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Dynamical Systems without Separatrices*

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1. Introduction

1.1. Deep results in the classification of two-dimensional differentiable flows were obtained by Markus [2] using a definition of a separatrix. Such results can be extended to the category of compact surfaces. In the two-dimensional theory, it is the separatrices which almost completely determine the flow (cf. also [11]). The rest of the trajectories seems to take care of themselves. Markus [3] has recently given a definition (see definition 3.1), for the set of separatrices for a limited class of flows (the unstable ones) in spaces of dimension greater than two, and developed some results for completely unstable flows without separatrices (Theorem 3.2). Another main result in [3] is an example of a differentiable flow in R^4 which is parallelizable but not parallel (see section 2 for definitions).

1.2. In this paper we introduce a definition (definition 3.3) of separatrices for dynamical systems defined on Hausdorff spaces [4], [5]. We show that our definition 3.3 is equivalent to the Markus definition 3.1 for completely unstable dynamical systems (proposition 3.5), which is the class of differentiable flows discussed by Markus in [3]. Several results are then established for the class of dynamical systems defined on locally compact spaces which include the Markus results as well as some new ones (see section 3). In particular, we show that for the Markus results one need not confine oneself to differentiable flows, and we prove the referees' conjecture that separatrix free is equivalent to a flow of characteristic $0^\pm$ ([10]). In section 4 we study twice times differentiable flows in the 3-dimensional euclidean space R^3 and attempt to classify those without separatrices (Conjecture 4.4). This will unify and extend results of Montgomery and Zippin [6] (Theorem 4.3) and Markus [3] (Theorem 4.1).

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2. Definitions and Elementary Results

2.1. The relationship between vector-fields and local one-parameter transformation groups on a manifold is well established [1]. The local flow so defined is called a system of integral curves of the vector field. If the domain of definition of each such curve is the interval $(-\infty, +\infty)$ we obtain a dynamical system. More generally one has

2.2. Definition. A dynamical system is a triple (X, R, π) where X is a topological space, R is the real interval $(-\infty, +\infty)$, and π is a continuous mapping from the product space $X \times R$ into X satisfying:

$$2.2.1. \quad \pi(x, 0) = x \text{ for each } x \in X,$$

$$2.2.2. \quad \pi(\pi(x, t), s) = \pi(x, t+s) \text{ for all } x \in X \text{ and all } t, s \in R.$$

If X is a differentiable manifold and the map π is differentiable, then the dynamical system is called a differentiable flow (or simply a flow). ([4] contains a general introduction to the theory of dynamical systems).

2.3. Notation. A dynamical system on a Hausdorff space X is always assumed given. We shall denote the image of (x, t) under π by xt , if it is not possible for confusion to arise. If $A \subset R$ and $M \subset X$, then $\pi(M \times A)$ shall be denoted by MA . R^+ and R^- denote the half lines $[0, +\infty)$ and $(-\infty, 0]$, respectively. The closure, boundary, and complement of a set $M \subset X$ are denoted by $\overline{M}$, ∂M , and $X - M$, respectively.

We recall the following definitions [4].

2.4. Definition. With each $x \in X$ we associate the set $C^+(x), L^+(x), D^+(x)$, and $J^+(x)$ as follows:

$$2.4.1. \quad C^+(x) = xR^+,$$

$$2.4.2. \quad L^+(x) = \cap \{\overline{yR^+} \mid y \in C^+(x)\},$$

$$2.4.3. \quad D^+(x) = \cap \{\overline{UR^+} \mid U \text{ is a neighborhood of } x\}, \text{ and}$$

$$2.4.4. \quad J^+(x) = \cap \{\overline{UR^+} \mid U \text{ is a neighborhood of some } y \in C^+(x)\}.$$

The sets $C^+(x), L^+(x), D^+(x)$, and $J^+(x)$ are respectively called the positive trajectory, the positive limit set, the positive prolongation, and the positive prolongational limit set of x . Using R^- instead of R^+ in the definitions we also define the sets $C^-(x), L^-(x), D^-(x)$, and $J^-(x)$. Finally $C(x), L(x), D(x)$, and $J(x)$ are defined as the unions of the positive and negative versions.

