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Forced Nonoscillations in Fourth Order Functional Equations

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1. Introduction.

A great deal of literature exists on the oscillation and nonoscillation of the equation

$$(1)\qquad (r(t)y^{(n-1)}(t))'+a(t)y(t)=0$$

both for even and odd integer $n\geq 1$, and $a(t)>0$ on some positive half line. For this see [9] and the references cited in them. When $n=2$, the literature is filled with all kinds of oscillation and nonoscillation criteria for equation (1), see [1, 2, 3, 6, 7, 10, 16]. However not much is known about equations of the type

$$(2)\qquad y^{(iv)}(t)+a(t)y_{\tau}(t)=f(t),\qquad y_{\tau}(t)\equiv y(t-\tau(t))$$

where

τ, τ' are nonnegative, continuous, bounded on R, $0\leq\tau'(t)<1$,

a: $R\to R$, continuous, R being the real line.

These assumptions will hold throughout this paper.

In what follows, the term "solution" will apply only to continuously extendable solutions of equations under consideration over some positive half line. We shall call a function $h(t)\in C[t_0, \infty)$ as oscillatory if $h(t)$ has a sequence of zeros converging to infinity. Otherwise $h(t)$ will be called nonoscillatory.

Recently Hammett [6] studied the equation

$$(3)\qquad (r(t)y'(t))'+a(t)h(y(t))=f(t)$$

and proved that if $r(t)\geq k_1>0$, $a(t)\geq k_2>0$ and $f(t)$ was integrable on some positive half line, then nonoscillatory solutions of (3) approach zero asymptotically. We will prove similar results about the general equation (2). Our assumption about $a(t)$ is milder than Hammett's and the method is different.

In fact, Hammett's method is based on a theorem of Bhatia [1] which does not extend to delay equations as the following example due to Travis [16] indicates. Consider the equation

(4) $$y''(t)+\frac{\sin t}{2-\sin t}y(t-\pi)=0$$

which has $y(t)=2+\sin t$ as a nonoscillatory solution, even though

(5) $$\int^{\infty}(\sin t)/(2-\sin t)dt=\infty.$$

But according to Bhatia's theorem all solutions of the equation

(6) $$y''(t)+\frac{\sin t}{2-\sin t}y(t)=0$$

are oscillatory.

2. Main results.

Theorem (2.1). *Suppose*

(7) $$\int^{\infty}|f(t)|\,dt<\infty$$

and there exists a $K>0$ *such that*

(8) $$a(t)\geq K \qquad \text{for } t\geq T>0.$$

Let (t) *be a nonoscillatory solution of equation* (2). *Then* $y''(t)\to 0$ *and* $y'''(t)\to 0$, *as* $t\to\infty$.

Proof. Since $y(t)$ is nonoscillatory, we can assume the existence of $t_0'>T$ such that both $y(t)$, $y_\tau(t)>0$ for $t\geq t_0'$. The case when $y(t)<0$ for $t\geq t_0'$ can be handled similarly. Integrating equation (2) between t_0' and t we get

(9) $$y'''(t)-y'''(t_0')+\int_{t_0'}^{t}a(s)y_\tau(s)ds\leq\int_{t_0'}^{t}|f(s)|\,ds.$$

From this we must have

(9a) $$\int_{t_0'}^{\infty}a(s)y_\tau(s)ds<\infty,$$

because otherwise $y'''(t)\to-\infty$ forcing $y(t)\to-\infty$, a contradiction since $y(t)>0$. Hence (9a) holds. From 9(a), $0\leq\tau'(t)<1$ and (8), we get

(10) $$\int_{T_1}^{\infty}y(t)dt<\infty, \qquad T_1>t_0'.$$

From (10) we have

(11) $$\liminf_{t\to\infty} y(t)=0.$$

If $y'''(t)$ is nonoscillatory, then

(12) $$\liminf_{t\to\infty} |y'''(t)|=0.$$

In fact if

$$\liminf_{t\to\infty} |y'''(t)|>\alpha>0$$

then $y(t)\to\pm\infty$ contradicting (11).

