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(Citation)

Funkcialaj Ekvacioj, 25(2):185-197

(Issue Date)

1982-08

(Resource Type)

journal article

(Version)

Version of Record

(JaLCD0I)

<https://doi.org/10.24546/0100499269>

(URL)

<https://hdl.handle.net/20.500.14094/0100499269>



Series Expansions of Painlevé Transcendents in the Neighbourhood of a Fixed Singular Point

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§ 0. Introduction.

Consider Painlevé equations

$$(III) \quad \ddot{\lambda} = \frac{\dot{\lambda}^2}{\lambda} - \frac{\dot{\lambda}}{t} + \frac{1}{t}(\alpha\lambda^2 + \beta) + \gamma\lambda^3 + \frac{\delta}{\lambda}, \quad \left(\cdot = \frac{d}{dt}\right)$$

$$(V) \quad \ddot{\lambda} = \left(\frac{1}{2\lambda} + \frac{1}{\lambda-1}\right)\dot{\lambda}^2 - \frac{\dot{\lambda}}{t} + \frac{(\lambda-1)^2}{t^2}\left(\alpha\lambda + \frac{\beta}{\lambda}\right) + \frac{\gamma}{t}\lambda + \delta\frac{\lambda(\lambda+1)}{\lambda-1},$$

$$(VI) \quad \ddot{\lambda} = \frac{1}{2}\left(\frac{1}{\lambda} + \frac{1}{\lambda-1} + \frac{1}{\lambda-t}\right)\dot{\lambda}^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{\lambda-t}\right)\dot{\lambda} \\ + \frac{\lambda(\lambda-1)(\lambda-t)}{t^2(t-1)^2}\left(\alpha + \beta\frac{t}{\lambda^2} + \gamma\frac{t-1}{(\lambda-1)^2} + \delta\frac{t(t-1)}{(\lambda-t)^2}\right).$$

These equations have a fixed singular point at $t=0$.

R. Garnier [1] studied equation (VI) in the neighbourhood of $t=0$, and obtained a family of solutions $\{\lambda(t)\}$ such that, for every real constant ω satisfying $0 \leq \omega < 1$, $t^\omega/\lambda(t)$ is bounded along a curve C_ω tending to $t=0$. On the other hand, B. M. McCoy et al. [2] obtained formal solutions of (III) as $t \rightarrow 0$ along the positive real axis.

The purpose of this paper is to construct a family of solutions of (J) (J=III, V, VI) expanded into convergent series in the neighbourhood of $t=0$.

To do this, in Section 1, a majorant series for a certain kind of formal series is defined, and some preparatory lemmas are proved.

In Section 2, we treat an equation of the form

$$(E) \quad x(xu')' = F_0(x, e^{-u}, xe^u) + F_1(x, e^{-u}, xe^u)(xu') + F_2(x, e^{-u}, xe^u)(xu')^2, \quad \left(' = \frac{d}{dx}\right),$$

in the neighbourhood of $x=0$, and give a family of solutions expanded into convergent series (See Theorem 2.1.).

In Section 3, we show that, in the neighbourhood of $t=0$, Painlevé equations

(J) (J=III, V, VI) are reduced to equation (E). Applying Theorem 2.1, we arrive at the main result which is Theorem 3.1.

Our solutions for equation (VI) correspond to Garnier's solutions $\lambda(t)$ along the curves C_ω for $0 < \omega < 1$ (See Remark 3.1.).

§ 1. Majorant series.

Let ω, κ be complex parameters, and $C(\omega)[[x, Y, Z]]$ the formal power series ring. We consider a set

$$(1.1) \quad \mathfrak{R} = \{f(x, e^{-\kappa}x^\omega, e^{\kappa}x^{1-\omega}) \mid f(x, Y, Z) \in C(\omega)[[x, Y, Z]]\}.$$

Proposition 1.1. *Every element φ of $\mathfrak{R}$ has a unique expression*

$$(1.2) \quad \varphi = \sum_{i \geq 0} a_i x^i + \sum_{\substack{i \geq 0 \\ j \geq 1}} b_{ij} x^i (e^{-\kappa}x^\omega)^j + \sum_{\substack{i \geq 0 \\ j \geq 1}} c_{ij} x^i (e^{\kappa}x^{1-\omega})^j,$$

where the coefficients a_i 's etc. are elements of $C(\omega)$.

Proof. Let φ be an element of $\mathfrak{R}$ such that

$$\varphi = f(x, e^{-\kappa}x^\omega, e^{\kappa}x^{1-\omega}) = \sum_{i, j, k \geq 0} c_{ijk} x^i (e^{-\kappa}x^\omega)^j (e^{\kappa}x^{1-\omega})^k.$$

Note that

$$(1.3) \quad x^i (e^{-\kappa}x^\omega)^j (e^{\kappa}x^{1-\omega})^k = \begin{cases} x^{i+j} (e^{\kappa}x^{1-\omega})^{k-j}, & \text{if } j \leq k, \\ x^{i+k} (e^{-\kappa}x^\omega)^{j-k}, & \text{if } j \geq k. \end{cases}$$

Then we see that the formal series φ is equivalent to

$$\varphi = \sum_{i \geq 0} A_i x^i + \sum_{\substack{i \geq 0 \\ j \geq 1}} B_{ij} x^i (e^{-\kappa}x^\omega)^j + \sum_{\substack{i \geq 0 \\ j \geq 1}} C_{ij} x^i (e^{\kappa}x^{1-\omega})^j,$$

where

$$A_I = \sum_{\substack{i+j=I \\ k-j=0}} c_{ijk}, \quad B_{IJ} = \sum_{\substack{i+k=I \\ j-k=J}} c_{ijk}, \quad C_{IJ} = \sum_{\substack{i+j=I \\ k-j=J}} c_{ijk}$$

are finite sums of c_{ijk} 's.

