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Perturbation of Linear Ordinary Differential Equations at Irregular Singular Points

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Abstract

A block-diagonalization theorem for the equation $\varepsilon^q dy/dx = x^r A(x, \mu, \varepsilon)y$ according to the Jordan canonical form of $A(\infty, 0, 0)$ is proved. Whereas $A(x, \mu, \varepsilon)$ is holomorphic in μ, a transformation matrix, which is holomorphic in μ as well as admits asymptotic expansions in both x^{-1} and ε, is constructed.

I. Introduction

0. An example.

In order to explain our problem, we shall study a very special case as an example. Let us consider a system of two equations:

(I) $$dy/dx = x^r A(x)y,$$

where y is a two-dimensional vector, $A(x)$ is a two-by-two matrix whose components are convergent power series in x^{-1}, and r is a nonnegative integer. Put $A(x) = \sum_{k=0}^{\infty} A_k x^{-k}$ and assume that the matrix A_0 has two distinct eigenvalues λ_1 and λ_2. Then it is well known that, for any given direction $\arg x = \theta$ in the x-plane, there exist two linearly independent solutions $y_1(x)$ and $y_2(x)$ such that

$$y_j(x) \cong x^{\sigma_j}\left(\sum_{k=0}^{\infty} p_{jk}x^{-k}\right)\exp Q_j(x)$$

as x tends to infinity in the direction $\arg x = \theta$, where $Q_j(x) = \lambda_j(r+1)^{-1}x^{r+1} + \sum_{k=1}^{r} q_{jk}x^k$. The quantities λ_j, σ_j, q_{jk}, and p_{jk} can be found in a purely algebraic manner, while the solutions $y_j(x)$ can be constructed essentially by the method of successive approximations. Therefore, the following generalization is rather obvious: Let us consider a system of the form

(II) $$dy/dx = x^r A(x, \mu)y,$$

where y, A and r are the same as in (I), except that components of $A(x, \mu)$ are convergent power series in x^{-1} and parameters $\mu = (\mu_1, \cdots, \mu_m)$. Put $A(x, \mu)$

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$=\sum_{k=0}^{\infty} A_k(\mu)x^{-k}$ and assume that the matrix $A_0(0)$ has two distinct eigenvalues. Then the matrix $A_0(\mu)$ has two eigenvalues $\lambda_1(\mu)$ and $\lambda_2(\mu)$ such that $\lambda_1(0) \neq \lambda_2(0)$. Now it is easily proved that, for any given direction $\arg x=\theta$ in the x-plane, there eixst two linearly independent solutions $y_1(x,\mu)$ and $y_2(x,\mu)$ such that

$$y_j(x,\mu) \cong x^{\sigma_j(\mu)}\left(\sum_{k=0}^{\infty} p_{jk}(\mu)x^{-k}\right)\exp Q_j(x,\mu)$$

as x tends to infinity in the direction $\arg x=\theta$, where $Q_j(x,\mu)=\lambda_j(\mu)(r+1)^{-1}x^{r+1}+\sum_{k=1}^{r} q_{jk}(\mu)x^n$. We can verify further that the quantities $y_j(x,\mu)$, $\lambda_j(\mu)$, $\sigma_j(\mu)$, $q_{jk}(\mu)$ and $p_{jk}(\mu)$ are convergent power series in $\mu_1,\cdots,\mu_m$. Let us now relax the condition on the dependence of A on parameters. To do this, let us consider a system of the following form:

(III) $$dy/dx=x^r A(x,\varepsilon)y,$$

where y, A and r are again the same as in (I), except that components of $A(x,\varepsilon)$ are convergent power series in x^{-1} and that they admit asymptotic expansions in powers of a parameter ε as ε tends to zero. Put $A(x,\varepsilon)=\sum_{k=0}^{\infty} A_k(\varepsilon)x^{-k}$, where matrices $A_k(\varepsilon)$ admit asymptotic expansions in powers of ε as ε tends to zero. Assume that $A_0(\varepsilon)$ has two eigenvalues $\lambda_1(\varepsilon)$ and $\lambda_2(\varepsilon)$ such that $\lambda_1(\varepsilon)-\lambda_2(\varepsilon)\neq 0$ as ε tends to zero. Then it can be again proved that, for any given direction $\arg x=\theta$ in the x-plane, there exist two linearly independent solutions $y_1(x,\varepsilon)$ and $y_2(x,\varepsilon)$ such that

