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Random Solutions of a System of Functional-Differential Equations in a Banach Space

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§ 1. Introduction.

In the present paper we shall prove a random fixed point theorem for the system of condensing transformations which generalizes the result of Shigeru Itoh ([5], Theorem 2.1).

Moreover, we give an application of the obtained theorem to the existence of a random solutions of a system of functional-differential equations in a Banach space. Our result generalizes theorems of A. Ambrosetti [1], R. Nussbaum [8] (Theorem 2, p. 102) and S. Itoh [5] (Theorem 4.3).

§ 2. Preliminaries.

Let $(T, \mathcal{A})$ be a measurable space. Let (X, d) be a metric space and 2^X be the family of all subsets of X . A mapping $F: T \rightarrow 2^X$ is called measurable (weakly measurable) if for any closed (open) subset B of X , $F^{-1}(B) = \{t \in T: F(t) \cap B \neq \emptyset\}$ is measurable. If $F(t)$ is compact for all $t \in T$ then F is measurable if and only if F is weakly measurable (cf. [4]).

K. Kuratowski ([7], p. 413, G. Darbo [3]) defined the measure of noncompactness, $\alpha(A)$, of a bounded subset A of a metric space X :

$$\alpha(A) = \inf \{ \varepsilon > 0: A \text{ can be covered by a finite number of sets of diameter less than or equal to } \varepsilon \}.$$

We shall need the following results.

Lemma 1 ([8], p. 100). *Let (X, d) be a compact metric space and (Y, d_1) a metric space. Let $C(X, Y) = U$ denote the continuous functions from X to Y with the metric $\rho: \rho(f, g) = \sup \{d_1[f(x), g(x)]: x \in X\}$. Let $H \subset U$ be an equicontinuous family of functions and let $H(x) = \{h(x): h \in H\}$, $x \in X$. Then we have $\alpha(H) = \sup \{\alpha_1[H(x)]: x \in X\}$, where α, α_1 is the measure of noncompactness in U and Y respectively.*

Lemma 2 ([8], p. 10). *Let (X_i, d_i) , $i = 1, 2$, be metric spaces and $f: X_1 \rightarrow X_2$ fulfils:*

$$d_2[f(x_1), f(x_2)] \leq k d_1(x_1, x_2), \quad x_1, x_2 \in X_1, \quad k\text{-constant},$$

then f is a α -set-contraction, i.e.

$$\alpha_2[f(A)] \leq k\alpha_1(A),$$

A -bounded set in X_1 .

§ 3. Random fixed point theorem.

We assume:

- (1) $(T, \mathcal{A})$ is a measurable space.
- (2) $(X_i, |\cdot|_i)$, $i=1, \dots, n$, are a Banach spaces.
- (3) Let G_i , $i=1, \dots, n$, be a separable closed convex bounded subset of X_i , and $F_i: T \times G_1 \times \dots \times G_n \rightarrow G_i$, $i=1, \dots, n$. For each $(x_1, \dots, x_n) \in G_1 \times \dots \times G_n$ and $i=1, \dots, n$, $F_i(\cdot, x_1, \dots, x_n)$ is measurable and for each $t \in T$, $i=1, \dots, n$, $F_i(t, \dots)$ is continuous.
- (4) If $A_i \subset G_i$, $i=1, \dots, n$, $t \in T$, then

$$\alpha_i[F_i(t, A_1, \dots, A_n)] < \sum_{k=1}^n a_{i,k}(t) \alpha_k(A_k) \quad \text{for } \sum_{k=1}^n \alpha_k(A_k) > 0,$$

where $a_{i,k}: T \rightarrow \langle 0, \infty \rangle$, $i, k=1, \dots, n$ and α_i , $i=1, \dots, n$, denotes the measure of noncompactness in X_i .

Let us denote

$$(5) \quad c_{i,k}^0(t) = \begin{cases} a_{i,k}(t), & i \neq k, \\ 1 - a_{i,k}(t), & i = k, \end{cases} \quad i, k=1, \dots, n, \quad t \in T,$$

$$(6) \quad c_{i,k}^s(t) = \begin{cases} c_{1,1}^{s-1}(t) c_{i+1,k+1}^{s-1}(t) + c_{i+1,1}^{s-1}(t) c_{1,k+1}^{s-1}(t), & i \neq k, \\ c_{1,1}^{s-1}(t) c_{i+1,k+1}^{s-1}(t) - c_{i+1,1}^{s-1}(t) c_{1,k+1}^{s-1}(t), & i = k, \end{cases}$$

