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Convergent Solution of Ordinary Nonlinear Differential Equations

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0. Introduction.

The singular points $x=0$ of a system of ordinary nonlinear differential equations of the form

$$(0.1) \qquad xy'=f(x, y) \qquad \left({}'=\frac{d}{dx}\right)$$

are usually said to be of Briot-Bouquet type. Here, x is a complex variable, y is an n-dimensional column vector, $f(x, y)$ is an n-dimensional column vector whose components are functions holomorphic and bounded in (x, y) for

$$(0.2) \qquad |x|<a, \quad ||y||<b \qquad (||y||=\max_j |y_j|)$$

and vanishes at $(0, 0)$, y_j being the j^{th} component of y.

Such singular points have been studied by diverse authors since C. H. Briot-J. C. Bouquet. However, it has not yet been studied, except for the case $n=1$, when the eigenvalues of the matrix $F\equiv\partial f_y(0, 0)$ are all zero.

1°. In 1937, M. Hukuhara [1] studied first the case for $n=1$. In this case the differential equation is written as

$$(\mathrm{A}) \qquad xy'=y^m f^{(0)}(y)y+\sum_{k=1}^{\infty} f^{(k)}(y)x^k,$$

where m is a positive integer and $f^{(k)}(y)$ $(k\geqq 0)$ are functions holomorphic and bounded in y for

$$(\mathrm{A}.1) \qquad |y|<b.$$

The detailed statement of M. Hukuhara's result is as follows:

Theorem A. *Suppose that the power series in the right-hand member of the equation* (A) *is uniformly convergent in*

$$(\mathrm{A}.2) \qquad |x|<a, \quad |y|<b$$

and that

$$(\mathrm{A}.3) \qquad f^{(0)}(0)\equiv\alpha\neq 0.$$

Then, there exists a solution $y=S(x, U)$ with the properties that

i) *$S(x, u)$ is a function holomorphic and bounded in (x, u) for*

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(A. 4) $$|x|<a_0,\quad 0<|u|<b_0,\quad \Theta_-<\arg u<\Theta_+,$$

where

(A. 5) $$\Theta_\pm=\frac{1}{m}\left(-\arg\alpha\pm\frac{3\pi}{2}\right),$$

and developable there in a uniformly convergent power series of x:

(A. 6) $$S(x,u)=u(1+P^{(0)}(u))+\sum_{k=1}^{\infty}P^{(k)}(u)x^k.$$

ii) *The coefficient* $P^{(0)}(u)$ *is a function holomorphic and bounded in* u *for*

(A. 7) $$|u|<b_0$$

and vanishing at $u=0$.

iii) *The coefficients* $P^{(k)}(u)$ $(k\geqq 1)$ *are functions holomorphic, bounded and asymptotically developable in powers of* u *as* u *tends to* 0 *in the domain*

(A. 8) $$0<|u|<b_0,\quad \Theta_-<\arg u<\Theta_+.$$

iv) $U=U(x,x_0,u^0)$ *is the holomorphic solution of a simplified equation of* (A) *satisfying the initial condition* $U=u^0$ *at* $x=x_0$, *where* (x_0,u^0) *is an arbitrary point in the domain* (A. 4). *The simplified equation has the form*

(A. 9) $$xu'=u^m(\alpha+\alpha^*u^m)u,$$

where α^* *is a complex constant which may be zero.*

Moreover, he investigated precisely the property of the general solution of (A. 9). After eighteen years, his result played an important role in the theory of T. Kimura [3, 4]. So, it is of some interest, I think, to discuss a system of differential equations such that the eigenvalues of the matrix F are all zero.

2°. Recently, the author (M. Iwano [5]) has studied a system of two ordinary nonlinear differential equations of the form

(B) $$xy'=y^{\mathcal{M}}\mathbf{1}_2(f^{(0)}(y))y+\sum_{k=1}^{\infty}f^{(k)}(y)x^k,$$

where y is a 2-dimensional column vector and $\mathbf{1}_2(y)$ is a 2-dimensional diagonal matrix such that its diagonal elements coincide with the components of y, $f^{(k)}(y)$ $(k\geqq 0)$ are 2-dimensional column vector functions holomorphic and bounded in y for

(B. 1) $$\|y\|<b,$$

$\mathcal{M}$ is a 2-dimensional row vector whose components m_1 and m_2 are nonnegative integers not simultaneously zero and the symbol $y^{\mathcal{M}}$ stands for the scalar expression

(B. 2) $$y^{\mathcal{M}}=y_1^{m_1}y_2^{m_2}.$$

Actually, in this case, F is the 2-by-2 zero matrix.

He introduced the following assumption:

Assumption B. *The real parts of the components of the vector* $\dfrac{1}{\mathcal{M}\cdot\alpha}\alpha$ *are* *both positive, where*

(B. 3) $$f^{(0)}(0)=\alpha,$$

i. e. we have the inequalities

(B. 4) $$\mathrm{Re}\frac{\alpha_j}{\mathcal{M}\cdot\alpha}>0 \qquad (j=1,2).$$

The symbol $\cdot$ means the inner product.