In view of (9), (9a) and (12), (by a proper choice of large t_0') we see that

(13) $$\lim_{t\to\infty} y'''(t)=0.$$

If $y'''(t)$ is oscillatory, then again (9) and (12) imply (13). We shall now show that $y''(t)\to 0$ as $t\to\infty$. Suppose to the contrary that $y''(t)\not\to 0$ as $t\to\infty$. In view of (11), we can assume, without any loss of generality,

(14) $$\liminf_{t\to\infty} |y''(t)|=0$$

and

(15) $$\limsup_{t\to\infty} |y''(t)|>3\beta>0.$$

As in Hammett [6], (14) and (15) imply that there exists a sequence $\{t_n\}$ such that $n\geq 0$ and

(i) $t_n\to\infty$ as $n\to\infty$, $t_0\geq T_1$, $t_{n+1}>t_n$,
(ii) for each n, $|y''(t_n)|>2\beta$,
(iii) for each n, there exists a number t_n' such that

$$t_{n-1}<t_n'<t_n \quad \text{and} \quad |y''(t_n')|<\beta.$$

Let a_n be the largest number less than t_n such that $|y''(a_n)|=\beta$ and b_n be the smallest number larger than t_n such that $|y''(b_n)|=\beta$. Thus

(16) $$|y''(t)|\geq\beta, \qquad t\in[a_n, b_n]$$

for each n. We will show that

(17) $$\lim_{n\to\infty}(b_n-a_n)=\infty.$$

To see this consider the interval $[a_n, t_n]$. There exists $a_n<\xi<t_n$ such that

$$y'''(\xi)=\frac{y''(t_n)-y''(a_n)}{t_n-a_n}$$

from which

$$
\begin{aligned}
|y'''(\xi)| &= \frac{|y''(t_n)-y''(a_n)|}{t_n-a_n} \\
&\geq \frac{|y''(t_n)|-|y''(a_n)|}{b_n-a_n} \\
&\geq \frac{2\beta-\beta}{b_n-a_n} \\
&= \frac{\beta}{b_n-a_n}.
\end{aligned}
\tag{18}
$$

Since $\beta>0$ and $|y'''(\xi)|\to 0$ as $\xi\to\infty$, we have proved (17). Let N be a large positive integer so that

$$b_n-a_n\geq 8, \qquad n\geq N \tag{19}$$

and

$$a_n>T_1.$$

In view of (10) and (19) we can choose N sufficiently large so that

$$\int_{a_n}^{b_n} y(t)dt\leq \beta/2; \qquad n\geq N. \tag{20}$$

Now by mean value theorem

$$y(c)=y(a)+(c-a)y'(a)+\frac{(c-a)^2}{2}y''(\delta), \qquad a<\delta<c. \tag{21}$$

Taking

$$c=\frac{a_n+b_n+2}{2}, \qquad a=\frac{a_n+b_n}{2}, \qquad n\geq N.,$$

in (21) we get

$$
\begin{aligned}
y\left(\frac{a_n+b_n+2}{2}\right) &> \left(\frac{a_n+b_n+2}{2}-\frac{a_n+b_n}{2}\right)y'\left(\frac{a_n+b_n}{2}\right) \\
&\quad +\left(\frac{a_n+b_n+2}{2}-\frac{a_n+b_n}{2}\right)^2\frac{y''(\delta)}{2}
\end{aligned}
\tag{22}
$$

since $y\left(\frac{a_n+b_n}{2}\right)>0$. Choose a particular $n_0>N$. To simplify notations a_{n_0}, b_{n_0} are thereafter replaced by a, b.

Case 1. $y''(t)\geq\beta$ in $[a, b]$.
Then from (22) we have

$$y\left(\frac{a+b+2}{2}\right)\geq y'\left(\frac{a+b}{2}\right)+\beta/2. \tag{23}$$

