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Semidynamical Systems with Nonunique Global Backward Extensions II; the Negative Aspects

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§ 1. Introduction.

In [5] the author considered the class of semidynamical systems with nonunique global backward extensions. For a semidynamical system in this class, all negative solutions are principal [1] and through each point in the phase space there exists at least one principal negative solution that passes through it. Furthermore, in such systems there are no start points [1] and the negative escape time (McCann's sense [9]) is ∞ for each point in the phase space of the system [5]. Furthermore, such systems arise naturally from normal control systems [7], [5].

In this paper we continue the study started in [5]. We deal here with the negative version of some important dynamical properties, such as stability, attraction, invariance and asymptotic stability. Following Roxin [10] and Kloeden [8], a weak version and a strong version are defined for each dynamical property. The main purpose of this paper is to carry over, as much as possible, basic results from the theory of dynamical systems [2] to the class of semidynamical systems with nonunique global backward extensions. Differences between the two classes will be demonstrated through the examples of section 6. We have been influenced by the work in [10] and [8] in constructing our examples and in introducing definitions and ideas.

Let (X, π) be a semidynamical system [1]. It will be assumed throughout this paper that (i) X is locally compact, Hausdorff and first countable and (ii) (X, π) has a global backward extension. A function $\sigma: (-\infty, 0] \rightarrow X$ is called a principal negative solution of (X, π) if $\sigma(-t+s) = \sigma(-t)\pi s$ whenever s and t in R^+ (R^+ denotes the set of nonnegative reals) and $t \geq s$. The negative solution σ is said to pass through a point $a \in X$ if $\sigma(0) = a$. The σ -negative trajectory $C_\sigma^-(a)$ through a is defined to be the set $\sigma(-\infty, 0]$.

The following sets are of special interest.

- (1) The funnel through $a \in X$ [1];

$$F(a) = \{b \mid b\pi t = a \text{ for some } t \in R^+\}.$$

- (2) The attainability set of $a \in X$ at $-t_0 \in R^-$;

$$F(a, -t_0) = \{b \mid b\pi t_0 = a\} \text{ [6], [8].}$$

(3) The attainability set of $A \subset X$ and $S \subset R^-$;

$$F(A, S) = \{b \mid b\pi s = a, a \in A, -s \in S\}.$$

A point $a \in X$ is said to be a start point if $F(a) = \phi$ [1]. A subset A of X is said to be {positively invariant} {weakly negatively invariant} {strongly negatively invariant} if, respectively, $\{A\pi R^+ \subset A\}$ {for each $a \in A$, $C_\sigma^-(a) \subset A$, for some negative solution through a } $\{F(A) \subset A\}$.

In section 2 we study weak and strong versions of the negative limit sets, the negative prolongation sets and the negative prolongation limit sets. Part of these notions were introduced in [3] and [4].

In section 3, negative strong stabilities are investigated. Three kinds of such stabilities are introduced; (1) negative strong Liapunov stability (2) strong D^- -stability and (3) strong J^- -stability, where D^- and J^- denotes respectively, the negative prolongation relation and the negative prolongation limit relation. We also introduce the notions of characteristic O^- and stable characteristic O^- .

In section 4 negative weak stabilities are considered. Results similar to those in section 3 are obtained.

In section 5 the notions of negative attraction and negative asymptotic stability are defined and studied.

In section 6 we give several examples to illustrate some of the differences between dynamical systems and semidynamical systems.

For a subset A of X , $\bar{A}$ will denote the closure of A and ∂A will denote the boundary of A . For any other unexplained notation we refer the reader to [1].

§ 2. Negative sets.

Definition 2.1. Let $a \in X$ and let σ be a principal negative solution through a . Then we have the following definitions.

- (i) The σ -negative trajectory through a ;
 $C_\sigma^-(a) = \sigma(-\infty, 0]$ and its closure $\overline{C_\sigma^-(a)} = K_\sigma^-(a)$.
- (ii) The σ -negative limit set of a ;
 $L_\sigma^-(a) = \{b \in X \mid \sigma(-t_i) \rightarrow b \text{ for some sequence } \{t_i\} \text{ in } R^+ \text{ with } t_i \rightarrow \infty\}$.
- (iii) The negative limit set of a ;
 $L^-(a) = \{b \in X \mid \text{there exist sequences } \{t_i\} \text{ in } R^+ \text{ with } t_i \rightarrow \infty \text{ and } \{b_i\} \text{ in } X \text{ with } b_i \in F(a, -t_i) \text{ and } b_i \rightarrow b\}$.
- (iv) The negative prolongation set of a ;
 $D^-(a) = \{b \in X \mid \text{there exist a sequence of principal negative solutions } \{\sigma_i\} \text{ with } \sigma_i(0) \rightarrow a \text{ and a sequence } \{t_i\} \text{ in } R^+ \text{ with } \sigma_i(-t_i) \rightarrow b\}$.

- (v) The negative prolongation limit set of a ;
 $J^-(a) = \{b \in X \mid \text{there exist a sequence of principal negative solutions } \{\sigma_i\} \text{ with } \sigma_i(0) \rightarrow a \text{ and a sequence } \{t_i\} \text{ in } R^+ \text{ with } t_i \rightarrow \infty \text{ and } \sigma_i(-t_i) \rightarrow b\}.$
- (vi) $K^-(a) = \overline{F(a)}$

The following two results from [5] are employed throughout this paper and we state them here for the convenience of the reader.

Theorem 2.2 [5; 1.8, 1.10]. *Let (X, π) be a semidynamical system such that all the negative solutions in X are principal and there is at least one principal negative solution passing through each point in X . Then the following hold:*

- (i) *The attainability set $F(A, P)$ is compact whenever $A \subset X$ and $P \subset R^-$ are compact in their respective spaces.*
- (ii) *The attainability set $F(A, [-t, 0])$ is connected whenever A is compact and connected and $t \in R^+$.*
- (iii) *(X, π) can be extended to the semidynamical system (X^*, π^*) , (where $X^* = X \cup \{\infty\}$ is the one point compactification of X and $\pi^*(x, t) = \pi(x, t)$ for $x \neq \infty$, $\pi^*(\infty, t) = \infty$ for $t \in R^+$).*

Theorem 2.3 [5, 4.2]. *Let (X, π) be a semidynamical system that have the same properties mentioned in Theorem 2.2. Let $\{\sigma_i\}$ be a sequence of principal negative solutions for which $\sigma_i(0) \rightarrow a \in X$. Then there exist a negative solution σ with $\sigma(0) = a$ and a subsequence $\{\sigma_{i_j}\}$ of $\{\sigma_i\}$ such that $\sigma_{i_j}(-t) \rightarrow \sigma(-t)$ for all $t \in R^+$.*

