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PDF issue: 2026-11-20

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(Citation)
Funkcialaj Ekvacioj,20(1):39-47

(Issue Date)
1977-04

(Resource Type)
journal article

(Version)
Version of Record

(JaLCDOI)
https://doi.org/10.24546/0100499012

(URL)
https://hdl.handle.net/20.500.14094/0100499012

KOBE UNIVERSITY

Funkcialaj Ekvacioj, **20** (1977) 39-47

Negative Escape Time in Semidynamical Systems

By

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1. Introduction

R^+ will denote the nonnegative reals. A semidynamical system on a space X is a continuous mapping $\pi: X\times R^+\to X$ such that

(i) $\pi(x, 0)=x$ for all $x\in X$

(ii) $\pi(\pi(x, t), s)=\pi(x, t+s)$ for all $x\in X$ and all $s, t\in R^+$.

If $A\subset X$ and $B\subset R^+$, $\pi(A, B)$ and $F(A, B)$ will denote the sets $\{\pi(y, t): y\in A, t\in B\}$ and $\{y: \pi(y, t)\in A$ for some $y\in A$ for some $t\in B\}$ respectively. A point $x\in X$ is called a start point if $x\notin\pi(y, (0, \infty))$ for any $y\in X$.

Let π and ρ be semidynamical systems on topological spaces X and Y respectively. π is said to be isomorphic to ρ in the sense of Gottschalk and Hedlund (abreviated as *GH*-isomorphic) if there exist a homeomorphism $h: X\to Y$ and a continuous mapping $\phi: X\times R^+\to R^+$ such that

(a) $\phi(x, 0)=0$ for each $x\in X$,

(b) $\phi(x, \cdot): R^+\to R^+$ is a homeomorphism for each $x\in X$,

(c) $h(\pi(x, t))=\rho(h(x), \phi(x, t))$ for each $(x, t)\in X\times R^+$.

Isomorphisms of local dynamical systems have been studied in depth by T. Ura, [6] and [7].

Henceforth, π will denote a semidynamical system on a Hausdorff space X. Intuitively the "negative escape time" of a point $x\in X$ should be the minimum "time" length of all negative trajectories through x. Negative trajectories are defined and discussed at length in [1]. Since only an intuitive concept of negative trajectory is required in this article, we omit a precise definition and refer the interested reader to [1]. In order to make the concept of "negative escape time" precise we need to consider the set M_x of all negative trajectories through x which originate at start points and the set N_x of negative trajectories through x which do not originate at start points. We first define a "negative escape time" with respect to each set. Set $m(x)=\inf\{t\geq 0: \pi(y, t)=x$ for some start point $y\}$ if $M_x\neq\emptyset$ and $m(x)=+\infty$ if $M_x=\emptyset$. Set $n(x)=\inf\{t\geq 0:$ there exist sequences $\{t_i\}$ in R^+ and $\{x_i\}$ in X such that $t_i\to t^-$, $\pi(x_i, t_i)=x$, $x_i\in\pi(x_{i+1}, R^+)$, and $\{x_i\}$ has no convergent subsequence$\}$ if $N_x\neq\emptyset$ and $n(x)=+\infty$ if $N_x=\emptyset$ ($t_i\to t^-$ means that $t_i\to t$ with each $t_i\leq t$). If $M\subset X$, then $n(M)=\inf\{n(x): x\in M\}$ and $m(M)=\inf\{m(x): x\in M\}$. The following lemma

can be verified directly.

Lemma 1.1. *Let $x \in X$ and $\{x_i\}$, $\{t_i\}$ be sequences in X, R^+ respectively such that $t_i \to t^-$, $\pi(x_i, t_i)=x$, $x_i \in \pi(x_{i+1}, R^+)$, and $\{x_i\}$ has no convergent subsequence. Then $n(x_i) \leq t-t_i$.*

Lemma 1.2. *Let X, $\{x_i\}$, and $\{t_i\}$ be as in Lemma 1.1. Then $x_i \notin \pi(x_i, (0, \infty))$ for all i sufficiently large.*