By virtue of Proposition 1.1, $\mathfrak{R}$ is a ring.

Let ρ be a positive constant satisfying

$$(1.4) \quad 0 < \rho < 1/2.$$

For $\varphi \in \mathfrak{R}$ expressed as (1.2), we set

$$(1.5) \quad \|\varphi\|_\rho = \sum_{i \geq 0} |a_i| |x|^i + \sum_{\substack{i \geq 0 \\ j \geq 1}} (|b_{ij}| + |c_{ij}|) |x|^{i+\rho j},$$

and call the *majorant series* of φ . The notation $\|\cdot\|_\rho$ is often simplified as $\|\cdot\|$. If the majorant series $\|\varphi\|_\rho$ converges for $|x| < r$, then the formal series φ also converges absolutely and satisfies

$$(1.6) \quad |\varphi| \leq \|\varphi\|_\rho$$

in the domain defined by $|x| < r$, $|e^{-\kappa x^\omega}| < |x|^\rho$, $|e^{\kappa x^{1-\omega}}| < |x|^\rho$.

We put

$$(1.7) \quad \mathfrak{R}(r) = \{\varphi \in \mathfrak{R} \mid \|\varphi\|_\rho < \infty \text{ for } |x| < r, \rho \leq \operatorname{Re} \omega \leq 1 - \rho\}.$$

Throughout this paper, we assume that $r < 1$.

Proposition 1.2. *Let φ and ψ be elements of $\mathfrak{R}$ with the properties $\|\varphi\|, \|\psi\| < \infty$ for $|x| < r$, $\rho \leq \operatorname{Re} \omega \leq 1 - \rho$. Then we have*

$$\text{i) } \|c\varphi\| = |c| \|\varphi\|, \quad c \in \mathbb{C},$$

$$\text{ii) } \|\varphi + \psi\| \leq \|\varphi\| + \|\psi\|,$$

$$\text{iii) } \|\varphi\psi\| \leq \|\varphi\| \|\psi\|$$

for $|x| < r$.

Proof. The verification of i) and ii) is not difficult. We only show iii). Recalling (1.3), we have

$$\|x^i (e^{-\kappa x^\omega})^j x^k (e^{\kappa x^{1-\omega}})^l\| = \begin{cases} |x|^{i+k+j} |x|^{\rho(l-j)}, & \text{if } l \geq j, \\ |x|^{i+k+l} |x|^{\rho(j-l)}, & \text{if } l < j. \end{cases}$$

On the other hand, it is clear that

$$\|x^i (e^{-\kappa x^\omega})^j\| \|x^k (e^{\kappa x^{1-\omega}})^l\| = |x|^{i+k} |x|^{\rho(j+l)}.$$

Since $0 < \rho < 1/2$, $r < 1$, we obtain

$$\|x^i (e^{-\kappa x^\omega})^j x^k (e^{\kappa x^{1-\omega}})^l\| \leq \|x^i (e^{-\kappa x^\omega})^j\| \|x^k (e^{\kappa x^{1-\omega}})^l\|,$$

for $|x| < r$. From this inequality, iii) follows immediately.

Proposition 1.2 implies that $\mathfrak{R}(r)$ is a subring of $\mathfrak{R}$.

Assume that a power series $f(x, Y_1, \dots, Y_n)$ expressed as

$$f(x, Y_1, \dots, Y_n) = \sum_{i, j, \dots, k \geq 0} c_{ij \dots k} x^i Y_1^j \dots Y_n^k$$

converges in the domain $|x| < r$, $|Y_\nu| < r_\nu$ ($\nu = 1, \dots, n$). The majorant series of $f(x, Y_1, \dots, Y_n)$ is defined by

$$\sum_{i,j,\dots,k \geq 0} |c_{ij\dots k}| |x|^i |Y_1|^j \dots |Y_n|^k,$$

which we denote by

$$\|f\|(x, Y_1, \dots, Y_n).$$

Then we have the following proposition.

