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A Singular Cauchy Problem with a Small Parameter

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1. Introduction.

Perturbations (singular and regular) have been extensively studied for initial value and boundary value problems for linear ordinary differential equations with variable coefficients. For example, see Wasow [15]. Although Hadamard [9, pp.103-104] has shown that the fundamental solution of the heat equation can be obtained as the limit of the Riemann function for the telegrapher's equation with a small parameter, the study of perturbation problems for partial differential equations obtained its impetus from a very important paper by N. Levinson [12] who considered the Dirichlet problem for the elliptic partial differential equation

$$(1.1) \quad \varepsilon \Delta u^\varepsilon + A(x, y) u_x^\varepsilon + B(x, y) u_y^\varepsilon + C(x, y) u^\varepsilon = D(x, y)$$

in an open simply or multiply connected region R with a boundary BR consisting of a finite number of simple closed curves where

$$(1.2) \quad u^\varepsilon(x, y) = \phi(x, y) \quad \text{on } BR.$$

Levinson showed that under certain conditions the family of solutions u^ε of (1.1) converges to a function $V(x, y)$ where V satisfies

$$(1.3) \quad A(x, y) V_x + B(x, y) V_y + C(x, y) V = D(x, y),$$

the equation obtained by setting $\varepsilon=0$ in (1.1), along with the same boundary conditions as in (1.2).

Zlámal, in a series of paper [16], [17], [18], has studied both the initial value problem and the mixed boundary value problem for the telegrapher's equations

$$(1.4) \quad \varepsilon u_{tt}^\varepsilon + B(t) u_t^\varepsilon - \Delta u^\varepsilon = 0,$$

where $B(t)$ satisfies the following conditions,

$$(1.5) \quad 0 < B_1 \leq |B(t)| \leq B_2, \quad 0 \leq |B'(t)| \leq B_3$$

where B_1, B_2 , and B_3 are three positive constants.

Although the order of this equation does not change as in (1.1) when the

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parameter is set equal to zero, the type of the equation changes from hyperbolic to parabolic and for this reason the solution U of the reduced equation

$$(1.6) \quad B(t)U_t - AU = 0$$

cannot satisfy all the initial conditions satisfied by u^ε . Bobisud [2] has given a quite general treatment of singular perturbations for the Cauchy problem governed by a system of constant coefficient partial differential equations. Classical Fourier analysis plays a key role in obtaining the results in the papers of Bobisud and Zlámal. The equations in their papers require certain regularity conditions on the coefficients... the coefficients are constants or they satisfy conditions similar to those in (1.5).

In this paper we develop a perturbation theory (regular and singular) for a hyperbolic partial differential equation with singular coefficients. It is believed that this is the first instance where the perturbation problem has been studied for a hyperbolic partial differential equation with unbounded coefficients.

Let $D = \{(x, t): x \in R = (-\infty, \infty), t > 0\}$. In the sequel we will understand by a solution of Cauchy's problem always the classical one in the following sense. For the Cauchy problem involving the hyperbolic equation,

$$(1.7) \quad u_{tt} + \left(\frac{\rho}{t}\right)u_t = u_{xx}, \quad (x, t) \in D$$

$$(1.8) \quad u(x, 0) = \phi(x), \quad u_t(x, 0) = 0, \quad x \in R$$

$u(x, t)$ is continuous in $R \times [0, \infty)$; the derivatives of the function $u(x, t)$ appearing in (1.7) are continuous everywhere within D ; the derivative $u_t(x, t)$ is, in addition, continuous for $t=0$, $x \in R$; and $u(x, t)$ satisfies the initial condition (1.8). In the parabolic case, a classical solution of the Cauchy problem

$$(1.9) \quad \left(\frac{\rho}{t}\right)V_t = V_{xx}, \quad (x, t) \in D$$

$$(1.10) \quad V(x, 0) = \phi(x), \quad x \in R$$

is a function which is continuous in $R \times [0, \infty)$; possesses a continuous first order derivative with respect to t and a second order continuous derivative with respect to x within D , and satisfies the initial condition in the usual sense.