Lemma 2.4. *The following statements are valid for a point $a \in X$ and $t \in R^+$.*

- (i) $L^+(a\pi t) = L^+(a)\pi t = L^+(a).$
- (ii) $D^+(a)\pi t \subset D^+(a); D^+(a)\pi t \subset D^+(a\pi t).$
- (iii) $J^+(a)\pi t = J^+(a) \subset J^+(a\pi t).$

Proof. Let $b \in L^+(a\pi t)$. Then $(a\pi t)\pi t_i \rightarrow b$ for some sequence $\{t_i\}$ in R^+ with $t_i \rightarrow \infty$. Let $A = \{a\pi(t + t_i) \mid i = 1, 2, \dots\} \cup \{b\}$. Then A is compact. This implies by Theorem 2.2 that $F(A, -t)$ is compact. Since the sequence $\{a\pi t_i\}$ is in $F(A, -t)$, we may assume, without loss of generality, that $a\pi t_i \rightarrow c$. Hence $c \in L^+(a)$. Furthermore, $a\pi(t_i + t) = (a\pi t)\pi t_i \rightarrow c\pi t$ and thus $c\pi t = b$. Consequently, $b \in L^+(a)\pi t$ and hence $L^+(a\pi t) \subset L^+(a)\pi t$. From [1] it follows that $L^+(a)\pi t \subset L^+(a)$. Now let $b \in L^+(a)$. Then $a\pi t_i \rightarrow b$ for some sequence $\{t_i\}$ on R^+ with $t_i \rightarrow \infty$. There exists a positive integer i_0 such that $t_i > t$ for all $i \geq i_0$. Thus $(a\pi t)\pi(t_i - t) = a\pi t_i \rightarrow b$ and $(t_i - t) \rightarrow \infty$. This implies that $b \in L^+(a\pi t)$ and hence $L^+(a) \subset L^+(a\pi t)$. Consequently, $L^+(a\pi t) = L^+(a)\pi t = L^+(a)$.

(ii) The proof of the first part can be found in [1]. The proof of the second part is easy and will be omitted.

(iii) According to [1] $J^+(a)\pi t \subset J^+(a)$. So let $b \in J^+(a)$. Then there are sequences $\{a_i\}$ in X and $\{t_i\}$ in R^+ with $t_i \rightarrow \infty$ such that $a_i \rightarrow a$ and $a_i\pi t_i \rightarrow b$. Since the set $A = \{a_i\pi t_i \mid i=1, 2, \dots\} \cup \{b\}$ is compact, it follows from Theorem 2.2 that $F(A, -t)$ is compact. There exists a positive integer i_0 such that $t_i > t$ for all $i \geq i_0$. Hence for $i \geq i_0$ $\{a_i\pi(t_i - t)\}$ is in $F(A, -t)$. Then we may assume that $a_i\pi(t_i - t) \rightarrow c$. Hence $c\pi t = b$ and consequently, $b \in J^+(a)\pi t$. Therefore $J^+(a) \subset J^+(a)\pi t$ and thus $J^+(a) = J^+(a)\pi t$. Furthermore, $a_i\pi t \rightarrow a\pi t$ and $a_i\pi t_i = (a_i\pi t)\pi(t_i - t) \rightarrow b$ for $i \geq i_0$. Hence $b \in J^+(a\pi t)$ and thus $J^+(a) \subset J^+(a\pi t)$. This completes the proof of the lemma.

Remark 2.5. Let $a \in X$ and let σ be a negative solution through a . Let $b \in C_\sigma^-(a)$. Then $b\pi s = a$ for some $s \geq 0$. An associated negative solution σ_b through b can be defined by letting $\sigma_b(-t) = \sigma(-t - s)$ for $t \geq 0$. Now suppose that $b \in C^+(a)$. Then $a\pi s = b$ for some $s \geq 0$. An associated negative solution σ_b through b can be defined by letting

$$\sigma_b(-t) = \begin{cases} a\pi(s - t) & \text{if } 0 \leq t \leq s \\ \sigma(-t + s) & \text{if } t \geq s. \end{cases}$$

In either case the sets $C_\sigma^-(b)$, $K_{\sigma_b}^-(b)$ and $L_{\sigma_b}^-(b)$ will be denoted, for simplicity, by $C_\sigma^-(b)$, $K_\sigma^-(b)$ and $L_\sigma^-(b)$, respectively.

Lemma 2.6. *Let $a \in X$ and σ be a principal negative solution through a . Then the following statements are valid.*

- (i) $L_\sigma^-(a\pi t) = L_\sigma^-(a)\pi t = L_\sigma^-(a) = L_\sigma^-(\sigma(-t))$ for $t \geq 0$,
- (ii) $b \in D^-(a)$ iff $a \in D^+(b)$; $b \in J^-(a)$ iff $a \in J^+(b)$,
- (iii) $D^-(a) \subset D^-(a)\pi t \subset D^-(a\pi t)$,
- (iv) $J^-(a)\pi t = J^-(a) \subset J^-(a\pi t)$,
- (v) $L^-(a) = L^-(a)\pi t \subset L^-(a\pi t)$.

Proof. (i) Let $b \in L_\sigma^-(a\pi t)$. Then there exists a sequence $\{t_i\}$ in R^+ with $t_i \leq t$, $t_i \rightarrow \infty$ and such that $\sigma(-t_i + t) \rightarrow b$. Using Theorem 2.2 we may assume that $\sigma(-t_i) \rightarrow c \in X$. Then $c \in L_\sigma^-(a)$ and $c\pi t = b$. Hence $b \in L_\sigma^-(a)\pi t$ and consequently, $L_\sigma^-(a\pi t) \subset L_\sigma^-(a)\pi t$. According to [4] $L_\sigma^-(a)\pi t \subset L_\sigma^-(a)$. Now let $d \in L_\sigma^-(a)$. Then $\sigma(-t_i) \rightarrow d$ for some sequence $\{t_i\}$ in R^+ with $t_i \rightarrow \infty$. From Remark 2.5 we have $\sigma(-t_i) = \sigma_{a\pi t}(-t_i - t) \rightarrow d$ and thus $d \in L_\sigma^-(a\pi t)$. Therefore $L_\sigma^-(a\pi t) = L_\sigma^-(a)\pi t = L_\sigma^-(a)$. The last statement in (i) may be verified similarly.

