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# On Existence and Asymptotic Behavior of Solutions of Some Functional Equations

By

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## Introduction.

In recent years there appeared a lot of papers concerning the linear functional equation

$$x(\varphi(t))=g(t)x(t)+h(t) \tag{1}$$

or its generalization

$$x(t)=F(t, x(\varphi(t))), \tag{2}$$

where $x(t)$ is an unknown function, $t \in \langle 0, a)$ and $a \leqslant +\infty$ [5, 6, 7, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20] (cf. also bibliography in the well known monograph [13]). Equations of such type belong to basic ones considered in the theory of functional equations in a single variable [13]. Particular cases of equation (1) or (2) are well known Abel's, Schröder's and Gauss's equations, among others [13]. It is to be noticed that several mathematical problems lead us to the functional equations of the type mentioned above. For instance, the problem of Goursat or Kisyński for partial differential equations leads us to the equation (1) or (2) [4, 9]. Moreover, let us notice that some special functions such as Euler's Gamma function and Riemann's Zeta function satisfy such type of functional equations [13].

In order to solve the equation (1) or (2) many different methods have been applied. Most frequently used one was the method of successive approximation. Roughly speaking it was proved that if the function $F(t, x)$ (the right side of equation (2)) is continuous and satisfies the Lipschitz condition with constant less than 1 with respect to the second variable, the equation (2) posseses unique continuous solution defined on a certain interval [12]. This solution is the limit of a sequence $(x_n(t))$, defined inductively by

$$x_{n+1}(t)=F(t, x_n(\varphi(t))), \qquad n=0, 1, 2, \cdots$$

where $x_0(t)$ is a function satisfying certain conditions. Proof relies upon Banach fixed point theorem. If we don't require uniqueness, the existence of the solution

can be proved under weaker assumptions with the help of Schauder fixed point principle [13, 15].

Let us pay attention to the fact that in the "pure" theory of functional equations, equation in question is solved in various classes of functions. For example, by adding appropriate assumptions, equation (2) was proved to have monotonic solutions [13], convex solutions [10], solutions of class $C^n$ [5, 14, 15] or even analytic solutions [7, 18]. There has been considered also the problem of finding integrable solutions [16]. Up to the present, methods used in such theory are rather typical. The survey of various methods of such type may be found in the well known monograph of Aczél [1].

In this paper we shall consider equations more general than (1) and (2) (cf. sections 3 and 4). These equations are rather operator-functional equations and actually contain equations (1) and (2) as particular cases. By means of some modified version of the fixed point theorem of Darbo type [8, 3] we will prove the existence of solutions of equations which will be considered further on. This theorem uses the notion of so-called "relative" measure of noncompactness [3] and allows us not only to obtain the existence of solutions but also to give some information about the structure of these solutions.

By means of this theorem we obtain also some asymptotic characterization of solutions of our equations (see sections 3 and 4). Let us observe that the asymptotic behavior of solutions of equation (2) has been studied by Choczewski in [7] by means of rather difficult methods and under very strong assumptions (analyticity). Choczewski's method depends on expansion of functions appeared in the equation in power series.

In our paper we require only the continuity instead of analyticity together with some technical and natural assumptions. Because of this the obtained results will generalize those which have been mentioned before. Moreover, we shall show some new method of proving existence theorems for some functional equations. Some rather general details of this approach may be found in the paper [3] mentioned above.

## § 1. Notations and definitions.

Let $(E, \| \ \|)$ be a given Banach space with the zero element $\Theta$. In the sequel we will denote by $K(x, r)$ the closed ball centered at $x$ and with radius $r$. To denote the closure and the convex closure of the set $X$ we will use the standard symbols $\overline{X}$, Conv $X$, respectively. For given sets $X$, $Y$ and real numbers $\lambda_1$, $\lambda_2$ we will denote by $\lambda_1 X+\lambda_2 Y$ the linear combination of these sets, i.e.

$$\lambda_1 X+\lambda_2 Y=[\lambda_1 x+\lambda_2 y: x \in X, y \in Y].$$

Further let us denote by:

$\mathfrak{M}_E$: the family of all nonempty and bounded subsets of the space $E$,
$\mathfrak{N}_E$: the family of all nonempty and relatively compact subsets of $E$.