Proposition 1.3. *Let φ_ν 's ($\nu=1, \dots, n$) be elements of $\mathfrak{R}(r)$ satisfying $\|\varphi_\nu\| < r$, for $|x| < r$, $\rho \leq \operatorname{Re} \omega \leq 1 - \rho$. Then we have*

$$(1.8) \quad f(x, \varphi_1, \dots, \varphi_n) \in \mathfrak{R},$$

$$(1.9) \quad \|f(x, \varphi_1, \dots, \varphi_n)\| \leq \|f\|(|x|, \|\varphi_1\|, \dots, \|\varphi_n\|).$$

Proof. If we denote by $a_0^{(\nu)}$ the constant term of φ_ν , then $|a_0^{(\nu)}| \leq \|\varphi_\nu\| < r$. Hence we have

$$f(x, \varphi_1, \dots, \varphi_n) = f(x, a_0^{(1)}, \dots, a_0^{(n)}) + F(x, \bar{\varphi}_1, \dots, \bar{\varphi}_n),$$

where

$$\bar{\varphi}_\nu = \varphi_\nu - a_0^{(\nu)},$$

and

$$F(x, Y_1, \dots, Y_n) = f(x, a_0^{(1)} + Y_1, \dots, a_0^{(n)} + Y_n) - f(x, a_0^{(1)}, \dots, a_0^{(n)}).$$

Let I be any positive integer. Consider a triplet $(\alpha, \beta, \gamma) \in (N \cup \{0\})^3$ satisfying

$$(x^i)^\alpha (x^j (e^{-\varepsilon} x^\omega)^k)^\beta (x^l (e^\varepsilon x^{1-\omega})^m)^\gamma = x^I,$$

for some integers $i, k, m (\geq 1), j, l (\geq 0)$. Then the number of such triplets (α, β, γ) is finite. Since $F(0, 0, \dots, 0) = 0$, this fact implies that the coefficient of x^I in $F(x, \bar{\varphi}_1, \dots, \bar{\varphi}_n)$ is finite. In a similar way, we can show that the coefficients of $x^j (e^{-\varepsilon} x^\omega)^k$ and $x^l (e^\varepsilon x^{1-\omega})^m$ are also finite. Thus we obtain (1.8). Once (1.8) is obtained, inequality (1.9) is verified without difficulty. In fact we have

$$\begin{aligned} \|f(x, \varphi_1, \dots, \varphi_n)\| &\leq \sum_{i,j,\dots,k \geq 0} |c_{ij\dots k}| |x|^i \|\varphi_1\|^j \dots \|\varphi_n\|^k \\ &= \|f\|(|x|, \|\varphi_1\|, \dots, \|\varphi_n\|). \end{aligned}$$

Corollary 1.4. *Assume that φ_ν 's satisfy the same condition as in Lemma 1.3. Then $f(x, \varphi_1, \dots, \varphi_n) (\in \mathfrak{R}(r))$ converges uniformly and absolutely in the domain defined by*

$$|x| < r, \quad |e^{-\varepsilon} x^\omega| < |x|^\rho, \quad |e^\varepsilon x^{1-\omega}| < |x|^\rho.$$

§ 2. Solutions of a reduced equation.

Consider an equation of the form

$$(E) \quad x(xu')' = F_0(x, e^{-u}, xe^u) + F_1(x, e^{-u}, xe^u)(xu') + F_2(x, e^{-u}, xe^u)(xu')^2.$$

Here we assume that

(A.1) $F_j(x, \xi, \eta)$'s ($j=0, 1, 2$) are holomorphic and bounded in the polydisk defined by

$$|x| < r_0, \quad |\xi|, |\eta| < r_1,$$

where r_0, r_1 are positive constants with $r_0 < 1$,

and that

$$(A.2) \quad F_j(0, 0, 0) = 0.$$

Then we have the following theorem.

Theorem 2.1. *Let $D(r, A_1, A_2)$ ($0 < r < r_0$; $A_1, A_2 \in \mathbf{R}$) be a domain defined by*

$$(2.1) \quad |x| < r, \quad A_1 < \frac{\arg x}{\log |x|} < A_2,$$

and ρ a constant such that

$$(2.2) \quad 0 < \rho < 1/2, \quad r_1/r_0^\rho > 1.$$

Assume that the constant r is sufficiently small and that complex parameters ω, κ satisfy the following conditions:

$$(2.3) \quad \rho < \min(\operatorname{Re} \omega, 1 - \operatorname{Re} \omega),$$

$$(2.4) \quad |e^{-\kappa} x^\omega| < |x|^\rho, \quad |e^{\kappa} x^{1-\omega}| < |x|^\rho,$$

for every $x \in D(r, A_1, A_2)$.

Then equation (E) has a family of solutions $\{u(\omega, \kappa; x)\}$ expanded into the convergent series

$$u(\omega, \kappa; x) = -\omega \log x + \kappa + \sum_{i \geq 1} a_i(\omega) x^i + \sum_{\substack{i \geq 0 \\ j \geq 1}} b_{ij}(\omega) x^i (e^{-\kappa} x^\omega)^j + \sum_{\substack{i \geq 0 \\ j \geq 1}} c_{ij}(\omega) x^i (e^{\kappa} x^{1-\omega})^j,$$

in $D(r, A_1, A_2)$.

Remark 2.1. Relation (2.4) means that

$$\rho - 1 + \operatorname{Re} \omega < \frac{1}{\log |x|} (\operatorname{Im} \omega \cdot \arg x + \log |e^\kappa|) < \operatorname{Re} \omega - \rho,$$

in $D(r, A_1, A_2)$.

Before going into our proof, we give the following lemmas.

