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Sequential active learning for estimating small failure probabilities in high-dimensional problems: Application to nonlinear vessel responses

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ABSTRACT

A new method for high-dimensional structural reliability analysis is proposed, with particular attention to estimating small failure probabilities. The proposed method is built upon an active learning framework, in which an active subspace for supervised dimensionality reduction and a surrogate model for bypassing the performance function are simultaneously updated. Heteroscedastic Gaussian process (hGP) modeling is employed for this purpose. To effectively address rare event problems, the method further incorporates a sequential sampling strategy based on the subset simulation. The resulting Sequential Active Learning with Active Subspace (SALAS) method is first demonstrated using the Sobol function to illustrate its accuracy and computational efficiency. Following this, its application is extended to specific high-dimensional engineering problems involving nonlinear vessel responses in waves. Two subject vessel responses are studied: vertical bending moment and roll motion of a vessel. A nonlinear strip theory and a two-degree-of-freedom roll motion model are used to analyze these responses, respectively. Comprehensive comparisons with crude Monte Carlo simulation, first order reliability method, and the adaptive active subspace-based heteroscedastic Gaussian process (AaS-hGP) method demonstrates the efficiency and accuracy of the proposed SALAS method, even in estimating rare event probabilities in high-dimensional stochastic spaces.

1. Introduction

In many practical engineering problems, estimating the exceedance probability of a given threshold, hereafter referred to as the “failure probability”, is of critical importance. For instance, amongst various maritime-related problems, the prediction of excessive wave-induced vessel responses is of particular concern, as such responses can lead to serious marine accidents [1]. Thus, many scholars have devised numerical models for wave-induced vessel responses, which are generally called “seakeeping models”, see e.g. Ref. [2]. As introduced in Ref. [2], a wide variety of seakeeping methodologies has been established and practically utilized, such as the strip theory and the three dimensional Boundary Element Method (BEM). In recent developments, the Computational Fluid Dynamics (CFD) has been employed to accurately simulate wave-induced hydrodynamic loads, while utilizing the Finite Element Analysis (FEA) for structural response computation, see e.g. Refs. [3,4]. This approach is generally referred to as “CFD-FEA co-simulation”.

Having established well-validated numerical methods for wave-induced response computation, however, probabilistic estimation of excessive nonlinear responses in a certain sea state remains a challenge.

This is inevitably attributed to the requirement for extensive numerical computations to estimate the failure probability, especially for small probability levels. The growing computational demand hinders the practical use of high-fidelity numerical methods, such as CFD-FEA co-simulation, in the design stage. Various efficient methodologies for failure probability estimation have been devised in the realm of reliability engineering. The First Order Reliability Method (FORM) and Second Order Reliability Method (SORM) are representative examples [5]. In estimating the failure probability, FORM/SORM searches for a design point on the associated limit state surface. However, the accuracy of FORM/SORM deteriorates if the response of interest exhibits strong nonlinearity, and in fact, nonlinear wave-induced vessel responses are no exception to this issue [6]. To enhance the accuracy and efficiency, a surrogate model-based approach incorporating an active learning framework using the Gaussian Process Regression (GPR) or Kriging was devised by Echard et al. [7]. The method is known as AK-MCS, and several variants of AK-MCS have been proposed thus far, which can be categorized as AK-based approaches, as exemplified by AK-SS [8], AK-IS [9], and AK-MCMC [10]. Moreover, the Bayesian Active Learning (BAL) approaches have been recently developed; representative examples include Refs. [11,12]. Evidence is mounting that the

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BAL approach may outperform the AK-based approaches. However, both the AK- and BAL-based methods suffer from the well-known “curse-of-dimensionality” issue, i.e., the computational burden for GPR meta-model construction increases as the dimensionality of the stochastic space grows. In solving the wave-induced vessel responses, the wave field is to be modeled in a stochastic manner. This entails a high-dimensional stochastic space, cf. Ref. [13]. Consequently, the direct application of AK- and BAL-based methods is impractical.

The authors’ research group recently incorporated the Karhunen–Loève (KL) expansion of the stochastic wave [14] to reduce the dimensionality, see Ref. [15]. By incorporating such an *unsupervised* dimensionality reduction technique, highly efficient estimations of extreme nonlinear vertical bending moment (VBM) in a ship were demonstrated, in conjunction with AK-MCMC [10]. The KL expansion of wave, however, does possess a drawback in that it can represent only short-time wave sequences within a reduced stochastic space. To address this unavoidable issue, the authors recently ascertained the effectiveness of the “active subspace” method proposed by Constantine et al. [16], to attain *supervised* dimensionality reduction. That is to say, the reduced dimensionality is adaptively identified with reference to a limited number of computational results, see Refs. [17–19]. Using the active subspace, Wang et al. [20] proposed an interval squeezing method (ISM) for high-dimensional reliability analysis. Albeit their method demonstrated reasonable efficiency and accuracy, the active subspace must be identified prior to the adaptive training of the Kriging-based surrogate model. Another research effort to take advantage of the active subspace for high-dimensional reliability analysis was recently made by Kim et al. [21]. In Ref. [21], the active subspace identification and surrogate modeling are conducted *simultaneously*, by incorporating the heteroscedastic Gaussian Process (hGP) to account for the noise induced by input variables, named as an adaptive active subspace-based heteroscedastic Gaussian process (AaS-hGP).

In the AaS-hGP method, the low-dimensional feature space is identified with reference to samples of the performance function and associated gradient vectors. Thus, if the direct finite difference method (FDM) is used for gradient vector estimation, finite-difference calculations at the sampled points are required, leading to a large number of numerical simulation runs in a high-dimensional stochastic space. The authors have recently proposed a gradient vector approximation (GVA) methodology using a low-fidelity surrogate, with the aim of resolving the excessive vessel roll motion problem [22]. It has been demonstrated that the use of GVA can bypass high-fidelity model computations in gradient estimation by employing a low-fidelity, physics-informed model as a surrogate. However, the AaS-hGP method, per se, is subject to a practical limitation in that it is only practically applicable to *moderate* failure probability levels (i.e. $10^{-3} \sim 10^{-5}$), since a large population of Monte Carlo (MC) samples is required to address small probability levels. Handling a large population of MC samples can result in memory issues and prolonged computational times. Hence, the next phase of our research is to develop a generic method to replace AaS-hGP, with the objective of enabling failure probability estimation in high-dimensional spaces and under low-probability conditions.

With this in mind, this paper proposes a new method combining the active subspace, hGP, and subset simulation [23]. The present method, named Sequential Active Learning with Active Subspace (SALAS), employs the simultaneous and adaptive training phase of active subspace and hGP model founded on the AaS-hGP approach, while the sequential sampling technique is further incorporated to tackle with small failure probability problems. The specific procedure in SALAS is partially inspired by AaS-hGP; however, it differs by incorporating a Markov chain Monte Carlo (MCMC) sampling phase along with appropriate convergence criteria. By taking this means, the sample pool for estimating the (intermediate) failure probability is progressively updated, thereby achieving a substantial reduction in CPU memory usage for failure probability estimation even for small probability cases, compared with AaS-hGP. Numerical demonstrations are first made by

using the Sobol function to illustrate the efficiency and accuracy of SALAS. The applicability of SALAS to nonlinear vessel responses is then examined through numerical demonstrations using two distinct wave-induced vessel responses in stochastic waves; the VBM and roll motion. Nonlinear strip theory [24] is used for computing VBM, while the 2 degree-of-freedom (DOF) model is employed for the calculation of nonlinear roll motion. The efficiency and accuracy of SALAS are examined by means of a comparison with other reliability methods, including the crude Monte Carlo Simulation (MCS) as the authentic solution.

2. Sequential active learning with active subspace

2.1. Supervised dimensionality reduction by active subspace

Let the D -dimensional stochastic variables be $\mathbf{X} = [X_1, X_2, \dots, X_D]$. Referring to the realization of $\mathbf{X}$ as $\mathbf{x}$, the associated performance function is considered as a continuous and differentiable function $y(\mathbf{x})$. Its gradient vector is denoted by $\nabla_{\mathbf{x}} y(\mathbf{x}) = \left[\frac{\partial y}{\partial x_1}, \frac{\partial y}{\partial x_2}, \dots, \frac{\partial y}{\partial x_D} \right]^T$. In this case,

$$y(\mathbf{x}) = 1 - \frac{\mathcal{M}(\mathbf{x})}{\mathcal{M}_f} \quad (1)$$

where $\mathcal{M}(\mathbf{x})$ denotes the response value under the stochastic excitation of $\mathbf{x}$, and $\mathcal{M}_f$ the target extreme value. The probability corresponding to $P_f = \text{Prob}(y(\mathbf{X}) \leq 0)$ is called the failure probability. To retrieve the active subspace [16] with reference to N_{tr} training samples, the following $\mathbf{C}$ matrix is computed based on the Monte Carlo integral [19]:

$$\hat{\mathbf{C}} = \frac{1}{N_{tr}} \sum_{i=1}^{N_{tr}} (\nabla_{\mathbf{x}} y)_i (\nabla_{\mathbf{x}} y)_i^T \quad (2)$$

$\mathbf{C}$ matrix is a positive semi-definite $D \times D$ matrix, admitting the following eigenvalue decomposition,

$$\mathbf{C} = \mathbf{W} \mathbf{\Lambda} \mathbf{W}^T \quad (3)$$

where $\mathbf{\Lambda}$ is the eigenvalue vector $\mathbf{\Lambda} = \text{diag}\{\lambda_1, \lambda_2, \dots, \lambda_D\}$, meeting $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_D \geq 0$. Matrices $\mathbf{W}$ and $\mathbf{\Lambda}$ can be split into two parts as:

$$\mathbf{\Lambda} = \begin{bmatrix} \mathbf{\Lambda}_r & \\ & \mathbf{\Lambda}_s \end{bmatrix}, \quad \mathbf{W} = [\mathbf{W}_r \quad \mathbf{W}_s] \quad (4)$$

where the largest d_r eigenvalues are populated into $\mathbf{\Lambda}_r$, i.e. $\mathbf{\Lambda}_r = \text{diag}\{\lambda_1, \lambda_2, \dots, \lambda_{d_r}\}$. The projection matrix $\mathbf{W}_r$, with a size of $D \times d_r$, is referred to as the active subspace. $y(\mathbf{X})$ can be then approximated by using the low-dimensional feature vector $\boldsymbol{\Psi} \in \mathbb{R}^{d_r}$ defined by

$$\boldsymbol{\Psi} = \mathbf{W}_r^T \mathbf{X} \quad (5)$$

using the active subspace $\mathbf{W}_r$.