If $\mathfrak{Z}$ is a given family of sets, then its subfamily consisting of all closed sets will be denoted by $\mathfrak{Z}^c$.

We will adopt the following definition of a measure of noncompactness [3].

**Definition 1.** A function $\mu: \mathfrak{M}_E \to \langle 0, +\infty)$ will be called a measure of noncompactness if it satisfies the following conditions:

1° the family $\mathfrak{P}=[X \in \mathfrak{M}_E: \mu(X)=0]$ is nonempty and $\mathfrak{P} \subset \mathfrak{N}_E$,
2° $X \subset Y \Rightarrow \mu(X) \leqslant \mu(Y)$,
3° $\mu(\overline{X})=\mu(X)$,
4° $\mu(\operatorname{Conv} X)=\mu(X)$,
5° $\mu(\lambda X+(1-\lambda)Y) \leqslant \lambda\mu(X)+(1-\lambda)\mu(Y)$ for every $\lambda \in \langle 0, 1\rangle$,
6° if $X_n \in \mathfrak{M}_E^c$, $X_{n+1} \subset X_n$, $n=1, 2, \cdots$ and if $\lim_{n\to\infty} \mu(X_n)=0$ then the set $X_\infty = \bigcap_{n=1}^{\infty} X_n \neq \phi$.

The family $\mathfrak{P}$ described in condition 1° is said to be the kernel of the measure $\mu$ and will be denoted by $\ker \mu$. It may be shown that $(\ker \mu)^c$ forms a closed set in the space $\mathfrak{M}_E^c$ with respect to the topology generated by Hausdorff distance [3]. Moreover, the function $\mu$ is continuous (and even locally Lipschitzian) with respect to the Hausdorff distance [3].

If a measure of noncompactness $\mu$ satisfies in addition the following two conditions

7° $\mu(X+Y) \leqslant \mu(X)+\mu(Y)$,
8° $\mu(\lambda X)=|\lambda|\,\mu(X)$, $\lambda \in R$,

it will be called a sublinear measure.

Let us mention that the well known Kuratowski's measure $\alpha$ and Hausdorff's measure $\chi$ are sublinear in our sense and such that $\ker \alpha = \ker \chi = \mathfrak{N}_E$. The simplest examples of measures such that their kernels are different from $\mathfrak{N}_E$ are the norm of set, $\|X\|=\sup [\|x\|: x \in X]$ and the diameter, $\operatorname{diam} X$. For other examples we refer to [3].

In what follows let $M \subset E$ be a given nonempty set and let $\mu$ be a measure of noncompactness on $E$.

**Definition 2** [3]. A continuous transformation $T: M \to E$ is referred to as a contraction with respect to $\mu$ (shortly $\mu$-contraction) if $X \subset M$, $X \in \mathfrak{M}_E$, then $TX \in \mathfrak{M}_E$ and

$$\mu(TX) \leqslant k\mu(X)$$

where $k \in \langle 0, 1)$.

Now we recall the modified version of Darbo type fixed point theorem [3].

**Theorem 1.** *Let $C$ be a nonempty, bounded, convex and closed subset of a Banach space $E$. Suppose, $T: C \to C$ is a $\mu$-contraction. Then $T$ has at least one fixed point in $C$ and the set* Fix $T=[x \in C: Tx=x]$ *belongs* to $\ker \mu$.

It is worth to mention that the information that the set of fixed points of a transformation $T$ belongs to $\ker \mu$ plays an important role in characterization of solutions of some functional equations. It will be shown in the following sections.

## § 2. Some measures of noncompactness in the space $C(\langle 0, +\infty), p(t))$.

Let $p(t)$ be a given function defined and continuous on the interval $\langle 0, +\infty)$ with real positive values. Denote by $C(\langle 0, +\infty), p(t))=C_p$ the set of all real continuous functions defined on the interval $\langle 0, +\infty)$ such that

$$\sup [|x(t)|\, p(t): t \geqslant 0] < +\infty.$$

It is easy to check that $C_p$ forms a real Banach space with respect to the norm

$$\|x\| = \sup [|x(t)|\, p(t): t \geqslant 0]$$

(cf. [21]).