(ii) Let $b \in D^-(a)$. Then there are sequences $\{\sigma_i\}$ of principal negative solutions and $\{t_i\}$ in R^+ with $\sigma_i(0) \rightarrow a$ and $\sigma_i(-t_i) \rightarrow b$. Since $\sigma_i(0) = \sigma_i(-t_i + t_i) = \sigma_i(-t_i)\pi t_i$, it follows that $b \in D^+(a)$. Conversely, let $a \in D^+(b)$. Then there are sequences $\{b_i\}$ in X and $\{t_i\}$ in R^+ with $b_i \rightarrow b$ and $b_i\pi t_i \rightarrow a$. For each i there exists a principal negative solution σ_i through b_i . According to Remark 2.5, for each i there exists an associated principal negative solution σ_i through $b_i\pi t_i$; that is $\sigma_i(0) =$

$b_i \pi t_i$ and $\sigma_i(-t_i) = b_i$. This implies that $b \in D^-(a)$. The second part of (ii) can be proved similarly.

(iii) Let $b \in D^-(a)$. Then there exist sequences $\{\sigma_i\}$ of principal negative solutions and $\{t_i\}$ in R^+ with $\sigma_i(0) \rightarrow a$ and $\sigma_i(-t_i) \rightarrow b$. By Theorem 2.2 we may assume that $\sigma_i(-t_i - t) \rightarrow c \in X$, where $c \pi t = b$. Hence $c \in D^-(a)$ and thus $b = c \pi t \in D^-(a) \pi t$. Consequently, $D^-(a) \subset D^-(a) \pi t$. Now let $d \in D^-(a) \pi t$. Then there exists $c \in X$ with $c \pi t = d$ and $c \in D^-(a)$. Then it follows from (ii) that $a \in D^+(c)$. This implies by Lemma 2.4 that $a \pi t \in D^+(c) \pi t \subset D^+(c \pi t) = D^+(b)$. Hence $b \in D^-(a \pi t)$. Consequently, $D^-(a) \pi t \subset D^-(a \pi t)$.

(iv) It is known from [4] that $J^-(a) \pi t \subset J^-(a)$. So let $b \in J^-(a)$. Then there are sequences $\{\sigma_i\}$ of principal negative solutions and $\{t_i\}$ in R^+ with $t_i \rightarrow \infty$ such that $\sigma_i(0) \rightarrow a$ and $\sigma_i(-t_i) \rightarrow b$. By Theorem 2.2 we may assume that $\sigma_i(-t_i - t) \rightarrow c \in X$. Hence $c \pi t = b$. Furthermore, $c \in J^-(a)$. Thus $b = c \pi t \in J^-(a) \pi t$ and consequently, $J^-(a) \subset J^-(a) \pi t$. It follows that $J^-(a) = J^-(a) \pi t$. Now let $d \in J^-(a)$. Then from (ii) it follows that $a \in J^+(d)$. Thus $a \pi t \in J^+(d) \pi t = J^+(d)$ (Lemma 2.4). This implies that $d \in J^-(a \pi t)$. Consequently, $J^-(a) \subset J^-(a \pi t)$.

(v) Let $b \in L^-(a)$. Then there are sequences $\{t_i\}$ in R^+ with $t_i \rightarrow \infty$ and $\{b_i\}$ in X such that $b_i \in F(a, -t_i)$ and $b_i \rightarrow b$. Let σ_i be a principal negative solution through b_i for each i . Then by Theorem 2.2 we may assume that $\sigma_i(-t) \rightarrow c \in X$. Thus $c \pi t = b$. Since $\sigma_i(-t) \subset F(a)$, it follows that $c \in L^-(a)$. This implies that $b = c \pi t \in L^-(a) \pi t$ and hence $L^-(a) \subset L^-(a) \pi t$. Let $d \in L^-(a) \pi t$. Then there exists $e \in L^-(a)$ with $e \pi t = d$. Furthermore, there are sequences $\{t_i\}$ in R^+ with $t_i \rightarrow \infty$ and $e_i \in F(a, -t_i)$ such that $e_i \rightarrow e$. There exists a positive integer i_0 such that $t_i - t > 0$ for $i \geq i_0$. For $i \geq i_0$, $e_i \pi t \in F(a, -t_i + t) \subset F(a \pi t, -t_i)$ and $e_i \pi t \rightarrow e \pi t = d$. Hence $d \in L^-(a) \cap L^-(a \pi t)$ and consequently $L^-(a) \pi t \subset L^-(a \pi t)$ and $L^-(a \pi t) \subset L^-(a)$. It follows that $L^-(a) = L^-(a) \pi t$. The proof of the lemma is now complete.

Lemma 2.7. Let $a \in X$ and σ be a principal negative solution through a . Then the following hold:

- (i) $L_\sigma^-(a) = \bigcap_{s \geq 0} \{\bigcup_{t \geq 0} \sigma(-s - t)\}$,
- (ii) $L^-(a) = \bigcap_{t \geq 0} \{K^-(b) \mid b \in F(a, -t)\}$
- (iii) $D^-(a) = \bigcap \{\overline{F(U)} \mid U \text{ is an open neighborhood of } a\}$,
- (iv) there exists a principal negative solution ν through a such that $J^-(a) = \bigcap \{D^-(b) \mid b \in C_\nu^-(a)\}$,
- (v) $K_\sigma^-(a) = C_\sigma^-(a) \cup L_\sigma^-(a)$,
- (vi) $D^-(a) = F(a) \cup J^-(a)$,
- (vii) $K^-(a) = F(a) \cup L^-(a)$.

Proof. (i) Let $b \in L_\sigma^-(a)$. Then there exists a sequence $\{t_i\}$ in R^+ with $t_i \rightarrow \infty$ and such that $\sigma(-t_i) \rightarrow b$. Let $s \geq 0$. Then there exists a positive integer i_0 such that $t_i > s$ for $i \geq i_0$. Hence $\sigma(-t_i) = \sigma(-t_i + s - s) = \sigma(-s - (t_i - s)) \rightarrow b$. Thus

$b \in \bigcup_{t \geq 0} \overline{\sigma(-s-t)}$. Since s was arbitrary chosen, $b \in \bigcap_{s \geq 0} \overline{\bigcup_{t \geq 0} (-s-t)}$. Hence $L_{\sigma}^{-}(a) \subset \bigcap_{s \geq 0} \overline{\bigcup_{t \geq 0} \sigma(-s-t)}$. Now let $b \in \bigcap_{s \geq 0} \overline{\bigcup_{t \geq 0} \sigma(-s-t)}$ and let U be an open neighborhood of b . Then for $s \geq 0$, $b \in \bigcup_{t \geq 0} \overline{\sigma(-s-t)}$. Hence there exists $r = r(s) \geq 0$ such that $\sigma(-s-r) \in U$. This implies that the net $\{\sigma(-t) \mid t \in R^+\}$ is frequently in U . Hence there exists a subnet $\sigma(-t_i)$ with $t_i \rightarrow \infty$ such that $\sigma(-t_i) \rightarrow b$. Consequently, $b \in L_{\sigma}^{-}(a)$. The conclusion of part (i) now follows:

(ii) Proof of (ii) is similar to the proof of (i) and will be omitted.