Proof. Suppose that $x_i=\pi(x_i, t)$ for some $t>0$. Then $\pi(x_i, R^+)=\pi(x_i, [0, t])$ is compact. Let $s>0$ and set $y=\pi(x_i, s)$. Evidently $\pi(y, R^+) \subset \pi(x_i, R^+)$. There is an integer k such that $kt>s$ so that $x_i=\pi(x_i, kt)=\pi(\pi(x_i, s), kt-s)=\pi(y, kt-s)$. It follows that $\pi(x_i, R^+) \subset \pi(y, R^+)$. Hence if $x_i \in \pi(x_i, (0, \infty))$ and $y \in \pi(x_i, (0, \infty))$, then $\pi(x_i, R^+)=\pi(y, R^+)$ is compact. If $\{x_j\}$ is a subsequence such that $x_j \in \pi(x_j, (0, \infty))$, then $x_j \in \pi(x_j, R^+)=\pi(x_{j_0}, R^+)$ where j_0 is any element of the set indexing $\{x_j\}$. Thus $\{x_j\}$ must have a convergent subsequence since $\pi(x_{j_0}, R^+)$ is compact. This is impossible. The desired result follows.

Definition 1.3. The negative escape time, $N(x)$, of $x \in X$ is given by $N(x)=\min\{n(x), m(x)\}$. If $M \subset X$, $N(M)=\inf\{N(x) : x \in M\}$.

It should be noted that this negative escape time is not equivalent to the negative escape time found in [1] on page 33.

The following lemma is easily verified.

Lemma 1.4. *For $x \in X$ and $t \in R^+$, $N(\pi(x, t)) \leq N(x)+t$.*

2. Representation

In [2], necessary and sufficient conditions are given for a local (semi) dynamical system on X to be GH-isomorphic to a global (semi) dynamical system on X. We modify the techniques used in [2] to obtain sufficient conditions for a semidynamical system on X to be GH-isomorphic to a semidynamical system on X which has infinite negative escape time for each $x \in X$.

The following lemma is a special case of [5, § 2, Lemma 3] and is easily verified.

Lemma 2.1. *If $\phi : X \times R^+ \to R^+$ is a continuous mapping such that $\phi(x, 0)=0$ and $\phi(x, \cdot) : R^+ \to R^+$ is a homeomorphism for each $x \in X$, then the mapping $h : X \times R^+ \to X \times R^+$ defined by $h(x, t)=(x, \phi(x, t))$ is a homeomorphism.*

Theorem 2.2. *If there exists a continuous function $f : X \to (0, 1]$ such that $f(x) \leq N(x)$ for all $x \in X$, then π is GH-isomorphic to a semidynamical system ρ on X which has infinite negative escape time for each $x \in X$.*

Proof. The mapping $\phi : X \times R^+ \to R_+$ defined by

$$\phi(x, t)=\int_0^t \frac{1}{f(\pi(x, s))}ds$$

will be the reparameterization. Clearly ϕ is continuous and $\phi(x, 0)=0$ for each $x \in X$. Since f is positive, the mapping $\phi(x, \cdot): R^+ \to R^+$ is a homeomorphism of R^+ onto R^+ for each $x \in X$. Recalling that $f: X \times R^+ \to (0, 1]$ we have that

$$\phi(x, t)=\int_0^t \frac{1}{f(\pi(x, s))}ds \geq \int_0^t ds=t.$$

It follows that $\phi(x, t) \to \infty$ as $t \to \infty$ for each $x \in X$. Hence, $\phi(x, \cdot)$ is a homeomorphism of R^+ into R^+. By Lemma 2.1, the mapping $h: X \times R^+ \to X \times R^+$ defined by $h(x, t)=(x, \phi(x, t))$ is a homeomorphism. A semidynamical system ρ on X may be defined by

$$\rho(x, \theta)=\pi(h^{-1}(x, \theta))$$

Continuity of ρ and the initial value property $\rho(x, 0)=x$ are obvious. Let $x \in X$ and $\theta_1, \theta_2 \in R^+$. If $t_1, t_2 \in R^+$ are such that $\phi(x, t_1)=\theta_1$, and $\phi(\pi(x, t_1), t_2)=\theta_2$, then

$$\begin{aligned}\rho(\rho(x, \theta_1), \theta_2)&=\rho(\pi(h^{-1}(x, \theta_1)), \theta_2)=\rho(\pi(x, t_1), \theta_2)\\ &=\pi(h^{-1}(\pi(x, t_1), \theta_2))=\pi(\pi(x, t_1), t_2)=\pi(x, t_1+t_2).\end{aligned}$$