Under these assumptions, M. Iwano [5] obtained formal solutions of diverse types of the equation (B) and investigated the analytical meaning of each of these formal solutions.

3°. We consider a system of n differential equations of the form

(C) $$xy'=y^{\mathcal{M}}\mathbf{1}_n(f^{(0)}(y))y+\sum_{k=1}^{\infty}f^{(k)}(y)x^k,$$

where y is an n-dimensional column vector and $\mathbf{1}_n(y)$ is an n-by-n diagonal matrix whose diagonal elements coincide with those of y, the n-dimensional row vector $\mathcal{M}$ has components m_j $(j=1,\cdots,n)$ which are nonnegative integers not simultaneously zero, the symbol $y^{\mathcal{M}}$ stands for the scalar expression

(C. 1) $$y^{\mathcal{M}}=y_1^{m_1}y_2^{m_2}\cdots y_n^{m_n},$$

$f^{(k)}(y)$ $(k\geqq 0)$ are n-dimensional column vector functions holomorphic and bounded in y for

(C. 2) $$\|y\|<b.$$

We assume that the power series in the right-hand member of the equation (C) is uniformly convergent in

(C. 3) $$|x|<a, \quad \|y\|<b$$

and introduce the following assumption, similar to the assumption B:

Assumption C. *The real parts of the components of the vector* $\frac{1}{\mathcal{M}\cdot\alpha}\alpha$, *where* $\alpha=f^{(0)}(0)$, *are all positive or, what is the same thing,*

(C. 4) $$\mathrm{Re}\frac{\alpha_j}{\mathcal{M}\cdot\alpha}>0 \qquad (j=1,\cdots,n),$$

α_j *being the* j^{th} *component of the vector* α.

The purpose of the present note is to prove the existence of a solution of the eqution (C) which contains n parameters and is developable in a uniformly convergent power series of x.

However, the analysis which is developed here will be almost exactly the same as that in [5]. We omit therefore the proof of Theorems and refer the reader to a forthcoming paper [6] of mine.

1. Formal transformation.

The result concerning a formal transformation can be stated as follows:

Theorem 1.1. *There exists a formal transformation of the form*

(f) $$y \sim u + \mathbf{1}_n(P^{(0)}(u))u + \sum_{k=1}^{\infty} P^{(k)}(u)x^k$$

by which the equation (C) *is formally transformed into an equation of the form*

(R) $$xu' = u^{\mathcal{M}}\mathbf{1}_n(\alpha + \sum_{\mathcal{H}\in S} \beta_{\mathcal{H}} u^{\mathcal{H}})u.$$

i) $P^{(k)}(u)$ $(k \geqq 0)$ *are n-dimensional column vectors whose components are formal power series of* u:

$(1.1)^k$ $$P^{(k)}(u) \sim \sum_{\mathcal{R}} P^{(k)}_{\mathcal{R}} u^{\mathcal{R}}.$$

ii) S *is the set of vectors* $\mathcal{H}$ *satisfying the equation*

(1.2) $$(\mathcal{H} - \mathcal{M})\cdot\alpha = 0.$$

iii) *If the n-vector constant* $\beta_{\mathcal{H}}$ *is different from zero, we have*

(1.3) $$(\mathcal{R} - \mathcal{M})\cdot\beta_{\mathcal{H}} = 0 \qquad \textit{for any } \mathcal{R}\in S.$$

The proof of this theorem will be carried out in quite a similar way to that of Theorem 1 in Section 2 of M. Iwano [5] and will be found in [6].

Remarks. 1°. By virtue of the Assumption C, S is a finite set. If any two components of the vector α are incommensurable, the set S consists of only one element $\{\mathcal{M}\}$.

2°. The formal transformation (f) is obtained by combining two kinds of formal transformations.

The formal transformation of the first kind is of the form

(1.4) $$y \sim z + \sum_{k=1}^{\infty} p^{(k)}(z)x^k$$

by which the equation (C) is formally transformed into

(1.5) $$xz' = z^{\mathcal{M}}\mathbf{1}_n(z)f^{(0)}(z).$$

Here, the n-vectors $p^{(k)}(z)$ are formal power series of z.

The formal transformation of the second kind is of the form

(1.6) $$z \sim u + \mathbf{1}_n(u)\sum_{\mathcal{R}} p_{\mathcal{R}} u^{\mathcal{R}}$$

by which the equation satisfied by u takes the form (R). Here, $p_{\mathcal{R}}$ are n-dimensional constant column vectors.

2. Integration of the equation (R).

Let $U = U(x, x_0, u^0)$ be the holomorphic solution of (R) satisfying the initial condition $U = u^0$ at $x = x_0$. A direct solution of (R) by quadratures is generally impossible. However, if we introduce an auxiliary variable

(2.1) $$w = U^{\mathcal{M}},$$

we can obtain a parametric repesentation of the solution $U(x, x_0, u^0)$.