(iii) Let $b \in D^{-}(a)$. Then there are sequences $\{\sigma_i\}$ of principal negative solutions and $\{t_i\}$ in R^+ with $\sigma_i(0) \rightarrow a$ and $\sigma_i(-t_i) \rightarrow b$. Let U be an open neighborhood of a . Then $\sigma_i(0) \in U$ for $i \geq i_0$. Hence $\sigma_i(-t_i) \in F(U)$. Consequently, $b \in \overline{F(U)}$. Thus $b \in \bigcap \{\overline{F(U)} \mid U \text{ is an neighborhood of } a\}$. It follows that $D^{-}(a) \subset \bigcap \{\overline{F(U)} \mid U \text{ is an open neighborhood of } a\}$. Now let $c \in \bigcap \{\overline{F(U)} \mid U \text{ is an open neighborhood of } a\}$. Let $\{U_i \mid i = 1, 2, 3, \dots\}$ and $\{V_i \mid i = 1, 2, 3, \dots\}$ be local bases at a and c , respectively. Then for each i , $V_i \cap F(U_i) \neq \emptyset$. Let $c_i \in V_i \cap F(U_i)$. Then $c_i \rightarrow c$. Furthermore, there exists $a_i \in U_i$ with $c_i \pi t_i = a_i$ for some $t_i \geq 0$ and for each i . Then clearly $a_i \rightarrow a$. There exist principal negative solutions σ_i such that $\sigma_i(0) = a_i$ and $\sigma_i(-t_i) = c_i$ (Remark 2.5). This implies that $b \in D^{-}(a)$. This completes the proof of part (iii).

(iv) Let $c \in J^{-}(a)$. Then there are sequences $\{\sigma_i\}$ of principal negative solutions and $\{t_i\}$ in R^+ with $t_i \rightarrow \infty$ such that $\sigma_i(0) \rightarrow a$ and $\sigma_i(-t_i) \rightarrow c$. According to Theorem 2.3 there exists a principal negative solution ν passing through a such that $\sigma_i(-t) \rightarrow \nu(-t)$ for all $t \in R^+$. Let $b \in \nu(-s)$ for some $s \in R^+$. Then $\sigma_i(-s) \rightarrow \nu(-s) = b$ and $\sigma_i(-s - (t_i - s)) = \sigma_i(-t_i) \rightarrow c$ for $i \geq i_0$, where $t_i > s$ whenever $i \geq i_0$. This implies, following Remark 2.5, that $c \in D^{-}(b)$. Hence $J^{-}(a) \subset \bigcap \{D^{-}(b) \mid b \in C_{\nu}^{-}(a)\}$. Now let $c \in \bigcap \{D^{-}(b) \mid b \in C_{\nu}^{-}(a)\}$. Then from Lemma 2.6, it follows that $b \in D^{+}(c)$ for all $b \in C_{\nu}^{-}(a)$. For each $t \in R^+$ there exists $d = \nu(-t)$. Thus $a \in \bigcap \{D^{+}(c) \pi t \mid t \in R^+\} \subset \bigcap \{D^{+}(c \pi t) \mid t \in R^+\} = J^{+}(c)$. This implies by Lemma 2.6 that $c \in J^{-}(a)$. The conclusion of this part now follows.

Parts (v), (vi) and (vii) are simple observations and their proofs will be omitted.

Remark 2.8. (i) It is known from [3] and [4] that the sets $L_{\sigma}^{-}(x)$ and $J^{-}(x)$ are closed positively and weakly negatively invariant. The same properties hold for $L^{-}(x)$. One can also remark from Lemma 2.7 that $D^{-}(a)$ is closed. Furthermore, it follows from Theorem 2.3 that $D^{-}(a)$ is weakly negatively invariant. However $D^{-}(a)$ is not positively invariant as it can be easily seen in the examples of section 6.

(ii) We observe also that if a set H is strongly negatively invariant, then $\overline{H}$ is weakly negatively invariant. This is an immediate consequence of Theorem 2.3. This result cannot be strengthened as is shown in Example 6.3.

Lemma 2.9. *If there are nets $\{a_i\}$ and $\{b_i\}$ in X with $a_i \rightarrow a$ and $b_i \rightarrow b$, then the following statements are valid.*

- (i) If $b_i \in D^-(a_i)$ for each i , then $b \in D^-(a)$,
- (ii) If $b_i \in J^-(a_i)$ for each i , then $b \in J^-(a)$.

Proof. The proof is easy and will be omitted.

Lemma 2.10. For each $x \in X$, $L^-(x) \neq \emptyset$ whenever $J^-(x) \neq \emptyset$ and compact.

Proof. Suppose that $J^-(x) \neq \emptyset$ and compact and $L^-(x) = \emptyset$ for a point $x \in X$. Then $F(x)$ is closed. If $F(x) \cap J^-(x) \neq \emptyset$, then it follows from Remark 2.8 that for $y \in F(x) \cap J^-(x)$, $C_\sigma^-(y) \subset J^-(x)$ for some principal negative solution through y . Since $J^-(x)$ is compact and $K_\sigma^-(y) \subset J^-(x)$, it follows that $L_\sigma^-(y) \neq \emptyset$. This implies by Lemma 2.6 that $L_\sigma^-(x) \neq \emptyset$. Consequently $L^-(x) \neq \emptyset$ and we thus have a contradiction. Suppose that $F(x) \cap J^-(x) = \emptyset$. Then there exists an open neighborhood U of $J^-(x)$ with $\bar{U}$ compact such that $\bar{U} \cap F(x) = \emptyset$. Let $z \in J^-(x)$. Then there are sequences $\{\sigma_i\}$ of principal negative solutions and $\{t_i\}$ in R^+ with $t_i \rightarrow \infty$ such that $\sigma_i(0) \rightarrow x$ and $\sigma_i(-t_i) \rightarrow z$. Hence for some positive integer i_0 , $\sigma_i(-\infty, 0] \cap U \neq \emptyset$ and $\sigma_i(-\infty, 0] \cap X - U \neq \emptyset$ for $i \geq i_0$. Thus $\sigma_i(-\infty, 0] \cap \partial U \neq \emptyset$ for $i \geq i_0$. Let $\sigma_i(-s_i) \in \partial U$ for $i \geq i_0$. Since ∂U is compact, we may assume that $\sigma_i(-s_i) \rightarrow b \in \partial U$. Hence $b \in D^-(x)$. Since $\partial U \cap F(x) = \emptyset$ and $\partial U \cap J^-(x) = \emptyset$, we then have a contradiction. The proof of the lemma is now complete.