Lemma 2.2. *If ω, κ and x satisfy*

$$|x| < r, \quad |e^{-\kappa} x^\omega|, \quad |e^\kappa x^{1-\omega}| < r^\rho,$$

then the transformation

$$(2.5) \quad u = -\omega \log x + \kappa + v$$

takes equation (E) into

$$(\mathcal{E}) \quad x(xv')' = \mathcal{F}(x, e^{-\kappa} x^\omega, e^\kappa x^{1-\omega}, v, xv') + \Phi(x, e^{-\kappa} x^\omega, e^\kappa x^{1-\omega}),$$

with the properties:

(1) $\Phi(x, \xi, \eta)$ is holomorphic and bounded for

$$|x| < r, \quad |\xi|, |\eta| < r^\rho,$$

and $\mathcal{F}(x, \xi, \eta, v, w)$ is holomorphic and bounded for

$$|x| < r, \quad |\xi|, |\eta| < r^\rho, \quad |v| < \log(r_1/r^\rho), \quad |w| < R,$$

(2) $\Phi(0, 0, 0) = 0$,

(3) $\mathcal{F}(x, \xi, \eta, 0, 0) = 0$ and $\mathcal{F}(0, 0, 0, v, w) = 0$.

Proof. By (2.5), equation (E) is transformed into

$$(2.6) \quad \begin{aligned} x(xv')' = & G_0(x, e^{-\kappa} x^\omega e^{-v}, e^\kappa x^{1-\omega} e^v) + G_1(x, e^{-\kappa} x^\omega e^{-v}, e^\kappa x^{1-\omega} e^v)(xv') \\ & + G_2(x, e^{-\kappa} x^\omega e^{-v}, e^\kappa x^{1-\omega} e^v)(xv')^2, \end{aligned}$$

where $G_0 = F_0 - \omega F_1 + \omega^2 F_2$, $G_1 = F_1 - 2\omega F_2$, $G_2 = F_2$. Clearly, G_j 's are defined in the domain given by

$$|x| < r, \quad |e^{-\kappa} x^\omega|, |e^\kappa x^{1-\omega}| < r^\rho, \quad |e^v|, |e^{-v}| < r_1/r^\rho.$$

Then we set

$$\begin{aligned} \Phi(x, \xi, \eta) &= G_0(x, \xi, \eta), \\ \mathcal{F}(x, \xi, \eta, v, w) &= G_0(x, \xi e^{-v}, \eta e^v) - G_0(x, \xi, \eta) \\ &\quad + G_1(x, \xi e^{-v}, \eta e^v)w + G_2(x, \xi e^{-v}, \eta e^v)w^2. \end{aligned}$$

Taking (A.2) into consideration, we arrive at the conclusion.

Lemma 2.3. Assume that $\mathcal{F}(x, \xi, \eta, v, w)$ satisfies (1) and (3) in Lemma 2.2. Then we have

$$\begin{aligned} & \mathcal{F}(x, \xi, \eta, v_2, w_2) - \mathcal{F}(x, \xi, \eta, v_1, w_1) \\ &= (v_2 - v_1) \mathcal{F}_1(x, \xi, \eta, v_1, v_2, w_1, w_2) + (w_2 - w_1) \mathcal{F}_2(x, \xi, \eta, v_1, v_2, w_1, w_2), \end{aligned}$$

if $|x| < r$, $|\xi|, |\eta| < r^\rho$, $|v_i|, |w_i| < 2^{-1} \log(r_1/r^\rho)$, ($i=1, 2$). Here $\mathcal{F}_1, \mathcal{F}_2$ are holomorphic and bounded in the domain given above, and satisfy

$$\mathcal{F}_i(0, 0, 0, v_1, v_2, w_1, w_2) = 0, \quad (i=1, 2).$$

Proof.

$$\begin{aligned} & \mathcal{F}(x, \xi, \eta, v_2, w_2) - \mathcal{F}(x, \xi, \eta, v_1, w_1) \\ &= \int_0^1 \frac{\partial}{\partial \theta} \mathcal{F}(x, \xi, \eta, v_2\theta + v_1(1-\theta), w_2\theta + w_1(1-\theta)) d\theta \\ &= (v_2 - v_1) \int_0^1 \mathcal{F}_v(x, \xi, \eta, v_2\theta + v_1(1-\theta), w_2\theta + w_1(1-\theta)) d\theta \\ & \quad + (w_2 - w_1) \int_0^1 \mathcal{F}_w(x, \xi, \eta, v_2\theta + v_1(1-\theta), w_2\theta + w_1(1-\theta)) d\theta. \end{aligned}$$

Thus we obtain the lemma.

As an immediate consequence of this lemma, we have

Lemma 2.4. Let φ_i, ψ_i ($i=1, 2$) be elements of $\mathfrak{R}(r)$ satisfying $\|\varphi_i\|, \|\psi_i\| < 2^{-1} \log(r_1/r^\rho)$, for $|x| < r$, $\rho \leq \operatorname{Re} \omega \leq 1 - \rho$, where $\|\cdot\| = \|\cdot\|_\rho$. Then, under the same condition as in Lemma 2.3, for some positive constant c , we have

$$\begin{aligned} & \|\mathcal{F}(x, e^{-\varepsilon}x^\omega, e^\varepsilon x^{1-\omega}, \varphi_2, \psi_2) - \mathcal{F}(x, e^{-\varepsilon}x^\omega, e^\varepsilon x^{1-\omega}, \varphi_1, \psi_1)\| \\ & \leq c|x|^\rho (\|\varphi_2 - \varphi_1\| + \|\psi_2 - \psi_1\|), \end{aligned}$$

if $|x| < r$, $|e^{-\varepsilon}x^\omega| < |x|^\rho$, $|e^\varepsilon x^{1-\omega}| < |x|^\rho$, $\rho \leq \operatorname{Re} \omega \leq 1 - \rho$.