$$y_j(x,\varepsilon) \cong x^{\sigma_j(\varepsilon)}\left(\sum_{k=0}^{\infty} p_{jk}(\varepsilon)x^{-k}\right)\exp Q_j(x,\varepsilon)$$

as x tends to infinity in the direction $\arg x=\theta$, where $Q_j(x,\varepsilon)=\lambda_j(\varepsilon)(r+1)^{-1}x^{r+1}+\sum_{k=1}^{r} q_{jk}(\varepsilon)x^k$. We can further verify that the quantities $\lambda_j(\varepsilon)$, $\sigma_j(\varepsilon)$, $q_{jk}(\varepsilon)$ and $p_{jk}(\varepsilon)$ admit asymptotic expansions in powers of ε as ε tends to zero. However, the solutions $y_j(x,\varepsilon)$ do not admit such asymptotic representations unless we specify them more precisely. This difficulty is simply due to non-uniqueness of asymptotic solutions $y_j(x,\varepsilon)$. To see this, let us assume that $y_j(x,\varepsilon)$ *do have* asymptotic representations in ε as ε tends to zero. Further assume that $\operatorname{Re} Q_1 < \operatorname{Re} Q_2$ along the direction $\arg x=\theta$. Then

$$y(x,\varepsilon)=y_2(x,\varepsilon)+(\varepsilon\log\varepsilon)\,y_1(x,\varepsilon)$$

admits the same asymptotic representation in x^{-1} as y_2 does. However, $y(x,\varepsilon)$ *does not* admit an asymptotic representation in powers of ε as ε tends to zero. Then the problem is to specify y_j so that they admit also asymptotic representations in ε. To find an answer for this question, let us assume that

there exist two linearly independent solutions $\tilde{y}_1(x,\varepsilon)$ and $\tilde{y}_2(x,\varepsilon)$ such that

$$x^{-\sigma_j(\varepsilon)}\exp\{-Q_j(x,\varepsilon)\}\tilde{y}_j(x,\varepsilon)\cong\sum_{h=0}^{\infty}\tilde{p}_{jh}(x)\varepsilon^h$$

as ε tends to zero uniformly with respect to x in a small sector around the given direction $\arg x=\theta$, where $\tilde{p}_{jh}(x)$ admit asymptotic expansions in powers of x^{-1} as x tends to infinity in this small sector. Then we can prove that $\tilde{y}_j$ also admit asymptotic representations in x^{-1} as x tends to infinity. To see this, assume that $\operatorname{Re}Q_1<\operatorname{Re}Q_2$ in this small sector. Consider the relations between $\{y_1,y_2\}$ and $\{\tilde{y}_1,\tilde{y}_2\}$, and put

$$\tilde{y}_j(x,\varepsilon)=\gamma_{j1}(\varepsilon)y_1(x,\varepsilon)+\gamma_{j2}(\varepsilon)y_2(x,\varepsilon)\qquad(j=1,2)$$

Then we get

$$\tilde{y}_1(x,\varepsilon)=\gamma_{11}(\varepsilon)y_1(x,\varepsilon),\qquad\tilde{y}_2(x,\varepsilon)\cong\gamma_{22}(\varepsilon)y_2(x,\varepsilon)$$
$$\text{as } x \text{ tends to infinity.}$$

This means that $\tilde{y}_j$ also admit asymptotic representations in x^{-1}. In case when $\operatorname{Re}Q_1=\operatorname{Re}Q_2$ along the given direction $\arg x=\theta$, we must modify our method to get the same conclusion. In this paper, we shall discuss this problem in a very general case.

1. Main theorem.

Consider a system of linear ordinary differential equations

$$\varepsilon^\sigma dy/dx=x^rA(x,\mu,\varepsilon)y,\tag{1.1}$$

where r and σ are nonnegative integers, x is a complex independent variable, ε is a complex parameter, μ is an m-dimensional vector whose components are also parameters, y is an n-dimensional vector whose components are unknown functions of (x,μ,ε), and $A(x,\mu,\varepsilon)$ is an n-by-n matrix whose components are single-valued, bounded and holomorphic functions of (x^{-1},μ,ε) for

$$|x|\geqq R_0,\quad|\mu|\leqq\mu_0,\quad 0<|\varepsilon|\leqq\varepsilon_0,\quad|\arg\varepsilon|\leqq\rho_0,\tag{1.2}$$