$$s=1, \dots, n-1, \quad i, k=1, \dots, n-s, \quad t \in T.$$

Now we shall prove the following

Theorem 1. Let the hypotheses (1)–(4) be fulfilled, and $a_{i,k}(t) > 0$, $c_{i,k}^0(t) > 0$ for $t \in T$, $i, k=1, \dots, n$. If the numbers $c_{i,k}^s(t)$, $s=1, \dots, n-1$, $i, k=1, \dots, n-s$, $t \in T$, defined by (5) and (6), fulfil the inequalities

$$(7) \quad c_{i,i}^s(t) > 0 \text{ for } s=1, \dots, n-2, i=1, \dots, n-s, t \in T \text{ and } c_{1,1}^{n-1}(t) \geq 0$$

for $t \in T$,

then there exists a measurable mapping $\xi: T \rightarrow G_1 \times \dots \times G_n$, $\xi = (\xi_1, \dots, \xi_n)$, such that

$$(8) \quad F_i[t, \xi_1(t), \dots, \xi_n(t)] = \xi_i(t), \quad t \in T, \quad i=1, \dots, n.$$

Proof. Let $G = G_1 \times \dots \times G_n$ and $X = X_1 \times \dots \times X_n$. We define the map $F: T \rightarrow 2^G$ by

$$(9) \quad F(t) = \{x = (x_1, \dots, x_n) \in G : F_i(t, x_1, \dots, x_n) = x_i, i = 1, \dots, n\}.$$

Then, in view of Theorem 10.3 of [2], $F(t)$ is nonempty for any $t \in T$. It is easily seen that $F(t)$ is closed for each $t \in T$. Hence by Theorem 10.1 of [2], $F(t)$ is compact for all $t \in T$.

Now we shall prove that F is the measurable mapping. Let $A_i = \{x_i^k\}_{k=1}^\infty$, $i = 1, \dots, n$, be a countable dense subset of G_i . Then $A = A_1 \times \dots \times A_n$ is countable dense subset of G . Let B be a closed subset of G . We denote

$$B^m = \left\{x \in G : D(x, B) < \frac{1}{m}\right\},$$

$$V(B) = \bigcap_{m=1}^\infty \bigcup_{(x_1, \dots, x_n) \in B^m \cap A} \left\{t \in T : |F_i(t, x_1, \dots, x_n) - x_i| < \frac{2}{m}, i = 1, \dots, n\right\}.$$

At first we show that $F^{-1}(B) \subset V(B)$. Let $t \in F^{-1}(B)$. Then there exists $x = (x_1, \dots, x_n) \in B$ such that $F_i(t, x_1, \dots, x_n) = x_i$ for $i = 1, \dots, n$. Let $x_i^{i(k)} \rightarrow x_i$, $i = 1, \dots, n$, $k \rightarrow \infty$. There exists K such that for $k > K$ we have

$$|x_i^{i(k)} - x_i| < \frac{1}{m} \quad \text{and} \quad |F_i(t, x_1^{1(k)}, \dots, x_n^{n(k)}) - x_i| < \frac{1}{m}.$$

So we obtain for $k > K$

$$|F_i(t, x_1^{1(k)}, \dots, x_n^{n(k)}) - x_i^{i(k)}| < \frac{2}{m}, \quad i = 1, \dots, n.$$

Since $x \in B$ and $x_i^{i(k)} \rightarrow x_i$, for some $k > K$ we obtain $(x_1^{1(k)}, \dots, x_n^{n(k)}) \in B^m$. Therefore $t \in V(B)$.

Now we shall prove that $V(B) \subset F^{-1}(B)$. Let $t \in V(B)$. Then for each m there exists $x = (x_1^{1(m)}, \dots, x_n^{n(m)})$ such that $D(x, B) < 1/m$ and

$$(10) \quad |F_i(t, x) - x_i^{i(m)}| < \frac{2}{m}, \quad i = 1, \dots, n.$$

Let $U = \{(x_1^{1(m)}, \dots, x_n^{n(m)}) : m = 1, 2, \dots\}$. Then it follows from Theorem 10.1 of [2] that $\alpha(U) = 0$, where α is the measure of noncompactness in the Banach space X . Therefore U is precompact and there exists a subsequence of U converging to $u = (u_1, \dots, u_n) \in G$. Since B is closed, $u \in B$. From (10) we obtain that $F_i(t, u) = u_i$, $i = 1, \dots, n$. This means that $y \in F(t) \cap B$ and therefore $t \in F^{-1}(B)$. In view of measurability of $F_i(\cdot, x_1, \dots, x_n)$ for $i = 1, \dots, n$, we obtain that $V(B)$ is measurable and hence F too.