Proof. By virtue of Lemma 1.3, we have

$$\begin{aligned} & \|\mathcal{F}_i(x, e^{-\varepsilon}x^\omega, e^\varepsilon x^{1-\omega}, \varphi_1, \varphi_2, \psi_1, \psi_2)\| \\ & \leq \|\mathcal{F}_i\| (|x|, |x|^\rho, |x|^\rho, \|\varphi_1\|, \|\varphi_2\|, \|\psi_1\|, \|\psi_2\|) = O(|x| + |x|^\rho + |x|^\rho) \\ & \leq c|x|^\rho, \quad (i=1, 2), \end{aligned}$$

for $|x| < r$, since $\mathcal{F}_i(x, \dots)$ converges absolutely in the domain given by $|x| < r$, $|\xi|, |\eta| < r^\rho$, $|v_i|, |w_i| < 2^{-1} \log(r_1/r^\rho)$. Thus we obtain the lemma.

For $x = |x| \exp(\phi\sqrt{-1}) \in D(r, A_1, A_2)$, denote by $\gamma(x)$ the path defined by

$$(2.7) \quad \gamma(x) = \left\{ t = |t| \exp(\psi\sqrt{-1}) \mid \frac{\psi}{\log|t|} = \frac{\phi}{\log|x|} = \sigma(x), |t| \leq |x| \right\}$$

which starts from x and tends to zero.

Lemma 2.5. *If $a > 0$, then*

$$(2.8) \quad \int_{\gamma(x)} |t|^{a-1} |dt| = |1 + \sigma(x)\sqrt{-1}| |x|^a/a,$$

and, if $|t^\omega| \rightarrow 0$ as $t \rightarrow 0$ along $\gamma(x)$, then

$$(2.9) \quad \int_{\gamma(x)} t^{\omega-1} dt = x^\omega/\omega.$$

Proof. Note that $x \in D(r, A_1, A_2)$ implies $\gamma(x) \subset D(r, A_1, A_2)$. Hence by putting

$$t = \exp((1 + \sigma(x)\sqrt{-1})\tau), \quad \tau = \log |t|,$$

for $t \in \gamma(x)$, we have

$$\begin{aligned} \int_{\gamma(x)} |t|^{a-1} |dt| &= \int_{-\infty}^{\log |x|} e^{(a-1)\tau} |1 + \sigma(x)\sqrt{-1}| e^\tau d\tau \\ &= |1 + \sigma(x)\sqrt{-1}| |x|^a/a, \end{aligned}$$

and

$$\begin{aligned} \int_{\gamma(x)} t^{\omega-1} dt &= \int_{-\infty}^{\log |x|} \exp((\omega-1)(1 + \sigma(x)\sqrt{-1})\tau) \\ &\quad \times (1 + \sigma(x)\sqrt{-1}) \exp((1 + \sigma(x)\sqrt{-1})\tau) d\tau \\ &= x^\omega/\omega. \end{aligned}$$

Lemma 2.6. *Let $f(x, Y_1, \dots, Y_n)$ be a convergent power series in the domain $|x| < r, |Y_\nu| < r_\nu$, and φ_ν 's ($\nu = 1, \dots, n$) elements of $\mathfrak{R}(r)$ satisfying $\|\varphi_\nu\| < r_\nu$, for $|x| < r, \rho \leq \operatorname{Re} \omega \leq 1 - \rho$. Assume that ω satisfies (2.3) and that*

$$(2.10) \quad f(0, 0, \dots, 0) = 0,$$

$$(2.11) \quad \varphi_\nu|_{x=0} = 0.$$

Then the function

$$\int_{\gamma(x)} t^{-1} f(t, \varphi_1(t), \dots, \varphi_n(t)) dt$$

belongs to $\mathfrak{R}(r)$, and is estimated as

$$\left\| \int_{\gamma(x)} t^{-1} f(t, \varphi_1(t), \dots, \varphi_n(t)) dt \right\| \leq \int_{\gamma(x)} |t|^{-1} \|f\|(|t|, \|\varphi_1(t)\|, \dots, \|\varphi_n(t)\|) |dt|$$

in $D(r, A_1, A_2)$, if (2.4) is satisfied.