Since the active subspace is searched based on the limited number of training samples, the active subspace $\mathbf{W}_r$ identified during the training process is not necessarily *perfect* [21]. Thus, given the low-dimensional projected variables $\boldsymbol{\Psi} = \mathbf{W}_r^T \mathbf{x}$, the hGP is introduced, accounting for the input-dependent noise. By the hGP, the performance function y is modeled as:

$$\hat{y} = f(\boldsymbol{\Psi}) + \varepsilon(\boldsymbol{\Psi}), \quad \varepsilon(\boldsymbol{\Psi}) \sim \mathcal{N}(0, r(\boldsymbol{\Psi})) \quad (6)$$

where

$$f(\boldsymbol{\Psi}) \sim \mathcal{GP}(\mu_f(\boldsymbol{\Psi}), k_f(\boldsymbol{\Psi}, \boldsymbol{\Psi}')), \quad r(\boldsymbol{\Psi}) = \exp f_g(\boldsymbol{\Psi}) \quad (7)$$

where $f(\boldsymbol{\Psi})$ denotes a latent function following a Gaussian process with mean $\mu_f(\boldsymbol{\Psi})$ and covariance kernel $k_f(\boldsymbol{\Psi}, \boldsymbol{\Psi}')$, and $r(\boldsymbol{\Psi})$ is the variance of the heteroscedastic noise. $f_g(\boldsymbol{\Psi})$ in Eq. (7) follows a Gaussian process, i.e. $f_g(\boldsymbol{\Psi}) \sim \mathcal{GP}(\mu_0(\boldsymbol{\Psi}), k_g(\boldsymbol{\Psi}, \boldsymbol{\Psi}'))$. In the present study, the mean of $f(\boldsymbol{\Psi})$ is assumed to be zero, i.e., $\mu_f(\boldsymbol{\Psi}) = 0$, whereas the noise

mean is assumed to be a constant μ_0 , independent of ψ . Under this assumption, a variational approximation scheme proposed by Lázaro-Gredilla et al. [25] can be employed. The hGP is parametrized by using the hyperparameters $\{\theta_f, \theta_g, \mu_0\}$, where θ_f and θ_g are the hyperparameters for the kernel $k_f(\psi, \psi')$ and $k_g(\psi, \psi')$, respectively. Details of the hyperparameter identification are given in Appendix A. Using the identified hyperparameters, the hGP model yields the predictive mean $\mu_{\hat{y}}(\psi_*)$ and standard deviation $\sigma_{\hat{y}}(\psi_*)$ for a given input sample ψ_* .

By leveraging the hGP meta-model for the performance function the dimensionality d_r is then identified via a screening process referring to the root mean squared error between the computed $y(x_i)$ for N_{tr} training samples x_i , $i = 1, 2, \dots, N_{tr}$, and corresponding hGP-based predictions [21]:

$$\varepsilon_d = \sqrt{\frac{1}{N_{tr}} \sum_{i=1}^{N_{tr}} \{y(x_i) - \mu_{\hat{y}}(\psi_i)\}^2} \quad (8)$$

where $\psi_i = \mathbf{W}_r^T x_i$, cf. Eq. (5). Referring to ε_d , d_r is identified via the following iterative process:

1. Acquire $y(x_i)$ and $\nabla_x y(x_i)$ for N_{tr} training samples.
2. Compute C and $\mathbf{W}$ according to Eqs. (2) and (3), and initialize $d_r = 1$.
3. Train the hGP model using ψ_i and $y(x_i)$ in the low-dimensionality space d_r .
4. If $\varepsilon_d < \varepsilon_d^t$ or $d_r = D$, stop the iteration and accept the current d_r value. Otherwise, increment d_r by 1 and return to 3.

In the present study, $\varepsilon_d^t = 0.001$ is employed. Note that the above screening process for d_r is to be executed each time a new training sample is added.

2.2. Active learning by sequential sampling

SALAS employs an adaptive training process to train the hGP model guided by the sequential sampling, based on the subset simulation procedure. The specific training procedure draws upon the AK-MCMC procedure proposed by Ref. [10], which utilizes Markov chain Monte Carlo (MCMC) samples, while SALAS introduces appropriate convergence criteria for training hGP instead of the conventional GP. The procedure as a whole is given below:

1. Set $k = 1, m = 1, C = 1$ and generate an initial sample population (i.e., $\mathcal{W}_{k=1}^D = \{x_{1,k=1}, \dots, x_{N,k=1}\}$) consisting of N pairs of the stochastic variables $\mathbf{X}$.
2. Arbitrarily select N_0 initial training samples (i.e., $\mathcal{W}_{tr}^D = \{x_{1,k=1}, \dots, x_{N_0,k=1}\}$) from $\mathcal{W}_{k=1}^D$, and then compute the corresponding values of the performance function $y(x)$ and gradient vectors $\nabla_x y(x)$ by running numerical simulations.
3. Determine the reduced dimensionality d_r by the procedure mentioned in sub-Section 2.1 and trained hGP models.
4. Estimate $\hat{y}(x)$ of non-training samples in $\mathcal{W}_k = \{\mathbf{W}_{r,m}^T x_{1,k}, \dots, \mathbf{W}_{r,m}^T x_{N,k}\}$ via the hGP model predictions, and then determine or update a subspace threshold $b_k^{(m)}$ such that $p_0 N$ samples in $\mathcal{W}_k$ satisfy $y(x) \leq b_k^{(m)}$.
5. Determine the next training sample ψ^* according to a procedure described later.
6. Check the convergence of $b_k^{(m)}$ and the intermediate probability $Pr(F_k|F_{k-1})^{(m)}$ corresponding to the k th subspace by referencing the following four quantities:

$$\varepsilon_{b1}^{(m)} = \frac{|b_k^{(m)} - b_k^{(m-1)}|}{b_k^{(m)}} \quad (9)$$

$$\varepsilon_{b2}^{(m)} = |\varepsilon_{b1}^{(m)} - \varepsilon_{b1}^{(m-1)}| \quad (10)$$

$$\varepsilon_{p1}^{(m)} = \frac{|Pr(F_k|F_{k-1})^{(m)} - Pr(F_k|F_{k-1})^{(m-1)}|}{Pr(F_k|F_{k-1})^{(m)}} \quad (11)$$

$$\varepsilon_{p2}^{(m)} = |\varepsilon_{p1}^{(m)} - \varepsilon_{p1}^{(m-1)}| \quad (12)$$

where $\varepsilon_{b1}^{(m)}$ and $\varepsilon_{p1}^{(m)}$ indicate the relative changes of $b_k^{(m)}$ and $Pr(F_k|F_{k-1})^{(m)}$, respectively, while $\varepsilon_{b2}^{(m)}$ and $\varepsilon_{p2}^{(m)}$ indicate their respective convergence trends. Note that the intermediate probability is given by:

$$Pr(F_k|F_{k-1})^{(m)} = \frac{\sum_{i=1}^N \mathbb{I}(\mu_{\hat{y}}(\psi) \leq b_k)}{N} \quad (13)$$

where $\mathbb{I}(\mu_{\hat{y}}(\psi) \leq b_k) = 1$ if $\mu_{\hat{y}}(\psi) \leq b_k$, otherwise equal to 0. If $\varepsilon_{b1}^{(m)}, \varepsilon_{b2}^{(m)}, \varepsilon_{p1}^{(m)}$, and $\varepsilon_{p2}^{(m)}$ are less than predefined thresholds $\varepsilon_{b1}^{tol}, \varepsilon_{b2}^{tol}, \varepsilon_{p1}^{tol}$, and ε_{p2}^{tol} , respectively, update $C = C + 1$ and proceed to step 7. Otherwise, update $m = m + 1$, $N_{tr} = N_{tr} + 1$, $C = 1$, and compute the performance function at the new training sample $y(x^*)$ and gradient vectors $\nabla_x y(x^*)$; noting that x^* denotes the high-dimensional stochastic variables corresponding to ψ^* . Then, return to step 3.

7. Generate an MCMC sample pool of size N in the d_r -dimensional subspace defined by b_k , by using the hGP model and modified Metropolis-Hastings (M-H) algorithm [23]. Corresponding high-dimensional (D -dimensional) samples are approximated by:

$$\mathbf{X} = (\mathbf{W}_r^T)^+ \psi \quad (14)$$

using the current projection matrix $\mathbf{W}_r$, where $(\cdot)^+$ indicates the Moore–Penrose pseudo inverse. Update $k = k + 1$ and name the generated sample population as $\mathcal{W}_{k+1}$. Following this, reset $m = 1$ and return to step 3.

8. Compute the failure probability P_f using the trained hGP model and terminate the SALAS process.

In SALAS procedure, the convergence criteria $\varepsilon_{b1}^{(m)}, \varepsilon_{b2}^{(m)}, \varepsilon_{p1}^{(m)}$, and $\varepsilon_{p2}^{(m)}$ are monitored to ensure sufficient convergence of both the subset threshold $b_k^{(m)}$ and the intermediate probability $Pr(F_k|F_{k-1})^{(m)}$. Furthermore, in the above procedure, a “delayed judgment” is further employed; that is, the conditions $\varepsilon_{b1}^{(m)} < \varepsilon_{b1}^{tol}, \varepsilon_{b2}^{(m)} < \varepsilon_{b2}^{tol}, \varepsilon_{p1}^{(m)} < \varepsilon_{p1}^{tol}$, and $\varepsilon_{p2}^{(m)} < \varepsilon_{p2}^{tol}$ must be satisfied for C^{tol} consecutive times. In step 5, given a whole N sample pool undergoing dimensionality reduction $\mathcal{W}_k$ and an existing N_{tr} training sample pool (i.e., $\mathcal{W}_{tr} = \{\mathbf{W}_r^T x_1, \dots, \mathbf{W}_r^T x_{N_{tr}}\}$), the next training sample ψ^* is determined by the following two steps:

- a. From the whole sample pool $\mathcal{W}_k$, identify a subset $\mathcal{W}_c$ for which the so-called “U-function” obtained by the hGP predictions is less than 2. That is,

$$\mathcal{W}_c = \left\{ \psi \in \mathcal{W}_k : U(\psi) = \frac{|\mu_{\hat{y}}(\psi) - b_k|}{\sigma_{\hat{y}}(\psi)} \leq 2 \right\} \quad (15)$$

- b. Among the samples in $\mathcal{W}_c$, compute the minimum norm distance to the current training samples $\psi' (\in \mathcal{W}_{tr})$. Then, a sample in $\mathcal{W}_c$ that maximizes this minimum distance, denoted as ψ^* , is selected as the next training sample. It means,

$$\psi^* = \arg \max_{\psi \in \mathcal{W}_c} \left(\min_{\psi' \in \mathcal{W}_{tr}} \|\psi - \psi'\| \right) \quad (16)$$

Note that $U(\psi) \leq 2$ is practically useful for various types of failure probability estimations, see e.g. Refs. [7,10].

In computing the failure probability P_f based on the trained hGP model at step 8., the present study employs the following two ways:

- (1) P_f is estimated by the product of the intermediate (conditional) probabilities at each subset. That is, $P_{f,SS}$ is computed as per