From the theorem of Kuratowski-Nardzewski [6], there exists a measurable selector $\xi : T \rightarrow G$ of F , i.e. $\xi(t) = (\xi_1(t), \dots, \xi_n(t)) \in F(t)$ for all $t \in T$. By (9) $(\xi_1, \dots, \xi_n)$ fulfils the system of equations (8), which completes the proof.

Remark 1. The condition (7) is equivalent with the condition that the characteristic roots λ_i , $i=1, \dots, n$ of the matrix $[a_{i,k}(t)]_{i,k=1}^n$, $t \in T$, fulfil the inequalities $|\lambda_i| \leq 1$, $i=1, \dots, n$ (see [2]).

Now we note

$$(11) \quad c_{i,i}^s(t) > 0 \quad \text{for } s=0, 1, \dots, n-1, i=1, \dots, n-s, t \in T.$$

Remark 2. The condition (11) is equivalent with the condition $|\lambda_i| < 1$, $i=1, \dots, n$ (λ_i -as in Remark 1; see [2]).

Remark 3. Theorem 1 is also true if the condition (7) is replaced by the condition (11) and (4) by

$$\alpha_i[F_i(t, A_1, \dots, A_n)] \leq \sum_{k=1}^n a_{i,k}(t) \alpha_k(A_k),$$

(see Theorem 9.2 and the proof of the Theorem 10.1 of [2]).

§ 4. Existence theorem.

Let $(X, \|\cdot\|)$ be a separable real Banach space, θ the zero in X and $B=B(r, \theta)$ a closed ball of radius r about θ , $r>0$. Let $a>0$ and $b \geq 0$ be real numbers. Let T be a measurable space. Let $C(U, Y)$ denote the continuous functions from U to Y with the metric sup.

We assume:

(i) The maps $h_i: T \times \langle 0, a \rangle \times C(\langle -b, 0 \rangle, B)^n \rightarrow X$, $i=1, \dots, n$, are continuous, for every fixed $t \in T$. There exist functions $a_{i,k}(t) \geq 0$, $i, k=1, \dots, n$, $t \in T$, such that for every $x \in \langle 0, a \rangle$ and $t \in T$ and every equicontinuous family $K_i \subset C(\langle -b, 0 \rangle, B)$, $i=1, \dots, n$ we have

$$\alpha[h_i(t, A_1, \dots, A_n)] \leq \sum_{k=1}^n a_{i,k}(t) \alpha_k(A_k), \quad i=1, \dots, n,$$

where $A_i \subset K_i$, $i=1, \dots, n$ (α , α_i -measure of noncompactness in X and $C(\langle -b, 0 \rangle, B)$ respectively).

(ii) For each $x \in \langle 0, a \rangle$, $y_1, \dots, y_n \in C(\langle -b, 0 \rangle, B)$, $h_i(\cdot, x, y_1, \dots, y_n)$ is measurable, $i=1, \dots, n$.

(iii) If K_i , $i=1, \dots, n$, is any equicontinuous family contained in $C(\langle -b, 0 \rangle, B)$, then for each $t \in T$, $h_i(t, \cdot, \dots, \cdot)$, $i=1, \dots, n$ is uniformly continuous on $\langle 0, a \rangle \times K_1 \times \dots \times K_n$.

(iv) There exists a constant $M \geq 0$ such that

$$\|h_i(t, x, y_1, \dots, y_n)\| \leq M \quad \text{for } i=1, \dots, n,$$

$(t, x, y_1, \dots, y_n) \in T \times \langle 0, a \rangle \times C(\langle -b, 0 \rangle, B)^n$.

(v) The functions $f_i: \langle -b, 0 \rangle \rightarrow \langle -b, 0 \rangle$, $g_i: \langle 0, a \rangle \rightarrow \langle 0, a \rangle$, $i=1, \dots, n$ are continuous.