Proof. Since $\rho < \operatorname{Re} \omega$, we have, for $i \geq 0, j \geq 1$,

$$\begin{aligned}
 \|(i+j\omega)^{-1}x^i(e^{-\epsilon}x^\omega)^j\| &= |(i+j\omega)^{-1}| |x|^{i+\rho j} \\
 (2.12) \quad &\leq (i+\rho j)^{-1} |1+\sqrt{-1}\sigma(x)| |x|^{i+\rho j} \\
 &= \int_{r(x)} |t|^{-1} |t|^{i+\rho j} |dt| = \int_{r(x)} |t|^{-1} \|t^i(e^{-\epsilon}t^\omega)^j\| |dt|.
 \end{aligned}$$

In a similar way, using $\rho < 1 - \operatorname{Re} \omega$, we have

$$(2.13) \quad \|(i+j(1-\omega))^{-1}x^i(e^\epsilon x^{1-\omega})^j\| \leq \int_{r(x)} |t|^{-1} \|t^i(e^\epsilon t^{1-\omega})^j\| |dt|.$$

By (2.10), (2.11) and Lemma 1.3, the series $f(x, \varphi_1, \dots, \varphi_n)$ ($\in \mathfrak{R}(r)$) is expressed as

$$(2.14) \quad f(x, \varphi_1, \dots, \varphi_n) = \sum_{i \geq 1} a_i x^i + \sum_{\substack{i \geq 0 \\ j \geq 1}} b_{ij} x^i (e^{-\epsilon} x^\omega)^j + \sum_{\substack{i \geq 0 \\ j \geq 1}} c_{ij} x^i (e^\epsilon x^{1-\omega})^j,$$

which converges uniformly and absolutely, if $|x| < r$, $|e^{-\epsilon} x^\omega| < |x|^\rho$, $|e^\epsilon x^{1-\omega}| < |x|^\rho$. On the other hand, consider a series defined by

$$\begin{aligned}
 F(x, \varphi_1, \dots, \varphi_n) &= \sum_{i \geq 1} i^{-1} a_i x^i + \sum_{\substack{i \geq 0 \\ j \geq 1}} (i+j\omega)^{-1} b_{ij} x^i (e^{-\epsilon} x^\omega)^j \\
 (2.15) \quad &+ \sum_{\substack{i \geq 0 \\ j \geq 1}} (i+j(1-\omega))^{-1} c_{ij} x^i (e^\epsilon x^{1-\omega})^j.
 \end{aligned}$$

From inequalities (2.12) and (2.13), we derive

$$\begin{aligned}
 \|F(x, \varphi_1, \dots, \varphi_n)\| &= \sum_{i \geq 1} |a_i| \|i^{-1}x^i\| + \sum_{\substack{i \geq 0 \\ j \geq 1}} |b_{ij}| \|(i+j\omega)^{-1}x^i(e^{-\epsilon}x^\omega)^j\| \\
 &+ \sum_{\substack{i \geq 0 \\ j \geq 1}} |c_{ij}| \|(i+j(1-\omega))^{-1}x^i(e^\epsilon x^{1-\omega})^j\| \\
 (2.16) \quad &\leq \int_{r(x)} |t|^{-1} \|f(t, \varphi_1(t), \dots, \varphi_n(t))\| |dt| \\
 &\leq \int_{r(x)} |t|^{-1} \|f\|(|t|, \|\varphi_1(t)\|, \dots, \|\varphi_n(t)\|) |dt| \\
 &= O(|x|^\rho),
 \end{aligned}$$

for $x \in D(r, A_1, A_2)$, which implies that $F(x, \varphi_1, \dots, \varphi_n)$ belongs to $\mathfrak{R}(r)$ and converges uniformly and absolutely in $D(r, A_1, A_2)$, if (2.4) is satisfied. Hence, by virtue of Weierstrass' theorem, we have

$$\frac{d}{dx} F(x, \varphi_1, \dots, \varphi_n) = x^{-1} f(x, \varphi_1, \dots, \varphi_n),$$

and

$$F(x, \varphi_1, \dots, \varphi_n) = \int_{r(x)} t^{-1} f(t, \varphi_1(t), \dots, \varphi_n(t)) dt,$$

in $D(r, A_1, A_2)$, if (2.4) is satisfied.

Combining this with (2.16), we arrive at the conclusion.

Now, using the lemmas obtained above, we can prove Theorem 2.1.

Proof of Theorem 2.1. Assume that x belongs to $D(r, A_1, A_2)$ and that the parameters ω, κ satisfy (2.3) and (2.4). In view of Lemma 2.2, it is sufficient to show that equation (ℰ) has a solution $v(\omega, \kappa; x)$ expanded into the convergent series

$$(2.17) \quad v(\omega, \kappa; x) = \sum_{i \geq 1} a_i x^i + \sum_{\substack{i \geq 0 \\ j \geq 1}} b_{ij} x^i (e^{-\kappa} x^\omega)^j + \sum_{\substack{i \geq 0 \\ j \geq 1}} c_{ij} x^i (e^\kappa x^{1-\omega})^j,$$

in $D(r, A_1, A_2)$, if r is sufficiently small.

We shall construct such a solution by the method of successive approximation.