$$P_{f,SS} = Pr(F_1) \prod_{i=2}^k Pr(F_i|F_{i-1}) \quad (17)$$

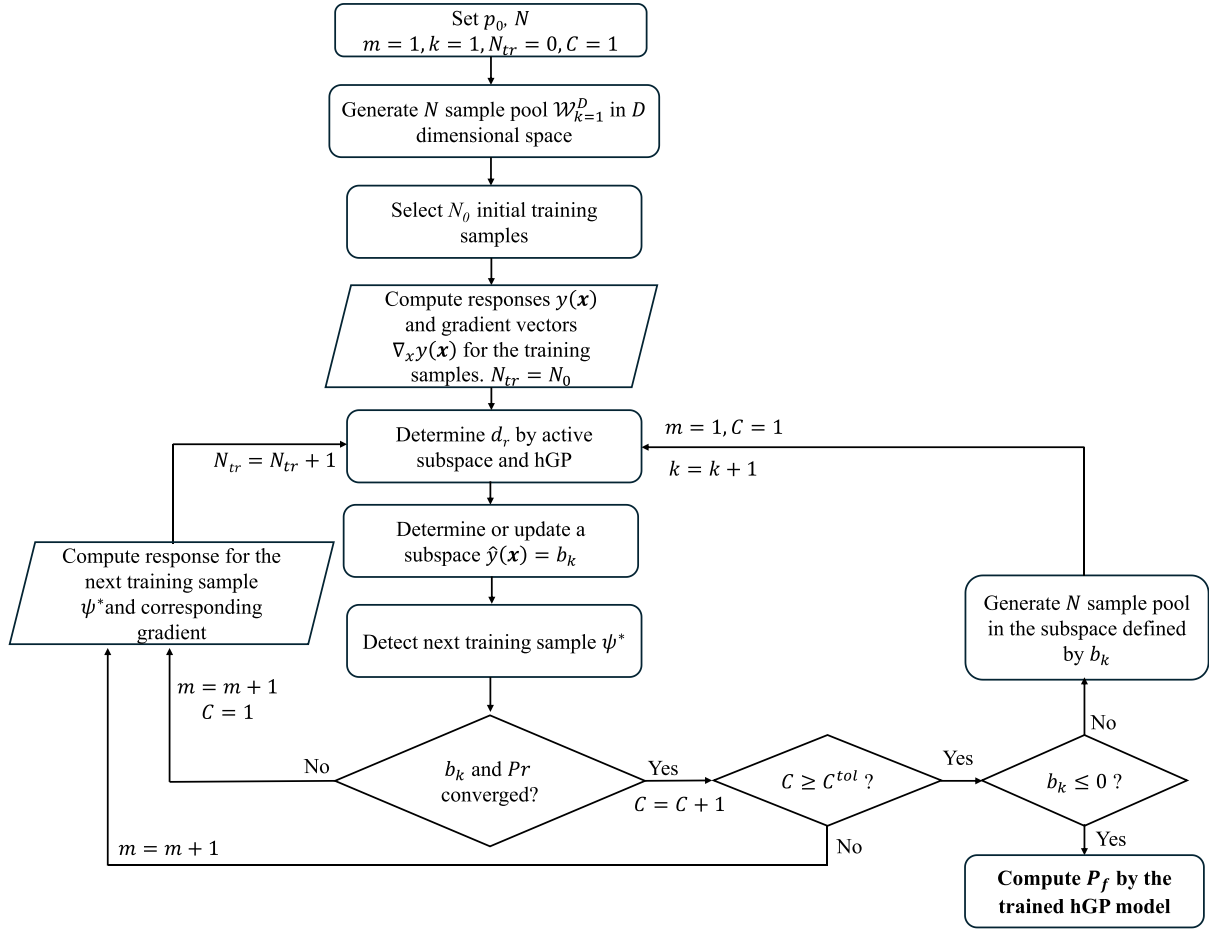


Fig. 1. Overall workflow of the SALAS.

with reference to the intermediate probability Pr at i th step. Notice that $Pr(F_1)$ and $Pr(F_i|F_{i-1})$ for $i < k$ are equal to p_0 in the present framework.

- (2) P_f is estimated by a newly generated MC samples. That is, by generating a MC sample pool $\mathcal{W}_{MC}^D = \{\mathbf{x}_{1,k=1}, \dots, \mathbf{x}_{N_{MC},k=1}\}$ consisting of N_{MC} pairs of the stochastic variables $\mathbf{X}$, the low dimensional sample pool $\mathcal{W}_{MC} = \{\psi_1, \dots, \psi_{N_{MC}}\} = \{\mathbf{W}_{r,m}^T \mathbf{x}_{1,k}, \dots, \mathbf{W}_{r,m}^T \mathbf{x}_{N_{MC},k}\}$ is identified, followed by the $P_{f,MCS}$ estimation according to

$$P_{f,MCS} = \frac{\sum_{i=1}^{N_{MC}} \mathbb{I}(\mu_{\hat{y}}(\psi) \leq 0)}{N_{MC}} \quad (18)$$

In the normal application of the subset simulation, and also in the present study, $Pr(F_i|F_{i-1})$ is evaluated in a sample-basis, i.e.,

$$Pr(F_i|F_{i-1}) \approx \frac{n_i}{N} \quad (19)$$

where n_i denotes the number of samples that exceed b_i (note that for $i < k$, Pr is equal to p_0). However, as is obvious from Eq. (19), if N is small, the accuracy in $Pr(F_i|F_{i-1})$ may be unreliable. In addition, it is inevitable that the effective sample size is reduced as the subset is updated, which also deteriorates the accuracy in $P_{f,SS}$. Increasing N could inevitably enhance the accuracy of $Pr(F_i|F_{i-1})$, however, it also increases the CPU memory requirement and thus impairs the computational advantage of SALAS. This is an essential issue of the subset simulation in estimating P_f , see Li et al. [26]. The introduction of Eq. (18) is motivated to address issue. That is, a dedicated MCS is conducted using the trained hGP model while discarding $Pr(F_i|F_{i-1})$ records, to evaluate P_f via newly generated MC samples. Estimating

$P_{f,MCS}$ requires additional computation, not to mention the fact that it involves a separate MC process; nevertheless, the computation time is relatively short (within 5 min on the authors' PC) and does not demand large CPU memory.

Fig. 1 shows an overall workflow of the SALAS method. In the AK-MCMC framework by Ref. [10], $N = 50/p_0 \sim 100/p_0$ is suggested while $p_0 = 0.001$ is employed for problems with P_f of $10^{-6} \sim 10^{-9}$ and $p_0 = 0.01$ for P_f of $10^{-3} \sim 10^{-5}$ problems [27]. In line with these previous works, this study employs two combinations:

Case 1: $p_0 = 0.01$, $N = 10,000$

Case 2: $p_0 = 0.001$, $N = 100,000$

Results obtained from these two cases will be compared in the subsequent numerical demonstration sections. The gradient vectors at the training samples are derived by direct FDM with finite-difference calculations. For reference, the AaS-hGP procedure [21], in which the MC sample pool is utilized to train the hGP model, is described in Appendix B.

3. Numerical demonstration using Sobol function

To demonstrate the effectiveness of the present SALAS method, the following Sobol function [19,21] is used:

$$\mathcal{M}(\mathbf{X}) = - \prod_{j=1}^D \frac{4X_j - 2 + \tau_j}{1 + \tau_j} \quad (20)$$

where the stochastic variables X_j are uniformly distributed over the range of $[0, 1]$. τ_j is the model parameter that characterizes the

Table 1

Comparison of failure probabilities and requisite number of computations. P_f and N_{PF} values by AaS-hGP and SALAS are the averages over 20 independent realizations. Sobol function (Eq. (20)) case, with $\mathcal{M}_f = 0.65$.

Dimension	Method	Failure probability		N_{PF}	$N_{grad} = N_{PF} D$
		$P_{f,SS}$	$P_{f,MCS}$		
$D = 30$	MCS	5.80×10^{-3}		10^6	–
	AaS-hGP ($N_p = 10^6$)	5.78×10^{-3}		126	3780
	SALAS (Case 1, $C^{tol} = 1$)	5.76×10^{-3}	5.73×10^{-3}	173	5190
$D = 100$	MCS	3.33×10^{-3}		10^6	–
	AaS-hGP ($N_p = 10^6$)	3.26×10^{-3}		148	14 800
	SALAS (Case 1, $C^{tol} = 1$)	3.55×10^{-3}	3.43×10^{-3}	172	17 200

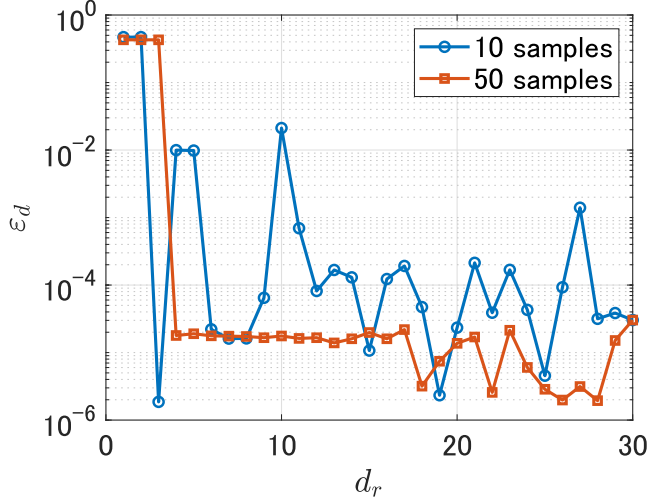


Fig. 2. ε_d values by Eq. (8) with respect to the reduced dimension d_r taken from 10 and 50 samples of the performance function. $D = 30$ and $\mathcal{M}_f = 0.65$ case.

influence of X_j , and in the present study, τ_j is set as follows [21]:

$$\tau_j = \begin{cases} 1 & \text{for } j = 1, 2, \dots, 4, \\ 500 & \text{otherwise} \end{cases} \quad (21)$$

Thus, an effective dimension of Eq. (20) is 4.

Taking $D = 30$ and $\mathcal{M}_f = 0.65$ as an example, the ε_d values with respect to d_r are examined. In Fig. 2, the ε_d values obtained from randomly selected 10 and 50 samples of the performance function are compared. Note that the ε_d values are plotted on a logarithmic scale. From the figure, ε_d exhibits occasional increases, notably in the 10-sample case. Increasing the number of samples to 50 mitigates these fluctuations. Therefore, it would be preferable to avoid choosing an initial training sample size N_0 that is too small. It is also noteworthy that an alternative measure for determining d_r , such as the “activity score”, see Constantine et al. [28], could be used instead of ε_d .

Failure probabilities for the target extreme values of $\mathcal{M}_f = 0.65$ and 1.25 are estimated and compared. Cases with dimensionalities of $D = 30$ and $D = 100$ are examined. The crude MCS is computed to serve as the authentic solution. For comparison, the failure probabilities are also computed by AaS-hGP. The settings for AaS-hGP computation are summarized below (see Appendix B):

- The sample size $N_p = 10^6$.
- The tolerances $\varepsilon_1^{tol} = 0.0001$ and $\varepsilon_2^{tol} = 0.0001$.
- The number of initial training samples $N_0 = 50$.

Meanwhile, the SALAS computations are carried out with:

- The tolerances $\varepsilon_{b1}^{tol} = \varepsilon_{b2}^{tol} = \varepsilon_{p1}^{tol} = \varepsilon_{p2}^{tol} = 0.001$.
- The number of initial training samples $N_0 = 50$.

Note that two different cases (Case 1 and Case 2) are adopted for p_0 and N selection, see sub-Section 2.2.

To obtain gradient vectors during the AaS-hGP/SALAS procedures, a simple FDM is employed. It means, suppose that $\nabla_{\mathbf{x}} y(\mathbf{x}^*) = [\Delta_1(\mathbf{x}^*), \Delta_2(\mathbf{x}^*), \dots, \Delta_D(\mathbf{x}^*)]^T$,

$$\Delta_i(\mathbf{x}^*) = \frac{y(\mathbf{x}^* + \delta \mathbf{e}_i) - y(\mathbf{x}^*)}{\delta} \quad (22)$$

where $\mathbf{e}_i$ denotes i th standard basis vector, and δ is the perturbation magnitude. In the present specific examples, $\delta = 10^{-5}$ is used.