Let $\varphi_i \in C(\langle -b, a \rangle, B)$, $i=1, \dots, n$. We denote

$$P_x^i(\varphi_i)(s) = \varphi_i[g_i(x) + f_i(s)], \quad x \in \langle 0, a \rangle, \quad s \in \langle -b, 0 \rangle, \quad i=1, \dots, n.$$

Let $\varphi_i^0 \in C(\langle -b, 0 \rangle, B)$, $\varphi_i^0(0) = \theta$, $i=1, \dots, n$.

We have the following

Theorem 2. Let the hypotheses (i)–(v) be fulfilled. Let the functions $c_{i,k}^s(t)$, $s=0, 1, \dots, n-1$, $i, k=1, \dots, n$, $t \in T$, defined by

$$c_{i,k}^0(t) = \begin{cases} a a_{i,k}(t), & i \neq k, \\ 1 - a a_{i,k}(t), & i = k, \end{cases} \quad i, k=1, \dots, n, \quad t \in T,$$

and (6) fulfil the inequalities (11) and $Ma \leq r$. Then there exist functions $\varphi_i: T \times \langle -b, a \rangle \rightarrow B$, $i=1, \dots, n$ such that

- 1) $\varphi_i|_{T \times \langle -b, 0 \rangle} = \varphi_i^0$, $i=1, \dots, n$;
- 2) for each $t \in T$, $i=1, \dots, n$, $\varphi_i(t, \cdot)$ is continuous;
- 3) for each $x \in \langle 0, a \rangle$, $\varphi(\cdot, x)$ is measurable, where $\varphi = (\varphi_1, \dots, \varphi_n)$;
- 4) for any $t \in T$

$$\frac{d\varphi_i(t, x)}{dx} = h_i(t, x, P_x^1\varphi_1(t, \cdot), \dots, P_x^n\varphi_n(t, \cdot)), \quad x \in \langle 0, a \rangle, \quad i=1, \dots, n.$$

Proof. We define the maps:

$$F_i: T \times C(\langle -b, a \rangle, B)^n \rightarrow C(\langle 0, a \rangle, X), \quad i=1, \dots, n \quad \text{by}$$

$$F_i(t, \varphi_1, \dots, \varphi_n)(x) = h_i(t, x, P_x^1\varphi_1, \dots, P_x^n\varphi_n), \quad x \in \langle 0, a \rangle.$$

Let $(y_1, \dots, y_n) \in C(\langle -b, a \rangle, B)^n$. Then for any $t \in T$, $F_i(t, y_1, \dots, y_n)(\cdot)$ is continuous function from $\langle 0, a \rangle$ into X and for any $x \in \langle 0, a \rangle$, $F_i(\cdot, y_1, \dots, y_n)(x)$ is measurable by (ii). Therefore, applying Theorem 4.2 of [5] we see that $F_i(\cdot, y_1, \dots, y_n)$ is measurable for $i=1, \dots, n$.

Now we define S_i , $i=1, \dots, n$ as the space of all functions $\varphi_i \in C(\langle -b, a \rangle, B)$ which fulfil the conditions

$$\varphi_i|_{\langle -b, 0 \rangle} = \varphi_i^0, \quad i=1, \dots, n,$$

$$\|\varphi_i(x) - \varphi_i(z)\| \leq M|x - z|, \quad x, z \in \langle 0, a \rangle.$$

It is easy to see that S_i , $i=1, \dots, n$, is separable closed bounded and convex subset of a Banach space $C(\langle -b, a \rangle, X)$. Moreover, S_i , $i=1, \dots, n$, is an equicontinuous family of functions.

For any $t \in T$, $F_i(t, \cdot, \dots, \cdot)$, $i=1, \dots, n$, is continuous on $S_1 \times \dots \times S_n$ by (iii).

Next, we define $G: C(\langle 0, a \rangle, X) \rightarrow C(\langle 0, a \rangle, X)$ by

$$G(\varphi)(x) = \int_0^x \varphi(s) ds, \quad x \in \langle 0, a \rangle.$$

Then we have

$$(12) \quad \rho[G(\varphi), G(\psi)] \leq a\rho(\varphi, \psi), \quad \varphi, \psi \in C(\langle 0, a \rangle, X).$$