Now if $y'\left(\frac{a+b}{2}\right)\geq 0$, then $y''(t)\geq\beta>0$ implies that $y'(t)\geq 0$ in $\left(\frac{a+b}{2}, b\right)$ and then (23) reveals that

$$y(t)\geq\beta/2, \qquad t\in\left[\frac{a+b+2}{2}, b\right].$$

Therefore

$$\int_{[(a+b+2)/2]}^{b} y(t)dt\geq\frac{\beta}{2}\left(\frac{b-a-2}{2}\right)\geq\frac{3}{2}\beta \tag{24}$$

since $b-a\geq 8$ for $n\geq N$. Since (24) contradicts (20), we must have

$$y'\left(\frac{a+b}{2}\right)<0. \tag{25}$$

From (25) and $y''(t)\geq\beta$ for $t\in[a, b]$, we get

$$y'(t)<0, \qquad t\in\left[a, \frac{a+b}{2}\right]. \tag{26}$$

By mean value theorem and the fact that $y(t)>0$ for $t\geq a$, we have

$$\begin{aligned}\int_{[(a+b)/2]}^{b} y(t)dt &\geq\int_{[(a+b)/2]}^{b}\left(t-\frac{a+b}{2}\right)y'\left(\frac{a+b}{2}\right)dt+\frac{\beta}{2}\int_{[(a+b)/2]}^{b}\left(t-\frac{a+b}{2}\right)^2 dt\\ &=\frac{1}{2}y'\left(\frac{a+b}{2}\right)\left(\frac{b-a}{2}\right)^2+\frac{\beta}{6}\left(\frac{b-a}{2}\right)^3\\ &\geq 8y'\left(\frac{a+b}{2}\right)+\frac{32\beta}{3}.\end{aligned} \tag{27}$$

Using (20) in (27) we get

$$\beta\geq 16y'\left(\frac{a+b}{2}\right)+\frac{64\beta}{3}$$

which gives

$$y'\left(\frac{a+b}{2}\right) \le -\frac{61\beta}{48} < -\beta. \tag{28}$$

From (26), (28) and $y''(t) > \beta > 0$ we have

$$y'(t) < -\beta, \qquad t \in \left[a, \frac{a+b}{2}\right]. \tag{29}$$

From (29), taking $t \in \left[\frac{a+b-2}{2}, \frac{a+b}{2}\right]$ we have

$$y(t) < y\left(\frac{a+b-2}{2}\right) - \beta\left(t - \frac{a+b-2}{2}\right)$$

and taking $t = \frac{a+b}{2}$ we get

$$0 < y\left(\frac{a+b}{2}\right) < y\left(\frac{a+b-2}{2}\right) - \beta.$$

Thus

$$y\left(\frac{a+b-2}{2}\right) > \beta$$

and since $y(t)$ is decreasing in $\left[a, \frac{a+b}{2}\right]$ we get

$$y(t) > \beta, \qquad t \in \left[a, \frac{a+b-2}{2}\right]. \tag{30}$$

From (20) and (30), we get

$$\frac{\beta}{2} \ge \int_a^{[(a+b-2)/2]} y(t)\,dt \ge \beta\left(\frac{b-a-2}{2}\right) \ge \beta.$$

This contradiction concludes Case 1.

Case 2. $y''(t) < -\beta$, $t \in [a, b]$.

Taking $c = b$ and $a = \frac{a+b}{2}$ in (21) we get

$$0 < y(b) < y\left(\frac{a+b}{2}\right) + \left(\frac{b-a}{2}\right) y'\left(\frac{a+b}{2}\right) - \frac{\beta}{2}\left(\frac{b-a}{2}\right)^2 \tag{31}$$

$$<y\left(\frac{a+b}{2}\right)+\left(\frac{b-a}{2}\right)y'\left(\frac{a+b-2}{2}\right)-\frac{\beta}{2}\left(\frac{b-a}{2}\right)^2.$$

Now if $y'\left(\frac{a+b-2}{2}\right)\leqq 0$, then $y'(t)\leqq 0$ for $t\in\left(\frac{a+b-2}{2}, b\right)$.