We have first

Proposition 2.1. *The function* $U^{\mathcal{H}-\mathcal{M}}$ *for any* $\mathcal{H}\in S$ *is independent of* x

and, consequently, we have

(2.2) $$U^{\mathcal{H}-\mathcal{M}}=(u^0)^{\mathcal{H}-\mathcal{M}}.$$

Proof. It is enough to prove that $x(U^{\mathcal{H}-\mathcal{M}})'=0$. However, this relation is verified by a direct calculation.

We define the n-vector α^* by

(2.3) $$\alpha^*=\sum_{\mathcal{H}\in S}\beta_{\mathcal{H}}(u^0)^{\mathcal{H}-\mathcal{M}}.$$

Then we have

Proposition 2.2. *The solution $U(x, x_0, u^0)$ is parametrically expressed as*

(2.4) $$\begin{cases} \tilde{x}(w)=c\left(1+\dfrac{\mathcal{M}\cdot\alpha}{\mathcal{M}\cdot\alpha^*}\dfrac{1}{w}\right)^{\frac{\mathcal{M}\cdot\alpha^*}{(\mathcal{M}\cdot\alpha)^2}}\exp\left(-\dfrac{1}{\mathcal{M}\cdot\alpha w}\right), \\ \tilde{U}(w)=\mathbf{1}_n\left(w^{\frac{1}{\mathcal{M}\cdot\alpha}\alpha}\right)\mathbf{1}_n\left((\mathcal{M}\cdot\alpha+\mathcal{M}\cdot\alpha^*w)^{\frac{1}{\mathcal{M}\cdot\alpha^*}\alpha^*-\frac{1}{\mathcal{M}\cdot\alpha}\alpha}\right)C. \end{cases}$$

Here, for the n-vector β and the scalar w,

$$w^{\beta}=(w^{\beta_1}, \cdots, w^{\beta_n}).$$

The constant c and the n-vector C must be so chosen that $\tilde{x}(w^0)=x_0$ and $\tilde{U}(w^0)=u^0$, where $w^0=(u^0)^{\mathcal{M}}$.

Proof. By differentiating both sides of (2.1), we have

$$xw'=w^2(\mathcal{M}\cdot\alpha+\mathcal{M}\cdot\alpha^*w).$$

From this it follows immediately that

(2.5) $$w^2\frac{dx}{dw}=\left(\frac{1}{\mathcal{M}\cdot\alpha+\mathcal{M}\cdot\alpha^*w}\right)x.$$

Therefore, the function U must satisfy the equation

(2.6) $$w\frac{dU}{dw}=\mathbf{1}_n\left(\frac{\alpha+\alpha^*w}{\mathcal{M}\cdot\alpha+\mathcal{M}\cdot\alpha^*w}\right)U.$$

The formula (2.4) is obtained by integrating the equations (2.5) and (2.6) with the initial conditions $\tilde{x}(w^0)=x_0$ and $\tilde{U}(w^0)=u^0$.

Remark. Let $\hat{U}(w)$ be a solution of the equation (2.6) satisfying the initial condition $\hat{U}=\hat{u}^0$. A short calculation shows that the expression $\hat{U}(w)^{\mathcal{M}}-w$ satisfies the linear differential equation $w\dfrac{dY}{dw}=Y$. From this we see that the relation $\hat{U}(w)^{\mathcal{M}}\equiv w$ holds if and only if we have $(u^0)^{\mathcal{M}}=w^0$.

On the other hand, if we use a certain transcendental function introduced by M. Hukuhara [1, 4], we can express the function U as a function of x.

Proposition 2.3. *Let $W=\mathfrak{H}(X)$ be defined implicitly by*

(2.7) $$X=W-\log(W+1).$$

and consider the branch of $\mathfrak{H}(X)$ such that $\mathfrak{H}(X)-X-\log X$ tends to 0 as X tends to the infinity. Then, the solution U is expressed as

$$(2.8) \qquad U=\mathbf{1}_n\left(\mathfrak{H}(X+c_1)^{-\frac{1}{\mathcal{M}\cdot\alpha}\alpha}\right)\mathbf{1}_n\left(\left(\mathcal{M}\cdot\alpha[1+\mathfrak{H}(X+c_1)^{-1}]\right)^{\frac{1}{\mathcal{M}\cdot\alpha^*}\alpha^*-\frac{1}{\mathcal{M}\cdot\alpha}\alpha}\right)C,$$

$$c^{\mathcal{M}}=1, \quad X=-\frac{(\mathcal{M}\cdot\alpha)^2}{\mathcal{M}\cdot\alpha^*}\log x.$$

Proof. Following Hukuhara, we put

$$(2.9) \qquad x=c\exp\left(-\frac{\mathcal{M}\cdot\alpha^*}{(\mathcal{M}\cdot\alpha)^2}X\right), \qquad w=\frac{\mathcal{M}\cdot\alpha}{\mathcal{M}\cdot\alpha^* W}.$$

Then, the equation (2.5) is transformed into

$$(2.10) \qquad \frac{dW}{dX}=1+\frac{1}{W}.$$

By integrating this, we have the equation $X+c_1-W-\log(W+1)$. Let $W=\mathfrak{H}(X+c_1)$ be defined implicitly by this relation. If we substitute the expression $\mathcal{M}\cdot\alpha/\mathcal{M}\cdot\alpha^*\mathfrak{H}(X+c_1)$ for w in the second equation of (2.4), we have the formula (2.8).