Theorem 2.11. For each $x \in X$, $D^-(x)$ is connected whenever it is compact.

Proof. Assume that $D^-(x) = A \cup B$ is a separation of $D^-(x)$, where A and B are disjoint nonempty compact sets. Suppose that $x \in A$. Let $y \in B$. Then $y \in D^-(x)$. There exists an open neighborhood U of A such that $\bar{U}$ is compact and $\bar{U} \cap B = \emptyset$. There exists a sequence of principal negative solutions $\{\sigma_i\}$ such that $\sigma_i(0) \rightarrow x$ and $\sigma_i(-t_i) \rightarrow y$ for some sequence $\{t_i\}$ in R^+ . We may assume, without loss of generality, that $\sigma_i(0) \in U$ and $\sigma_i(-t_i) \in X - \bar{U}$ for all i . Since $\sigma_i([-t_i, 0])$ is connected, $\sigma_i([-t_i, 0]) \cap \partial U \neq \emptyset$ for each i . Thus for each i , there exists $s_i \in R^+$ such that $\sigma_i(-s_i) \in \partial U$. Since ∂U is compact, may assume that $\sigma_i(-s_i) \rightarrow z \in \partial U$. Then $z \in D^-(x)$ and we thus have a contradiction. The proof of the lemma is now complete.

Theorem 2.12. If M is a connected subset of X , then $D^-(M)$ is connected whenever it is compact.

Proof. Assume that $D^-(M) = A \cup B$ is a separation of $D^-(M)$ into two disjoint nonempty compact sets. Since $M \subset D^-(M)$ is connected, either $M \subset A$ or $M \subset B$. Assume that $M \subset A$. There exists $x \in M$ with $D^-(x) \cap B \neq \emptyset$. Since it is also true that $D^-(x) \cap A \neq \emptyset$ and $D^-(x)$ is connected (Theorem 2.11), we thus have a contradiction. The proof of the theorem is now complete.

§ 3. Strong stability.

Definition 3.1. A subset M of X is said to be strongly negatively Liapunov stable if for each open neighborhood U of M there exists an open neighborhood V of M with $F(V) \subset U$.

Definition 3.2. A subset M of X is said to be strongly $\{D^-\text{-stable}\}\{J^-\text{-stable}\}$ if for each open neighborhood U of M there exists an open neighborhood V of M such that $\{D^-(V) \subset U\}\{J^-(V) \subset U\}$.

The semidynamical system (X, π) is said to have one of the properties in definitions 3.1 and 3.2 if for every $a \in X$, $K^-(a)$ possesses the property.

Definition 3.3. A point $a \in X$ is said to be of characteristic O^- if $D^-(a) = K^-(a)$. The system (X, π) is said to be of characteristic O^- if $D^-(a) = K^-(a)$ for each $a \in X$.

Theorem 3.4. A compact subset M of X is strongly D^- -stable iff it is strongly negatively Liapunov stable.

Proof. Assume that M is strongly D^- -stable. Then it follows from Lemma 2.7 that M is strongly negatively Liapunov stable. Conversely, assume that M is strongly negatively Liapunov stable. Suppose that M is not strongly D^- -stable. Then there exists an open neighborhood U of M such that $D^-(W) \not\subset U$ for all open neighborhood W of M . There exists an open neighborhood V of M with $\overline{F(V)} \subset U$. Lemma 2.7, $D^-(V) = \bigcup_{a \in V} \bigcap \{\overline{F(W)} \mid W \text{ is an open neighborhood of } a\} \subset \overline{F(V)} \subset U$ and we then have a contradiction. This completes the proof of the theorem.

Theorem 3.5. A compact subset M of X is strongly D^- -stable iff $D^-(M) = M$.

Proof. Assume that M is strongly D^- -stable. Then M is strongly negatively invariant. It follows from Lemma 2.7 that $M \subset D^-(M)$. Let $a \in D^-(M) - M$. Then there exists an open neighborhood U of M with $a \notin U$. Since M is strongly D^- -stable, there exists an open neighborhood W of M with $D^-(W) \subset U$. Hence $a \in D^-(M) \subset D^-(W) \subset U$ and thus we have a contradiction. Consequently, $D^-(M) = M$. Conversely, assume that $D^-(M) = M$ and M is not strongly D^- -stable. Then it follows from Theorem 2.1 that M is not strongly negatively Liapunov stable. Then there exists an open neighborhood U of M whose closure $\overline{U}$ is compact such that $F(W) \not\subset U$ for all open neighborhood W of M . Hence there are sequences $\{a_i\}$ in U and $\{t_i\}$ in R^+ such that $a_i \rightarrow a \in M$ and $F(a_i, -t_i) \not\subset U$ for each i . Hence for each i there exists a principal negative solution σ_i through a_i such that $\sigma_i(-\infty, 0] = C_{\sigma_i}^-(a_i) \cap U \neq \emptyset$ and $C_{\sigma_i}^-(a_i) \cap (X - U) \neq \emptyset$. Since each σ_i is continuous [1], it follows that $C_{\sigma_i}^-(a_i)$ is connected for each i . This implies that $C_{\sigma_i}^-(a_i) \cap \partial U \neq \emptyset$ for each i . Hence for each i there exists $s_i \in R^+$ such that $\sigma_i(-s_i) \in \partial U$. We may assume that

$\sigma_i(-s_i) \rightarrow d \in \partial U$. Thus $d \in D^-(M) = M$ and we thus have a contradiction. This completes the proof of the theorem.

Theorem 3.6. *A compact negatively invariant subset M of X is strongly J^- -stable iff $J^-(M) = M$.*

Proof. Assume that M is strongly J^- -stable. Let $b \in J^-(M) - M$. Then there exists an open neighborhood U of M with $b \notin U$. There is an open neighborhood W of M with $J^-(W) \subset U$. Thus $b \in J^-(M) \subset J^-(W) \subset U$ and hence we have a contradiction. Consequently, $J^-(M) \subset M$. Let $a \in M - J^-(M)$. Since M is compact, it follows easily from Lemma 2.9 that $J^-(M)$ is closed. Hence there exists an open neighborhood U whose closure $\bar{U}$ is compact and such that $a \notin U$. Let $b \in J^-(a)$. Then there are sequences $\{\sigma_i\}$ of principal negative solutions and $\{t_i\}$ in R^+ such that $\sigma_i(0) \rightarrow a$, and $\sigma_i(-t_i) \rightarrow b$. Hence there exists a sequence $\{s_i\}$ in R^+ with $\sigma_i(-s_i) \in \partial U$. Since ∂U is compact, we may assume that $\sigma_i(-s_i) \rightarrow c \in \partial U$. This implies that $c \in J^-(a) \subset J^-(M)$ and thus we then have a contradiction. Consequently, $M \subset J^-(M)$ and hence $M = J^-(M)$. Conversely, assume that $M = J^-(M)$. Then $D^-(M) = F(M) \cup J^-(M) = M$. This implies by Theorem 3.5 that M is strongly D^- -stable and consequently, M is strongly J^- -stable. The proof of the theorem is now complete.