Since

$$\theta_2=\int_0^{t_2} \frac{1}{f(\pi(\pi(x, t_1), s))}ds=\int_0^{t_2} \frac{1}{f(\pi(x, t_1+s))}ds=\int_{t_1}^{t_1+t_2} \frac{1}{f(\pi(x, s))}ds$$

and

$$\varphi(x, t_1+t_2)=\int_0^{t_1+t_2} \frac{1}{f(\pi(x, s))}ds=\int_0^{t_1} \frac{1}{f(\pi(x, s))}ds+\int_{t_1}^{t_1+t_2} \frac{1}{f(\pi(x, s))}ds=\theta_1+\theta_2$$

we have

$$\rho(x, \theta_1+\theta_2)=\pi(h^{-1}(x, \theta_1+\theta_2))=\pi(x, t_1+t_2)=\rho(\rho(x, \theta_1), \theta_2).$$

This shows that ρ is a semidynamical system on X. We next show that ρ has no start points. Since $0<f(x) \leq N(x)$ for each $x \in X$, π has no start points. Let $x, y \in X$ and $t>0$ be such that $\pi(y, t)=x$. Set $\theta=\phi(x, t)$. Then $\rho(y, \theta)=\pi(h^{-1}(y, \theta))=\pi(y, t)=x$. ρ has no start points so that, with respect to ρ, $m(x)=+\infty$ for every $x \in X$. We now show that, with respect to ρ, $n(x)=+\infty$ for every $x \in X$. Suppose that $n(x)<+\infty$ for some $x \in M$. Then there exist $\theta \in R^+$, a sequence $\{x_i\}$ in X, and a sequence $\{\theta_i\}$ in R^+ such that $\theta_i \to \theta^-$, $\rho(x_i, \theta_i)=x$, $x_i \in \rho(x_{i+1}, R^+)$, and $\{x_i\}$ has no

convergent subsequence. Let $t_i \in R^+$ be such that $\theta_i=\phi(x_i, t_i)$. Then $x=\rho(x_i, \theta_i)$ $=\pi(h^{-1}(x_i, \theta_i))=\pi(x_i, t_i)$ and $x_i \in \rho(x_{i+1}, R^+)=\pi(h^{-1}(x_{i+1}, R^+))=\pi(x_{i+1}, R^+)$. Without loss of generality, we may assume that the θ_i are such that $x \notin \rho(x_i, [0, \theta_i))=$ $\pi(x_i, [0, t_i))$. By Lemma 1.2, we may also suppose that $x_i \notin \pi(x_i, (0, \infty))$ for any i. Let $\tau_i \geq 0$ be the least number such that $\pi(x_{i+1}, \tau_i)=x_i$. Then $x=\pi(x_i, t_i)=\pi(x_{i+1}$, $t_i+\tau_i)$ and $x \notin \pi(x_i, [0, t_i))=\pi(x_{i+1}, [\tau_i, t_i+\tau_i))$. If $x \in \pi(x_{i+1}, [0, \tau_i])$, then there is a $s_i \in [0, \tau_i]$ such that $x=\pi(x_{i+1}, s_i)$ so that $\pi(x_i, t_i+\tau_i-s_i)=\pi(x, \tau_i-s_i)=\pi(\pi(x_{i+i}$, $s_i), \tau_i-s_i)=\pi(x_{i+1}, \tau_i)=x_i$. This contradicts our assumption that $x_i \notin \pi(x_i, (0, \infty))$. Hence, $x \notin \pi(x_{i+1}, [0, \tau_i])$ so that $x \notin \pi(x_{i+1}, [0, t_i+\tau_i))$ and $\pi(x_{i+1}, t_i+\tau_i)=\pi(x_i, t_i)=$ x. It follows that $t_{i+1}=t_i+\tau_i \geq t_i$. Hence, $\{t_i\}$ is an increasing sequence and is bound above since by (1), $t_i \leq \phi(x_i, t_i)=\theta_i \to \theta^-$. Let t denote the limit of $\{t_i\}$. By Lemma 1.1, $N(x_i) \leq t-t_i$ and we have

$$\theta_i=\phi(x_i, t_i)=\int_0^{t_i} \frac{1}{f(\pi(x_i, s))}\, ds \geq \int_0^{t_i} \frac{1}{N(\pi(x_i, s))} ds$$