Next, for an arbitrary $x \in C_p$, $X \in \mathfrak{M}_{C_p}$, $T>0$ and $\varepsilon>0$ let us denote:

$$\begin{aligned}
\omega^T(x, \varepsilon) &= \sup [|x(t)p(t)-x(s)p(s)|: t, s \in \langle 0, T\rangle, |t-s| \leqslant \varepsilon],\\
\omega^T(X, \varepsilon) &= \sup [\omega^T(x, \varepsilon): x \in X],\\
\omega_0^T(X) &= \lim_{\varepsilon \to 0} \omega^T(X, \varepsilon),\\
\omega_0(X) &= \lim_{T \to \infty} \omega_0^T(X),\\
a(X) &= \lim_{T \to \infty} \sup_{x \in X} \{\sup [|x(t)|\, p(t): t \geqslant T]\},\\
b(X) &= \lim_{T \to \infty} \sup_{x \in X} \{\sup [|x(t)p(t)-x(s)p(s)|: t, s \geqslant T]\}.
\end{aligned}$$

If $g(t)$ is a continuous real function defined on the interval $\langle 0, +\infty)$ then we will denote by $\nu^T(g, \varepsilon)$ the modulus of continuity of a function $g(t)$ on the interval $\langle 0, T\rangle$, i.e.

$$\nu^T(g, \varepsilon) = \sup [|g(t)-g(s)|: t, s \in \langle 0, T\rangle, |t-s| \leqslant \varepsilon].$$

Finally, for any set $X \in \mathfrak{M}_{C_p}$, let us denote

$$\begin{aligned}
\mu_0(X) &= \omega_0(X) + a(X),\\
\mu(X) &= \omega_0(X) + b(X).
\end{aligned}$$

The functions $\mu_0(X)$ and $\mu(X)$ defined above are the sublinear measures of noncompactness in the space $C_p$ [3]. The kernel $\ker \mu_0$ is the family of all bounded sets consisting of functions which are equicontinuous on each compact interval and such that $\lim_{t\to\infty} x(t)p(t)=0$ uniformly with respect to $x \in X$, while the kernel $\ker \mu$ is the family of all bounded sets consisting of functions equicontinuous on each compact interval and such that

$$\lim_{T\to\infty} \{\sup [|x(t)p(t)-x(s)p(s)|: t, s \geqslant T]\}=0$$

uniformly with respect to $x \in X$ [3].

For another properties of the measures $\mu$ and $\mu_0$ see [2].

## § 3. Some functional equation.

In this section we apply one of our measures of noncompactness to the study of existence and asymptotic behavior of solutions of functional equation which is more general than the equations (1) and (2). Namely, we consider the following equation

$$x(t)=A(x)F(t, x(\varphi(t))). \tag{3}$$

We will assume the following hypotheses:

(i) $\varphi: \langle 0, +\infty)\to\langle 0, +\infty)$ is continuous, $\varphi(t)\leqslant t$ and $\lim_{t\to\infty} \varphi(t)=\infty$,

(ii) the functional A is defined on the space $C(\langle 0, +\infty), e^{-\varphi(t)})$ (shortly $C_\varphi$), continuous and bounded, $|A(x)|\leqslant k_1$,

(iii) the function $F(t, x)=F: \langle 0, +\infty)\times R\to R$ is continuous and satisfies the Lipschitz condition with a constant $k_2$ with respect to the second variable. Moreover $F(t, 0)=o(e^{\varphi(t)})$ as $t\to\infty$.

Now we prove the following theorem.

**Theorem 2.** *Under the assumptions* (i)–(iii), *if* $k=k_1k_2<1$, *the equation* (3) *has at least one solution* $x \in C_\varphi$ *such that* $x(t)=o(e^{\varphi(t)})$ *as* $t\to\infty$.