Corollary 3.7. *A compact negatively invariant subset M of X is strongly D^- -stable iff it is strongly J^- -stable.*

Proof. This follows immediately from Theorems 3.5 and 3.6.

Theorem 3.8. *The semidynamical system (X, π) is of characteristic O^- iff the relation $S = \{(a, b) \in X \times X \mid b \in K^-(a)\}$ is closed, negatively invariant relation on X .*

Proof. Assume that (X, π) is of characteristic O^- . Let $\{(a_i, b_i)\}$ be a sequence in S which converges to (a, b) . Then $b_i \in K^-(a_i) = D^-(a_i)$, for each i . It follows from Lemma 2.9 that $b \in D^-(a) = K^-(a)$ and consequently, $(a, b) \in S$. Thus S is closed. Let $(a, b) \in S$ and $(c, d) \in X \times X$ with $c \in F(a, -t)$, $d \in F(b, -t)$. Suppose that $d \notin K^-(c) = D^-(c)$. Then $b = d\pi t \notin D^-(c)\pi t$. It follows from Lemma 2.6 that $b \notin D^-(c\pi t) = D^-(a) = K^-(a)$ which is false. Hence S is negatively invariant. Conversely, assume that S is closed and negatively invariant. Let $b \in D^-(a)$. Then there are sequences $\{\sigma_i\}$ of principal negative solutions and $\{t_i\}$ in R^+ with $\sigma_i(0) \rightarrow a$ and $\sigma_i(-t_i) \rightarrow b$. Since $(\sigma_i(0), \sigma_i(-t_i)) \in S$ for each i , it follows that $(a, b) \in S$. Thus $b \in K^-(a)$. Hence $D^-(a) \subset K^-(a)$. From Remark 2.8, it follows that $D^-(a) = K^-(a) \cup J^-(a)$ and hence $K^-(a) \subset D^-(a)$. This implies that $D^-(a) = K^-(a)$ and consequently, a is of characteristic O^- . The proof of the theorem is now complete.

Theorem 3.9. *If a semidynamical system (X, π) is strongly D^- -stable, then it is of characteristic O^- .*

Proof. The proof is easy and will be omitted.

Definition 3.10. A point $a \in X$ is said to be of stable characteristic O^- if it is of characteristic O^- and whenever there are sequences $\{a_i\}$ and $\{b_i\}$ in X with $a_i \rightarrow a$ and $b_i \in F(x_i)$, then $\{b_i\}$ has a convergent subsequence. The semidynamical system (X, π) is of stable characteristic O^- if every point in X is of stable characteristic O^- .

Remark 3.11. Let $a \in X$ be of stable characteristic O^- . Then it follows immediately from Definition 2.7 that $K^-(a)$ is compact.

Theorem 3.12. A semidynamical system (X, π) is of stable characteristic O^- if it is of characteristic O^- and $K^-(a)$ is compact for each $a \in X$.

Proof. Assume that (X, π) is of characteristic O^- and that $K^-(a)$ is compact for each $a \in X$. Suppose that there exist $x \in X$ which is not of stable characteristic O^- . Thus there exist sequences $\{x_i\}$ and $\{y_i\}$ in X with $x_i \rightarrow x$, $y_i \in F(x_i)$ and $\{y_i\}$ has no convergent subsequence. Let U be an open neighborhood of $K^-(x)$ such that $\bar{U}$ is compact. This implies that $y_i \notin U$ for all $i \geq i_0$. For each $i \geq i_0$ there exists a principal negative solution σ_i such that $\sigma_i(0) = x_i$ and $\sigma_i(-t_i) = y_i$. Thus $C_{\sigma_i}^-(x_i) \cap \partial U \neq \emptyset$ for $i \geq i_0$. Let $\sigma_i(-s_i) \in \partial U$ for $i \geq i_0$. Since ∂U is compact, $\sigma_i(-s_i) \rightarrow z \in \partial U$. Consequently $z \in D^-(x) = K^-(x)$ and we then have a contradiction. This shows that x is of stable characteristic O^- . The converse is clear.

Theorem 3.13. A semidynamical systems (X, π) is of stable characteristic O^- iff it is strongly D^- -stable and $K^-(a)$ is compact for each $a \in X$.

Proof. Assume that (X, π) is of stable characteristic O^- . Suppose that for some $x \in X$, $K^-(x)$ is not strongly D^- -stable. Then from Theorem 3.4 it follows that $K^-(x)$ is not strongly negatively Liapunov stable. Hence there exists an open neighborhood U of $K^-(x)$ such that $F(V) \not\subset U$ for all open neighborhoods V of $K^-(x)$. Then there exists a sequence $\{x_i\}$ in U which converges to $a \in K^-(x)$ and such that $F(x_i) \not\subset U$ for all i . Let $y_i \in F(x_i) - U$. Since x is of stable characteristic O^- , $\{y_i\}$ has a convergent subsequence $\{y_{i_n}\}$. Then $y_{i_n} \rightarrow y \in X - U$. Hence $y \in D^-(a) \subset D^-(x) = K^-(x)$ and thus we have a contradiction. Therefore (X, π) is strongly D^- -stable. The converse follows from Theorems 3.12 and 3.9.

Theorem 3.14. A compact subset M of X is strongly negatively Liapunov stable iff every component of M is strongly negatively Liapunov stable.

Proof. Suppose that M is strongly negatively Liapunov stable. Then M is negatively invariant. Furthermore, it follows from Theorem 3.4 and 3.5 that $D^-(M) = M$. Let M_i be a component of M . Then one can show that $D^-(M_i)$ is closed. Thus $D^-(M_i)$ is compact. Then according to Theorem 2.12, $D^-(M_i)$ is

connected. This implies that $D^-(M_i) = M_i$. It follows from Theorems 3.4 and 3.5 that M_i is strongly negatively Liapunov stable. The converse is trivial.

§ 4. Weak stability.