2.5. Definition. A non-empty set $M \subset X$ is

2.5.1. invariant, positively invariant, or negatively invariant whenever $M = MR, M = MR^+$, or $M = MR^-$, respectively,

2.5.2. minimal if it is closed and invariant and contains no nontrivial closed and invariant subset. Positive and negative minimality is defined using positive and negative invariance, respectively.

The following are some of the basic results regarding the above concepts [4].

2.6. Lemma.

2.6.1. For any $x \in X$, $D^+(x)$ is closed and positively invariant, $J^+(x)$ is closed and invariant, and $D^+(x) = C^+(x) \cup J^+(x)$.

2.6.2. For any $x, y \in X$ we have $x \in J^+(y)$ if and only if $y \in J^-(x)$.

2.6.3. If $y \in L^+(x)$ for some x , then $y \in J^+(y)$.

2.6.4. A compact minimal set is positively and negatively minimal. A positively minimal set is minimal and if the phase space is locally compact then it is compact minimal. (See [5] page 135).

In the study of dynamical systems we frequently encounter the notions of recursiveness, stability, and dispersiveness. It is possible to couch these notions using the concepts of prolongations and prolongational limit sets. The following definitions are equivalent to those usually found in the literature [4]. (for 2.7.7 see [5], [10]).

2.7. Definitions.

2.7.1. A point $x \in X$ is Poisson stable whenever $x \in L^+(x) \cap L^-(x)$.

2.7.2. A point $x \in X$ is Langrange stable whenever $\overline{C(x)}$ is compact.

2.7.3. A point $x \in X$ is wandering if $x \notin J^+(x)$.

2.7.4. A dynamical system is unstable if $\overline{C(x)}$ is not compact for any $x \in X$.

2.7.5. A dynamical system is completely unstable if each point $x \in X$ is wandering.

2.7.6. A dynamical system is dispersive if $J^+(x) = \emptyset$ for all $x \in X$.

2.7.7. A dynamical system is of characteristic $0^+(0^-)$ if for all $x \in X$ one has $D^+(x) = \overline{C^+(x)}$ ($D^-(x) = \overline{C^-(x)}$). It is of characteristic $0^\pm$ if it is of characteristic 0^+ and 0^- .

2.7.8. A dynamical system has an improper saddle if there are nets $x_i \rightarrow x$, $x_i t_i \rightarrow y$ with $t_i > 0$ such that no subnet of the net $x_i[0, t_i]$ remains in a compact subset of X . (In metric spaces this definition can be easily seen to be equivalent to the usual one, namely, there are sequences $x_n \rightarrow x$, $x_n t_n \rightarrow y$ with $t_n \rightarrow +\infty$, such that $x_n \tau_n$ has no convergent subsequence for some choice of $\tau_n, 0 \leq \tau_n \leq t_n$).

2.7.9. A compact set M in a locally compact space X is stable whenever $D^+(M) = M$.

2.8. Definition. Two dynamical systems (X_1, R, π_1) and (X_2, R, π_2) are said to be equivalent in case there is a homeomorphism h of X_1 upon X_2 for which h and h^{-1} both carry trajectories to trajectories. The two systems are

isomorphic if under such an h the following diagram is commutative.

$$\begin{array}{ccc}
 X_1 \times R & \xrightarrow{\pi_1} & X_1 \\
 \downarrow h \times 1 & & \downarrow h \\
 X_2 \times R & \xrightarrow{\pi_2} & X_2
 \end{array}$$

In what follows we shall be interested in the notion of equivalence which is more relaxed than that of isomorphism.

2.9. Parallelizable and Parallel dynamical systems:

Given a topological space Y , we define a dynamical system on $Y \times R$ as follows: $(y, s)\pi t = (y, t+s)$ for every $(y, s) \in Y \times R$ and $t \in R$. It is easily verified that this defines a dynamical system on $Y \times R$. We call this the additive or product flow on $Y \times R$. A dynamical system (X, R, π) is said to be *parallelizable* whenever it is equivalent to a product flow. A more restricted version of parallelizable flows are *parallel flows* on n -dimensional euclidean spaces. A flow on a n -dimensional euclidean space R^n is *parallel* whenever it is equivalent to a product flow on $R^{n-1} \times R$. Recent work of Markus [3] has shown that the notions of parallelizability and being parallel are equivalent for $n \leq 3$, and he has given an example of a product flow on R^4 which is not parallel.