*Proof.* Consider the transformation defined on the space $C_\varphi$ by the formula

$$(\mathfrak{F}x)(t)=A(x)F(t, x(\varphi(t))).$$

Obviously the function $(\mathfrak{F}x)(t)$ is continuous on the interval $\langle 0, +\infty)$. Also, using our assumptions, for an arbitrary $x \in C_\varphi$ we obtain

$$\begin{aligned}|(\mathfrak{F}x)(t)|\, e^{-\varphi(t)} &= |A(x)|\,|F(t, x(\varphi(t)))|\, e^{-\varphi(t)}\\ &\leqslant k_1 |F(t, x(\varphi(t)))-F(t, 0)|\, e^{-\varphi(t)}+k_1 |F(t, 0)|\, e^{-\varphi(t)}\\ &\leqslant k_1k_2 |x(\varphi(t))|\, e^{-\varphi(t)}+k_1 |F(t, 0)|\, e^{-\varphi(t)}\end{aligned}$$

$$= k\,|x(\varphi(t))|\,e^{-\varphi(\varphi(t))}e^{\varphi(\varphi(t))-\varphi(t)} + k_1\,|F(t,0)|\,e^{-\varphi(t)}$$
$$\leqslant k\,\|x\| + k_1 \sup\,[|F(t,0)|\,e^{-\varphi(t)} : t \geqslant 0] < +\infty$$

so that the transformation $\mathfrak{F}$ maps the space $C_\varphi$ into itself. Moreover, from the above evaluation follows, that $\mathfrak{F}$ transforms the ball $K(\Theta, r)$ into itself, where

$$r = k_1(1-k)^{-1} \sup\,[|F(t,0)|\,e^{-\varphi(t)} : t \geqslant 0].$$

Now taking $x, y \in K(\Theta, r)$ and calculating in the similar way as above we obtain

$$\|\mathfrak{F}x - \mathfrak{F}y\| \leqslant \frac{r}{k_1}\,|A(x)-A(y)| + k\,\|x-y\|$$

so that in view of assumption (ii) we conclude that $\mathfrak{F}$ is continuous on the ball $K(\Theta, r)$. Furthermore, fix $X \subset K(\Theta, r)$, $x \in X$, $T \geqslant 0$ and $t \geqslant T$. Then we have

$$|(\mathfrak{F}x)(t)|\,e^{-\varphi(t)} \leqslant k_1 k_2\,|x(\varphi(t))|\,e^{-\varphi(\varphi(t))}e^{\varphi(\varphi(t))-\varphi(t)} + k_1\,|F(t,0)|\,e^{-\varphi(t)}$$
$$\leqslant k\,|x(\varphi(t))|\,e^{-\varphi(\varphi(t))} + k_1\,|F(t,0)|\,e^{-\varphi(t)}$$
$$\leqslant k \sup\,[|x(t)|\,e^{-\varphi(t)} : t \geqslant \inf_{s \geqslant T} \varphi(s)] + k_1\,|F(t,0)|\,e^{-\varphi(t)}.$$

Hence in virtue of assumptions (i) and (iii) we get

$$a(\mathfrak{F}X) \leqslant k\,a(X), \tag{4}$$

where $a(X)$ was defined in section 2.

Now, let us take $\varepsilon > 0$, $T > 0$ and $t, s \in \langle 0, T\rangle$ such that $|t-s| \leqslant \varepsilon$. Utilizing the assumptions (i)–(iii) we have

$$|(\mathfrak{F}x)(t)e^{-\varphi(t)} - (\mathfrak{F}x)(s)e^{-\varphi(s)}| \leqslant k_1\,|F(t, x(\varphi(t)))e^{-\varphi(t)} - F(t, x(\varphi(t)))e^{-\varphi(s)}|$$
$$+ k_1\,|F(t, x(\varphi(t)))e^{-\varphi(s)} - F(s, x(\varphi(s)))e^{-\varphi(s)}|$$
$$\leqslant k_1\,|e^{-\varphi(t)} - e^{-\varphi(s)}|\,|F(t, x(\varphi(t)))| + k_1\,|F(t, x(\varphi(t))) - F(t, x(\varphi(s)))|\,e^{-\varphi(s)}$$
$$+ k_1\,|F(t, x(\varphi(s))) - F(s, x(\varphi(s)))|\,e^{-\varphi(s)}$$
$$\leqslant re^T\nu^T(e^{-\varphi(t)}, \varepsilon) + k\,|x(\varphi(t)) - x(\varphi(s))|\,e^{-\varphi(s)}e^{\varphi(\varphi(s))-\varphi(s)}$$
$$+ k_1\,|F(t, x(\varphi(s))) - F(s, x(\varphi(s)))|\,e^{-\varphi(s)}$$