2. Degeneration of a Singular Hyperbolic Equation to a Regular Hyperbolic Equation.

Let us consider the Cauchy problem given by (1.7) and (1.8) and make the additional assumptions that $\rho > 0$ and that $\phi(x)$ is an element of the domain of $(\partial^2/\partial x^2)$. Equation (1.7), the Euler-Poisson-Darboux equation, has variable

coefficients; it depends upon a parameter ρ . Since a coefficient of this equation becomes infinite for $t=0$, we meet a singular Cauchy problem when the initial data are given at $t=0$. In [6] it is shown that the unique solution of this Cauchy problem is given by

$$(2.1) \quad u(x, t) = \int_{-1}^1 f(\rho, \sigma) \phi(x + \sigma t) d\sigma,$$

where

$$(2.2) \quad f(\rho, \sigma) = C(\rho)(1 - \sigma^2)^{(\rho/2) - 1} \quad \text{and}$$

$$(2.3) \quad C(\rho) = \frac{\Gamma((\rho+1)/2)}{\Gamma(\rho/2)\Gamma(1/2)}.$$

This solution representation is basic in the investigations that will be done in this paper.

We state the results of this section as

Theorem 1. *The solution $u^\varepsilon(x, t)$ of the initial value problem*

$$(2.4) \quad (I) \quad \begin{cases} u_{tt}^\varepsilon + \left(\frac{\varepsilon\rho}{t}\right) u_t^\varepsilon = u_{xx}^\varepsilon, \\ u^\varepsilon(x, 0) = \phi(x), \quad u_t^\varepsilon(x, 0) = 0, \end{cases}$$

$$(2.5)$$

converges to $V(x, t) = [\phi(x+t) + \phi(x-t)]/2$ as $\varepsilon \searrow 0$ uniformly for $t \in [0, T]$, any $T > 0$ and $x \in A$ where A is any bounded closed set in R . Furthermore, $V(x, t)$ satisfies the initial value problem for the reduced equation,

$$(2.6) \quad V_{tt} = V_{xx},$$

$$(2.7) \quad V(x, 0) = \phi(x), \quad V_t(x, 0) = 0.$$

Proof. The proof of the second part of the theorem is self evident since one recognizes immediately that $[\phi(x+t) + \phi(x-t)]/2$ is D'Alembert's solution representation of the one dimensional wave equation. We prove now the first part of the theorem. From (2.1) it is seen that the solution of (I) is given by

$$\begin{aligned} u^\varepsilon(x, t) &= \int_{-1}^1 f(\varepsilon\rho, \sigma) \phi(x + \sigma t) d\sigma \\ &= \int_0^1 f(\varepsilon\rho, \sigma) \left(\frac{\phi(x + \sigma t)}{2} \right) d\sigma + \int_0^1 f(\varepsilon\rho, \sigma) \left(\frac{\phi(x - \sigma t)}{2} \right) d\sigma \\ &= I_1 + I_2. \end{aligned}$$

Let $M_1(\varepsilon) = |I_1 - (\phi(x+t)/2)|$ and $M_2(\varepsilon) = |I_2 - \phi(x-t)/2|$. It will be demonstrated that $M_1(\varepsilon)$ and $M_2(\varepsilon)$ converge to zero as $\varepsilon \searrow 0$. Since the same argument is used for $M_1(\varepsilon)$ and $M_2(\varepsilon)$, we show only that $M_1(\varepsilon)$ has this property. Therefore, let $\alpha > 0$ and let $K = \sup \phi(x+t)$ for $(x, t) \in A \times [0, T]$. The continuity of