Definition 4.1. Let $a \in X$ and σ be a principal negative solution through a . Then we have the following:

- (i) The σ -negative prolongation set of a ;
 $D_\sigma^-(a) = \{b \in X \mid \text{there exists a sequence } \{\sigma_i\} \text{ of principal negative solutions such that } \sigma_i(-t) \rightarrow \sigma(-t) \text{ for each } t \in R^+ \text{ and } \sigma_i(-t_i) \rightarrow b \text{ for some sequence } \{t_i\} \text{ in } R^+\}.$
- (ii) The σ -negative prolongation limit set of a ;
 $J_\sigma^-(a) = \{b \in X \mid \text{there exists a sequence } \{\sigma_i\} \text{ of principal negative solutions such that } \sigma_i(-t) \rightarrow \sigma(-t) \text{ for each } t \in R^+ \text{ and } \sigma_i(-t_i) \rightarrow b \text{ for some sequence } \{t_i\} \text{ in } R^+ \text{ with } t_i \rightarrow \infty\}.$

Lemma 4.2. Let $a \in X$ and σ be a principal negative solution through a . Then the following statements are valid:

- (i) $D_\sigma^-(a) \subset D_\sigma^-(a)\pi t$ and $D_\sigma^-(a) \subset D^-(a)$,
- (ii) $J_\sigma^-(a) = J_\sigma^-(a)\pi t \subset J^-(a)$.

Proof. The proof is similar to that of Lemma 2.6.

Lemma 4.3. Let $a \in X$ and σ be a principal negative solution through a . Then

- (i) $J_\sigma^-(a)$ is closed, positively invariant and weakly negatively invariant.
- (ii) $D_\sigma^-(a)$ is closed and weakly negatively invariant.
- (iii) $D_\sigma^-(a) = C_\sigma^-(a) \cup J_\sigma^-(a)$.

Proof. The proof follows easily from Theorem 2.3 and Lemma 4.2.

Definition 4.4. A subset M of X is said to be weakly negatively Liapunov stable if for each open neighborhood V of M there exists an open neighborhood U of M such that for each $a \in U$, $C_\sigma^-(a) \subset V$ for some principal negative solution σ through a .

Definition 4.5. A subset M of X is weakly $\{D^-$ -stable $\}\{J^-$ -stable $\}$ if for each open neighborhood U of M there exists an open neighborhood V of M such that for each $a \in V$ there exists a principal negative solution σ through a with $\{D_\sigma^-(a) \subset U\} \{J_\sigma^-(a) \subset U\}$.

Definition 4.6. A point $a \in X$ is said to be of weakly characteristic O^- if $D_\sigma^-(a) = K_\sigma^-(a)$ for some principal negative solution through a .

Theorem 4.7. A compact subset M of X is weakly negatively Liapunov stable iff it is weakly D^- -stable.

Proof. The proof is similar to that of Theorem 3.4 and will be omitted.

Theorem 4.8. *A compact subset M of X is weakly D^- -stable iff for each $a \in M$ there exists a principal negative solution σ_a such that $\bigcup \{D_{\sigma_a}^-(a) \mid a \in M\} = M$.*

Proof. The proof is similar to that of Theorem 3.5 and will be omitted.

Theorem 4.9. *A compact negatively invariant subset M of X is weakly J^- -stable iff for each $a \in M$ there exists a principal negative solution σ_a such that $\bigcup \{J_{\sigma_a}^{-1}(a) \mid a \in M\} = M$.*

Proof. The proof is similar to that of Theorem 3.6,

Corollary 4.10. *A compact negatively invariant subset M of X is weakly D^- -stable iff it is weakly J^- -stable.*

Proof. This follows immediately from Theorems 4.8 and 4.9.

§ 5. Attraction and saymptotic stability.

Definition 5.1. For a nonempty subset M of X we have the following:

- (i) The region of weak negative attraction of M ;
 $A_w^-(M) = \{x \in X \mid L^-(x) \cap M \neq \emptyset\},$
- (ii) The region of negative attraction of M ;
 $A^-(M) = \{x \in X \mid L^-(x) \subset M, L^-(x) \neq \emptyset\},$
- (iii) The region of strong negative attraction of M ;
 $A_s^-(M) = \{x \in X \mid J^-(x) \subset M, J^-(x) \neq \emptyset\}.$

A point $x \in X$ is said to be {weakly negatively attracted} {negatively attracted} {strongly negatively attracted} to M if $\{x \in A_w^-(M)\} \{x \in A^-(M)\} \{x \in A_s^-(M)\}$, respectively.

Definition 5.2. A nonempty subset M of X is said to be {a negative weak attractor} {a negative attractor} {a negative strong attractor} if $\{A_w^-(M)\} \{A^-(M)\} \{A_s^-(M)\}$ is a neighborhood of M , respectively. The adjective global is added to the above if the corresponding region of attraction is the whole phase space of X .

Lemma 5.3. *For a nonempty subset M of X , the following hold:*

- (i) *The sets $A_w^-(M)$, $A^-(M)$ and $A_s^-(M)$ are weakly invariant.*
- (ii) *$A_s^-(M) \subset A^-(M) \subset A_w^-(M)$.*

Proof. Statement (i) follows from Lemma 2.6 and statement (ii) follows from Lemma 2.10.

Theorem 5.4. *A compact subset M of X is strongly negatively Liapunov stable iff it is negatively invariant and $M \subset A_s^-(M)$.*

Proof. Assume that M is strongly negatively Liapunov stable. Let $z \in F(M) - M$. Then there exists an open neighborhood U of M with $z \notin U$. There exists an open neighborhood V of M with $F(V) \subset U$. Thus $z \in F(M) \subset F(V) \subset U$ and hence we have a contradiction. Consequently $F(M) \subset M$ and M is thus negatively invariant. Let $x \in M$. Suppose that $x \notin A_s^-(M)$. Then there exists $y \in J^-(x) - M$. Let W be open neighborhood of M with $y \notin \bar{W}$. Then there exists an open neighborhood G of M with $F(G) \subset W$. From Lemma 2.7 it follows that $D^-(x) \subset \overline{F(G)} \subset \bar{W}$. This implies that $y \in J^-(x) \subset D^-(x) \subset \bar{W}$ and we thus have a contradiction. Hence $M \subset A_s^-(M)$. Conversely, assume that $M \subset A_s^-(M)$ and M is negatively invariant. If M is not strongly negatively Liapunov stable, then there exists an open neighborhood U of M with $\bar{U}$ compact and such that $F(V) \not\subset U$ for all open neighborhoods V of M . Hence there exist sequences $\{x_i\}$ in U and $\{t_i\}$ in R^+ such that $x_i \rightarrow x \in M$ and $F(x_i, -t_i) \not\subset U$. Since $F(x_i, [-t_i, 0])$ is connected (Theorem 2.2) for each i , $F(x_i, [-t_i, 0]) \cap \partial U \neq \emptyset$. Let $b_i \in F(x_i, [-t_i, 0]) \cap \partial U$ for each i . Then $b_i \in F(x_i, -s_i)$ for some $s_i \in R^+$. Since ∂U is compact, we may assume that $b_i \rightarrow b \in \partial U$. Hence $b \in D^-(x)$. Since $D^-(x) = F(x) \cup J^-(x)$ (Lemma 2.7) and $F(x) \subset M$, $b \in J^-(x)$. This leads to a contradiction since $J^-(x) \subset M$ for each $x \in M$. This completes the proof of the theorem.