the quantities R_0, μ_0, ε_0 and ρ_0 being positive constants. Assume further that A admits an asymptotic expansion

$$A(x,\mu,\varepsilon)\cong\sum_{h=0}^{\infty}A_h(x,\mu)\varepsilon^h\tag{1.3}$$

in powers of ε as $\varepsilon\to0$ in the sector: $|\arg\varepsilon|\leqq\rho_0$ uniformly for $|x|\geqq R_0$ and $|\mu|\leqq\mu_0$, where the coefficients $A_h(x,\mu)$ are holomophic in (x^{-1},μ) for $|x|\geqq R_0$ and $|\mu|\leqq\mu_0$.

Let $\lambda_1,\cdots,\lambda_s$ be the distinct eigenvalues of $A_0(\infty,0)$ and let $n_1,\cdots,n_s$ denote their respective multiplicities. Assume that

$$A_0(\infty,0)=\begin{bmatrix} \overset{\circ}{A}_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \overset{\circ}{A}_2 & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 & \overset{\circ}{A}_s \end{bmatrix}, \tag{1.4}$$

where

$$\overset{\circ}{A}_j=\begin{bmatrix} \lambda_j & d_{j1} & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \lambda_j & d_{j2} & 0 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & \lambda_j & d_{jn_j-1} \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \lambda_j \end{bmatrix} \tag{1.5}$$

the quantities d_{jk} being 0 or 1.

Put

$$\omega_{jk}=\arg(\lambda_j-\lambda_k) \qquad (j\neq k) \tag{1.6}$$

and

$$\omega_+=\max_{(j,k)}\omega_{jk}, \quad \omega_-=\min_{(j,k)}\omega_{jk}. \tag{1.7}$$

Let us denote by $\mathcal{S}(\rho_1,\rho_2)$ the sector in the x-plane which is defined by

$$\rho_1\leqq\arg x\leqq\rho_2. \tag{1.8}$$

In this paper we shall prove the following theorem.

Theorem. *Assume that the sector $\mathcal{S}(\rho_1,\rho_2)$ is contained in the sector*

$$-\frac{3\pi}{2}-\omega_-+\sigma\rho_0<(r+1)\arg x<\frac{3\pi}{2}-\omega_+-\sigma\rho_0 \tag{1.9}$$

for a suitable choice of $\omega_{jk}(j\neq k)$. Then there exist a sufficiently large positive number N, sufficiently small positive numbers ε_1 and μ_1, and an n-by-n matrix $P(x,\mu,\varepsilon)$ satisfying the following conditions:

(i) *for every nonnegative integer h_0, the matrix P can be written in a form*:

$$P(x,\mu,\varepsilon)=\sum_{h=0}^{h_0}P_h(x,\mu)\varepsilon^h+\varepsilon^{h_0+1}Q_{h_0}(x,\mu,\varepsilon), \tag{1.10}$$

where $P_h(x,\mu)$ are holomorphic for $|x|\geqq N$, $x\in\mathcal{S}(\rho_1,\rho_2)$, $|\mu|\leqq\mu_1$ and admit for $|\mu|\leqq\mu_1$ uniform asymptotic expansions

$$P_h(x,\mu)\cong\sum_{k=0}^{\infty}P_{hk}(\mu)x^{-k} \tag{1.11}$$

as x tends to infinity through the sector $\mathcal{S}(\rho_1,\rho_2)$, and Q_{h_0} is holomorphic for $|x|\geqq N$, $x\in\mathcal{S}(\rho_1,\rho_2)$, $|\mu|\leqq\mu_1$, $0<|\varepsilon|\leqq\varepsilon_1$, $|\arg\varepsilon|\leqq\rho_0$ and admits for $|\mu|<\mu_1$, $0<|\varepsilon|\leqq\varepsilon_1$, $|\arg\varepsilon|\leqq\rho_0$ a uniform asymptotic expansion

$$Q_{h_0}(x,\mu,\varepsilon)\cong\sum_{k=0}^{\infty}Q_{h_0k}(\mu,\varepsilon)x^{-k} \tag{1.12}$$

as x tends to infinity through the sector $S(\rho_1, \rho_2)$;