3. Estimation of the growth of the solution U near the origin $x=0$.

We see by the first equation of (2.4) that the function $\tilde{x}(w)$ tends to 0 exponentially as w tends to 0 in the sector

$$(3.1) \qquad \theta_- < \arg w < \theta_+, \qquad \theta_\pm = -\arg(\mathcal{M}\cdot\alpha)\pm\frac{\pi}{2}.$$

Let

$$(3.2) \qquad \Theta_1=-\pi-\min_j \arg\alpha_j+4\delta_0, \quad \Theta_2=\pi-\max_j \arg\alpha_j-4\delta_0,$$

where δ_0 is a sufficiently small positive constant. We consider a domain of the form

$$(3.3) \qquad \mathfrak{D}(c)=\{w:\Theta_1<\arg w<\Theta_2, \quad 0<|w|<d(\arg w, c)\},$$

where

$$(3.4) \qquad d(\varphi, c)=c\exp\int_{\theta_0}^{\varphi}\cot A(\tau)d\tau.$$

Here

$$(3.5) \qquad A(\tau)=\begin{cases}\max(\varphi-\theta_++2\delta_0, \delta_0) & \text{for} \quad \varphi\in[\theta_+-2\delta_0, \Theta_2],\\ \pi/2 & \text{for} \quad \varphi\in[\theta_-+2\delta_0, \theta_+-2\delta_0],\\ \min(\varphi-\theta_-+\pi-2\delta_0, \pi-\delta_0) & \text{for} \quad \varphi\in[\Theta_1, \theta_-+2\delta_0],\end{cases}$$

and $ce^{i\theta_0}$ is a boundary point of the domain $\mathfrak{D}(c)$.

By the definition of θ_-, θ_+, Θ_1 and Θ_2, it is easy to see that $\delta_0\leqq A(\varphi)\leqq\pi-\delta_0$ for $\Theta_1\leqq\varphi\leqq\Theta_2$.

By virtue of the Assumption C, we can assume without loss of generality that

$$(3.6) \qquad \left\|\arg\frac{1}{\mathcal{M}\cdot\alpha}\alpha\right\|\leqq\frac{\pi}{2}-3\delta_0, \qquad (\|\arg\beta\|=\max_j|\arg\beta_j|),$$

for the same δ_0 as before. Then, we can easily verify that

(3.7) $$\Theta_1 < \theta_- + 2\delta_0 < \theta_+ - 2\delta_0 < \Theta_2.$$

Let x_0 and u^0 be arbitrary values such that

(3.8) $$|x_0| < a_0, \quad 0 < \| u^0 \| < b_0, \quad (u^0)^{\mathcal{M}} \in \mathfrak{D}(c_0)$$

and let $\tilde{x}(w)$ and $\tilde{U}(w)$ be the holomorphic solutions of the equations (2.5) and (2.6) satisfying the initial conditions $\tilde{x}(w^0) = x_0$ and $\tilde{U}(w^0) = u^0$. Of course, $w^0 = (u^0)^{\mathcal{M}}$. Then, we have the relations

(3.9) $$\tilde{U}(\tilde{x}^{-1}(x)) \equiv U(x, x_0, u^0), \quad \tilde{U}(w)^{\mathcal{M}} \equiv w,$$

because of the Remark following Proposition 2.2.

Then, we have the following

Theorem 3.1. *There exists, in the domain $w \in \mathfrak{D}(c_0)$, a path $\Gamma^*_{w^0}$, which starts form $w = w^0$ and approaches the origin in the sector $\theta_- + 2\delta_0 \leqq \arg w \leqq \theta_+ - 2\delta_0$, such that we have the inequalities*

(3.10) $$\frac{d|\tilde{x}(w)|}{ds} \geqq \frac{\sin \delta_0}{2|\mathcal{M}\cdot\alpha|\,|w|^2} |\tilde{x}(w)|,$$

(3.11) $$\frac{d\| \tilde{U}(w) \|}{ds} \geqq \frac{\| \alpha \|' \sin \delta_0}{2|\mathcal{M}\cdot\alpha|\,|w|} \| \tilde{U}(w) \|, \qquad (\| \alpha \|' = \min_j |\alpha_j|),$$

(3.12) $$\left| \frac{dw}{ds} \right| = 1$$

on the path $\Gamma^*_{w^0}$, where s is the length of the curve $\Gamma^*_{w^0}$ measured from the origin to the variable point w.