and consequently

$$|(\mathfrak{F}x)(t)e^{-\varphi(t)} - (\mathfrak{F}x)(s)e^{-\varphi(s)}| \leqslant re^T\nu^T(e^{-\varphi(t)}, \varepsilon)$$
$$+ k\,|x(\varphi(t))e^{-\varphi(\varphi(t))} - x(\varphi(s))e^{-\varphi(\varphi(s))}| + k\,|x(\varphi(t))e^{-\varphi(\varphi(s))} - x(\varphi(t))e^{-\varphi(\varphi(t))}|$$
$$+ k_1\,|F(t, x(\varphi(s))) - F(s, x(\varphi(s)))| \leqslant re^T\nu^T(e^{-\varphi(t)}, \varepsilon) + k\omega^T(x, \nu^T(\varphi(t), \varepsilon))$$
$$+ kre^T\nu^T(e^{-\varphi(\varphi(t))}, \varepsilon) + k_1\tilde{\nu}_r^T(F(t, x), \varepsilon),$$

where we have denoted

$$\tilde{\nu}_r^T(F(t, x), \varepsilon)=\sup [|F(t, x)-F(s, x)|: t, s \in \langle 0, T\rangle, |t-s|\leqslant\varepsilon, |x|\leqslant re^T].$$

Finally, taking into consideration that

$$\lim_{\varepsilon\to 0} \nu^T(\varphi(t), \varepsilon)=\lim_{\varepsilon\to 0} \nu^T(e^{-\varphi(t)}, \varepsilon)=\lim_{\varepsilon\to 0} \nu^T(e^{-\varphi(\varphi(t))}, \varepsilon)$$
$$=\lim_{\varepsilon\to 0} \tilde{\nu}_r^T(F(t, x), \varepsilon)=0,$$

from the above estimation we deduce the following inequality

$$\omega_0^T(\mathfrak{F}X)\leqslant k\omega_0^T(X),$$

for an arbitrary set $X\subset K(\Theta, r)$. Thus we get

$$\omega_0(\mathfrak{F}X)\leqslant k\omega_0(X). \tag{5}$$

Now, combining (4) and (5) we obtain

$$\mu_0(\mathfrak{F}X)\leqslant k\mu_0(X)$$

which means that $\mathfrak{F}$ is $\mu_0$-contraction. Finally, applying Theorem 1 we complete the proof.

## § 4. Existence theorem for another functional equation.

Now we are going to work in the same space as in the previous section and study the same equation perturbed by an integral operator

$$x(t)=A(x)F(t, x(\varphi(t)))+\int_0^t x(s)e^{-s}ds. \tag{6}$$

We will apply the second measure of noncompactness defined in section 2.

Suppose that:

(i) The function $\varphi: \langle 0, +\infty)\to\langle 0, +\infty)$ is continuous, $\varphi(t)\leqslant t$, $\lim_{t\to\infty}(t-\varphi(t))=\infty$, $\lim_{t\to\infty}\varphi(t)=\infty$, $M=\sup [te^{-\varphi(t)}: t\geqslant 0]<\infty$, $\sup [\varphi(t)-\varphi(s): 0\leqslant t\leqslant s]<\infty$ and the inegral $\int_0^{+\infty} e^{\varphi(t)-t}dt$ converges,

(ii) the functional $A: C_\varphi\to R$ is continuous and bounded, $|A(x)|\leqslant k_1$,

(iii) $F(t, x)=F: \langle 0, +\infty)\times R\to R$ is continuous and Lipschitzian with respect to $x$, i.e.

$$|F(t, x)-F(t, y)|\leqslant k_2|x-y|,$$

(iv) there exists a finite limit $\lim_{t\to\infty} F(t, 0)e^{-\varphi(t)}$.

We shall prove the following theorem.