20 independent realizations of AaS-hGP and SALAS are executed, each with a different initial sample population. For SALAS computation, Case 1 is adopted to $\mathcal{M}_f = 0.65$ cases, while Cases 1 and 2 are adopted to $\mathcal{M}_f = 1.25$ cases, and $C^{tol} = 1$. Fig. 3 ($\mathcal{M}_f = 0.65$) and Fig. 4 ($\mathcal{M}_f = 1.25$) show the mean and standard deviation of the estimated failure probabilities $P_{f,SS}$ and $P_{f,MCS}$, in comparison with that derived from AaS-hGP. The red dotted lines in the figures indicate the MCS results from a sufficient number of computations (10^6 for $\mathcal{M}_f = 0.65$ and 10^8 for $\mathcal{M}_f = 1.25$). From Fig. 3 in which the moderate P_f cases, it is evident that both AaS-hGP and SALAS offer highly accurate failure probabilities. However, remarkable deterioration of AaS-hGP observed in the $\mathcal{M}_f = 1.25$ case ($D = 30$), where the P_f level is on the order of 10^{-5} . The underlying cause of this issue appears to lie in the inherent failure of AaS-hGP solution, as all the samples often satisfy $U > 2$ at the very early training stage. Essentially, increasing N_p could ameliorate the errors in P_f by AaS-hGP, however, as previously noted, increasing N_p demands a substantial memory capacity and, the computation of P_f by Eq. (B.1) becomes computationally prohibitive. In fact, if $N_p = 5 \times 10^6$ is used, the execution of AaS-hGP became infeasible on the authors' PC environment (Intel(R) Core(TM) i7-14700, 2.10 GHz, 16 GB RAM). Conversely, SALAS functions well also in such a case, see Figs. 4a and 4b, thereby underscoring its notable computational tractability over AaS-hGP. One may notice that the standard deviations from $P_{f,SS}$ (cf. error bars in the figures) are relatively larger than $P_{f,MCS}$. This is conspicuous in cases with smaller failure probabilities, see Fig. 4. The large variation in $P_{f,SS}$ may be ascribed to the cumulative errors of the intermediate probability Pr in Eq. (17). In the event that the Case 2 setting is employed to SALAS computations, see Figs. 4b and 4d, the standard deviation of $P_{f,SS}$ is reduced. This is attributed to the sufficient number of samples being employed and to the number of subsets (i.e., k) being smaller than in Case 1 owing to the smaller p_0 value. Therefore, $P_{f,SS}$ can be selected with less concern regarding its variation when a large N and small p_0 are employed. Nonetheless, all in all, $P_{f,MCS}$ exhibits superior accuracy and smaller standard deviation in comparison to $P_{f,SS}$, indicating that the trained hGP models effectively capture the limit state surface. The accuracy of failure probability estimates by SALAS for $\mathcal{M}_f = 1.25$ is somewhat inferior to the corresponding estimates from $\mathcal{M}_f = 0.65$ case, and the variation of P_f is relatively large. A potential cause of this is that, under the present convergence criteria, the SALAS procedure may terminate even if the hGP model has not yet been sufficiently trained. Fig. 5 presents the results on $\mathcal{M}_f = 1.25$ from 20 independent realizations, with the introduction of a tolerance $C^{tol} = 2$ serving to facilitate a more stringent convergence decision. As found from the figures, the variations of P_f are apparently smaller in comparison with $C^{tol} = 1$

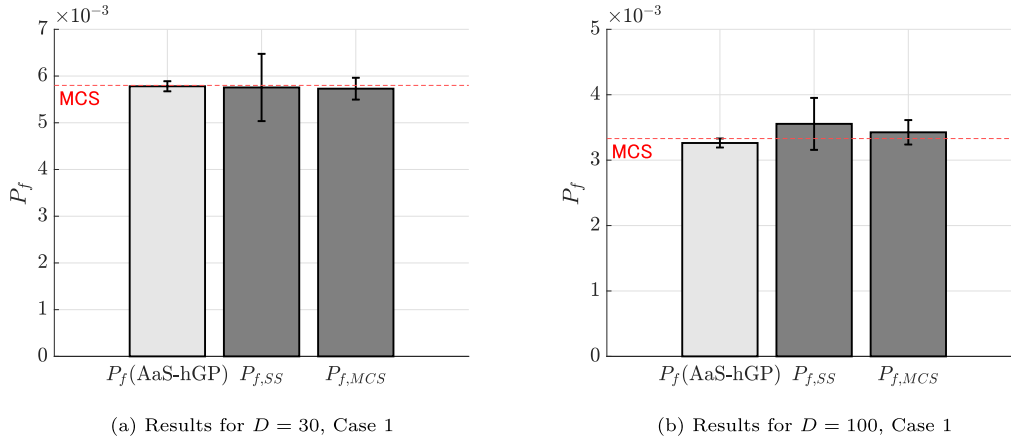


Fig. 3. Mean and standard deviation of $P_{f,SS}$ and $P_{f,MCS}$ derived from 20 realizations of AaS-hGP and SALAS. Sobol function (Eq. (20)) case with $\mathcal{M}_f = 0.65$. The bars indicate the mean values, while the error bars indicate ± 1 standard deviation on either side. Red dotted lines indicate the MCS results with 10^6 computations. $C^{tol} = 1$ is used.

cases. Nevertheless, increasing C^{tol} value inevitably leads to a rise in the required number of computations.

Table 1 presents a comparison of the results obtained from crude MCS, AaS-hGP, and SALAS for $\mathcal{M}_f = 0.65$ case, where $C^{tol} = 1$ is introduced for SALAS computations. The tables present the derived failure probabilities P_f and the requisite number of computations of performance function, denoted by N_{PF} . Note that the number of performance function computations for obtaining the gradient vectors $\nabla_{\mathbf{x}} y(\mathbf{x}^*)$ by FDM has been eliminated from N_{PF} , and is instead indicated separately as N_{grad} in the table, i.e., $N_{grad} = N_{PF} D$. The averages of P_f (for SALAS, $P_{f,SS}$ and $P_{f,MCS}$) over 20 independent realizations, each with a distinct initial sample pool, are tabulated in the table.

From Table 1, it is apparent that both AaS-hGP and SALAS exhibit fairly accurate P_f with the limited number of performance function computations, N_{PF} . All in all, N_{PF} by SALAS is slightly larger than AaS-hGP, as SALAS introduces double-checking of b_k and $Pr(F_k|F_{k-1})$. That said, the difference is marginal and the efficiency of SALAS remains evident regardless of the dimension D . Fig. 6 shows the history of identified d_f and the convergence histories of b_k and $Pr(F_k|F_{k-1})$, for $D = 30$ and $\mathcal{M}_f = 0.65$ case, obtained from the SALAS execution. The figures are plotted with respect to the number of trainings, i.e., $N_{tr} - N_0$. As found in the d_f history, d_f exhibits dynamic changes in the initial training process, but eventually converges to $d_f = 4$, which is consistent with the effective dimension denoted by Eq. (21). As described in sub-Section 2.2, the parameter subject to convergence checking is initially set to b_k . Subsequent to ensuring that $b_k = 0$, the procedure transitions to checking $Pr(F_k|F_{k-1})$. As found from the right-hand figure in Fig. 6, both b_k and $Pr(F_k|F_{k-1})$ exhibit clear convergence trends, indicating that the stopping criteria given in Eqs. (9), (10), (11), and (12) function effectively. The computational effort associated with FDM is significant, as indicated by N_{grad} in the table; nevertheless, it remains more efficient than MCS. Employing an alternative method for computing the gradient vectors would be preferable when applying SALAS to practical engineering problems. This point will be further discussed in Section 4.

Table 2 presents a comparison of the results for $\mathcal{M}_f = 1.25$ case. As mentioned earlier, AaS-hGP is no longer applicable to the $\mathcal{M}_f = 1.25$ cases, and thus the corresponding results are not included in the table. In contrast to AaS-hGP, the P_f estimates obtained by SALAS exhibit significantly better agreement with the MCS results, albeit some discrepancies caused by the variation in results. Both Case 1 and Case 2 yield reasonable estimates of P_f . Increasing C^{tol} improves the accuracy of the estimated mean value of P_f , but it is apparently accompanied by an increase in N_{PF} . Despite the demonstrated effectiveness of SALAS, the calculation of gradient vectors incurs a significant computational burden, as the direct FDM is leveraged in this study. As such, if a

high-fidelity numerical model is referenced, the finite-difference calculations are computationally prohibitive, impeding the efficiency of the SALAS method. Thus, an alternative to FDM would be ideal, such as the automatic differentiation [29] or a low-fidelity surrogate model-based approach [22].

According to Chan et al. [30], when the limit state surface is discontinuous, the standard subset simulation may behave erratically, causing the solution to get stuck and fail to correctly reach the failure events. In the present cases such a problem is not observed, but to verify whether the samples in each subset during the execution of SALAS appropriately transition into the failure domain, the samples at each subset are examined. Fig. 7 presents comparisons of failure and non-failure classification results within the sample pool for each subset (projected onto the X_1 - X_2 plane), referencing the $D = 30$, $\mathcal{M}_f = 1.25$ case (Case 1 with $C^{tol} = 1$ is used). In Fig. 7, the left-hand side figures show the true solutions, where “failure” samples satisfy $y(\mathbf{x}) \leq b_i$ at the i th subset, while the right-hand side figures show the hGP model predictions by SALAS, where “failure” samples satisfy $\mu_{\hat{y}}(\psi_i) \leq b_i$. Note that the classification is made with reference to hGP models at the final stage of training in each subset. As can be seen from Fig. 7, the samples are progressively transitioned into the failure domain as the subsets are updated. The hGP model predictions also accurately capture the true solutions. This underscores the effectiveness of SALAS even when the adaptive updating of the hGP model and the active subspace are incorporated. Notwithstanding the effectiveness of SALAS, it would be necessary to carefully investigate its applicability to problems involving a discontinuous limit state surface, as discussed in Ref. [30]. In addition, updating the hGP model during the simulation can in principle alter the definition of the intermediate events, i.e., F_i , since the events are inherently tied to the surrogate model at a given stage [31]. Although this issue is not expected to significantly affect the results, it is worth acknowledging.

4. Application to wave-induced vessel responses

4.1. Vertical bending moment

The VBM in a container vessel is taken as a first example to illustrate the applicability of SALAS. The present example has been also used in our preceding studies, see Refs. [12,15]. Although the present example is limited to moderate P_f level of approximately 10^{-4} , a comprehensive comparison with other reliability methods incorporating the KL expansion-based dimensionality reduction technique can be made. The performance function in the present example is defined as:

$$y(\mathbf{x}) = 1 - \frac{\mathcal{M}(\mathbf{x})}{\mathcal{M}_f} = 1 - \frac{M_v(t = t_f | \mathbf{x})}{\mathcal{M}_f} \quad (23)$$

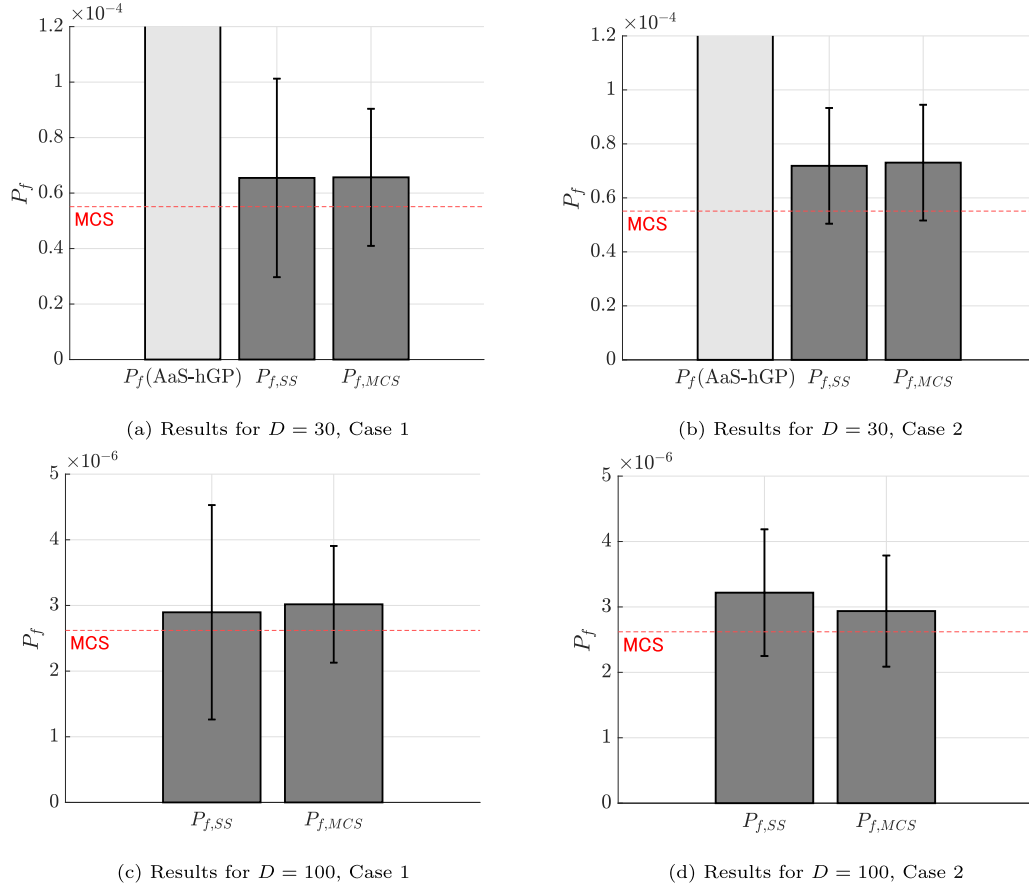


Fig. 4. Mean and standard deviation of $P_{f,SS}$ and $P_{f,MCS}$ derived from 20 realizations of AaS-hGP and SALAS. Sobol function (Eq. (20)) case with $\mathcal{M}_f = 1.25$. The error bars indicate ± 1 standard deviation on either side. Red dotted lines indicate the MCS results with 10^8 computations. $C^{tol} = 1$ is used.