The curve $\Gamma^*_{w^0}$ is defined as follows: Let $w^0 = re^{i\theta}$ and $w = \rho e^{i\varphi} \in \Gamma^*_{w^0}$. If $\theta_- + 2\delta_0 \leqq \theta \leqq \theta_+ - 2\delta_0$, the path $\Gamma^*_{w^0}$ is the segment joining w^0 with the origin. If $\theta_+ - 2\delta_0 \leqq \theta < \Theta_2$ or $\Theta_1 < \theta \leqq \theta_- + 2\delta_0$, the curve $\Gamma^*_{w^0}$ consists of a curvilinear part Γ' defined by

$$\rho = r \exp \int_\theta^\varphi \cot A(\tau) d\tau, \qquad \theta_+ - 2\delta_0 \leqq \varphi \leqq \theta \quad or \quad \theta \leqq \varphi \leqq \theta_- + 2\delta_0$$

and of a rectilinear part Γ''

$$0 \leqq \rho \leqq r \exp \int_\theta^{\theta_+ - 2\delta_0} \cot A(\tau) d\tau \quad or \quad 0 \leqq \rho \leqq r \exp \int_\theta^{\theta_- - 2\delta_0} \cot A(\tau) d\tau.$$

Since the proof of this theorem is almost exactly the samc as that of Lemma 3.2 in Section 12 (M. Iwano [5]), we omit the proof and refer the reader to M. Iwano [6].

We denote by Γ_{x_0} a curve in the complex x-plane obtained by mapping $\Gamma^*_{w^0}$ by the first equation of (2.4), where the integration constant c must be so chosen that the point $w = w_0$ is mapped into $x = x_0$. Clearly, Γ_{x_0} is spiral-shaped.

Since the correspondence between the points on these two curves is one-to-one, the solution $U(x, x_0, u^0)$ and the variable point x on Γ_{x_0} can be considered as functionsof $w\in\Gamma^*_{w^0}$. We denote them by $\tilde{x}(w)$ and $\tilde{U}(w)$. Explicit representation of these functions is given by the formula (2.4) for the suitably chosen constant c and the n-vector C.

By virtue of the relations (3.9) and Theorem 3.1, we have

Theorem 3.2. *There exists a spiral-shaped path Γ_{x_0} which starts from $x=x_0$ and approaches $x=0$, turning aroud the origin in the complex x-plane, such that when x moves on this curve the values of x and $U(x, x_0, u^0)$ always remain in the domain*

$$(3.8)^* \qquad |x|<a_0, \quad 0<\|u\|<b_0, \quad u^{\mathcal{M}}\in\mathfrak{D}(c_0).$$

Proof. The inequalities (3.10) and (3.11) imply that $|\tilde{x}(w)|$ and $\|\tilde{U}(w)\|$ are monotonously decreasing functions as w tends to 0 along $\Gamma^*_{w^0}$, whence we have $|x|\leqq|x_0|$ and $\|U\|\leqq\|u^0\|$ for $x\in\Gamma_{x_0}$.

4. Formal solution.

Let x_0 and u^0 be arbitrary values satisfying the inequalities (3.8) and define the solution $U(x, x_0, u^0)$ as before. Then, by virtue of Theorem 1.1, if we replace u by U, we have a formal solution of the form

$$(\mathrm{F}) \qquad y\sim U+\mathbf{1}_n(U)P^{(0)}(U)+\sum_{k=1}^{\infty}P^{(k)}(U)x^k,$$

where the n-vectors $P^{(k)}(u)$ $(k\geqq 0)$ are the power series of u given by $(1.1)^k$.

In order to give the formal solution (F) an analytical meaning, it is first necessary that the coefficients defined by the power series $(1.1)^k$ should be given some analytical interpretation.

We have the following theorem.

Theorem 4.1. *The n-vector function $P^{(0)}(U)$ can be uniquely determined as a solution of the non-linear differential equation*

$$(4.1) \qquad x\frac{dP^{(0)}}{dx}=U^{\mathcal{M}}(\mathbf{1}+P^{(0)})^{\mathcal{M}}\mathbf{1}_n(\mathbf{1}+P^{(0)})f^{(0)}(U+\mathbf{1}_n(U)P^{(0)})$$
$$-(\alpha+\alpha^* U^{\mathcal{M}})-\mathbf{1}_n(\alpha+\alpha^* U^{\mathcal{M}})P^{(0)}$$

in such a way that the n-vector $P^{(0)}(u)$ is a function holomorphic and bounded in u for

$$(4.2) \qquad \|u\|<b_0'$$

and developable there in the uniformly convergent power series $(1.1)^0$. *Here,* $\mathbf{1}$ *is an n-dimensional column vector whose entries are all equal to* 1.