(ii) *the quantities $P_{hk}(\mu)$ are holomorphic for $|\mu| \geqq \mu_1$, and the quantities $Q_{h_0k}(\mu, \varepsilon)$ are holomorphic for $|\mu| \leqq \mu_1$, $0 < |\varepsilon| \leqq \varepsilon_1$, $|\arg \varepsilon| \leqq \rho_0$ and admit for $|\mu| \leqq \mu_1$ uniform asymptotic expansions*

$$Q_{h_0k}(\mu, \varepsilon) \cong \sum_{h=0}^{\infty} Q_{h_0kh}(\mu)\varepsilon^h \tag{1.13}$$

as ε tends to zero in the sector $|\arg \varepsilon| \leqq \rho_0$, where the coefficients are holomorphic in μ for $|\mu| \leqq \mu_1$;

(iii) *the matrix $P_{00}(0)$ is nonsingular;*

(iv) *the transformation*

$$y = P(x, \mu, \varepsilon)z \tag{1.14}$$

reduces System (1.1) to

$$\varepsilon^\sigma dz/dx = x^r B(x, \mu, \varepsilon)z \tag{1.15}$$

where

$$B(x, \mu, \varepsilon) = \operatorname{diag}(B_1(x, \mu, \varepsilon), \cdots, B_s(x, \mu, \varepsilon)), \tag{1.16}$$

and $B_j(x, \mu, \varepsilon)$ is an n_j-by-n_j matrix such that

$$B_j(x, \mu, \varepsilon) = \overset{\circ}{A}_j + O(|x^{-1}| + |\mu| + |\varepsilon|). \tag{1.17}$$

Remarks. (1) Previously we studied System (1.1) in case when A does not depend on μ. [2] By using the same method, we can construct P which satisfies all conditions of our theorem for *a fixed* h_0. Our main concern is to construct P in such a manner that P is independent of h_0.

(2) The theorem of P. F. Hsieh and Y. Sibuya [1] is a special case of our theorem in which $\sigma = 0$ and A does not depend on ε.

2. Lemmas on nonlinear systems.

In the same way as in our previous paper [2], we can reduce our main problem to the study of a nonlinear system

$$\varepsilon^\sigma dy/dx = x^r f(x, y, \mu, \varepsilon), \tag{2.1}$$

where r and σ are nonnegative integers, ε is a complex parameter, x is a complex independent variable, μ is an m-dimensional vector whose components are also parameters, y is an n-dimensional vector, and $f(x, y, \mu, \varepsilon)$ is an n-dimensional vector whose components are single-valued, bounded and holomorphic functions of $(x^{-1}, y, \mu, \varepsilon)$ for

$$|x| \geqq R_0, \quad |y| \leqq \delta_0, \quad |\mu| \leqq \mu_0, \quad 0 < |\varepsilon| \leqq \varepsilon_0, \quad |\arg \varepsilon| \leqq \rho_0, \tag{2.2}$$

the quantities R_0, δ_0, μ_0, ε_0 and ρ_0 being positive constants. In addition we shall suppose that f admits for

$$|x| \geqq R_0, \quad |y| \leqq \delta_0, \quad |\mu| \leqq \mu_0 \tag{2.3}$$

a uniform asymptotic expansion

$$f(x, y, \mu, \varepsilon) \cong \sum_{h=0}^{\infty} f_h(x, y, \mu)\varepsilon^h \tag{2.4}$$

as ε tends to zero through the sector

$$|\arg \varepsilon| \leqq \rho_0, \tag{2.5}$$

where the coefficients f_h are assumed to be single-valued, bounded and holomorphic in the domain (2.3).

Let

$$f(x, y, \mu, \varepsilon) = \hat{f}_0(x, \mu, \varepsilon) + A(x, \mu, \varepsilon)y + O(|y|^2), \tag{2.6}$$

where $\hat{f}_0$ is an n-dimensional vector, and A is an n-by-n matrix. It is easily seen that A admits for

$$|x| \geqq R_0, \quad |\mu| \leqq \mu_0 \tag{2.7}$$

a uniform asymptotic expansion

$$A(x, \mu, \varepsilon) \cong \sum_{h=0}^{\infty} A_h(x, \mu)\varepsilon^h \tag{2.8}$$

as ε tends to zero through the sector (2.5), where the coefficients $A_h(x, \mu)$ are single-valued, bounded and holomorphic in the domain (2.7).