Table 2

Comparison of failure probabilities and requisite number of computations. P_f and N_{PF} values by SALAS are the averages over 20 independent realizations. Sobol function (Eq. (20)) case, with $\mathcal{M}_f = 1.25$.

Dimension	Method	Failure probability		N_{PF}	$N_{grad} = N_{PF} D$
		$P_{f,SS}$	$P_{f,MCS}$		
$D = 30$	MCS	5.51×10^{-5}		10^8	–
	SALAS (Case 1, $C^{tol} = 1$)	6.55×10^{-5}	6.57×10^{-5}	183	5490
	SALAS (Case 2, $C^{tol} = 1$)	7.19×10^{-5}	7.31×10^{-5}	218	6540
	SALAS (Case 1, $C^{tol} = 2$)	5.75×10^{-5}	5.75×10^{-5}	193	5790
	SALAS (Case 2, $C^{tol} = 2$)	7.15×10^{-5}	6.87×10^{-5}	274	8220
$D = 100$	MCS	2.62×10^{-6}		10^8	–
	SALAS (Case 1, $C^{tol} = 1$)	2.90×10^{-6}	3.02×10^{-6}	188	18 800
	SALAS (Case 2, $C^{tol} = 1$)	3.22×10^{-6}	2.94×10^{-6}	247	24 700
	SALAS (Case 1, $C^{tol} = 2$)	2.93×10^{-6}	2.93×10^{-6}	281	28 100
	SALAS (Case 2, $C^{tol} = 2$)	3.05×10^{-6}	3.02×10^{-6}	308	30 800

where the VBM amidship at the time instant t , $M_v(t)$, is referenced. In the present example, the reference time $t_f = 80$ s is employed. The VBM is excited by the stochastic linear wave $\eta(\mathbf{x}, t)$ represented as:

$$\eta(\mathbf{x}, t) = \sum_{i=1}^{D/2} \sqrt{S(\omega_i)} d\omega_i \{x_i \cos(\omega_i t) - x_{i+D/2} \sin(\omega_i t)\} \quad (24)$$

where ω_i denotes the discretized wave frequency, $S(\omega_i)$ the incident wave spectrum, and $d\omega_i$ the frequency increment. In this case, $\mathbf{x}$ is uncorrelated and standard normal distributed stochastic variables. Note that $d\omega_i$ is taken equidistantly in the present study. In accordance with the Ref. [15], the Pierson–Moskowitz type wave spectrum with the significant wave height $H_s = 15$ m and the zero-upcrossing period $T_z = 12$ s is used as $S(\omega_i)$. The frequency discretization is made over the range of $\frac{\pi}{T_z} \leq \omega_i \leq \frac{3\pi}{T_z}$, with the dimensionality $D = 100$. A container

vessel that is used in Ref. [15] is selected as a subject ship, and the instantaneous VBM amidships is computed by the nonlinear strip theory-based code NMRIW-II [32]. The numerical model is illustrated in Fig. 8, where the hull form is discretized in 34 strips for hydrodynamic computation. One may consult Ref. [15] for further details of the VBM computation. The weak nonlinearity in the VBM, i.e., accounting only for the nonlinearities in the hydrostatic and Froude–Krylov forces in a rigid ship, is addressed. P_f corresponding to $\mathcal{M}_f = -9000$ MNm is computed by SALAS.

10 independent SALAS realizations are carried out with the different initial sample populations. The relevant parameters are set as:

- The tolerances $\epsilon_{b1}^{tol} = \epsilon_{b2}^{tol} = \epsilon_{p1}^{tol} = \epsilon_{p2}^{tol} = 0.005$.
- The number of initial training samples $N_0 = 50$.
- The tolerance $C^{tol} = 1$.

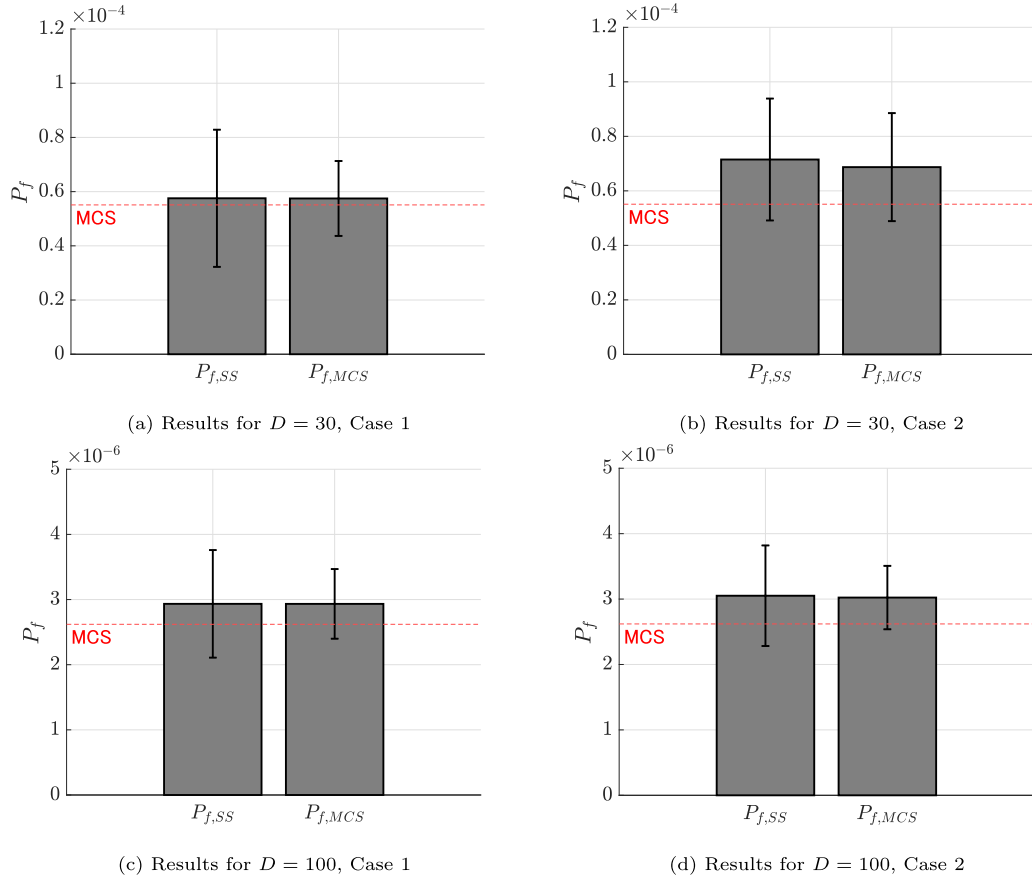


Fig. 5. Mean and standard deviation of $P_{f,SS}$ and $P_{f,MCS}$ derived from 20 realizations of SALAS. Sobol function (Eq. (20)) case with $\mathcal{M}_f = 1.25$. The error bars indicate ± 1 standard deviations on either side. Red dotted lines indicate the MCS results with 10^8 computations. $C^{tol} = 2$ is used.

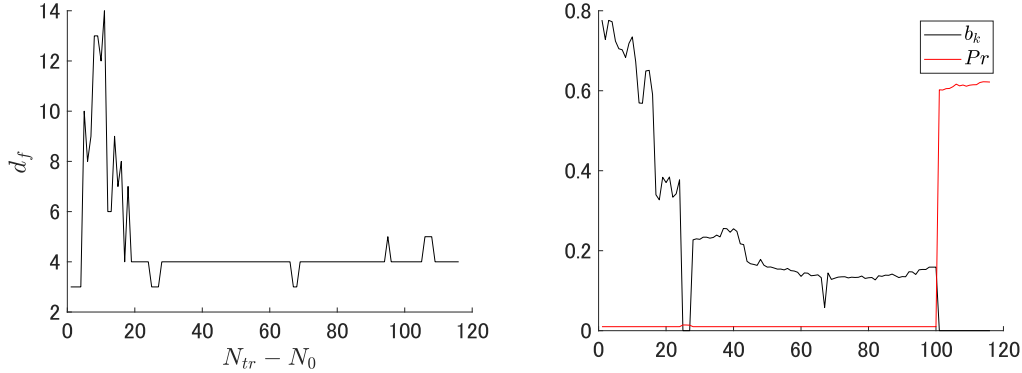


Fig. 6. Histories of identified low dimensionality d_f (left) and convergence histories of b_k and $Pr(F_k|F_{k-1})$ (right) during the SALAS process. Sobol function (Eq. (20)) case with $D = 30$ and $\mathcal{M}_f = 0.65$.

Notice from the first bullet point that the tolerances have been slightly relaxed, as the solution convergence is found to be inefficient when a tolerance of 0.001 is used. Case 1 is used for p_0 and N selection. Due to the large variation observed in $P_{f,SS}$ as outlined in Section 3, only $P_{f,MCS}$ is referenced henceforth.