The n-vector functions $P^{(k)}(U)$ $(k\geqq 1)$ can be uniquely determined as solution of the linear differential equations

$$(4.3)^k \qquad x\frac{dP^{(k)}}{dx}=-kP^{(k)}+F(U+\mathbf{1}_n(U)P^{(0)}(U))P^{(k)}+R^{(k)}(U),$$

$$F(y) \equiv \frac{\partial}{\partial y}(y^{\mathfrak{M}} \mathbf{1}_n(f^{(0)}(y))y), \qquad F(0)=0$$

in such a way that the n-vector functions $P^{(k)}(u)$ *are holomorphic, bounded and asymptotically developable in the power series* $(1.1)^k$ *as* u *tends to* 0 *in the domain*

(4.4) $$0<\|u\|<b_0', \quad u^{\mathfrak{M}} \in \mathfrak{D}(c_0').$$

Here, the n-vector $R^{(k)}(u)$ *is a known function admitting an asymptotic expansion in powers of* u *as* u *tends to* 0 *in the domain* (4.4).

The proof of this theorem is almost exactly the same as that of Theorem 4.3.1 in Section III in Part II (M. Iwano [5]). We omit therefore the proof and refer the reader to M. Iwano [6].

5. Statement of main theorem.

By virtue of Theorem 4.1, we have obtained a formal solution (F) which is arranged in the form of a single power series of x. Our main theorem will be stated as follows:

Theorem 5.1 (Main Theorem). *Suppose that the Assumption* C *is satisfied. Then, the differential equation* (C) *admits a solution* $y=S(x, U)$, *where the n-vector* $S(x,u)$ *is a function holomorphic and bounded in* (x,u) *for*

(5.1) $$|x|<a'', \quad 0<\|u\|<b'', \quad u^{\mathfrak{M}} \in \mathfrak{D}(c'')$$

and admit there the uniformly convergent expansion

(f) $$S(x,u)=u+\mathbf{1}_n(u)P^{(0)}(u)+\sum_{k=1}^{\infty} P^{(k)}(u)x^k.$$

The coefficients $P^{(k)}(u)$ *are the same as those that appeared in Theorem* 4.1 *and* U *is the holomorphic solution of the simplified equation* (R) *satisfying the initial condition* $U=u^0$ *at* $x=x_0$, (x_0, u^0) *being an arbitrary point in the domain* (5.1).

Remark. The equation (C) has the form similar to the equation (A) and the expansions of both solutions have the same form except for the difference of the dimension. But, there exists an essential distinction between the domains (5.1) and (A.4) where the solutions are defined. In fact, if $n \geqq 2$, the central angle of the angular domain $\mathfrak{D}(c'')$ within which the variable $w=U^{\mathfrak{M}}$ is restricted is less than 2π, while if $n=1$ the corresponding angle is almost equal to 3π. The reason is as follows: In the case $n=1$, if $\|U\|$ is small, then $|w|$ is too and the converse is also true. But, if $n \geqq 2$, the converse is not true. Therefore, in order that the converse is also true, we must impose restrictions on the domain of w so that $\|U\|$ is small.

6. Proof of the main theorem.

We put

$$(6.1)\qquad P_N(x, u) = u + \mathbf{1}_n(u)P^{(0)}(u) + \sum_{k=1}^{N-1} P^{(k)}(u)x^k$$

and apply a transformation of the form

$$(6.2)\qquad y = z + P_N(x, U)$$

to the equation (C). By an elementary calculation, we see that the transformed equation is written as

$$(6.3)\qquad xz' = g(x, U, z)$$

with

$$(6.4)\qquad g(x, u, z) = f(x, z + P_N(x, u)) - x\frac{\partial P_N(x, u)}{\partial x} - \frac{u^{\mathcal{M}}\partial P_N(x, u)}{\partial u}\mathbf{1}_n(\alpha + \sum_{\mathcal{H}\in S}\beta_{\mathcal{H}}u^{\mathcal{H}})u.$$

Hence, we can assume without loss of generality that the n-vector $g(x, u, z)$ is a function holomorphic and bounded in (x, u, z) for

$$(6.5)\qquad |x| < a_1, \quad 0 < \|u\| < b_1, \quad u^{\mathcal{M}} \in \mathfrak{D}(c_1), \quad \|z\| < d_1$$

for suitably chosen positive constants a_1, b_1, c_1 and d_1. Clearly, the equation (6.3) admits the formal solution

$$(\mathrm{F})_N\qquad z \sim \sum_{k=N}^{\infty} P^{(k)}(U)x^k.$$

From this it is concluded that the function $g(x, u, z)$ satisfies the inequalities

$$(6.6)\qquad \|g(x, u, z)\| \leqq A\|z\| + B_N|x|^N,$$

$$(6.7)\qquad \|g(x, u, z) - g(x, u, \tilde{z})\| \leqq A\|z - \tilde{z}\|$$

if (x, u, z) and $(x, u, \tilde{z})$ are contained in the domain (6.5). A is a certain positive constant independent of (x, u, z, N), while B_N may depend on N.