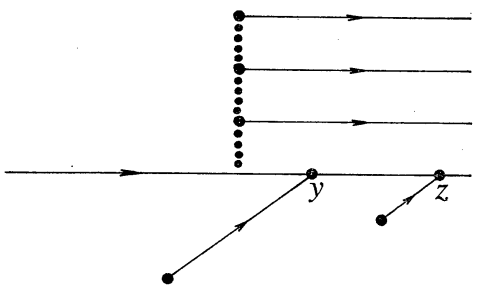
$$\geq \int_0^{t_i} \frac{1}{N(x_i)+s}\, ds \geq \int_0^{t_i} \frac{1}{t-t_i+s} ds=\log t-\log (t-t_i).$$

It follows directly that $\theta_i \to \infty$ as $i \to \infty$ which contradicts our assumption that $\theta_i \to \theta^-$. Hence, we must have $n(x)=+\infty$ so that every $x \in X$ has infinite negative escape time with respect to the semidynamical system ρ. Clearly $\pi(x, t)=\rho(x, \phi(x, t))$. This completes the proof.

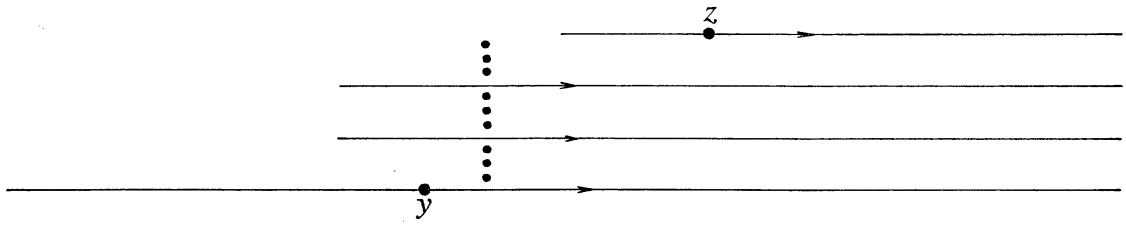
3. Properties of the negative escape time function

The following examples show that if π has start points ($M_x \neq \emptyset$) or if X is not locally compact, then $N(\cdot)$ does not possess any "nice" continuity properties. In each example π will be the semidynamical system indicated in the diagram where $\pi(x, t)$ is the point at a distance t from x along the trajectory through x and the phase spaces are subspaces of R^2. "Dots" indicate that the region is filled by trajectories which are "similar" to the indicated trajectories which do not pass through the points y or z.

Example 1



Example 2



In each example there are sequences $\{y_i\}$, $y_i \to y$, and $\{z_i\}$, $z_i \to z$, such that

$$\limsup_{i\to\infty} N(y_i) < N(y) \quad \text{and} \quad \liminf_{i\to\infty} N(z_i) > N(z).$$

In light of these examples, we will restrict our attention to the situation that the semidynamical system π has no start points and the phase space X is locally compact.

Since we are requiring X to be locally compact, it is natural to consider the function $t: X \to R^+$ defined by $t(x) = \sup\{t: F(x, [0, t]) \text{ is compact}\}$. The principal results are that $N(\cdot)$ is lower semicontinuous and that if $A \subset X$ is compact, then $N(A) = t(A)$.

Lemma 3.3. *If $M \subset X$ is compact, then $F(M, [a, t])$ and $F(M, t)$ are closed for all $t \geq a \geq 0$.*

Proof. The proofs of the two cases are similar. Hence, we will only prove that $F(M, [a, t])$ is closed. Let $\{y_i\}$ be a net in $F(M, [a, t])$ which converges to some $y \in X$. Then there are $s_i \in [a, t]$ such that $\pi(y_i, s_i) \in M$. Since M and $[a, t]$ are compact, there is a subnet $\{y_j\}$ of $\{y_i\}$ such that $s_j \to s \in [a, t]$ and $\pi(y_j, s_j) \to z \in M$. Then by the continuity of π, we have that $z \leftarrow \pi(y_j, s_j) \to y\pi s$. Hence, $y \in F(M, [a, t])$. The desired result follows.

Lemma 3.4. *If $M \subset X$ is compact, then there is an $\alpha > 0$ such that $F(M, (0, t])$ is compact whenever $t < \alpha$. In particular, $n(M) > 0$ whenever M is compact.*

Proof. Let U be a compact neighborhood of M. By Proposition 4.4 of [1], there is an $\alpha > 0$ such that $F(M, [0, \alpha]) \subset U$. The result follows from the previous lemma.

The following may be verified directly.