Table 3 presents a comparison of P_f and the required number of computations. Mean values of P_f and N_{PF} derived from SALAS are indicated in the table, together with the coefficient of variation (COV) of P_f . The COV by MCS in the table has been derived by

$$\text{COV}(P_f) = \sqrt{\frac{1 - P_f}{N_{LSF} P_f}}. \quad (25)$$

The table introduces the crude MCS results with 2.5×10^5 samples using the sinusoidal wave representation by Eq. (24). Furthermore, the other two results, i.e., AK-MCMC [15] and Sequential and Adaptive Probabilistic Integration (SAPI) [12], are introduced for comparison. Note that AK-MCMC and SAPI have been computed in KL-reduced wave [15]; that is, an *unsupervised* dimensionality reduction technique has been incorporated. This is indicated in the table as “KL wave”. As found in Table 3, SALAS provides favorable P_f estimates in comparison to MCS and other methods. The COV of P_f is favorably small. In the VBM case, the KL-reduced wave is of particular use due to the short memory time of the response [15], thereby both AK-MCMC and SAPI also exhibit high accuracy and efficiency. Nonetheless, SALAS

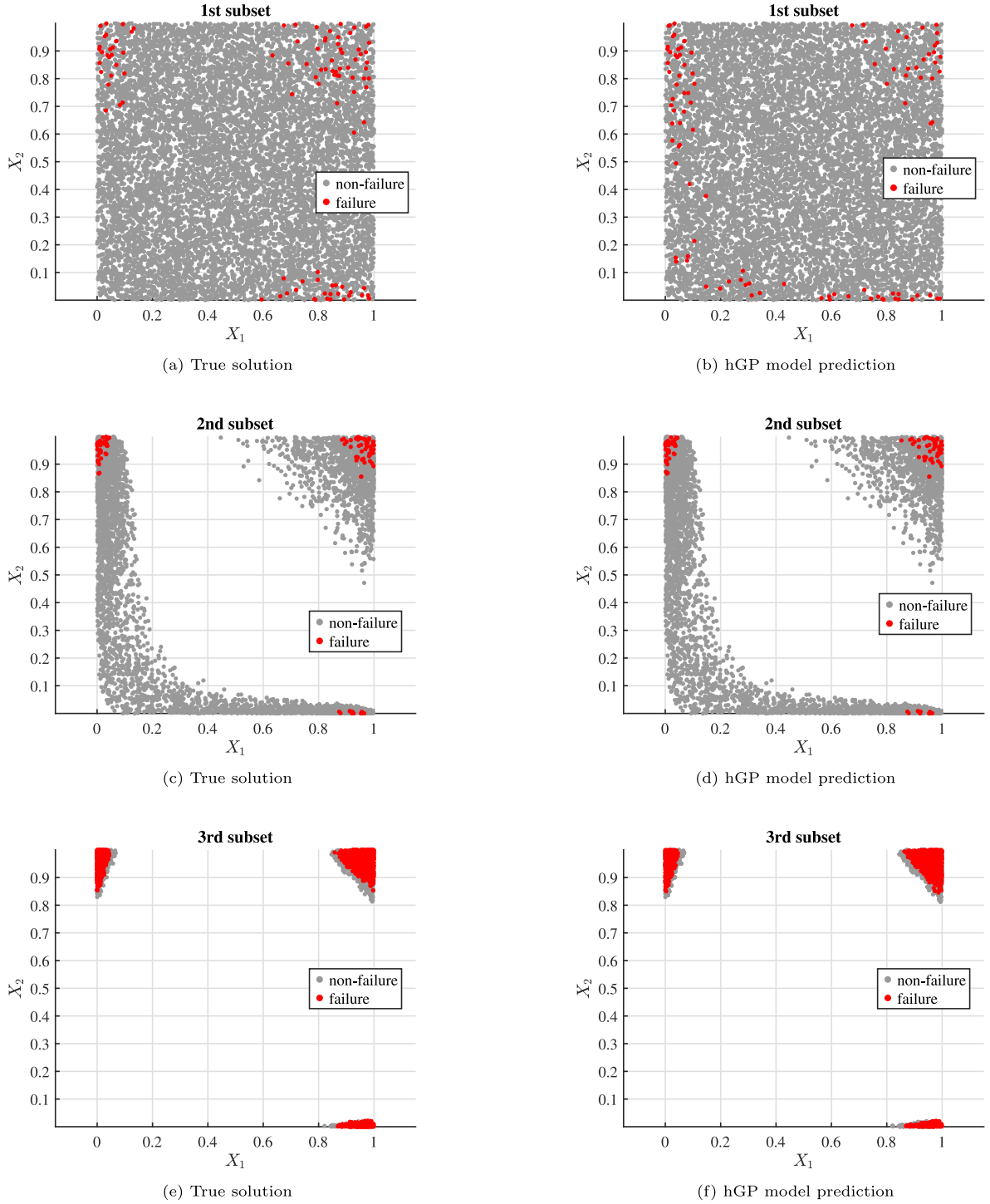


Fig. 7. Comparison of failure and non-failure classification results within the sample pool for each subset, projected onto the X_1 - X_2 plane. $D = 30$, $\mathcal{M}_f = 1.25$. Case 1 with $C_{tol} = 1$ is used. Plots are depicted with respect to X_1 and X_2 . The thresholds corresponding to each subspace are $b_1 = 0.5576$, $b_2 = 0.0563$, and $b_3 = 0.0$.

exhibits much smaller number of N_{PF} . Therefore, the introduction of higher-fidelity numerical models, such as CFD-FEA co-simulation, can be expected in future research. Note that N_{grad} for SALAS is significant, but a suitable low-fidelity surrogate for VBM would ideally improve efficacy [22], cf. the discussion in the subsequent sub-section.

As a side note, the KL expansion of wave in Refs. [12,15] is carried out using the Prolate Spheroidal Wave Functions (PSWF) [33], in which the Slepian frequency c_p is utilized to characterize the effective

dimensionality ($d_{f,KL}$) as follows:

$$d_{f,KL} \approx \frac{2c_p}{\pi} \quad (26)$$

where the Slepian frequency c_p is determined as:

$$c_p = T_p \Omega_p \quad (27)$$

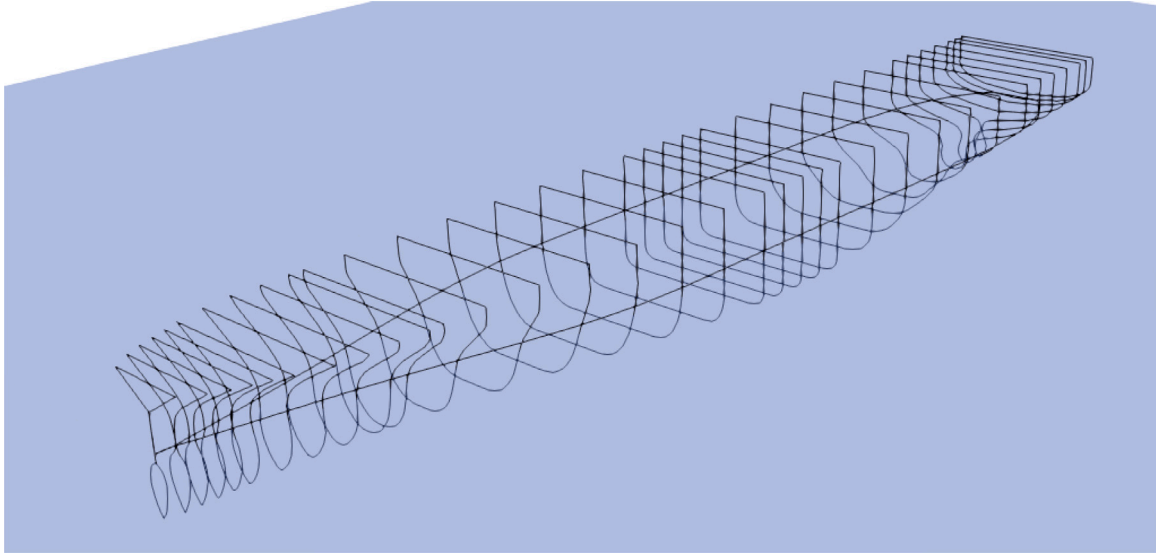


Fig. 8. Container vessel for VBM calculation by nonlinear strip theory.

Table 3

Comparison of failure probabilities and requisite number of computations. P_f and N_{PF} values by SALAS are averaged over 10 independent runs. VBM case with $M_f = -9000$ MNm, $D = 100$. MCS and AK-MCMC results are referenced from Ref. [15], while the SAPI result is taken from Ref. [12].

Method	Failure probability P_f	N_{PF}	$N_{grad} = N_{PF} D$
MCS	2.72×10^{-4} (COV: 0.121)	2.5×10^5	—
AK-MCMC (KL wave)	2.49×10^{-4}	563	—
SAPI (KL wave)	2.51×10^{-4}	235	—
SALAS (Case 1, $P_{f,MCS}$)	2.47×10^{-4} (COV: 0.058)	139	13 900

using the time duration of the generated wave elevation T_p by the KL wave, and the upper bound of the wave frequency Ω_p . One may notice that a small c_p value is ideal, but according to the relation Eq. (27), this requires narrowing the time and frequency ranges. This implies that KL reduction is not a versatile approach, because if the response of interest exhibits a long memory time, T_p must be taken sufficiently large, which in turn increases the effective dimensionality $d_{f,KL}$.

4.2. Nonlinear roll motion in beam seas

4.2.1. Numerical model and performance function

Nonlinear roll motion in a vessel is known to exhibit a longer memory time than that of the VBM. In the present study, a 2DOF model in beam seas which introduces an implicit coupling with heave motion is employed as a numerical model for roll motion, which has been also used in our recent study [22]. The model is based on Ref. [34], and is expressed in the form of the following nonlinear Ordinary Differential Equation (ODE):

$$\ddot{\phi} + 2\xi_{1,t}\omega_0\dot{\phi}_e + \xi_{2,t}|\dot{\phi}|\dot{\phi} + \frac{\xi_{3,t}}{\omega_0}\dot{\phi}_e^3 + \frac{\omega_0^2(1-\frac{h}{g})}{GM}GZ(\phi) = \frac{M_t(t)}{J_x} \quad (28)$$

A bulk carrier studied in Ref. [35] is used. The principal particulars are summarized in Table 4. The damping terms $\{\xi_{1,t}, \xi_{2,t}, \xi_{3,t}\} = \{0.01, 0.073, 0.51\}$ are introduced based on the measurements in Ref. [35]. J_x and ω_0 are calculated by

$$J_x = (k_{xx}B)^2 \rho_f L B T C_b \quad (29)$$

$$\omega_0 = \frac{\sqrt{gGM}}{k_{xx}B} \quad (30)$$

with the introduction of principal particulars in Table 4. h denotes the heave motion, and $\ddot{h}$ denotes the acceleration of heave. $\dot{h}$ and the roll

excitation moment $M_t(t)$ are computed by

$$\ddot{h}(t) = \sum_{i=1}^{D/2} -\omega_i^2 R_h(\omega_i) \sqrt{S(\omega_i)} d\omega_i \times \{x_i \cos(\omega_i t + \Gamma_h(\omega_i)) - x_{i+D/2} \sin(\omega_i t + \Gamma_h(\omega_i))\} \quad (31)$$

$$M_t(t) = \sum_{i=1}^{D/2} R_\phi(\omega_i) \sqrt{S(\omega_i)} d\omega_i \left\{ x_i \cos(\omega_i t - \frac{\pi}{2}) - x_{i+D/2} \sin(\omega_i t - \frac{\pi}{2}) \right\} \quad (32)$$

where the transfer function (TRF) $R_\phi(\omega)$ is computed by a 3D panel theory-based code SPREME-web [36], with the 3D hull profile used as input. As like the VBM case, $\mathbf{x}$ is uncorrelated and standard normal distributed stochastic variables, and the frequency discretization is made over the range of $\frac{\pi}{T_z} \leq \omega_i \leq \frac{3\pi}{T_z}$, with the dimensionality $D = 100$. As the (relative) wave direction is beam seas, the phase of TRF for roll excitation moment is set $-\pi/2$. The TRF for heave motion $\{R_h(\omega), \Gamma_h(\omega)\}$ is derived from the Closed-form expression (CFE) presented by Jensen et al. [37]. The obtained TRFs in the present vessel are shown in Fig. 9. The righting lever, GZ , is set by:

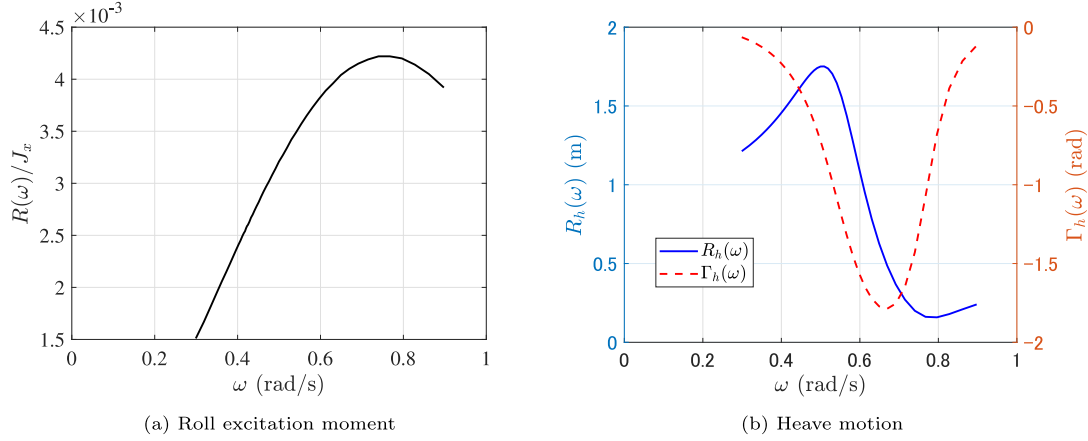
$$GZ(\phi) = (GM - 1) \sin(\phi) + \phi + 15\phi^3 - 15\phi^5 \quad (33)$$

The Pierson–Moskowitz type wave spectrum with the significant wave height $H_s = 9$ m and the zero-upcrossing period $T_z = 10.5$ s is employed as the wave spectrum $S(\omega_i)$. The frequency increment $d\omega_i$ is taken non-equidistantly to prevent repetition of wave sequences. The Runge–Kutta fourth-order method is used for solving Eq. (28), with the time increment of 0.5 s and the initial states $\phi(0) = 0$ rad, $\dot{\phi}(0) = 0$ rad/s.