Assume that the matrix $A_0(\infty, 0)$ has the Jordan canonical form with non-zero eigenvalues $\lambda_1, \cdots, \lambda_n$, and that

$$\hat{f}_0(0, \mu, \varepsilon) \equiv 0. \tag{2.9}$$

Put

$$\omega_j = \arg \lambda_j \qquad (j = 1, 2, \cdots, n) \tag{2.10}$$

and

$$\omega_+ = \max_j \omega_j, \quad \omega_- = \min_j \omega_j. \tag{2.11}$$

Let us denote by $\mathcal{S}(\rho_1, \rho_2)$ the sector in the x-plane which is defined by

$$\rho_1 \leqq \arg x \leqq \rho_2. \tag{2.12}$$

Now we make the following assumption:

Assumption (A). *The sector $\mathcal{S}(\rho_1, \rho_2)$ is contained in the sector*

$$-\frac{3\pi}{2} - \omega_- + \sigma\rho_0 < (r+1) \arg x < \frac{3\pi}{2} - \omega_+ - \sigma\rho_0 \tag{2.13}$$

for a suitable choice of ω_j.

We shall state four lemmas *under Assumption (A).*

Lemma 1. *There exists a solution of System* (2.1):

$$y = \varphi(x, \mu, \varepsilon) \tag{2.14}$$

such that

(i) *φ is single-valued, bounded and holomorphic for*

$$|x| \geqq N, \quad x \in \mathcal{S}(\rho_1, \rho_2), \quad |\mu| \leqq \mu_1, \quad 0 < |\varepsilon| \leqq \varepsilon_1, \quad |\arg \varepsilon| \leqq \rho_0 \tag{2.15}$$

if $1/N$, μ_1, ε_1 *are sufficiently small*;

(ii) φ *admits for*

$$0<|\varepsilon|\leqq\varepsilon_1,\ |\arg\varepsilon|\leqq\rho_0,\ |\mu|\leqq\mu_1 \tag{2.16}$$

a uniform asymptotic expansion

$$\varphi(x,\mu,\varepsilon)\cong\sum_{k=1}^{\infty}\varphi_k(\mu,\varepsilon)x^{-k} \tag{2.17}$$

as x *tends to infinity in* $S(\rho_1,\rho_2)$, *where the coefficients* $\varphi_k(\mu,\varepsilon)$ *are single-valued, bounded and holomorphic for* (2.16) *and they admit for* $|\mu|\leqq\mu_1$ *uniform asymptotic expansions*

$$\varphi_k(\mu,\varepsilon)\cong\sum_{h=0}^{\infty}\varphi_{kh}(\mu)\varepsilon^h \tag{2.18}$$

as ε *tends to zero in the sector* $|\arg\varepsilon|\leqq\rho_0$, *the coefficients* $\varphi_{kh}(\mu)$ *being holomorphic for* $|\mu|\leqq\mu_1$.

Lemma 2. *Suppose that there exists a solution of System* (2.1):

$$y=\psi(x,\mu,\varepsilon) \tag{2.19}$$

such that

(i) ψ *is single-valued, bounded and holomorphic for* (2.15);

(ii) ψ *admits for*

$$|x|\geqq N,\ x\in S(\rho_1,\rho_2),\ |\mu|\leqq\mu_1 \tag{2.20}$$

a uniform asymptotic expansion

$$\psi(x,\mu,\varepsilon)\cong\sum_{h=0}^{\infty}\psi_h(x,\mu)\varepsilon^h \tag{2.21}$$

as ε *tends to zero in the sector* $|\arg\varepsilon|\leqq\rho_0$, *where the coefficients* $\psi_h(x,\mu)$ *are single-valued, bounded and holomorphic for* (2.20) *and they admit for* $|\mu|\leqq\mu_1$ *uniform asymptotic expansions*

$$\psi_h(x,\mu)\cong\sum_{k=1}^{\infty}\psi_{hk}(\mu)x^{-k} \tag{2.22}$$

as x *tends to infinity through* $S(\rho_1,\rho_2)$, *the coefficients* $\psi_{hk}(\mu)$ *being holomorphic for* $|\mu|\leqq\mu_1$. *Then*

$$\psi(x,\mu,\varepsilon)-\varphi(x,\mu,\varepsilon)\cong 0 \tag{2.23}$$

as x *tends to infinity through* $S(\rho_1,\rho_2)$ *uniformly for* (μ,ε) *in the domain* (2.16).