3. Separatrix Theory

As earlier mentioned, the most complete work in classifying equivalence classes of dynamical systems has been the work of Markus [2] on differentiable flows on R^2 . What we would like to do is to take a dynamical system and select a region of the phase space which looks simple. We would then like to select another such region, and so forth. These regions will be separated from each other by the set of trajectories called separatrices. In a sense a trajectory will not be a separating trajectory if it behaves nicely and all its neighbors do exactly the same thing. The following definition and theorem were given for differentiable flows by Markus [3].

3.1. Definition. (Markus) Consider an unstable differentiable flow. Two trajectories $C(x)$ and $C(y)$ are inseparable in case any invariant neighborhoods of $C(x)$ and $C(y)$ have non-empty intersection. A trajectory is a separatrix if its image in the orbit space is in the closure of the inseparable elements. (Note: The orbit space of a dynamical system on a space X is the quotient space X/C , where C is the equivalence relation defined by "point $x \in X$ is C -related to point $y \in X$ if $x \in C(y)$ ").

3.2. Theorem. (Markus) Consider a completely unstable differentiable flow. Then the orbit space is a T_1 -space. Further, the following three assertions are equivalent.

- 3.2.1. the flow has no separatrices.
- 3.2.2. the orbit space is Hausdorff.
- 3.2.3. the flow has no improper saddle.

One drawback of 3.1 is that a flow be unstable before a separatrix can even be defined. We observe that the requirement of differentiability in the Markus definition and theorem is also redundant. In the following we will use the Markus definition 3.1 without the assumption of differentiability for arbitrary unstable dynamical systems. We first offer the following alternative definition. A dynamical system on a Hausdorff space X is assumed given.

3.3. Definition. A trajectory $C(x)$ does not belong to the set of separating trajectories provided that x has a neighborhood U such that

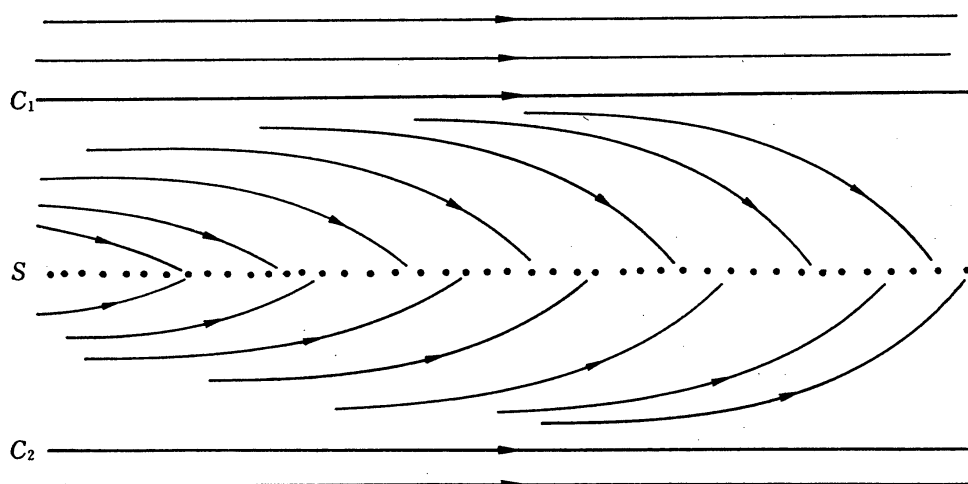
$$J^+(y) = L^+(y) = \phi \text{ for all } y \in U, \text{ or } J^+(y) = L^+(y) \neq \phi \text{ for all } y \in U,$$

and

$$J^-(y) = L^-(y) = \phi \text{ for all } y \in U, \text{ or } J^-(y) = L^-(y) \neq \phi \text{ for all } y \in U.$$

Note that the definition implies that $L(y) = J(y)$ for all $y \in U$. But more importantly we are requiring that these sets do not vary throughout the region. We give an example to illustrate this point.