From (31) and this fact we get

$$y(t)\geqq y\left(\frac{a+b-2}{2}\right)\geqq\frac{\beta}{2}\left(\frac{b-a}{2}\right)^2\geqq 8\beta,\quad t\in\left(\frac{1}{2}(a+b)-1, \frac{1}{2}(a+b)\right) \tag{32}$$

which contradicts (20). Hence $y'\left(\frac{a+b-2}{2}\right)>0$ and since $y'(t)$ is decreasing in (a, b), we get

$$y'(t)>0, \qquad t\in\left(a, \frac{a+b-2}{2}\right). \tag{33}$$

Again taking $c=\frac{a+b-2}{2}$, $a=\frac{a+b-4}{2}$ in (21), we get

$$0<y\left(\frac{b+a-2}{2}\right)\leqq y\left(\frac{b+a-4}{2}\right)+y'\left(\frac{b+a-4}{2}\right)-\frac{\beta}{2}. \tag{34}$$

Now (33) implies

$$y\left(\frac{b+a-4}{2}\right)\leqq y\left(\frac{b+a-2}{2}\right). \tag{35}$$

(34) and (35) give

$$y'\left(\frac{b+a-4}{2}\right)\geqq\frac{\beta}{2}. \tag{36}$$

From (36) and $y''(t)\leq -\beta$ in $[a, b]$ we have

$$y'(t)>\beta/2, \qquad t\in\left[a, \frac{b+a-4}{2}\right]. \tag{37}$$

From (37), we have

$$y(t)>y(a)+\frac{\beta}{2}(t-a),$$

which gives

$$y\left(\frac{b+a-4}{2}\right)>\frac{\beta}{2}\left(\frac{b+a-4}{2}-a\right)\geq\frac{\beta}{2}\left(\frac{b-a-4}{2}\right)>\beta, \tag{38}$$

since $b-a\geq 8$. From (33) and (38)

$$y(t)\geq\beta, \qquad t\in\left(\frac{1}{2}(a+b)-2, \frac{1}{2}(a+b)-1\right) \tag{39}$$

which contradicts (20).

This shows that $\beta=0$ and the proof is complete.

Theorem (2.2). *Suppose the conditions of Theorem* (2.1) *hold. Let $y(t)$ be a nonoscillatory solution of equation* (2). *Then $y'(t)\to 0$ as $t\to\infty$.*

Proof. Suppose not. We proceed as in the proof of Theorem (2.1) and arrive at conclusions (10) and (11). Suppose first that $y'(t)$ is nonoscillatory. Then $\liminf_{t\to\infty}|y'(t)|=0$, because otherwise (11) will be contradicted. If $y'(t)$ is oscillatory then we automatically have $\liminf_{t\to\infty}|y'(t)|=0$. Thus if $\lim_{t\to\infty}y'(t)\neq 0$, we must have

$$\liminf_{t\to\infty}|y'(t)|=0 \tag{40}$$

and

$$\limsup_{t\to\infty}|y'(t)|>3m>0. \tag{41}$$

Since by Theorem (2.1), $y''(t)\to 0$, as $t\to\infty$, in a manner of Hammett again, we can find sequences $\{p_n\}$ and $\{q_n\}$ such that $q_n>p_n$ for each $n\geq 1$, $p_n\to\infty$, $(q_n-p_n)\to\infty$ as $n\to\infty$ and

$$|y'(t)|\geq m, \qquad t\in[p_n, q_n]. \tag{42}$$

Again let $N>0$ be large integer so that

$$q_n-p_n\geq 4 \tag{43}$$

and

$$\int_{p_n}^{q_n} y(t)dt\leq m/2 \tag{44}$$

for $n\geq N$. In what follows we shall choose an $n_0>N$ and consider the interval $[p_{n_0}, q_{n_0}]$ which we write simply as $[p, q]$.

If $y'(t)\geq m$ for $t\in[p, q]$ then from (44) and mean value theorem

$$(45)\qquad \frac{m}{2} \geq \int_p^q y(t)dt = (q-p)y(p) + \int_p^q (t-p)y'(\xi)dt, \qquad p \leq \xi \leq q.$$

(45) yields a contradiction to (44) since $y(p) > 0$.

Thus

$$(46)\qquad y'(t) \leq -m, \qquad t \in [p, q].$$

Again by mean value theorem

$$(47)\qquad 0 < \int_{[(p+q)/2]}^q y(t)dt = y\left(\frac{p+q}{2}\right)\left(\frac{q-p}{2}\right) + \int_{[(p+q)/2]}^q \left(t - \frac{p+q}{2}\right)y'(x)dt,$$
$$p \leq x \leq q$$

which gives

$$(48)\qquad 0 \leq y\left(\frac{p+q}{2}\right)\left(\frac{q-p}{2}\right) - \frac{1}{8}(q-p)^2 m.$$

From (48) and (43), we have

$$(49)\qquad y\left(\frac{p+q}{2}\right) \geq m.$$

Since

$$y'(t) \leq -m,$$

we have

$$(50)\qquad y(t) > m, \qquad t \in \left[p, \frac{p+q}{2}\right].$$

(50) now leads to a contradiction on (44), and the proof is complete.