ϕ at $x+t$ implies the existence of $\delta_1 > 0$ such that $|\phi(x+\sigma t) - \phi(x+t)| < \alpha/2$ whenever $|\sigma-1| < \delta_1/T$. Furthermore since $1/\Gamma(\varepsilon\rho)$ converges to zero as $\varepsilon \rightarrow 0$, there exists $\delta_2 > 0$ such that $C(\varepsilon\rho) < \alpha/4K$ whenever $0 < \varepsilon < \delta_2$. For $\delta = \min(\delta_1, \delta_2)$ we have

$$\begin{aligned} 2M_1(\varepsilon) &= \left| \int_0^1 f(\varepsilon\rho, \sigma) [\phi(x+\sigma t) - \phi(x+t)] d\sigma \right| \\ &< \int_0^{1-(\delta/T)} f(\varepsilon\rho, \sigma) |\phi(x+\sigma t) - \phi(x+t)| d\sigma \\ &\quad + \int_{1-(\delta/T)}^1 f(\varepsilon\rho, \sigma) |\phi(x+\sigma t) - \phi(x+t)| d\sigma \\ &< 2C(\varepsilon\rho) \int_0^{1-(\delta/T)} |\phi(x+\sigma t) - \phi(x+t)| d\sigma + \frac{1}{2} \int_{1-(\delta/T)}^1 \alpha f(\varepsilon\rho, \sigma) d\sigma \\ &< 2KC(\varepsilon\rho) + \frac{1}{2} \alpha \int_0^1 f(\varepsilon\rho, \sigma) d\sigma < \frac{\alpha}{2} + \frac{\alpha}{2} = \alpha \text{ whenever } 0 < \varepsilon < \delta. \end{aligned}$$

The proof may now be concluded by appealing to the inequality

$$|u^\varepsilon(x, t) - V(x, t)| < M_1(\varepsilon) + M_2(\varepsilon).$$

3. Degeneration of a Singular Hyperbolic Equation to a Singular Parabolic Equation.

We treat in this section the Cauchy problem for a singular hyperbolic equation with a small parameter. In our method we use an appropriate representation for the solution of

$$(3.1) \quad (II) \quad \begin{cases} \varepsilon u_{tt} + \left(\frac{\rho}{t}\right) u_t = u_{xx}, & (x, t) \in D \\ u^\varepsilon(x, 0) = \phi, u_t^\varepsilon(x, 0) = 0, & x \in R \end{cases}$$

and show directly that this solution converges to the solution of the Cauchy problem

$$(3.3) \quad (III) \quad \begin{cases} \left(\frac{\rho}{t}\right) V_t = V_{xx}, & (x, t) \in D \\ V(x, 0) = \phi(x), & x \in R. \end{cases}$$

It can be readily demonstrated that the solution of (II) is provided by

$$(3.5) \quad u^\varepsilon(x, t) = \int_{-1}^1 f\left(\frac{\rho}{\varepsilon}, \sigma\right) \phi\left(x + \frac{\sigma t}{\sqrt{\varepsilon}}\right) d\sigma$$

which may be rewritten in the form

$$(3.6) \quad u^\varepsilon(x, t) = \int_{-1/\sqrt{\varepsilon}}^{1/\sqrt{\varepsilon}} \sqrt{\varepsilon} f\left(\frac{\rho}{\varepsilon}, \xi\sqrt{\varepsilon}\right) \phi(x + \xi t) d\xi$$

upon making the substitution $\sigma = \xi\sqrt{\varepsilon}$.

Now set $K(\xi, t) = (4\pi t)^{-(1/2)} \exp(-\xi^2/4t)$, the fundamental solution of the one dimensional heat equation, and define $V(x, t)$ by

$$(3.7) \quad V(x, t) = \int_{-\infty}^{\infty} \phi(x + \xi t) K\left(\xi, \frac{1}{2\rho}\right) d\xi.$$

We can now state

Theorem 2. Let $u^\varepsilon(x, t)$ be the solution of the Cauchy problem (II). Then $u^\varepsilon(x, t)$ converges to $V(x, t)$ as $\varepsilon \searrow 0$.

Theorem 3. $V(x, t)$ is the solution of the Cauchy problem (III).

The proof of theorem 2 uses two lemmas.