Lemma 3. *Suppose that a solution* (2.19) *satisfies Conditions* (i) *and* (ii) *of Lemma* 2. *Then, for each* h_0 *the solution* ψ *can be written in a form*

$$\psi(x,\mu,\varepsilon)=\sum_{h=0}^{h_0}\psi_h(x,\mu)\varepsilon^h+\varepsilon^{h_0+1}\varphi_{h_0}(x,\mu,\varepsilon) \tag{2.24}$$

where $\varphi_{h_0}(x,\mu,\varepsilon)$ *satisfies Conditions* (i) *and* (ii) *of Lemma* 1.

Lemma 4. *There exists a solution* (2.19) *of System* (2.1) *which satisfies*

Conditions (i) *and* (ii) *of Lemma* 2.

Remarks. (1) Lemma 1 can be proved in the same manner as we proved a similar lemma in our previous paper. [2; page 42, Lemma 2]. Therefore, we shall omit the proof of Lemma 1.

(2) Lemma 3 can be derived from Lemma 2. In fact, put

$$y=\sum_{h=0}^{h_0}\psi_h(x,\mu)\varepsilon^h+\varepsilon^{h_0+1}z$$

and derive a system of differential equations for z. Then

$$z=\varepsilon^{-h_0-1}\Big[\psi(x,\mu,\varepsilon)-\sum_{h=0}^{h_0}\psi_h(x,\mu)\varepsilon^h\Big]$$

is a solution which satisfies Conditions (i) and (ii) of Lemma 2. Therefore, use Lemma 2 to complete the proof of Lemma 3.

(3) Since the proof of our main theorem can be reduced to the proofs of these four lemmas, it is only necessary to prove Lemmas 2 and 4. Lemma 4 will be proved in Chapter II, and Lemma 2 will be proved in Chapter III.

II. Proof of Lemma 4

3. A formal solution.

We shall construct a formal solution of System (2.1)

$$y=\sum_{h=0}^{\infty}\psi_h(x,\mu)\varepsilon^h \tag{3.1}$$

whose coefficients $\psi_h(x,\mu)$ satisfy the conditions of Lemma 4. We shall study only the case $\sigma=0$. The case $\sigma>0$ can be treated in a similar manner.

The quantity $\psi_0(x,\mu)$ must satisfy the system

$$dz/dx=x^r f_0(x,z,\mu), \tag{3.2}$$

where

$$f_0(x,z,\mu)=f_0(x,0,\mu)+A_0(x,\mu)z+O(|z|^2). \tag{3.3}$$

By using Lemma 1, we can construct $\psi_0(x,\mu)$. Then let us put

$$y=u+\psi_0(x,\mu) \tag{3.4}$$

to derive

$$\begin{aligned} du/dx&=x^r\{f(x,u+\psi_0(x,\mu),\mu,\varepsilon)-f_0(x,\psi_0(x,\mu),\mu)\} \\ &=x^r\{g_0(x,\mu,\varepsilon)+B(x,\mu,\varepsilon)u+O(|u|^2)\}, \end{aligned} \tag{3.5}$$

where

$$\begin{aligned} g_0(x,\mu,\varepsilon)&=f(x,\psi_0(x,\mu),\mu,\varepsilon)-f_0(x,\psi_0(x,\mu),\mu) \\ &\cong\sum_{h=1}^{\infty}f_h(x,\psi_0(x,\mu),\mu)\varepsilon^h, \end{aligned} \tag{3.6}$$

and $B(x,\mu,\varepsilon)$ is an n-by-n matrix such that

$$B(x,\mu,\varepsilon)\cong\sum_{h=0}^{\infty}B_h(x,\mu)\varepsilon^h \tag{3.7}$$

and

$$B_0(x,0)=A_0(\infty,0)+O(x^{-1}). \tag{3.8}$$

Now $\psi_h(x,\mu)$ $(h\geqq 1)$ must satisfy a linear system

$$d\psi_h/dx=x^r\{B_0(x,\mu)\psi_h+\mathcal{P}_h(x,\mu)\}, \tag{3.9}$$

where $\mathcal{P}_h$ is a polynomial in $\psi_k(k<h)$ and their derivatives. Thus we can construct a formal solution (3.1) whose coefficients ψ_h satisfy the conditions of Lemma 4.