Table 4

Principal particulars of a bulk carrier for numerical investigation. L : Length, B : Breadth, T : Draught, C_b : Block coefficient, GM : Transverse metacentric height, k_{xx} : Non-dimensional roll radius of gyration.

L (m)	B (m)	T (m)	C_b	GM (m)	k_{xx}
278	45	17.7	0.844	5.47	0.37

**Fig. 9.** Transfer functions of roll excitation moment and heave motion used for numerical computation of nonlinear roll motion.

The performance function of the nonlinear roll motion is defined as:

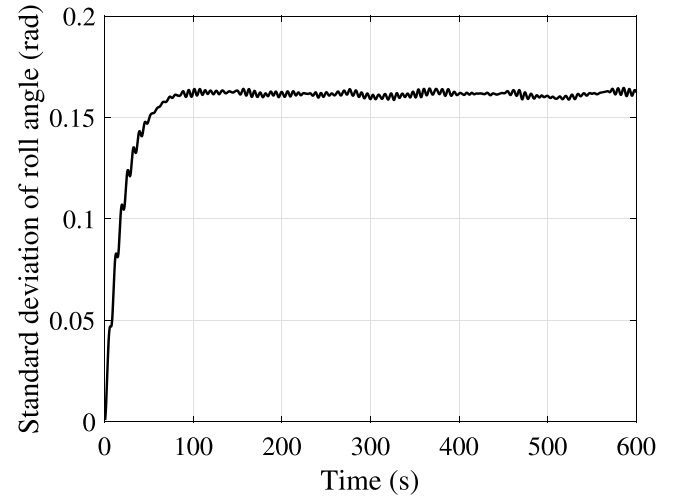
$$y(x) = 1 - \frac{\mathcal{M}(x)}{\mathcal{M}_f} = 1 - \frac{\phi(t = t_f | x)}{\mathcal{M}_f} \quad (34)$$

The memory time of the roll motion computed by the present 2DOF model is checked. To this end, time histories of standard deviations of instantaneous roll angles are examined by collecting 5000 realizations of the present 2DOF model. The result is presented in Fig. 10. The standard deviation converges after approximately 100 s of elapsed time, meaning that at least 100 s computations of Eq. (28) is needed to ensure that the resulting failure probability and associated reliability index can be reliably used for estimating the mean-out-crossing rate of the roll motion [15]. Recalling the constraints of the KL wave, i.e., Eqs. (26) and (27), together with e.g. $T_p = 100$ s and $\Omega_p = \frac{3\pi}{T_p}$, it follows that $c_p = 90$ is required, leading to the effective dimensionality $d_{f,KL} = 58$. Hence, it goes without saying that the present 2DOF model hardly benefits from using the KL wave. With reference to Fig. 10, the reference time t_f in Eq. (34) is set 600 s, so that the memory effects in the response are adequately covered.

4.2.2. Failure probability estimation results

The failure probabilities associated with the performance function given by Eq. (34) are examined. Moderate ($\mathcal{M}_f = 0.4$ rad) and rare ($\mathcal{M}_f = 0.47$ rad) events are analyzed. For comparison, crude MCS computations are carried out with a sample size of 1.3×10^7 . In addition, FORM analyses are made to exhibit the nonlinearity of the limit state surface in the present case. The Modified Hasofer–Lind (M-HL) algorithm [38] is utilized in the FORM analyses, and the relevant parameters are set in accordance with Ref. [15]. In the $\mathcal{M}_f = 0.4$ rad case, AaS-hGP results with reference to Ref. [22] are compared. 20 independent realizations of AaS-hGP and SALAS for $\mathcal{M}_f = 0.4$ rad case, while 20 independent realizations of only SALAS for $\mathcal{M}_f = 0.47$ rad case are carried out. In all cases, different initial sample populations are used for each realization. The relevant parameters for SALAS execution are set in the same manner as in the VBM case:

- The tolerances $\epsilon_{b1}^{tol} = \epsilon_{b2}^{tol} = \epsilon_{p1}^{tol} = \epsilon_{p2}^{tol} = 0.005$.
- The number of initial training samples $N_0 = 50$.
- The tolerance for count variable $C^{tol} = 1$.

**Fig. 10.** Time history of standard deviation of instantaneous roll angles.

Case 1 is used for p_0 and N selection. For AaS-hGP computations, in compliance with Ref. [22], the following settings are employed:

- The sample size $N_p = 10^6$.
- The tolerances $\epsilon_1^{tol} = 0.0005$ and $\epsilon_2^{tol} = 0.0005$.
- The number of initial training samples $N_0 = 50$.

It is worth recalling that the BAL approaches, such as SAPI [12], are not capable of solving such a high-dimensional problem, as discussed in Ref. [22].

Comparisons of estimated failure probabilities, the COV of P_f , and N_{PF} , and N_{grad} by using each method are given in Tables 5 and 6. Note that FORM also requires gradient vector estimation during the execution of the M-HL algorithm. The discrepancy in P_f derived from FORM is significant in both cases, highlighting the strong nonlinearity of the limit state surface. In Table 5, the AaS-hGP and SALAS results represent the averages of 20 independent realizations. From Table 5, SALAS provides a mean P_f value comparable to that obtained by MCS. AaS-hGP is not without merit, however, the standard deviation

Table 5

Comparison of failure probabilities, reliability indices, and the number of required computations among each method ($\mathcal{M}_f = 0.4$ rad). P_f and N_{PF} values by AaS-hGP and SALAS are averages from 20 independent realizations.

	MCS	FORM	AaS-hGP	SALAS (Case 1, $P_{f,MCS}$)
P_f	6.12×10^{-4}	2.30×10^{-3}	6.91×10^{-4}	5.90×10^{-4}
$COV(P_f)$	0.011	–	0.344	0.230
N_{PF}	1.3×10^7	30 530	277	289
$N_{grad} = N_{PF} D$	–		27 700	28 900

Table 6

Comparison of failure probabilities and the number of required computations among each method ($\mathcal{M}_f = 0.47$ rad). P_f and N_{PF} values by SALAS are the averages over 20 independent realization.

	MCS	FORM	SALAS (Case 1, $P_{f,MCS}$)
P_f	4.13×10^{-6}	2.09×10^{-5}	3.84×10^{-6}
$COV(P_f)$	0.135	–	0.224
N_{PF}	1.3×10^7	31 446	400
$N_{grad} = N_{PF} D$	–		40 000

is significantly larger than SALAS. This is evidently attributed to the sample population size of $N_p = 10^6$, which is somewhat ineffective in reliably capturing probability on the order of 10^{-4} . As indicated by the N_{PF} value in Table 5, the computational burden of SALAS is at least comparable to that of AaS-hGP. Table 6 also illustrates the effectiveness of SALAS in estimating a rare event with the probability on the order of 10^{-6} . The expected number of performance function evaluations (N_{PF}) is quite small, which makes it tempting to adopt a high-fidelity numerical model such as CFD. The total computational burden of SALAS, i.e., $N_{PF} + N_{grad}$, is comparable to that of FORM. Although the computation of gradient vectors entails extra computational cost, see N_{grad} values in the tables, our recent work [22] suggested that leveraging a low-fidelity model to approximate gradient evaluations can offer a practical solution. While Ref. [22] focused solely on the nonlinear roll motion, it is anticipated that using an ad hoc reduced-order model (ROM), see e.g. Sui et al. [39], to represent physical phenomena in waves could be applied to approximate gradient vectors in the same manner.

In the nonlinear roll motion cases, the COVs by SALAS are relatively high, see Tables 5 and 6, in comparison to that observed in the VBM case (0.058, see Table 3). It is inferred that the strong nonlinearity in the limit state surface arises from the nonlinear roll motion, despite the brevity of its expression (Eq. (28)). As demonstrated in Section 3, increasing the C_{tol} value would mitigate the variation in the resulting P_f , with the trade-off of increased N_{PF} . However, as far as examined by the authors, applying $C_{tol} = 2$ can result in N_{PF} exceeding 600, depending on the choice of initial samples, which in turn can compromise the computational efficiency of SALAS. Hence, for problems exhibiting strong nonlinearity, accompanied by high dimensionality and small failure probabilities, the present SALAS framework still leaves room for improvement. One potential remedy for this critical issue is to improve the learning function used for selecting training samples. Instead of the U-function, the Posterior Variance Contribution (PVC) employed in BAL approaches for low-dimensional problems may prove promising [12].

5. Conclusion

A new reliability analysis method for rare event probability in a high-dimensional stochastic space is proposed. The proposed method, termed Sequential Active Learning with Active Subspace (SALAS), employs an active learning framework that integrates active subspace-based dimensionality reduction with heteroscedastic Gaussian Process (hGP) surrogate modeling. To effectively handle small probability events, a sequential sampling strategy is also incorporated into the framework, along with appropriate convergence criteria to ensure the reliability of the hGP model as a surrogate.

The validity of SALAS is first demonstrated by a toy model based on the Sobol function. The computational efficiency and accuracy of SALAS, demonstrated to be on par with the AaS-hGP method by Kim et al. [21], are confirmed for moderate failure probability levels on the order of 10^{-3} . Furthermore, SALAS demonstrates clear advantages at small failure probability levels on the order of 10^{-6} , where AaS-hGP is found to be computationally intractable. It is demonstrated that the convergence criteria and Markov Chain Monte Carlo (MCMC) sampler employed in the present SALAS can effectively transition the samples toward the failure region alongside the update of the subset.

The applicability of SALAS to complex engineering problems is then demonstrated by referencing nonlinear vessel responses in high-dimensional (100 dimensions) stochastic waves. First example is the nonlinear vertical bending moment in a container vessel. Here, a comprehensive comparison is made with reliability analysis methods combining unsupervised dimensionality reduction, specifically, Karhunen–Loeve expansion, with adaptive Kriging and Bayesian active learning approaches. This case study demonstrates the high efficiency of SALAS in comparison with these methods. In addition, the versatility of SALAS is discussed against the backdrop of the limitations of the KL expansion.

The numerical demonstration is further extended to nonlinear vessel roll motion in beam seas, for which the KL expansion is no longer beneficial due to the long memory effect of the response. Using a 2 degree-of-freedom numerical roll motion model coupled with heave motion, the accuracy and efficiency of SALAS are further demonstrated, both for moderate (on the order of 10^{-4}) and rare (on the order of 10^{-6}) event estimations. These favorable results further highlight the versatility of SALAS over AaS-hGP in addressing practical engineering problems.