Let $T_i, i=1, \dots, n$, be defined by

$$T_i(t, \varphi_1, \dots, \varphi_n)(x) = \begin{cases} (G \circ F_i)(t, \varphi_1, \dots, \varphi_n)(x), & x \in \langle 0, a \rangle, \\ \varphi_i^0(x), & x \in \langle -b, 0 \rangle, \end{cases}$$

$$\varphi_i \in S_i, \quad i=1, \dots, n, \quad t \in T.$$

Then, for $t \in T, \varphi_i \in S_i, i=1, \dots, n$, we have

$$\|T_i(t, \varphi_1, \dots, \varphi_n)(x)\| \leq \int_0^x \|h_i(t, s, P_s^1 \varphi_1, \dots, P_s^n \varphi_n)\| ds \leq Ma \leq r,$$

and

$$\|T_i(t, \varphi_1, \dots, \varphi_n)(x) - T_i(t, \varphi_1, \dots, \varphi_n)(z)\| \leq M|x-z|, \quad x, z \in \langle 0, a \rangle, \quad t \in T,$$

which means that T_i maps $T \times S_1 \times \dots \times S_n$ into $S_i, i=1, \dots, n$.

Moreover, for each $(y_1, \dots, y_n) \in S_1 \times \dots \times S_n, T_i(\cdot, y_1, \dots, y_n), i=1, \dots, n$, is measurable as a substitution of a continuous and measurable functions and for each $t \in T, T_i(t, \cdot, \dots, \cdot), i=1, \dots, n$, is continuous as a substitution of a continuous functions.

Let $K_i \subset C(\langle -b, a \rangle, B), i=1, \dots, n$, be an equicontinuous family of functions. For each $i=1, \dots, n$,

$$\bigcup_{0 \leq x \leq a} P_x^i K_i = R_i,$$

is an equicontinuous family in $C(\langle -b, 0 \rangle, B)$. Therefore, since h_i is uniformly continuous on $\langle 0, a \rangle \times R_1 \times \dots \times R_n$, we see that $F_i(t, K_1, \dots, K_n), i=1, \dots, n, t \in T$, is an equicontinuous family. Consequently, by Lemma 1 and (i) for $A_i \subset K_i, i=1, \dots, n, t \in T$, we get

$$\begin{aligned} \alpha_1[F_i(t, A_1, \dots, A_n)] &= \sup_{s \in \langle 0, a \rangle} \alpha(\{h_i(t, s, P_s^1 \varphi_1, \dots, P_s^n \varphi_n) : \varphi_k \in A_k, k=1, \dots, n\}) \\ &\leq \sup_{s \in \langle 0, a \rangle} \sum_{k=1}^n a_{i,k}(t) \alpha_1(\{P_s^k \varphi_k : \varphi_k \in A_k\}). \end{aligned}$$

Since $P_s^k: C(\langle -b, a \rangle, B) \rightarrow C(\langle -b, 0 \rangle, B), k=1, \dots, n, s \in \langle 0, a \rangle$ is nonexpansive, in view of definition of the function α_1 , we obtain

$$\alpha_1(\{P_s^k \varphi_k : \varphi_k \in A_k\}) \leq \alpha_1(A_k), \quad k=1, \dots, n,$$

hence

$$(13) \quad \alpha_1[F_i(t, A_1, \dots, A_n)] \leq \sum_{k=1}^n a_{i,k}(t) \alpha_1(A_k), \quad A_i \subset K_i, \quad t \in T, \quad i=1, \dots, n.$$

In view of Lemma 2, (12) and (13) we obtain for $A_i \subset S_i$, $i=1, \dots, n$, $t \in T$

$$\alpha_1[G(F_i(t, A_1, \dots, A_n))] \leq a \alpha_1[F_i(t, A_1, \dots, A_n)] \leq \sum_{k=1}^n a a_{i,k}(t) \alpha_1(A_k).$$

Hence, in view of (11) and Remark 3, we can apply the Theorem 1 and we obtain the measurable function $\xi: T \rightarrow S_1 \times \dots \times S_n$, $\xi = (\xi_1, \dots, \xi_n)$, such that

$$(14) \quad T_i(t, \xi_1(t), \dots, \xi_n(t)) = \xi_i(t), \quad t \in T, \quad i=1, \dots, n.$$

Let

$$\varphi(t, x) = \xi(t)(x) \quad \text{i.e.} \quad \varphi_i(t, x) = \xi_i(t)(x), \quad i=1, \dots, n, \quad t \in T, \quad x \in \langle -b, a \rangle.$$

Then by Theorem 4.2 of [5] the conditions 2) and 3) are fulfilled. Moreover, by the definition of S_i , $i=1, \dots, n$, we see that also 1) is true. From (14) and the definition of T_i , $i=1, \dots, n$, we can easily verify that the system of equations 4) is satisfied. This completes the proof.

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