4. Existence of an actual solution.

By virtue of Borel-Ritt theorem [3; page 43], we can construct a vector-valued function $G(x,\mu,\varepsilon)$ such that

(i) G is single-valued, bounded and holomorphic for (2.15);

(ii) G and dG/dx admit for (2.20) uniform asymptotic expansions

$$\begin{aligned} G(x,\mu,\varepsilon)&\cong\sum_{h=0}^{\infty}\psi_h(x,\mu)\varepsilon^h,\\ dG(x,\mu,\varepsilon)/dx&\cong\sum_{h=0}^{\infty}\psi_h'(x,\mu)\varepsilon^h \end{aligned} \tag{4.1}$$

as ε tends to zero in the sector $|\arg\varepsilon|\leqq\rho_0$. Put

$$y=u+G(x,\mu,\varepsilon) \tag{4.2}$$

to derive

$$\varepsilon^q du/dx=x^r[c(x,\mu,\varepsilon)+C(x,\mu,\varepsilon)u+O(|u|^2)], \tag{4.3}$$

where

$$c(x,\mu,\varepsilon)=f(x,G(x,\mu,\varepsilon),\mu,\varepsilon)-x^{-r}\varepsilon^q dG(x,\mu,\varepsilon)/dx \tag{4.4}$$

and $C(x,\mu,\varepsilon)$ is an n-by-n matrix such that

$$C(x,\mu,\varepsilon)\cong\sum_{h=0}^{\infty}C_h(x,\mu)\varepsilon^h \tag{4.5}$$

and

$$C_0(x,0)=A_0(\infty,0)+O(x^{-1}). \tag{4.6}$$

Since (3.1) is a formal solution of System (2.1), we have

$$c(x,\mu,\varepsilon)\cong 0 \tag{4.7}$$

uniformly for (2.20) as ε tends to zero in the sector $|\arg\varepsilon|\leqq\rho_0$.

Then in the same way as we did in our previous paper [2], we can construct a solution of System (4.3):

$$u=\hat{G}(x,\mu,\varepsilon) \tag{4.8}$$

such that $\hat{G}$ is single-valued, bounded and holomorphic for (2.15) and $\hat{G}\cong 0$ uniformly for (2.20) as ε tends to zero in the sector $|\arg\varepsilon|\leqq\rho_0$. This completes the proof of Lemma 4.

III. Proof of Lemma 2

5. A linear system.

Let (2.14) be a solution of System (2.1) which satisfied Conditions (i) and (ii) of Lemma 1. Put

(5.1) $$y=u+\varphi(x,\mu,\varepsilon)$$

to derive

(5.2) $$\varepsilon^{\sigma}du/dx=x^{r}g(x,u,\mu,\varepsilon)$$

where

(5.3) $$g(x,u,\mu,\varepsilon)=f(x,u+\varphi,\mu,\varepsilon)-x^{-r}\varepsilon^{\sigma}d\varphi/dx.$$

Since (2.14) is a solution of System (2.1), we have

(5.4) $$g(x,0,\mu,\varepsilon)\equiv 0.$$

Hence we can write g in the form

(5.5) $$g(x,u,\mu,\varepsilon)=E(x,u,\mu,\varepsilon)u,$$

where $E(x,u,\mu,\varepsilon)$ is an n-by-n matrix whose components are single-valued, bounded and holomorphic for (x,μ,ε) in the domain (2.15) and $|u|\leqq\delta$, δ being a small positive constant. Furthermore

(5.6) $$E(x,0,\mu,\varepsilon)=A_0(\infty,0)+O(|x|^{-1}+|\mu|+|\varepsilon|).$$

Now let (2.19) be a solution of System (2.1) which satisfied Conditions (i) and (ii) of Lemma 2, and put

(5.7) $$u_0(x,\mu,\varepsilon)=\psi(x,\mu,\varepsilon)-\varphi(x,\mu,\varepsilon).$$

Then $u=u_0(x,\mu,\varepsilon)$ is a solution of System (5.2), which is single-valued bounded and holomorphic for (2.15), and