In order to obtain Theorem 5.1, it is sufficient to prove the following theorem.

Theorem 6.1 (Auxiliary theorem). *The equation* (6.3) *admits a solution* $z = \Phi_N(x, U)$ *such that the* n*-vector* $\Phi_N(x, u)$ *is a function holomorphic and bounded in* (x, u) *for*

$$(6.8)_N\qquad |x| < a''_N, \quad 0 < \|u\| < b''_N, \quad u^{\mathcal{M}} \in \mathfrak{D}(c''_N)$$

and satisfying there the inequality

$$(6.9)_N\qquad \|\Phi_N(x, u)\| \leqq K_N|x|^N,$$

where K_N *is a certain positive constant.*

Moreover, the solution of (6.3) *satisfying the condition*

$$(6.10)_N\qquad \Phi_N(x, u) = O(|x|^N)$$

is unique.

Suppose that this theorem has been established. Then, the n-vector

$$S_N(x, U) = \Phi_N(x, U) + P_N(x, U)$$

is a solution of the equation (C). It is obvious that *the function* $S_N(x, u)$ *is holomorphic in* (x, u) *for the domain* $(6.8)_N$. Moreover, we can prove that *this function is independent of* N.

In fact, for any $N' > N$, the expression $S_{N'}(x, U) - P_N(x, U)$ is also a solution of the equation (6.3) provided that (x, U) belongs to both domains $(6.8)_N$ and $(6.8)_{N'}$. And, this solution satisfies the condition similar to $(6.10)_N$. From this it is concluded that we have the identity $S_N(x, u) \equiv S_{N'}(x, u)$ if (x, u) is in the common part of these two domains. However, since both sides are analytic functions, by the analytic continuation this identity must hold for (x, u) in the domain (5.1) where $a'' = \sup a''_N$, $b'' = \sup b''_N$, $c'' = \sup c''_N$.

Let $S(x, u) \equiv S_N(x, u)$. Then, since $x = 0$ is an inner point of the domain (5.1), it is known by Cauchy's theorem that $S(x, u)$ is developable there in a uniformly convergent power series of x. On the other hand, $S(x, u)$ admits the asymptotic expansion (f). By virtue of the uniqueness of the asymptotic expansion, the asymptotic expansion (f) must coincide with the convergent expansion. This proves the main theorem.

7. Sketch of the proof of the auxiliary theorem.

Let $\mathfrak{F} = \{\varphi(x, u)\}$ be the family of the n-vectors $\varphi(x, u)$ which are functions holomorphic and bounded in (x, u) for the domain $(6.8)_N$ and satisfying there the inequality

$$(7.1)_N \qquad \| \varphi(x, u) \| \leqq K_N |x|^N.$$

Define the n-vector $\bar{\varphi}(x, u)$ by the integral

$$(7.2) \qquad \bar{\varphi}(x_0, u^0) = \int_{\Gamma_{x_0}} G(x, U) dx, \qquad \left(G(x, u) \equiv g(x, u, \varphi(x, u)) \frac{1}{x} \right),$$

where the path Γ_{x_0} is the same as in Theorem 3.2.

Then, the mapping $\mathfrak{T}$ is defined as follows:

$$(7.3) \qquad \varphi(x, u) \xrightarrow{\mathfrak{T}} \bar{\varphi}(x, u).$$

By virtue of Theorem 3.2, as x moves on the path Γ_{x_0}, the values of x and U are always in the domain $(6.8)_N$. Moreover, by using the inequality (6.6), we see that *the integral* (7.2) *is uniformly convergent with respect to any* u^0 *for every* x_0. Hence, the mapping $\mathfrak{T}$ has a well-defined meaning.

Our proof is based on the existence of a fixed point of this mapping.

Since $\{0\} \in \mathfrak{F}$, $\mathfrak{F}$ is not empty and the family $\mathfrak{F}$ is convex, closed and normal. Therefore, to obtain the auxiliary theorem, it is sufficient to prove the following four propositions.

1°. $\mathfrak{T}$ *transforms* $\mathfrak{F}$ *into itself*: $\mathfrak{T}(\mathfrak{F}) \subset \mathfrak{F}$ or what is the same thing, *the vector function* $\bar{\varphi}(x, u)$ *is holomorphic and bounded in* (x, u) *for* $(6.8)_N$ *and* *satisfies the inequality*

(7.4) $$\|\bar{\varphi}(x_0, u^0)\| \leqq K_N |x_0|^N.$$

2°. $\mathfrak{T}$ *is a continuous mapping of* $\mathfrak{F}$ *with respect to the topology of uniform convergence* or, by virtue of the inequality (6.7), *if the sequence* $\{\varphi^m(x, u)\}$ *tends to* 0 *with respect to the topology of* $\mathfrak{F}$, *the corresponding sequence* $\{\bar{\varphi}^m(x, u)\}$ *does too.*

If we suppose that these two propositions have been proved, it is concluded by means of a fixed point theorem (M. Hukuhara [2]) that there exists a fixed point of the mapping $\mathfrak{T}$ or a member $\varphi(x, u) \in \mathfrak{F}$ such that $\varphi(x, u) \equiv \bar{\varphi}(x, u)$. Since this function depends on N, we denote it by $\Phi_N(x, u)$.