Lemma 3.5. *Let $M \subset X$ and $I, J \subset R^+$. Then $F(F(M, I), J) = F(M, I + J)$.*

Proposition 3.6. *If M is closed, then $\lim_{s\to t} F(M, s) = F(M, t)$.*

Proof. Suppose $y \in F(M, t)$. For $0 < s < t$ set $y_s = \pi(y, t - s)$. Then $\pi(y_s, s)$

$=\pi(y, t) \in M$ so that $y_s \in F(M, s)$ and $\lim_{s\to t} y_s = y$. Hence $y \in \liminf_{s\to t-} F(M, s)$. Since y is not a start point, there is a $z \in X$ and an $s_0 \in (0, t)$ such that $\pi(z, s_0) = y$. For $s \in [0, s_0)$ set $z_s = \pi(z, s_0 - s)$. Then $\pi(z_s, t+s) = \pi(\pi(z, s_0), t = \pi(y, t) \in M$, so that $z_s \in F(M, t+s)$ and $\lim_{s\to t} z_s = y$. Hence, $y \in \liminf_{s\to t+} F(M, s)$ and we have that $y \in \liminf_{s\to t} F(M, s)$. It follows that $F(M, t) \subset \liminf_{s\to t} F(M, s)$. Now suppose that $y \in \limsup_{s\to t} F(M, s)$. Then there exist $s_i \to t$ and $y_i \in F(M, s_i)$ with $y_i \to y$. Since M is closed, we have that $\pi(y_i, s_i) \to \pi(y, t) \in M$. Thus $y \in F(M, t)$ and we have that $\limsup_{s\to t} F(M, s) \subset F(M, t)$. It follows that $\lim_{s\to t} F(M, s) = F(M, t)$.

Lemma 3.7. *If M is compact and $t(M) < +\infty$, then $F(M, t(M))$ is not compact.*

Proof. Suppose that $F(M, t(M))$ is compact. Let U be a compact neighborhood of $F(M, t(M))$. By Lemma 3.3 and Proposition 3.6 there is an $\varepsilon > 0$ such that $F(M, t(M) + \varepsilon) = F(F(M, t(M)), \varepsilon) \subset U$ is compact. This is impossible.

Theorem 3.8. *Let X be locally compact, π have no start points, and $M \subset X$ be compact. Then $N(M) = t(M)$.*

Proof. Since π has no start points $N(M) = n(M)$. Suppose that $N(M) < T(M)$. Then $F(M, n(M))$ is compact. Let U be a compact neighborhood of $F(M, n(M))$. There exist sequences $\{x_i\}$ in X and $\{t_i\}$ in R^+ such that $t_i \to n(M)^-$, $\pi(x_i, t_i) \in M$, and $\{x_i\}$ has no convergent subsequence. Thus, $x_i \in F(M, t_i)$ and, by Proposition 3.6, we have that $F(M, t_i) \subset U$ for i sufficiently large. Hence, $\{x_i\}$ must have an accumulation point. This contradiction implies that $n(M) \geq t(M)$. If $t(M) = +\infty$, then $n(M) = +\infty$ and $t(M) = n(M)$. Now suppose that $t(M) < +\infty$. Let $\{y_i\}$ be a net in $F(M, t(M))$ which has no convergent subnet. This is possible since $F(M, t(M))$ is not compact (Lemma 3.7). Then $\pi(y_i, 1/n) \in F(M, t(M) - 1/n)$ for integers $n > (t(M))^{-1}$. Moreover, $F(M, t(M) - 1/n)$ is compact for $n > (t(M))^{-1}$. For ease of notation, we will assume that $t(M) > 1$. We define a sequence inductively as follows. Let x_1 be any accumulation point of $\{\pi(y_i, 1)\}$ and $\{y_i^1\}$ a subnet of $\{y_i\}$ such that $\pi(y_i^1, 1) \to x_1$. Suppose that x_n and $\{y_i^n\}$ have been defined. Let x_{n+1} be any accumulation point of $\{\pi(y_i^n, 1/(n+1))\}$ and $\{y_i^{n+1}\}$ be any subnet of $\{y_i^n\}$ such that $\pi(y_i^{n+1}, 1/(n+1)) \to x_{n+1}$. We first show that $x_n \in \pi(x_{n+1}, R^+)$. Since $\pi(y_i^{n+1}, 1/(n+1)) \to x_{n+1}$ we have that $\pi(\pi(y_i^{n+1}, 1/(n+1)), 1/n - 1/(n+1)) \to \pi(x_{n+1}, 1/n - 1/(n+1))$. But $\pi(\pi(y_i^{n+1}, 1/(n+1)), 1/n - 1/(n+1)) = \pi(y_i^{n+1}, 1/n) \to x_n$ since $\{y_i^{n+1}\}$ is a subnet of $\{y_i^n\}$. Thus $x_n = \pi(x_{n+1}, 1/n - 1/(n+1))$. Next we show that $x_n \in F(M, t(M) - 1/n)$. Since $\pi(y_i^n, 1/n) \to x_n$ and $\pi(\pi(y_i^n, 1/n), t(M) - 1/n) \in M$ we have that $\pi(x_n, t(M) - 1/n) \in M$. Also $\pi(x_n, t(M) - 1/n) \leftarrow \pi(\pi(y_i^n, 1/n), t(M) - 1/n) = \pi(y_i^n, t(M)) = \pi(\pi(y_i^n, 1), t(M) - 1) \to \pi(x_1, t(M) - 1) \in M$ since $\{y_i^{n+1}\}$ is a subnet of $\{y_1^1\}$. Thus if we set $t_n = t(M) - 1/n$, we have $\pi(x_n, t_n) = \pi(x_1, t(M) - 1) \in M$. Hence $x_n \in F(M, t_n)$ and $t_n \to t(M)^-$. Finally we show that $\{x_n\}$ has no accumulation point. Suppose $\{x_m\}$