To do this, we write equation (ℰ) in the form

$$(S) \quad \begin{aligned} xv'(x) &= \varphi_1(x) + I_1(v(t), tv'(t); x), \\ v(x) &= \varphi(x) + I_0(v(t), tv'(t); x), \end{aligned}$$

where

$$\begin{aligned} \varphi_1(x) &= \int_{r(x)} t^{-1} \Phi(t, e^{-\kappa} t^\omega, e^\kappa t^{1-\omega}) dt, \\ \varphi(x) &= \int_{r(x)} t^{-1} \varphi_1(t) dt, \\ I_1(v(t), w(t); x) &= \int_{r(x)} t^{-1} \mathcal{F}(t, e^{-\kappa} t^\omega, e^\kappa t^{1-\omega}, v(t), w(t)) dt, \\ I_0(v(t), w(t); x) &= \int_{r(x)} t^{-1} I_1(v(\cdot), w(\cdot); t) dt, \end{aligned}$$

and define $\{v_n(x)\}_{n \geq 0}, \{v'_n(x)\}_{n \geq 0}$, recursively by

$$(I)_n \quad \begin{aligned} v_0(x) &= v'_0(x) = 0, \\ \begin{cases} xv'_n(x) &= \varphi_1(x) + I_1(v_{n-1}(t), tv'_{n-1}(t); x), \\ v_n(x) &= \varphi(x) + I_0(v_{n-1}(t), tv'_{n-1}(t); x). \end{cases} \end{aligned}$$

Furthermore, we put

$$V_n(x) = v_n(x) - v_{n-1}(x), \quad xV'_n(x) = xv'_n(x) - xv'_{n-1}(x), \quad (n = 1, 2, \dots).$$

Then these sequences have the following properties: For sufficiently small r ,

$$(2.18) \quad v_n(x), \quad xv'_n(x) \in \mathfrak{R}(r),$$

$$(2.19) \quad \|v_n(x)\|, \|xv'_n(x)\| < 2^{-1} \log(r_1/r^\rho),$$

$$(2.20) \quad \|V_n(x)\| \leq c_0(2c)^{n-1} \rho^{-2n} |1 + \sigma(x)\sqrt{-1}|^{2n} |x|^{n\rho}(n!)^{-1},$$

$$(2.21) \quad \|xV'_n(x)\| \leq c_0(2c)^{n-1} \rho^{-2n} |1 + \sigma(x)\sqrt{-1}|^{2n} |x|^{n\rho}(n!)^{-1},$$

where c_0 and c are some positive constants.

We verify these relations by induction.

By virtue of Lemma 2.6, the functions $\varphi_1(x)$ and $\varphi(x)$ belong to $\mathfrak{R}(r)$. Hence $v_1(x)(=\varphi(x))$ and $xv'_1(x)(=\varphi_1(x))$ belong to $\mathfrak{R}(r)$, which implies that (2.18) is true for $n=1$. If the constant c_0 is chosen so that

$$\|\Phi(x, \xi, \eta)\| < c_0(|x| + |\xi| + |\eta|)/3,$$

for $|x| < r_0$, $|\xi|, |\eta| < r_1$ (cf. (2) in Lemma 2.2), then we have

$$\|xV'_1(x)\| = \|xv'_1(x)\| = \|\varphi_1(x)\| \leq c_0 \rho^{-1} |1 + \sigma(x)\sqrt{-1}| |x|^\rho,$$

and

$$\|V_1(x)\| = \|v_1(x)\| = \|\varphi(x)\| \leq c_0 \rho^{-2} |1 + \sigma(x)\sqrt{-1}|^2 |x|^\rho.$$

This implies that (2.20) and (2.21) are true for $n=1$.

Take the constant r so small that

$$(2.22) \quad c_0(2c)^{-1} (\exp(2c\rho^{-2}(1+A_0)^2 r^\rho) - 1) < 2^{-1} \log(r_1/r^\rho),$$

with $A_0 = \max(A_1, A_2)$. Then (2.19) is true for $n=1$.

Suppose that (2.18), (2.19), (2.20) and (2.21) are true for $n \leq N-1$.

By virtue of Lemma 2.6, the functions $v_N(x)$ and $xv'_N(x)$ belong to $\mathfrak{R}(r)$. Using Lemmas 2.4 and 2.6, we derive from (I)_N the following:

$$(2.23) \quad \|xV'_N(x)\| \leq \int_{\gamma(x)} c |t|^{\rho-1} (\|V_{N-1}(t)\| + \|tV'_{N-1}(t)\|) |dt|,$$

$$(2.24) \quad \|V_N(x)\| \leq \int_{\gamma(x)} |s|^{-1} \int_{\gamma(s)} c |t|^{\rho-1} (\|V_{N-1}(t)\| + \|tV'_{N-1}(t)\|) |dt| |ds|.$$

From these relations and (2.20), (2.21) for $n=N-1$, the inequalities (2.20), (2.21) for $n=N$ are obtained. Moreover, using (2.22) and (2.20), (2.21) for $n \leq N$, we have

$$\|v_N(x)\|, \|xv'_N(x)\| < 2^{-1} \log(r_1/r^\rho).$$

In this way, we have proved that (2.18), $\dots$, (2.21) hold for every $n \geq 1$.

Inequality (2.20) implies that $V_n(x)$ ($\in \mathfrak{R}(r)$) converges uniformly and absolutely in $D(r, A_1, A_2)$, and that

$$\left| \sum_{n=1}^{\infty} V_n(x) \right| \leq \sum_{n=1}^{\infty} \|V_n(x)\| = O(|x|^{\rho}),$$

in $D(r, A_1, A_2)$. Consequently, $v(\omega, \kappa; x) = \sum_{n=1}^{\infty} V_n(x)$ is expressible in the form

$$v(\omega, \kappa; x) = \sum_{i \geq 1} a_i(\omega) x^i + \sum_{\substack{i \geq 0 \\ j \geq 1}} b_{ij}(\omega) x^i (e^{-\kappa} x^{\omega})^j + \sum_{\substack{i \geq 0 \\ j \geq 1}} c_{ij}(\omega) x^i (e^{\kappa} x^{1-\omega})^j,$$

which converges to a holomorphic solution of equation ($\mathcal{E}$) in $D(r, A_1, A_2)$.