Lemma 1.

$$\lim_{\varepsilon \rightarrow 0} (1 - \varepsilon \xi^2)^{(\rho/2\varepsilon) - 1} = \exp\left(-\frac{\rho \xi^2}{2}\right).$$

Lemma 2.

$$(3.8) \quad \frac{\Gamma(\nu + a)}{\Gamma(\nu + b)} = \nu^{a-b} \left[1 + \frac{(a-b)(a+b-1)}{2\nu} + O(|\nu|^{-2}) \right]$$

as $|\nu| \rightarrow \infty$ in $|\arg(\nu)| < (1/2)\pi - \delta_1 < (1/2)\pi$.

The proof of lemma 1 can be easily demonstrated. For a proof of lemma 2. see [5, pp. 60-62].

Now define $F_\varepsilon(\xi)$ as follows: $F_\varepsilon(\xi) = 0$ for $|\xi| > 1/\sqrt{\varepsilon}$, $F_\varepsilon(\xi) = \sqrt{\varepsilon} f(\rho/\varepsilon, \xi\sqrt{\varepsilon}) \phi(x + \xi t)$ for $|\xi| < 1/\sqrt{\varepsilon}$. Upon employing lemmas 1 and 2, it is seen that

$$\lim_{\varepsilon \rightarrow 0^+} F_\varepsilon(\xi) = K\left(\xi, \frac{1}{2\rho}\right) \phi(x + \xi t).$$

It follows by the Lebesgue convergence theorem that

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0^+} u^\varepsilon(x, t) &= \lim_{\varepsilon \rightarrow 0^+} \int_{-1/\sqrt{\varepsilon}}^{1/\sqrt{\varepsilon}} \sqrt{\varepsilon} f\left(\frac{\rho}{\varepsilon}, \xi\sqrt{\varepsilon}\right) \phi(x + \xi t) d\xi \\ &= \lim_{\varepsilon \rightarrow 0^+} \int_{-\infty}^{\infty} F_\varepsilon(\xi) d\xi = \int_{-\infty}^{\infty} K\left(\xi, \frac{1}{2\rho}\right) \phi(x + \xi t) d\xi = V(x, t). \end{aligned}$$

This concludes proof of theorem 2.

The continuity of $\phi(x)$ and the identity

$$\int_{-\infty}^{\infty} K\left(\xi, \frac{1}{2\rho}\right) d\xi = 1$$

insure that $V(x, t)$ satisfies the initial condition in (3.4). We make the substitution $\xi = (u - x)/t$ in the integral in (3.7) and obtain

$$(3.9) \quad V(x, t) = \int_{-\infty}^{\infty} G(x, t, u; \rho) \phi(u) du$$

where $G(x, t, u; \rho) = (1/t)K((u-x)/t, 1/2\rho)$. Now G satisfies the parabolic equation.

$$(3.10) \quad G_t = \left(\frac{t}{\rho}\right) G_{xx}.$$

That $V(x, t)$ satisfies the equation in (3.3) follows by differentiating under the integral symbol in (3.9) and employing the equation in (3.10). The uniform convergence of the integral in (3.9) (for a very wide class of functions ϕ) justifies the operation of reversing the order of differentiation and integration. This concludes the proof of theorem 3.

4. Concluding Remarks.

Among the authors who have treated abstract singular perturbation problems in Hilbert space, we mention the work of Bardos [1], Bobisud and Calvert [3], Friedman [7], Kisynski [10], Latil [11], and Smoller [14]. For a treatment of similar problems in Banach spaces, see Bobisud and Hersh [4] (where a very general presentation may be found), Griego and Hersh [8], and Schoene [13]. The coefficients in the equations in all of these papers are required to satisfy certain regularity conditions.

It is natural to ascertain whether the techniques of this paper can be extended so that the singular perturbation problem for the singular abstract operator equation

$$\varepsilon u_\varepsilon''(t) + \left(\frac{\rho}{t}\right) u_\varepsilon'(t) = A^2 u_\varepsilon(t)$$

where A is the infinitesimal generator of a strongly continuous group on a Banach space X , can be treated. Partial results analogous to those of this paper have been obtained when X is a reflexive Banach space.

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