is a convergent subsequence of $\{x_n\}$ such that $x \to z \in X$. Then $\pi(y_i^m, 1/m) \xrightarrow{i} x_m \xrightarrow{m} z$. Let U be a compact neighborhood of z. Then, by Proposition 4.4 of [1], there is a relatively compact neighborhood V of z and an $\alpha > 0$ such that $F(V, [0, \alpha]) \subset U$. For m sufficiently large $x_m \in V$ and, hence, for i and m sufficiently large $\pi(y_i^m, 1/m) \subset V$. Thus, for i and m sufficiently large $y_i^m \in U$. Since $\{y_i^m\}$ is a subnet of $\{y_i\}$ and U is compact, $\{y_i\}$ must have an accumulation point in U. This is impossible since $\{y_i\}$ has no convergent subnet. This contradiction implies that $\{x_n\}$ has no convergent subsequence. We have constructed sequences $\{x_n\}$ and $\{t_n\}$ such that $\pi(x_n, t_n) = \pi(x_1, t(M) - 1) \in M$, $x_n \in \pi(x_{n+1}, R^+)$, $t_n \to t(M)^-$, and $\{x_n\}$ has no convergent subsequence. Hence, $n(M) \leq t(M)$. The desired result follows.

Lemma 3.9. *Let $x \in X$, $0 \leq t \leq t(x)$, and $x_i \to x$. If U is a compact neighborhood of $F(x, [0, t])$, then eventually $F(x_i, [0, t]) \subset U$.*

Proof. Suppose the contrary. Then there are a subnet $\{x_j\}$ of $\{x_i\}$, $y_j \in F(x_j, [0, t])$, and $t_j \in [0, t]$ such that $y_j \in X - U$ and $\pi(y_j, t_j) = x_j \in U$. Thus, there are $s_j \in [0, t]$ such that $\pi(y_j, s_j) \in \partial U$. Since ∂U is compact, there is a subnet $\{y_k\}$ of $\{y_i\}$ such that $t_k \to t' \in [0, t]$, $s_k \to s \in [0, t']$, and $\pi(y_k, s_k) \to z \in \partial U$. Then $x \leftarrow x_k = \pi(y_k, t_k) = \pi(\pi(y_k, s_k), t_k - s_k) \to \pi(z, t' - s)$. Hence, $z \in F(x, [0, t])$. This is impossible because $z \subset \partial U \subset X -$ interior $U \subset X - F(x, [0, t])$. This contradiction implies the desired result.

Theorem 3.10. *Let X be locally compact and π have no start points. Then the negative escape time, $N(\cdot)$, is a lower semicontinuous function, i.e., $\liminf_{y \to x} N(y) \geq N(x)$ for all $x \in X$.*

Proof. By Theorem 3.8 is suffices to show that $\liminf_{y \to x} t(y) \geq t(x)$. Let $0 \leq t < t(x)$ and U be a compact neighborhood of $F(x, [0, t])$. For any net $x_i \to x$ eventually $F(x_i, t) \subset F(x_i, [0, t]) \subset U$ (Lemma 3.9). Hence, eventually $F(x_i, t)$ is compact (Lemma 3.3). Thus, $\liminf_{y \to x} t(y) \geq t$. Since this is true for all $t \in [0, t(x)]$ the desired result follows.