**Theorem 3.** *Let the assumptions* (i)–(iv) *be satisfied and let* $k=k_1k_2+M<1$. *Then the equation* (6) *has at least one solution* $x \in C_\varphi$ *such that* $\lim_{t\to\infty} x(t)e^{-\varphi(t)}$ *exists.*

*Proof.* For an arbitrary $x \in C_\varphi$ we define

$$(Hx)(t)=A(x)F(t, x(\varphi(t)))+\int_0^t x(s)e^{-s}ds.$$

From the well known facts it follows that the function $(Hx)(t)$, $t\geqslant 0$, is continuous. Besides in view of our assumptions we get the following estimate

$$\begin{aligned}
&|(Hx)(t)|\,e^{-\varphi(t)}\\
&\quad\leqslant k_1|F(t, x(\varphi(t)))-F(t, 0)|\,e^{-\varphi(t)}+k_1|F(t, 0)|\,e^{-\varphi(t)}+\left(\int_0^t |(x(s)|\,e^{-s}ds\right)e^{-\varphi(t)}\\
&\quad\leqslant k_1k_2|x(\varphi(t))|\,e^{-\varphi(\varphi(t))}+k_1|F(t, 0)|\,e^{-\varphi(t)}\\
&\qquad+te^{-\varphi(t)}\|x\|\leqslant k\|x\|+k_1\sup[|F(t, 0)|\,e^{-\varphi(t)}: t\geqslant 0]<+\infty.
\end{aligned}$$

Hence we see that $H$ transforms the space $C_\varphi$ into itself and moreover $H: K(\Theta, r)\to K(\Theta, r)$, where $r=k_1(1-k)^{-1}\sup[|F(t, 0)|\,e^{-\varphi(t)}: t\geqslant 0]$. Continuity of $H$ is easy to check and therefore is omitted.

Now let us notice that the transformation $H$ is the sum of two mappings

$$\begin{aligned}
(H_1x)(t)&=A(x)F(t, x(\varphi(t))),\\
(H_2x)(t)&=\int_0^t e^{-s}x(s)ds.
\end{aligned}$$

Thus if we want to apply in our considerations the measure $\mu$, which is sublinear, we may check both mappings separately. From the results of the previous section it follows immediately that

$$\omega_0(H_1X)\leqslant k_1k_2\omega_0(X) \tag{7}$$

for any $X\subset K(\Theta, r)$. Further, for any fixed $T>0$, $\varepsilon>0$ and $t, s \in \langle 0, T\rangle$, $|t-s|\leqslant\varepsilon$ we obtain

$$\begin{aligned}
&|(H_2x)(t)e^{-\varphi(t)}-(H_2x)(s)e^{-\varphi(s)}|\\
&\quad\leqslant\left|\left(\int_0^t x(\tau)e^{-\tau}d\tau\right)e^{-\varphi(t)}-\left(\int_0^t x(\tau)e^{-\tau}d\tau\right)e^{-\varphi(s)}\right|\\
&\qquad+\left|\left(\int_0^t x(\tau)e^{-\tau}d\tau\right)e^{-\varphi(s)}-\left(\int_0^s x(\tau)e^{-\tau}d\tau\right)e^{-\varphi(s)}\right|\\
&\quad\leqslant\nu^T(e^{-\varphi(t)}, \varepsilon)rT+\left|\int_s^t |x(\tau)|\,e^{-\tau}d\tau\right|\\
&\quad\leqslant rT\nu^T(e^{-\varphi(t)}, \varepsilon)+\varepsilon r.
\end{aligned}$$

Hence we easily conclude that

$$\omega_0(H_2X)=0. \tag{8}$$

Furthermore, let us assume that $t, s \geqslant T$, where $T>0$ is arbitrarily fixed. Then we get

$$\begin{aligned}
&|(H_2x)(t)e^{-\varphi(t)}-(H_2x)(s)e^{-\varphi(s)}| \\
&\quad\leqslant |e^{-\varphi(t)}-e^{-\varphi(s)}|\int_0^t |x(\tau)|\,e^{-\varphi(\tau)}e^{\varphi(\tau)-\tau}d\tau+\left|\int_s^t |x(\tau)|\,e^{-\tau}d\tau\right| e^{-\varphi(s)} \\
&\quad\leqslant 3r\left(\int_0^{+\infty} e^{\varphi(\tau)-\tau}d\tau\right)\sup\,[e^{-\varphi(t)}: t\geqslant T].
\end{aligned}$$