(5.8) $$u_0(x,\mu,\varepsilon)=O(|x|^{-1}+|\varepsilon|).$$

Put

(5.9) $$F(x,\mu,\varepsilon)=E(x,u_0(x,\mu,\varepsilon),\mu,\varepsilon).$$

Then $u=u_0(x,\mu,\varepsilon)$ is a solution of the linear system

(5.10) $$\varepsilon^{\sigma}du/dx=x^{r}F(x,\mu,\varepsilon)u.$$

Notice that

(5.11) $$F(x,\mu,\varepsilon)=A_0(\infty,0)+O(|x|^{-1}+|\mu|+|\varepsilon|).$$

6. A block-diagonalization theorem.

Let us assume that

(6.1) $$A_0(\infty,0)=\operatorname{diag}(\overset{\circ}{A}_1,\cdots,\overset{\circ}{A}_s),$$

where

(6.2) $$\overset{\circ}{A}_j=\nu_jI_j+N_j$$

and the quantities ν_j are distinct eigenvalues of $A_0(\infty,0)$, I_j is the n_j-by-n_j identity matrix, and N_j is an n_j-by-n_j nilpotent matrix. Let us fix a direction

(6.3) $$\arg x=\theta$$

so that

(6.4) $$\rho_1<\theta<\rho_2.$$

Let us classify $1, 2, \cdots, s$ into three classes J_1, J_2 and J_3 so that

(6.5) $$\mathrm{Re}\,[\nu_j x^{r+1}\varepsilon^{-\sigma}]\begin{cases}>0 & j\in J_1,\\ =0 & j\in J_2,\\ <0 & j\in J_3\end{cases}$$

for $\arg x=\theta$ and $\varepsilon>0$. Then choose a small sector $\mathcal{S}(\theta)$

(6.6) $$|\arg x-\theta|\leqq\rho$$

so that

(6.7) $$\mathrm{Re}\,[\nu_j x^{r+1}\varepsilon^{-\sigma}]\begin{cases}>0 & j\in J_1\\ <0 & j\in J_3\end{cases}$$

for $x\in\mathcal{S}(\theta)$ and $|\arg\varepsilon|\leqq\rho_0$, where we assume that ρ_0 be a small positive constant. Furthermore, we can assume that, if $j\in J_2$, we have

(6.8) $$\mathrm{Re}\,[\nu_j x^{r+1}\varepsilon^{-\sigma}]>0$$

for some direction in the sector $\mathcal{S}(\theta)$.

Now by choosing ρ and ρ_0 sufficiently small and by using the same method as in our previous paper [2], we can construct an n-by-n matrix $Q(x,\mu,\varepsilon)$ such that

(6.9) $$Q(x,\mu,\varepsilon)=I+O(|x|^{-1}+|\mu|+|\varepsilon|)$$

uniformly for $|\mu|\leqq\mu_1$, $|x|\geqq N$, $x\in\mathcal{S}(\theta)$, $0<|\varepsilon|\leqq\varepsilon_1$, $|\arg\varepsilon|\leqq\rho_0$, where I is the n-by-n identity matrix, and that the transformation

(6.10) $$v=Q(x,\mu,\varepsilon)u$$

reduces System (5.10) to

(6.11) $$\varepsilon^{\sigma}dv/dx=x^{r}G(x,\mu,\varepsilon)v,$$

where

(6.12) $$G(x,\mu,\varepsilon)=\mathrm{diag}\,(G_1(x,\mu,\varepsilon),\cdots,G_s(x,\mu,\varepsilon))$$

and $G_j(x,\mu,\varepsilon)$ is an n_j-by-n_j matrix such that

(6.13) $$G_j(x,\mu,\varepsilon)=\overset{\circ}{A}_j+O(|x|^{-1}+|\mu|+|\varepsilon|).$$

It is evident that

(6.14) $$v=Q(x,\mu,\varepsilon)\,u_0(x,\mu,\varepsilon)$$

is a bounded solution of System (6.11). Hence the components of this solution must be identically equal to zero except for $j\in J_3$. This implies that

(6.15) $$Q(x,\mu,\varepsilon)\,u_0(x,\mu,\varepsilon)\cong 0$$

uniformly for (μ,ε) as x tends to infinity in $\mathcal{S}(\theta)$. Since θ is an arbitrary direction in $\mathcal{S}(\rho_1,\rho_2)$, we get

(6.16) $$u_0(x, \mu, \varepsilon) \cong 0$$

uniformly for (μ, ε) as x tends to infinity in $\mathcal{S}(\rho_1, \rho_2)$. This completes the proof of Lemma 2.

Remark. In this proof of Lemma 2, we assumed that ρ_0 is small. If ρ_0 is not sufficiently small, then cover the sector $|\arg \varepsilon| \leqq \rho_0$ by a finite number of small sectors, and use the same proof for each sector.

References

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