3.4. Example. Consider a flow on R^2 as described in the figure below.



There are two parallel trajectories C_1 and C_2 which form the boundary of an infinite strip. Trajectories outside of the strip are parallel to C_1 (or C_2). There is a set S of stationary points along a line in the strip and all other trajectories

in the strip have positive limit sets consisting of a single stationary point and have empty negative limit sets. In fact the following relationships hold: $L^+(x) = J^+(x)$ for all $x \in R^2$ and $L^-(x) = J^-(x)$, for all $x \in R^2 - S$. According to definition 3.3, however, the trajectories C_1 and C_2 belong to the set of separating trajectories, and so are also the stationary points. It should be noted that if the Markus definition was adopted without the assumption of unstability of the system, then all trajectories in the strip as well as the trajectories C_1 and C_2 will constitute the set of separatrices. The two definitions are therefore not equivalent in general. We believe that they are not even equivalent for unstable systems. Finally, notice that $D^+(x) = \overline{C^+(x)}$ and $D^-(x) = \overline{C^-(x)}$ hold for all $x \in S$. Thus if we were to adopt the definition (as suggested by the notion of characteristic $0^\pm$) that $C(x)$ is a separatrix if either $D^+(x) \neq \overline{C^+(x)}$ or $D^-(x) \neq \overline{C^-(x)}$, then the only separatrices will be the points in S .

Since the Markus results (Theorem 3.2) concern completely unstable systems, the following proposition is relevant.

3.5 Proposition. *Consider a completely unstable dynamical system.*

Then the following are equivalent.

- 3.5.1. *The system has no separatrices in the sense of 3.3.*
- 3.5.2. *The system is dispersive.*
- 3.5.3. *The system has no saddle at infinity.*
- 3.5.4. *The orbit space is Hausdorff.*
- 3.5.5. *The system has no separatrices in the sense of Markus (definition 3.1).*

Before proving the above proposition we establish the equivalence of 3.5.2 and 3.5.3 under milder conditions. This generalizes a statement of Nemytskii [7] in locally compact separable metric spaces and also provides a simple and direct proof of the same.

3.6. Theorem. *A dynamical system is dispersive if and only if it is unstable and has no improper saddle.*

Proof. That a dispersive system is unstable and has no improper saddle is immediate. We prove the converse therefore. Let if possible $J^+(x) \neq \emptyset$ for some x . Let $y \in J^+(x)$. Then there are nets $x_i \rightarrow x$, $x_i t_i \rightarrow y$ with $t_i \rightarrow +\infty$. Since the system has no improper saddle, there is a subnet $x_j[0, t_j]$ of the net $x_i[0, t_i]$ which is contained in a compact set $K \subset X$. But then $xR^+ \subset K$ and hence $\overline{xR^+} \subset K$. To see this, note that for any $t \geq 0$, we have $x_j t \in x_j[0, t_j]$ ultimately, so that $x_j t \in K$. Since $x_j t \rightarrow xt$, we have $xt \in K$. Thus $L^+(x)$ is a non-empty compact invariant subset of X and so the given dynamical system is not unstable. This contradiction shows that $J^+(x) = \emptyset$ for every $x \in X$ and (X, R, π) is dispersive.

Proof of 3.5. That 3.5.2 implies 3.5.1, 3.5.4, and 3.5.5 is immediate, because $J^+(x) = \emptyset$ for all x implies $J^-(x) = \emptyset$ for all x by 2.6.2. Now let 3.5.1 hold. If $J^+(x) \neq \emptyset$ for some x , then $J^+(x) = L^+(x) \neq \emptyset$, and for a $y \in L^+(x)$ we have $y \in J^+(y)$, by 2.6.3. This contradicts complete instability of the system. Thus $J^+(x) = \emptyset$ for all x and 3.5.2 holds. Now let 3.5.4 hold. If $J^+(x) \neq \emptyset$ for some x , then indeed $x \notin J^+(x)$ by complete instability of the system. But then for a $y \in J^+(x)$, there exist disjoint invariant open sets U_x and U_y containing $C(x)$ and $C(y)$. Thus $y \notin \overline{U_x} \supset D(x) \supset J^+(x)$. This contradicts $y \in J^+(x)$. Thus $J^+(x) = \emptyset$ for all x and 3.5.2 holds. Finally assume 3.5.5. Since any distinct trajectories C_1 and C_2 can be separated by invariant neighborhoods, the orbit space is Hausdorff. Thus 3.5.4 holds and consequently 3.5.2 holds. This proves the proposition.

We now study the structure of dynamical systems having no separatrices in the sense 3.3.