Hence, taking into account that by assumption (i) the integral

$$\int_0^{+\infty} e^{\varphi(\tau)-\tau}d\tau$$

converges, we can easily deduce the following assertion

$$b(H_2X)=0. \tag{9}$$

Now we will evaluate $b(H_1X)$. In order to do it let us take a fixed $T>0$ and $t, s\geqslant T$. Without loss of generality we may assume that $s>t$. Then we get

$$\begin{aligned}
&|(H_1x)(t)e^{-\varphi(t)}-(H_1x)(s)e^{-\varphi(s)}| \\
&\quad\leqslant k_1|F(t, x(\varphi(t)))e^{-\varphi(t)}-F(s, x(\varphi(t)))e^{-\varphi(s)}|+k_1k_2|x(\varphi(t))-x(\varphi(s))|\,e^{-\varphi(s)} \\
&\quad\leqslant k_1|F(t, x(\varphi(t)))e^{-\varphi(t)}-F(s, x(\varphi(t)))e^{-\varphi(s)}| \\
&\qquad+k_1k_2|x(\varphi(s))e^{-\varphi(\varphi(s))}-x(\varphi(t))e^{-\varphi(\varphi(t))}| \\
&\qquad+k_1k_2|x(\varphi(t))e^{-\varphi(\varphi(t))}-x(\varphi(t))e^{-\varphi(\varphi(s))}|\,e^{\varphi(\varphi(s))-\varphi(s)} \\
&\quad\leqslant k_1|F(t, x(\varphi(t)))-F(t, 0)|\,e^{-\varphi(t)}+k_1|F(s, x(\varphi(t)))-F(s, 0)|\,e^{-\varphi(s)} \\
&\qquad+k_1|F(t, 0)e^{-\varphi(t)}-F(s, 0)e^{-\varphi(s)}|+k_1k_2|x(\varphi(s))e^{-\varphi(\varphi(s))}-x(\varphi(t))e^{-\varphi(\varphi(t))}| \\
&\qquad+k_1k_2(|x(\varphi(t))|\,e^{-\varphi(\varphi(t))}+|x(\varphi(t))|\,e^{-\varphi(\varphi(s))})e^{\varphi(\varphi(s))-\varphi(s)} \\
&\quad\leqslant k_1k_2|x(\varphi(t))|\,e^{-\varphi(t)}+k_1k_2|x(\varphi(t))|\,e^{-\varphi(s)}+k_1|F(t, 0)e^{-\varphi(t)}-F(s, 0)e^{-\varphi(s)}| \\
&\qquad+k_1k_2|x(\varphi(s))e^{-\varphi(\varphi(s))}-x(\varphi(t))e^{-\varphi(\varphi(t))}|+k_1k_2re^{\varphi(\varphi(s))-\varphi(s)} \\
&\qquad+k_1k_2re^{\varphi(\varphi(t))-\varphi(t)}e^{\varphi(t)-\varphi(s)}.
\end{aligned}$$

Hence by similar evaluation we obtain

$$\begin{aligned}
&|(H_1x)(t)e^{-\varphi(t)}-(H_1x)(s)e^{-\varphi(s)}| \\
&\quad\leqslant k_1k_2|x(\varphi(s))e^{-\varphi(\varphi(s))}-x(\varphi(t))e^{-\varphi(\varphi(t))}|+k_1k_2re^{\varphi(\varphi(t))-\varphi(t)}(1+2e^{\varphi(t)-\varphi(s)}) \\
&\qquad+k_1k_2re^{\varphi(\varphi(s))-\varphi(s)}+k_1|F(t, 0)e^{-\varphi(t)}-F(s, 0)e^{-\varphi(s)}|.
\end{aligned}$$

Because of the assumptions, (i) and (iv) the last three terms of the above inequality tend to zero as $T$ tends to infinity. Then finally we have

(10) $$b(H_1X)\leqslant k_1k_2b(X).$$

Next, combining (7), (8), (9) and (10) we get

$$\mu(HX)\leqslant k_1k_2\mu(X)\leqslant k\mu(X)$$

so that we have proved that $H$ is $\mu$-contraction. Thus by virtue of Theorem 1 the proof is complete.

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