Thus the proof is completed.

Theorem 2.1 leads to the following.

Corollary 2.7. $u(\omega, \kappa; x \exp(2\pi\sqrt{-1})) = u(\omega, \kappa - 2\pi\sqrt{-1}\omega; x)$.

§ 3. Painlevé equations.

In the neighbourhood of $t=0$, simple calculation shows that the J -th Painlevé equation (J) ($J=III, V, VI$) is reduced to (E) by the change of variables given by the following table:

For $J=III$, $\lambda = e^w$, $w = -u - \log t$, $t^2 = x$.

For $J=V$, $\lambda = 1 - e^{-u}$, $t = x$.

For $J=VI$, $\lambda = e^{-u}$, $t = x$.

Then, the constants r_0, r_1 appearing in assumption (A.1) are taken so that

$$0 < r_0 < 1, \quad 0 < r_1, \quad \text{if } J=III,$$

$$0 < r_0 < 1, \quad 0 < r_1 < 1, \quad \text{if } J=V,$$

$$0 < r_0 < 1, \quad 0 < r_1 < 1, \quad \text{if } J=VI.$$

Applying Theorem 2.1 to each reduced equation and going back to the original variables, we have the following theorem concerning Painlevé equations.

Theorem 3.1. *Let ρ be a constant with $0 < \rho < 1/2$, and r a sufficiently small positive constant. Assume that parameters ω, κ satisfy (2.3) and (2.4). Then, Painlevé equations (J) ($J=III, V, VI$) have the families of solutions represented by convergent series given as follows:*

For $J=III$,

$$\begin{aligned} \lambda_{III}(\omega, \kappa; t) = & e^{-\kappa} t^{2\omega-1} \left(\sum_{i \geq 0} a_i^{(3)}(\omega) t^{2i} + \sum_{\substack{i \geq 0 \\ j \geq 1}} b_{ij}^{(3)}(\omega) t^{2i} (e^{-\kappa} t^{2\omega})^j \right. \\ & \left. + \sum_{\substack{i \geq 0 \\ j \geq 1}} c_{ij}^{(3)}(\omega) t^{2i} (e^{\kappa} t^{2(1-\omega)})^j \right), \quad \text{if } t^2 \in D(r, A_1, A_2). \end{aligned}$$

For $J=V$,

$$\lambda_V(\omega, \kappa; t) = 1 - e^{-\kappa t^\omega} \left(\sum_{i \geq 0} a_i^{(5)}(\omega) t^i + \sum_{\substack{i \geq 0 \\ j \geq 1}} b_{ij}^{(5)}(\omega) t^i (e^{-\kappa t^\omega})^j \right. \\ \left. + \sum_{\substack{i \geq 0 \\ j \geq 1}} c_{ij}^{(5)}(\omega) t^i (e^{\kappa t^{1-\omega}})^j \right), \quad \text{if } t \in D(r, A_1, A_2).$$

For J=VI,

$$\lambda_{VI}(\omega, \kappa; t) = e^{-\kappa t^\omega} \left(\sum_{i \geq 0} a_i^{(6)}(\omega) t^i + \sum_{\substack{i \geq 0 \\ j \geq 1}} b_{ij}^{(6)}(\omega) t^i (e^{-\kappa t^\omega})^j \right. \\ \left. + \sum_{\substack{i \geq 0 \\ j \geq 1}} c_{ij}^{(6)}(\omega) t^i (e^{\kappa t^{1-\omega}})^j \right), \quad \text{if } t \in D(r, A_1, A_2).$$

Here $a_0^{(k)}(\omega) = 1$ ($k=3, 5, 6$).

Moreover, these solutions satisfy

$$\begin{aligned} \lambda_{III}(\omega, \kappa; t \exp(2\pi\sqrt{-1})) &= \lambda_{III}(\omega, \kappa - 4\pi\sqrt{-1}\omega; t), \\ \lambda_V(\omega, \kappa; t \exp(2\pi\sqrt{-1})) &= \lambda_V(\omega, \kappa - 2\pi\sqrt{-1}\omega; t), \\ \lambda_{VI}(\omega, \kappa; t \exp(2\pi\sqrt{-1})) &= \lambda_{VI}(\omega, \kappa - 2\pi\sqrt{-1}\omega; t). \end{aligned}$$

Remark 3.1. Since ω, κ satisfy (2.4), we have

$$c_1 |t|^{1-\rho} \leq |\lambda_{VI}(\omega, \kappa; t)| \leq c_2 |t|^\rho \quad (c_1, c_2 > 0, 0 < \rho < 1/2)$$

in the domain of convergence.

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(Ricevita la 24-an de octobro, 1981)
(Reviziita la 23-an de marto, 1982)