3.7. Theorem. *Let (X, R, π) have no separatrices. If the space X is connected, then exactly one of the following statements holds.*

3.7.1. *The dynamical system is dispersive.*

3.7.2. *Every trajectory closure is a positively and negatively minimal set.*

Proof. Let 3.7.1 not hold. Then there is an x with $J^+(x) \neq \emptyset$. We must have $x \in J^+(x)$. Otherwise, $x \notin J^+(x) = L^+(x)$ (the last equality holds by definition 3.3 for all $x \in X$). Then for a $y \in L^+(x)$, $x \in J^-(y)$ by 2.6.2 on one hand and $L^-(y) = J^-(y) \supset L^+(x)$. Consequently $x \in L^+(x)$ a contradiction. Now $x \in J^+(x) = L^+(x)$ implies $\overline{C(x)} = L^+(x)$. Now set $\tilde{X} = \{x : x \in J^+(x)\}$. This set is non-empty, is always closed, and by definition 3.3 is also open. Since X is connected, we have $\tilde{X} = X$. Lastly we prove, that $\overline{C(x)} = \overline{C^+(x)} = J^+(x) = L^+(x) = \overline{C^-(x)} = J^-(x) = L^-(x)$ for every $x \in X$ and that each $\overline{C(x)}$ is positively and negatively minimal. Indeed the equalities follow easily from the fact that $x \in J^+(x)$ and therefore by 2.6.2 also $x \in J^-(x)$, holds for all x , and that 3.3 holds. Finally, if $y \in \overline{C^+(x)} = L^+(x) = J^+(x)$, then $x \in J^-(y) = \overline{C^-(y)} = \overline{C(y)} = \overline{C^+(y)}$. This shows that $\overline{C^+(x)} = \overline{C^+(y)}$ and therefore $\overline{C^+(x)} = \overline{C(x)}$ is positively minimal. Similarly $\overline{C(x)}$ is negatively minimal, and the theorem is proved.

In locally compact spaces we can say more.

3.8. Theorem. *Let (X, R, π) have no separatrices, and let it be not dispersive. If X is locally compact and connected, then every trajectory closure is a positively and negatively stable compact minimal set.*

Proof. This is a direct consequence of 2.6.4 and 2.7.8 and their negative versions.

Finally we have the following theorem.

3.9. Theorem. *Let X be connected and either locally compact or complete metric. Let (X, R, π) have no separatrices and let it be not dispersive. Then the orbit space is Hausdorff if and only if each orbit is periodic.*

Proof. By 3.7 we immediately conclude that every orbit closure is positively minimal. If the orbit space is Hausdorff, then such a positively minimal set cannot contain two distinct orbits. Consequently, $\overline{C(x)} = C(x) = L^+(x)$. Now periodicity of $C(x)$ follows from Theorem 2.9, p.34 in [4]. The converse is trivial.

As a final result we prove a conjecture of the referees (this result was known to the authors).

3.10. Theorem. *Let X be locally compact. The flow on X is separatrix free if and only if it is of characteristic $0^\pm$.*

