $\psi(t^{-1}, a)$, where

$$\psi(t, a) = t^{6a-4}(a + at^2 - at^3 + t^4 - at^5), \tag{19}$$

with $a = (1 \pm (i\sqrt{3})^d)/2$, $d \in \mathbb{Z}_{\geq 0}$. Since a is an integer, d should be even. For $(d, a) = (0, 0)$ and $(0, 1)$ we have $\psi(t, 0) = 1$ and $\psi(t, 1) = f_4(t)$, respectively. For $(d, a) = (2, 2)$ we have $\psi(t, 2) = f_8(t)$. For $(d, a) = (2, -1)$ we have $\psi(t, -1) = t^{-10}(-1 - t^2 + t^3 + t^4 + t^5)$ and $\psi(-1, -1) = -3$, contradicting Eq. (8). If $d \geq 4$, then $\psi(-1, a) = 4a + 1 = 3 \pm 2 \cdot 3^{d/2} \not\equiv 0 \pmod{3^{d/2}}$, contradicting Eq. (8).

The proof for span 6 is similar, and so omit it. This completes the proof of Theorem 1.1.

Remark 3.1. There is no knot with up to 18 crossings whose Jones polynomial is $f_8(t)$ or $f_{19}(t)$ listed in Table 1. However, $V(12\text{n\_}850) = V(4_1)f_{19}(t)$.

A knot with Jones polynomial $f_8(t)$ or $f_{19}(t)$ should be neither almost alternating nor of Turaev genus one; for $f_8(t)$ this follows from [3] and for $f_{19}(t)$ from [10].

Question 3.2. *Does there exist a knot whose Jones polynomial is $f_8(t)$ or $f_{19}(t)$?*

4. Polynomials with span ≥ 7

For the polynomials with span ≥ 7 satisfying Eqs. (3)–(8) we have the following.

Theorem 4.1. *For each integer $n \geq 7$ there exist infinitely many polynomials $V(t) \in \mathbb{Z}[t^{\pm 1}]$ with span n satisfying Eqs.* (3)–(8).

Proof. First, we consider $n = 7$. The polynomial with span 7

$$V(t) = t^{6a}\left(1 + a(1-t)(1+t^2)(1-t+t^2)(1+t+t^2)\right), \tag{20}$$

$a \in \mathbb{Z} \setminus \{0, -1\}$, satisfies Eqs. (3)–(8). In fact, $V(i) = (-1)^a$, $V(e^{\pi i/3}) = 1$ and $V(-1) = 1 + 12a \equiv 1 \pmod 3$.

Suppose that $n \geq 8$. The polynomial

$$V(t) = 1 + (a + t + t^2 + \cdots + t^{n-8})(1-t)^2(1+t^2)(1-t+t^2)(1+t+t^2), \tag{21}$$

$a \in \mathbb{Z}$, satisfies Eqs. (3)–(8). In fact, $V(e^{\pi i/3}) = 1$ and $V(-1) = 24a + 12(-1)^n - 11 \equiv 1 \pmod 3$. Note that for $n = 8$ the span of $V(t)$ is 8 if $a \neq 0, -1$, and for $n \geq 9$ the span of $V(t)$ is n if $a \neq -1$. $\square$

Remark 4.2. We found the polynomial $V(t)$ in Eq. (20) in a similar way to the proof of Theorem 1.1, and $V(t)$ in Eq. (21) using Theorem 2 in [1].

The polynomial $V(t)$ in Eq. (20) with $a = -1$ is $f_{10}(t^{-1})$ in Table 1, which is $V(6_2!)$. The polynomials $V(t)$ in Eq. (21) with $(a, n) = (1, 8), (-2, 8), (-3, 9)$ are $V(12\text{n\_}838)$, $V(12\text{n\_}730)$, $V(11\text{n\_}178)$, respectively.

Question 4.3. *For each integer $n \geq 7$ do there exist infinitely many knot Jones polynomials with span n?*

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