This study employs the finite difference method for gradient vector computations, which imposes additional computational burdens. Nonetheless, this issue is expected to be alleviated by approximating the gradient vector using a low-fidelity surrogate, as demonstrated in our recent work [22] referring to nonlinear roll motion. In this context, it is anticipated that higher-fidelity numerical models, such as Computational Fluid Dynamics, can be incorporated within the SALAS framework alongside gradient vector approximation. When the limit state surface exhibits strong nonlinearity, specifically in the case of nonlinear roll motion, the present SALAS framework tends to induce a relatively high coefficient of variation (COV) in the estimated failure probability. Improving SALAS for strongly nonlinear cases remains a topic for future work, which could be addressed by improving the learning function for selecting training samples. The present work is limited to time-invariant reliability analyses. Future work should extend the SALAS framework to time-variant reliability problems [40].

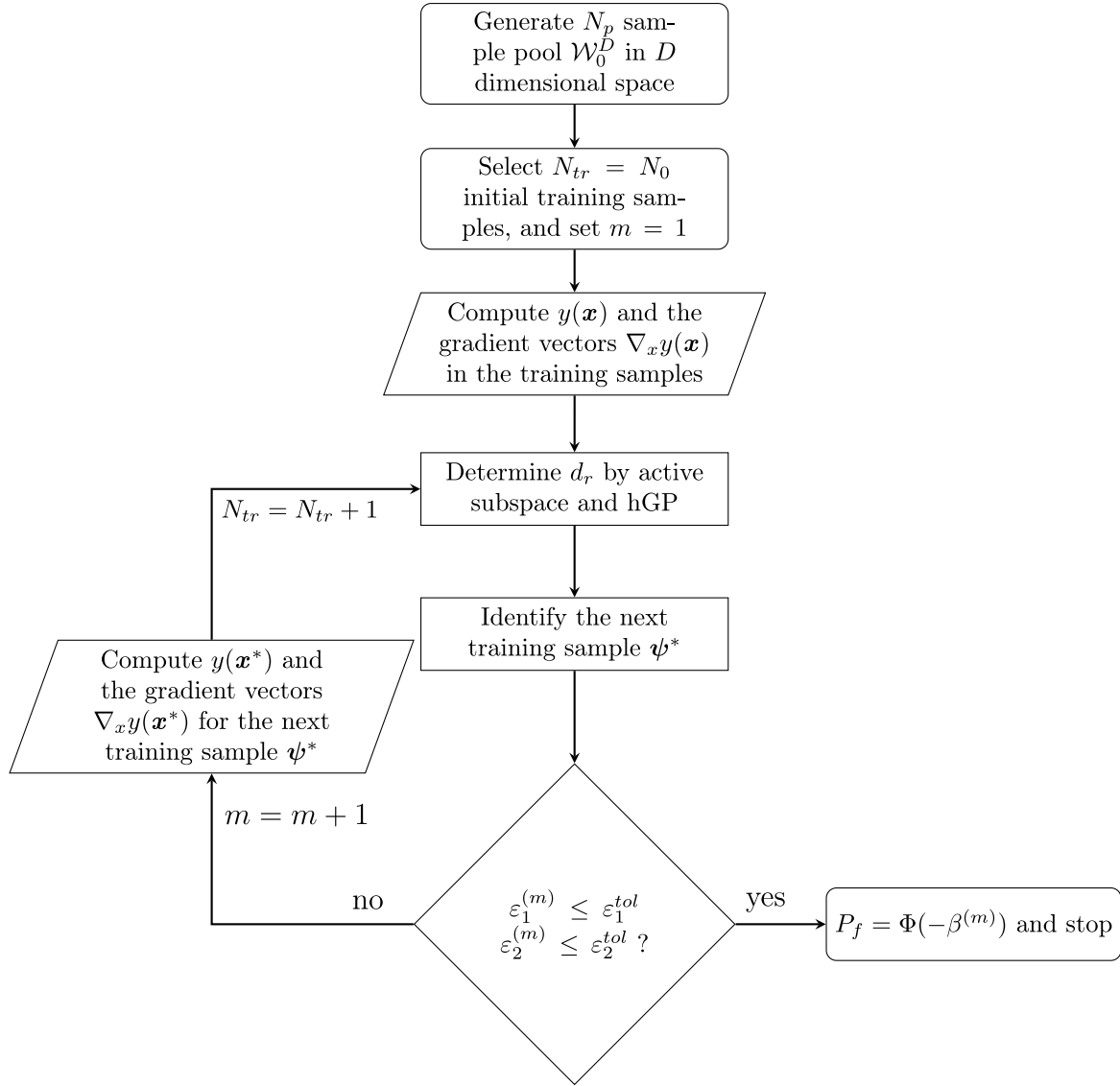


Fig. B.11. Overall workflow of AaS-hGP [21,22].

CRedit authorship contribution statement

Tomoki Takami: Writing – review & editing, Writing – original draft, Visualization, Validation, Software, Methodology, Investigation, Funding acquisition, Formal analysis, Conceptualization. **Masaru Kitahara:** Writing – review & editing, Writing – original draft, Validation, Software, Methodology, Investigation, Formal analysis, Conceptualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix A. Hyperparameter identification in heteroscedastic Gaussian process model

Lázaro-Gredilla et al. [25] proposed the following lower bound of the exact marginal likelihood using variational mean vector $\mathbf{m}$ and covariance matrix $\mathbf{V}$:

$$b_{MV}(\mathbf{m}, \mathbf{V}) = \ln \mathcal{N}(\mathcal{Y}_D; \mathbf{0}, \mathbf{K}_f + \mathbf{R}) - \frac{1}{4} \text{tr}(\mathbf{V}) - \text{KL}(\mathcal{N}(\mathbf{g}; \mathbf{m}, \mathbf{V}) \parallel \mathcal{N}(\mathbf{g}; \mu_0 \mathbf{1}, \mathbf{K}_g)) \quad (\text{A.1})$$

where $\mathcal{Y}_D = \{y(\mathbf{x}_1), y(\mathbf{x}_2), \dots, y(\mathbf{x}_{N_{tr}})\}$, $\text{tr}(\cdot)$ denotes a trace operator, $\text{KL}(\cdot \parallel \cdot)$ the Kullback–Leibler divergence between two PDFs, $\mathbf{K}_f$ and $\mathbf{K}_g$ respectively are the covariance matrices of $f(\boldsymbol{\psi})$ and $f_g(\boldsymbol{\psi})$ in Eqs. (6) and (7), and $\mathbf{R}$ is a diagonal matrix with elements $R_{i,j} = \exp m_i - V_{i,j}/2$, $i = 1, 2, \dots, N_{tr}$. The hyperparameters are to be identified so that this lower bound is maximized.

Upon identification of the hyperparameters, the predictive mean and variance of the performance function corresponding to a given sample $\boldsymbol{\psi}_*$ can be given as:

$$\mu_{\hat{y}}(\boldsymbol{\psi}_*) = \mathbf{k}_{f_*}^T (\mathbf{K}_f + \mathbf{R})^{-1} \mathcal{Y}_D, \quad (\text{A.2})$$

$$\sigma_{\hat{y}}^2(\boldsymbol{\psi}_*) = \exp \chi_* + \eta_*^2/2 + \gamma_*^2 \quad (\text{A.3})$$

where χ_* , η_*^2 , and γ_*^2 are:

$$\chi_* = \mathbf{k}_{g_*}^T \left(\Lambda_{MV} - \frac{1}{2} \mathbf{I} \right) \mathbf{1} + \mu_0, \quad (\text{A.4})$$

$$\eta_*^2 = \mathbf{k}_{g_*}^T - \mathbf{k}_{g_*}^T (\mathbf{K}_g + \Lambda_{MV}^{-1})^{-1} \mathbf{k}_{g_*}, \quad (\text{A.5})$$

$$\gamma_*^2 = \mathbf{k}_{f_*}^T - \mathbf{k}_{f_*}^T (\mathbf{K}_f + \mathbf{R})^{-1} \mathbf{k}_{f_*} \quad (\text{A.6})$$

where $\mathbf{k}_{f_*}$ and $\mathbf{k}_{g_*}$ are vectors of covariance functions between the location ψ_* and N_{tr} training samples $\mathcal{W}_{tr} = \{\mathbf{W}_r^T \mathbf{x}_1, \dots, \mathbf{W}_r^T \mathbf{x}_{N_{tr}}\}$ for latent functions $f(\psi)$ and $g(\psi)$, respectively. Λ_{MV} is a positive semi-definite diagonal matrix re-parameterizing $\mathbf{m}$ and $\mathbf{V}$ in a reduced order.

Appendix B. AaS-hGP method [21]

The AaS-hGP workflow [21] is illustrated in Fig. B.11. The overall procedure adopted in this study is as follows:

1. Set $m = 1$ and generate a sample population (i.e., $\mathcal{W}_0^D = \{\mathbf{x}_1, \dots, \mathbf{x}_{N_p}\}$) consisting of N_p pairs of the stochastic variables $\mathbf{X}$.
2. Arbitrarily select N_0 initial training samples (i.e., $\mathcal{W}_{tr}^D = \{\mathbf{x}_1, \dots, \mathbf{x}_{N_0}\}$) from $\mathcal{W}_0^D$, and then compute the corresponding values of the performance function $y(\mathbf{x})$ and gradient vectors $\nabla_{\mathbf{x}} y(\mathbf{x})$ by running numerical simulations.
3. Determine the reduced dimensionality d_r by the procedure mentioned in sub- Section 2.1 and trained hGP models.
4. Determine the next training sample ψ^* according to a procedure described in sub- Section 2.2. Note that $b_k = 0$ should be introduced into Eqs. (15) and (16).
5. Make a convergence check the failure probability P_f , which is derived by:

$$P_f \approx \frac{\sum_{i=1}^N \mathbb{I}(\mu_{\hat{y}}(\psi) \leq 0)}{N} \quad (\text{B.1})$$

where $\mathbb{I}(\mu_{\hat{y}}(\psi) \leq 0) = 1$ if $\mu_{\hat{y}}(\psi) \leq 0$, otherwise equal to 0. In Ref. [21], the relative changes of P_f are monitored to determine the convergence of the active learning process. However, if P_f is relatively small, an instability in convergence performance was observed, as the simple counting by Eq. (B.1) evokes large variations in the relative P_f changes. As such, in the present study, the reliability index β , i.e.

$$\beta = -\Phi^{-1}(P_f) \quad (\text{B.2})$$

where $\Phi(\cdot)$ denotes the cumulative distribution function of the standard normal distribution, is monitored. With reference to the reliability indices at the m th and $(m-1)$ th learning steps, the following criteria

$$\varepsilon_1^{(m)} = \frac{|\beta^{(m)} - \beta^{(m-1)}|}{\beta^{(m)}} \quad (\text{B.3})$$

$$\varepsilon_2^{(m)} = \left| \varepsilon_1^{(m)} - \varepsilon_1^{(m-1)} \right| \quad (\text{B.4})$$

are monitored, and if $\varepsilon_1^{(m)}$ and $\varepsilon_2^{(m)}$ are less than ε_1^{tol} and ε_2^{tol} , respectively, the AaS-hGP process is terminated. Otherwise, $m = m + 1$, $N_{tr} = N_{tr} + 1$ and compute the performance function at the new training sample $y(\mathbf{x}^*)$ and gradient vectors $\nabla_{\mathbf{x}} y(\mathbf{x}^*)$; noting that $\mathbf{x}^*$ denotes the high-dimensional stochastic variables corresponding to ψ^* . Then, return to step 3.

An overall workflow of AaS-hGP is illustrated in Fig. B.11. The procedure is largely congruent with that outlined in Ref. [21], with a difference in the convergence criteria; the reliability index β is monitored instead of P_f . In contradistinction to SALAS, the initial sample pool $\mathcal{W}_0^D$ undergoes no updates during the learning process.

Data availability

Data will be made available on request.

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