Proof. The "only if" part is obvious. So let the flow on X be of characteristic $0^\pm$. Let $x \in X$. Then indeed $D^+(x) = \overline{C^+(x)}$, and consequently $J^+(x) = L^+(x)$ (this is so because $L^+(x)$ and $J^+(x)$ are respectively the largest invariant sets in $\overline{C^+(x)}$ and $D^+(x)$). Now, first let $J^+(x) = L^+(x) = \phi$. We have to show that there is a neighborhood U of x such that $J^+(y) = L^+(y) = \phi$ for $y \in U$. If not, then there is a net $x_i \rightarrow x$ such that $J^+(x_i) = L^+(x_i) \neq \phi$ for each i . From local compactness of X there is a compact neighborhood V of x and either $J^+(x_i) \cap V = \phi$ ultimately or there is a subnet x_k of the net x_i with $J^+(x_k) \cap V \neq \phi$. In the former case, consider a point $x_i \in V$, with $J^+(x_i) = L^+(x_i) \neq \phi$ and $J^+(x_i) \cap V = \phi$. For any point $y \in J^+(x_i)$, we have $x_i \in J^-(y)$, but from the fact that the flow is of characteristic 0^- and $J^+(x_i)$ is closed and invariant, we get $L^-(y) = J^-(y) \subset J^+(x_i)$ which is a contradiction. In the later case there are points $y_i \in J^+(x_i)$ such that $y_i \in U$ ultimately and from compactness of U there is a convergent subnet y_j in U , again leading to the contradiction $U \cap J^+(x) \neq \phi$. Thus $J^+(y) = L^+(y) = \phi$ in some neighborhood U of x . Now assume that $J^+(x) = L^+(x) \neq \phi$. In this case we must have $x = J^+(x)$, for otherwise from $x \notin J^+(x)$ we will have for a point $y \in J^+(x)$ the contradicting conclusion $x \in J^-(y) = L^-(y) \subset J^+(x)$. Moreover, for each point $y \in J^+(x)$, $J^+(y) = L^+(y) = J^+(x)$, from which it follows using the weak attraction Theorem that $J^+(x)$ is a compact minimal set (see [5]). Now if for some net $x_i \rightarrow x$, $J^+(x_i) = L^+(x_i) = \phi$, then choosing a compact neighborhood V of $J^+(x)$, we see that there are points $x_i \tau_i \in \partial V$ with $\tau_i \in R^+$ and consequently $D^+(x) \cap \partial V \neq \phi$ which is a contradiction. We have shown that if $J^+(x) = L^+(x) = \phi$, then there is a neighborhood U of x such that $J^+(y) = L^+(y) = \phi$ for $y \in U$, and if $J^+(x) = L^+(x) \neq \phi$, then there is a neighborhood U of x such that $J^+(y) = L^+(y) \neq \phi$ for $y \in U$. By an analogous argument the same holds when we replace the sign

'+' by the sign '-'. Hence the flow on X is separatrix free and the theorem is proved.

4. Differentiable Flows in R^3

In this section we discuss differentiable flows in R^3 without separatrices. By Theorem 3.7 the flow is either dispersive or 3.7.2 holds. The dispersive case was studied by Markus [3] who proved the following theorem.

4.1. Theorem. (Markus) *A completely unstable flow in R^2 or R^3 without separatrices is parallel.*

It is known that in R^2 , a set is compact minimal if and only if it is the trajectory of a periodic point. Thus using 3.8 and 3.10, Ahmad's [10] characterization of flows of characters $0^\pm$ on R^2 shows that such a flow is either parallel or every point is a critical point or is a topological quasi-rotation group of R^2 . We therefore confine our attention to the flows on R^3 .

In the following we assume that a twice times differentiable flow on R^3 is given. This allows us to classify the possible minimal sets in R^3 using results of Schwartz [9]. We find that for those flow on R^3 is given. This together with the assumed structure should allow us to classify the possible minimal sets in R^3 using results of Schwartz [9]. If each set $\overline{C(x)}$ lies in an invariant 2-manifold, the only possible minimal sets are fixed points, periodic orbits, irrational slope. This suggests

4.2. Conjecture. For any non-dispersive flow in R^3 without separatrices, every trajectory is either a fixed point or is periodic.

We offer the following partial proofs;

Suppose that $\overline{C(x)}$ is a torus for some point $x \in R^3$. Since $\overline{C(x)}$ is stable (3.8), every neighborhood U of $\overline{C(x)}$ contains an invariant neighborhood V . Choose U so that $\overline{U}$ is compact. Since we are assuming that each trajectory closure lies on an invariant 2-manifold, we can get the manifold compact for points in a small enough V by a pasting-together process. If for each point $y \in V$ we have $\overline{C(y)}$ a torus we are done. Otherwise $\overline{C(y)}$ is a circle. Since the flow is differentiable, construct an annulus in U transversal to the flow (note the flow is twice times differentiable). Then each two manifold meets the annulus in an arc, using the invariance we paste them together into a circle. Thus each trajectory closure, in a sufficiently small neighborhood of $\overline{C(x)}$ lies on an invariant torus. We parametrize this family of tori with a map $\gamma : (-\epsilon, \epsilon) \rightarrow V$, so that $\overline{C(x)} = S_0 = \gamma(0)$ and we denote $\gamma(t)$ by S_t for $t \in (-\epsilon, \epsilon)$. The possible flows on a torus have been classified by Kneser, cf. Abraham [1]. Since all trajectory closures are minimal sets, we find that the