3°. $\Phi_N(x, U)$ *is a solution of the equation* (6.3), *or we have*

(7.5) $$\frac{d}{dx_1}\varphi(x_1, u^1) = G(x_1, u^1), \qquad (u^1 = U(x_1, x_0, u^0)).$$

4°. *The solution of* (6.3) *with the order of* $O(|x|^N)$ *is unique.*

Since the integral (7.2) is uniformly convergent, it is not so difficult to prove these propositions except for that of the inequality (7.4) (See Sections 31-35 in M. Iwano [5]). So we discuss the proof of the inequality (7.4) only.

We change the integration variable from x to w by the relation (2.5). Then, we have

$$\bar{\varphi}(x_0, u^0) = \int_{\Gamma^*_{w^0}} g(\tilde{x}(w), \tilde{U}(w), \varphi(\tilde{x}(w), \tilde{U}(w)))\frac{dw}{X(w)},$$

$$X(w) \equiv w^2(\mathcal{M}\cdot\alpha + \mathcal{M}\cdot\alpha^* w).$$

We can assume that $|X(w)| \geqq \frac{1}{2}|\mathcal{M}\cdot\alpha||w|^2$ for $\|U\| < b''_N$ and $w \in \mathfrak{D}(c''_N)$. Therefore, owing to the inequality (6.6), we have

(7.6) $$\|\bar{\varphi}(x_0, u^0)\| \leqq \int_0^{s_0} \frac{2(AK_N + B_N)}{|\mathcal{M}\cdot\alpha|}\frac{|\tilde{x}(w)|^N}{|w|^2}ds,$$

where s and s_0 are the length of the curve $\Gamma^*_{w^0}$ measured from the origin to the points w and w^0 respectively.

On the other hand, the inequality (3.10) in Theorem 3.1 implies that

$$\int_0^s \frac{|\tilde{x}(w)|^N}{|w|^2}ds \leqq \frac{2|\mathcal{M}\cdot\alpha|}{N\sin\delta_0}|\tilde{x}(w)|^N.$$

Hence, the right-hand member of (7.6) does not exceed the expression $(4(AK_N + B_N)/N\sin\delta_0)|x_0|^N$. Therefore, the inequality (7.4) is an immediate consequence of the inequality

$$4(AK_N + B_N) < K_N N \sin\delta_0.$$

We take first N so large that $4A < N\sin\delta_0$ and next K_N sufficiently large so that $B_N < K_N\ (N\sin\delta_0 - 4A)/4$. But, since $K_N(a''_N)^N \leqq d_1$ must be satisfied, we take finally a''_N so small that this inequality is satisfied

Added in proof

After the present author wrote this paper, he succeeded in improving the conclusion of the main theorem in the following way. Let

$$\hat{\Theta}_1=-3\pi/2-\arg \mathcal{M}\cdot\alpha+6\delta_0, \qquad \hat{\Theta}_2=3\pi/2-\arg \mathcal{M}\cdot\alpha-6\delta_0$$

and define the function $\hat{A}(\varphi)$ by (3.5) with $(\Theta_1, \Theta_2)=(\hat{\Theta}_1, \hat{\Theta}_2)$. Let

$$\hat{\mathfrak{D}}(c)=\{w: \hat{\Theta}_1<\arg w<\hat{\Theta}_2, \quad 0<|w|<\hat{d}(\arg w, c)\},$$

where $\hat{d}(\varphi, c)$ is to be defined by (3.4) with $A(\varphi)=\hat{A}(\varphi)$.

Theorem. *Under the Assumption* (C), *the equation* (C) *admits a solution* $\hat{S}(x, U)$ *such that the n-column vector* $\hat{S}(x, U)$ *admits the uniformly convergent expansion* (f) *whenever* (x, u) *is in the domain*

$$|x|<a'', \qquad 0<||u||<b'', \qquad u\mathcal{M}\in\hat{\mathfrak{D}}(c'').$$

The n-vectors $P^{(k)}(U)$ $(k\geqq 1)$ *can be uniquely determined as solutions of the equations* $(4.3)^k$ *so that* $P^{(k)}(u)$ *are asymptotically developable in powers of u for*

$$0<||u||<b_0', \qquad u\mathcal{M}\in\hat{\mathfrak{D}}(c_0').$$

It is clear that this theorem contains Theorem A as a corollary. The discussion for the proof of this improved theorem is quite different from what was developed here. The proof will be given in a forthcoming paper [6].

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