Theorem (2.3). *Under the conditions of theorem* (2.1), *all nonoscillatory solutions of equation* (2) *approach zero.*

Proof. Let $y(t)$ be a nonoscillatory solution of equation (2). Then (10) and (11) follow as in theorem (2.1). If $\lim_{t\to\infty} y(t) \neq 0$, then an argument similar to that used above will once again lead to a contradiction. This completes the proof.

3. Remarks and examples.

Example (1). The equation

(52) $$y^{(iv)}(t)+e^{-\pi}y(t-\pi)=2e^{-t}$$

has $y=e^{-t}$ as a solution. All conditions of theorem (2.1) are satisfied.

Remark 1. All the results of this paper remain true for the equation

(53) $$(r(t)y'''(t))'+a(t)y_{\tau}(t)=f(t)$$

if it is assumed that $r(t)\geq m_0\geq 0$. By this remark results of Hammett [6] are generalized.

Remark 2. Equation (2) may have nonoscillatory approaching zero even if (8) is violated. The following example illustrates this.

Example (2). Consider the equation

(53) $$y^{(iv)}(t)+e^{-t+\pi}y(t-\pi)=e^{-t}+e^{-2t}.$$

Condition (7) holds but not (8). Equation (53) has $y=e^{-t}$ as a nonoscillatory solution approaching zero.

Thus conditions found are only sufficient.

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References

[1] Bhatia, N. P., Some oscillation theorems for second order differential equation, J. Math. Anal., Appl., **15** (1966), 442–446.

[2] Burkowski, F. Oscillation theorems for a second order nonlinear functional differential equation, J. Math. Anal. Appl., **33** (1971), 258–263.

[3] Burkowski, F., Nonlinear oscillation of a second order sublinear functional differential equation, SIAM J. Appl. Math., **21** (1971), 486–490.

[4] Burton, T. and Grimmer, R., On the asymptotic behavior of solutions of $x''+a(t)f(x)=0$, Proc. Cambridge Philos. Soc., **70** (1971), 77–88.

[5] Burton, T. and Grimmer, R., On the asymptotic behavior of solutions of $x''+a(t)f(x)=e(t)$, Pac. J. Math., No. 1, **41** (1972), 43–55.

[6] Hammett, Michael E., Nonoscillation properties of a nonlinear differential equation, Proc. Amer. Math. Soc., No. 1, **30** (1971), 92–96.

[7] King, G. C. T., Oscillation and nonoscillation of differential equations with time lag, SIAM J. Appl. Math., **21** (1971), 207–213.

[8] Kreith, K., Oscillation Theory, Springer Lecture Notes in Mathematics, vol. **324**, Springer, New York, 1970.

[9] Kusano, T. and Onose, H., Oscillations of functional differential equations with retarded argument, J. Differential Equations, **15** (1974), 269–277.

[10] Londen, S., Some nonoscillation theorems for a second order nonlinear differential equation, SIAM J. Math. Anal., **4** (1973), 460–465.

[11] Onose, H., Oscillatory property of ordinary differential equations of arbitrary order, J. Differential Equations, **7** (1970), 454–458.

[12] Singh, Bhagat, Nonoscillations of forced fourth-order retarded equations, SIAM J. Appl. Math., **28** (1975), 265–269.

[13] Singh, B. and Dahiya, R. S., On oscillation of second-order retarded equations, J. Math. Anal. Appl., **47** (1974), 504–512.

[14] Swanson, C. A., *Comparison and Oscillation Theory of Linear Differential Equations,* Academic Press, New York, 1968.

[15] Travis, C. C., Oscillation theorems for second order differential equations with functional arguments, Proc. Amer. Math. Soc., **31** (1972), 199–201.

[16] Wong, J. S. W., A second order nonlinear oscillation theorem, Proc. Amer. Math. Soc., **40** (1973), 487–491.

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