flow on S_t is represented by a curve $y = cx + d$, where c is irrational if and only if S_t is a minimal set. As the flow is differentiable, the value of c is a continuous function of the parameter t , and must either be constant, or its rational and irrational values must both be dense subsets of the image. If there are periodic trajectories on some torus S_t , we find that each trajectory on S_t is a separatrix, for if $x \in S_t$, then $L^+(x) = C(x)$, but as x has arbitrarily close neighbors whose trajectory closures are tori, we have that $J^+(x) = S_0$. Thus, if any $\overline{C(x)}$ is a torus, every $\overline{C(y)}$ is a torus, in some neighborhood U of $C(x)$. Due to the way in which it was constructed, this neighborhood can be deformed onto $\overline{C(x)}$ by projection along a curve perpendicular to $\overline{C(x)}$. Such a deformation induces an isomorphism between the homology groups of $\overline{C(x)}$ and the neighborhood U (see for example Eilenberg and Steenrod [8]). Whether or not we have a flow in R^3 without separatrices depends upon whether or not R^3 can be written as the union of these neighborhoods U_i . One common technique for showing that this cannot happen is by computing the fundamental group of $R^3 - U_i$. These are all isomorphic under the homomorphism induced by the injection map, and passing to the limit we conclude that $R^3 - U_i \neq \emptyset$. Essentially the same technique involves the use of Čech homology groups, which commute with direct limits, in this case union of sets. We have $H_1(\overline{C(x)}) \neq 0$, and $H_1(U_i) \approx H_1(\overline{C(x)})$ by the above mentioned deformation. Thus $H_1(\lim U_i) = \lim H_1(U_i) \neq 0$ and so $\bigcup U_i \neq R^3$. We conclude that toral minimal sets do not exist.

It is clear in view of 3.8 that with the assumptions of Lemma 4.2 the conditions of the following theorem of Montgomery and Zippin [6] are satisfied.

4.3. Theorem. (*Montgomery and Zippin [6]*) *A one-parameter transformation group of R which is pointwise periodic, whose orbits are small near fixed points and whose period function is bounded in the neighborhood of moving points, is a topological quasi-rotation group of R^3 .*

Collecting results in 4.1, 4.2, and 4.3 we suggest the following classification result.

4.4. Conjecture. Consider a twice times differentiable flow in R^3 which has no separatrices. Then exactly one of the following holds:

- 4.4.1. the flow is equivalent to a translation.
- 4.4.2. the flow is equivalent to a rotation about an axis.
- 4.4.3. every point is a fixed point.

We close by presenting an example of a dynamical system without separatrices in R^4 which is not dispersive, but for which the orbit space is not Hausdorff.

Example. Consider first a flow on the sphere S^3 produced in the following manner: Write first $S^3 = X * X$ where $X = D^2 \times S^1$ is a solid torus, and $*$ denotes the operation of identification of boundary points. In the disc D^2 we let $S_r = \{(x, y) \mid x^2 + y^2 = r^2\}$ for $0 \leq r \leq 1$ then $S_r \times S^1$ is a torus for $0 < r$, and $S_0 \times S^1$ is a circle. On each $S_r \times S^1$ we place a flow with the same irrational slope, and make $S_0 \times S^1$ a periodic trajectory. We thus have a flow on $X = D^2 \times S^1$ for which there is one periodic orbit, and each other trajectory closure is one of the tori $S_r \times S^1$. It is clear that at this stage there are no separatrices, which is also true of the flow on S^3 obtained from the operation $S^3 = X * X$. It is thus possible to put a flow on S^3 so that there are precisely two periodic orbits, and each other trajectory closure is a two dimensional torus. Now by representing R^4 as a nested set of 3-spheres, S^3 , plus the origin we obtain a flow in R^4 for which the origin is fixed, and each three sphere surrounding the origin has the flow described above. It is easily seen that there are no separatrices, but that the flow does not have a Hausdorff orbit space. In particular, there is no 3-manifold which is a cross-section to the trajectories.

4.5. Remark. Conjecture 4.4 is definitely true if the additional assumption that the orbit space be Hausdorff is made. This follows from Theorem 3.9. The whole point of the conjecture is that such an assumption is redundant in R^3 . The example above shows that a similar result will not be available in R^4 .

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