Definition 5.5. A subset M of X is said to be negatively asymptotically stable if it is a negative attractor and strongly negatively Liapunov stable.

Theorem 5.6. Let M be a compact negatively asymptotically stable subset of X . Then a component of M is negatively asymptotically stable iff it is an isolated component.

Proof. Let M_1 be a negatively asymptotically stable component of M . If M_2 is any other component of M , then according to Theorem 3.14 the set M_2 is strongly negatively Liapunov stable. Hence M_2 is negatively invariant. Claim that $A^-(M_1) \cap M_2 = \emptyset$. To prove the claim, assume that there exists $x \in A^-(M_1) \cap M_2$. Then $L^-(x) \subset M_1$ and $L^-(x) \neq \emptyset$. Furthermore, $L^-(x) \subset M_2$ and thus we have a contradiction. Since M_1 is a negative attractor, there exists an open neighborhood U of M_1 with $U \subset A^-(M_1)$. Hence M_1 is isolated. Conversely, assume that M_1 is an isolated component of M . Then there exists an open neighborhood U of M_1 such that $\bar{U}$ is compact, $U \subset A^-(M)$ and $U \cap (X - M_1) = \emptyset$. According to Theorem 3.14 the set M_1 is negatively strongly Liapunov stable. Hence there exists an open neighborhood V of M_1 with $\overline{F(V)} \subset U$. It follows that for each $x \in V$, $L^-(x) \neq \emptyset$ and $L^-(x) \subset U$. Since $V \subset A^-(M)$, $L^-(x) \subset M$ for each $x \in V$. This implies that for each $x \in V$, $L^-(x) \subset M_1$. Thus $V \subset A^-(M_1)$ and consequently, M is an attractor. This completes the proof of the theorem.

§ 6. Examples.

The following examples demonstrate some of the basic differences between dynamical systems and semidynamical systems.

Example 6.1. Let X be the real line. A semidynamical system is defined graphically in Figure 1.

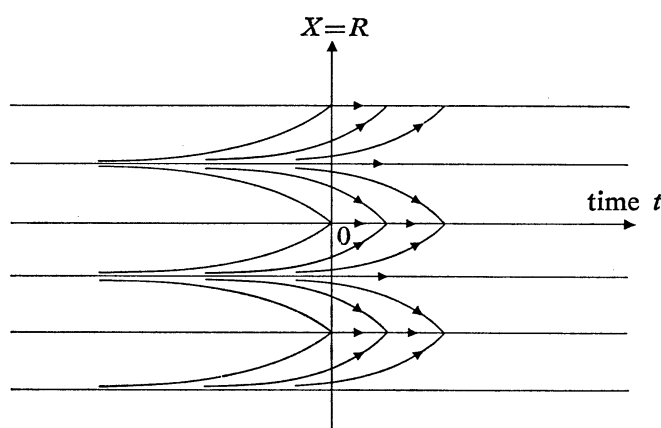


Figure 1.

All points in $X - \{0\}$ are points of negative uniqueness. Notice that $L^-(0) = \{-1, 0, 1\}$ is disconnected. Furthermore, $J^-(0) = L^-(0)$ and is thus disconnected. This is in contrast with the situation in dynamical systems. The set $\{0\}$ is positively Liapunov stable, weakly negatively Liapunov stable but not strongly negatively Liapunov stable. Let $H = \{-1, 1\}$. Then $A_w^-(H) = [-2, 2]$ and $A^-(H) = (-2, 0) \cup (0, 2) = A_s^-(H)$. Thus H is a negative strong attractor. Notice that $A_w^-(H)$ is not open, not negatively invariant and weakly negatively invariant. Furthermore, H is strongly negatively Liapunov stable and hence it is negatively asymptotically stable.

Example 6.2. Let $X = R$ be the real line. A semidynamical system is defined graphically in Figure 2. Notice that $L^-(0) = R$ while $L_\sigma^-(0) = \emptyset$ for all negative solutions σ through 0. Thus Lemma 2.6 cannot be improved. The set $\{0\}$ is globally positively asymptotically stable but it is not even weakly negatively invariant. We remark also that $F(0, [-t, -s])$ is connected iff $s = 0$.

Example 6.3. Let $X = R^2$. A semidynamical system is defined graphically in Figure 3. Notice that the set $G = \{(x, y) \mid -1 < x < 1, -1 < y < 1\}$ is negatively strongly invariant. However, its closure $\bar{G}$ is negatively weakly invariant and not strongly invariant (compare with Remark 2.8 (i)). The set $D^-(a)$ is not positively invariant for $a \in R^2$.

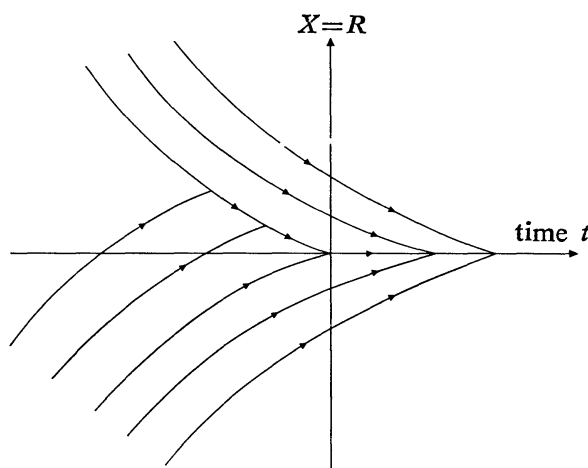


Figure 2.

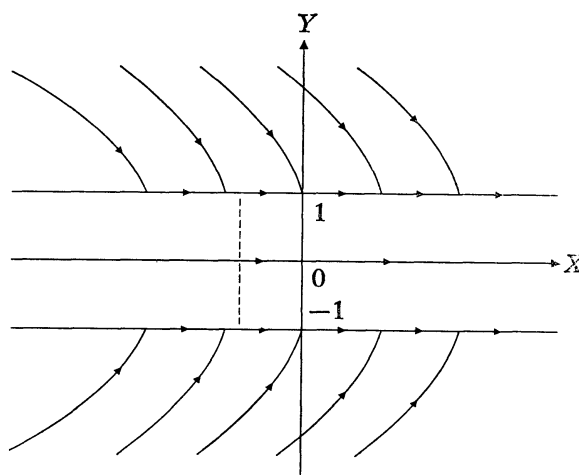


Figure 3.

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