Corollary 3.11. *If X and π are as in Theorem* 3.10, *then* $\lim_{t \to 0} N(x\pi t) = N(x)$.

Proof. The proof follows directly from Theorem 3.10, and Lemma 1.4.

4. Semidynamical systems on locally compact phase spaces.

Theorem 4.1. *Let π have no start points and X be locally compact and Lindelöf. Then π is GH-isomorphic to a semidynamical system ρ on X which has infinite negative escape time for each $x \in X$.*

Proof. Since $N(\cdot)$ is a lower semicontinuous function so is the function $g: X$

$\to R^+$ defined by $g(x)=\min\{1, N(x)\}$. By Lemma 3.4, $g(x)>0$ for each $x \in X$. The constant function $h(x)=0$ for all $x \in X$ is an upper semicontinuous function such that $h(x)<g(x)$. A locally compact, Lindelöf space is paracompact. By a well known result of Dowkar [3], (also [4], pg. 170), there is a continuous function $f: X\to R^+$ such that $h(x)<f(x)<g(x)$. Theorem 2.2 yields the desired result.

It is well known that a dynamical system on a locally compact space X can be extended to a dynamical system on the one point compactification X^* of X. This is not always possible for a semidynamical system (cf., [5, VII, 3.15]). Let $X^*=X\cup\{\infty\}$ be the one point compactification of the locally compact space X and define $\pi^*: X^*\times R^+\to X^*$ by

$$\pi^*(x,t)=\begin{cases}\pi(x,t) & \text{if } x \in X\\ \infty & \text{if } x=\infty.\end{cases}$$

Theorem 4.2. *Let X be locally compact and π have no start points. Then π can be extended to the semidynamical system π^* on X^* if and only if $N(x)=+\infty$ for every $x \in X$.*

Proof. Suppose $N(x)=+\infty$ for every $x \in X$. In order to show that π^* is a semidynamical system on X^* it suffices to show that π^* is continuous at ∞, i.e., to show that if $x_i\to\infty$ and $t_i\to t$, then $\pi^*(x_i, t_i)\to\infty$. Suppose $\pi^*(x_i, t_i)\not\to\infty$. Then there is a subnet $\{x_j\}$ such that $\pi^*(x_j, t_j)=\pi(x_j, t_j)\to y \in X$. Let U be a compact neighborhood of y. Eventually $t_j \in [0, t+1]$ and $\pi(x_j, t_j) \in U$. Thus, eventually $x_j \in F(U, [0, t+1])$. But $F(u, [0, t+1])$ is compact in X since $t(M)=N(M)=+\infty$ (Theorem 3.8). Hence, $x_j\not\to\infty$. This contradiction implies that $\pi(x_i, t_i)\to\infty$. Now suppose that π^* does define a semidynamical system on X^* and that $N(x)<+\infty$ for some $x \in X$. Then for any $\varepsilon>0$ there are sequences $\{x_i\}$ in X and $\{t_i\}$ in R^+ such that $t_i\to t^- \in [N(x)-\varepsilon, N(x)+\varepsilon]$, $\pi^*(x_i, t_i)=x$, and $\{x_i\}$ has no accumulation point in X. Hence, $x_i\to\infty$. Let U be a compact neighborhood of ∞ with $x \notin U$. Since π^* is continuous and ∞ is a critical point of π^*, there is a neighborhood V of ∞ such that $\pi(V, [0, N(x)+\varepsilon])\subset U$. But then $\pi(x_i, t_i)=x \in U$. This contradiction implies that if π^* is a semidynamical system on X^*, then $N(x)=+\infty$ for every $x \in X$.

Corollary 4.3. *Let X be locally compact and π have no start points. Then π is isomorphic to a semidynamical system which can be extended to a semidynamical system on X^*.*

Since a semidynamical system π on a manifold X has no start points [1, Theorem 11.8], the hypotheses of Theorems 3.8, 3.10, 4.1 and 4.2 are satisfied whenever the phase space X is a manifold.

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(Ricevita la 4-an de marto, 1976)
(Reviziita la 5-an de julio, 1976)