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PDF issue: 2026-11-06

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(Citation)
Funkcialaj Ekvacioj,27(2):157-172

(Issue Date)
1984-08

(Resource Type)
journal article

(Version)
Version of Record

(JaLCDOI)
https://doi.org/10.24546/0100499366

(URL)
https://hdl.handle.net/20.500.14094/0100499366

KOBE UNIVERSITY

Funkcialaj Ekvacioj, **27** (1984), 157–172

On the Existence of L^p-Solutions of Volterra Integral Equations in Banach Spaces

By

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Introduction.

This paper studies a nonlinear Volterra integral equation (the equation (1)) in a Banach space E. Most studies of (1) have been done in the space C of continuous functions over E. In contrast to this, in the paper we study (1) in the space L^p over E and establish an existence theorem and a Hukuhara-type and a Kneser-type theorems. In our earlier paper [11], similar results were established in C. The present paper extends these results to L^p under some additional restrictive conditions. To do so, we use the measure α_p of noncompactness in L^p, and clarify the relation between α_p and the measure of noncompactness in E in Section 2. This relation plays an essential role in our paper. Using it and the assumptions 2° and (12) on g, we prove that any sequence of approximate solutions of (1) has an L^p-convergent subsequence, so that the existence theorem (Theorem 1) follows immediately. The novelty here is the use of the assumption 3°, which is similar to that used in the study of Fredholm integral equations [5, p. 27], and this requires interesting calculations and an integral estimate (Lemma 2). In Section 4 we investigate the structure of the set of L^p-solutions of (1). The proof of Theorems 2 and 3 is based on a connectedness argument given by Hukuhara [4] for integral equations in R^n. We describe complete proofs of these theorems, because these detailed descriptions are necessary to understand the theorems clearly. In Section 5 we give an example of g which satisfies the assumption (12), i.e. $g=g_1+g_2$ and g_1 is compact and g_2 is Hölder continuous in x of some exponent.

§ 1. Let E, F be Banach spaces, $\Omega=[0, \omega]$, $Q=\Omega\times\Omega$, and let $B(F, E)$ be the space of continuous linear mappings of F into E. Denote by $L^p(\Omega, E)(p>1)$ the Lebesgue space of all strongly measurable functions $u\colon \Omega\to E$ with $\int_0^\omega \|u(t)\|^p\,dt<\infty$, provided with the norm $\|u\|_p=\left(\int_0^\omega \|u(t)\|^p\,dt\right)^{1/p}$.

In this paper we consider the nonlinear integral equation

$$x(t)=f(t)+\int_0^t K(t, s)g(s, x(s))ds, \tag{1}$$

where K is a function from Q into $B(F, E)$ and g is a function from $\Omega\times E$ into F. First, in Section 3, we present an existence theorem for L^p-solutions of (1). Next, in Section 4, we prove a Hukuhara-type theorem on the connectedness of the set of ε-approximate L^p-solutions and a Kneser-type theorem for (1).

In our existence criteria for solutions of (1) we use the measure of noncompactness α introduced by Kuratowski. For each bounded subset A of E the measure of noncompactness of A–denoted $\alpha(A)$–is defined to be infimum of positive number ε such that A can be covered by a finite number of sets of diameter $\leqslant\varepsilon$. Analogously we define the measure of noncompactness α_p in the space L^p. The fundamental properties of α are given in [6] and [1]. In Section 2 we establish some additional properties of α and α_p.

Throughout this paper we assume that

1° $f\in L^p(\Omega, E)$.

2° $(s, x)\to g(s, x)$ is a function from $\Omega\times E$ into F such that

(i) for every $x\in E$ the funciton $s\to g(s, x)$ is strongly measurable on Ω;

(ii) for almost every $s\in\Omega$ the function $x\to g(s, x)$ is continuous on E;

(iii) $\|g(s, x)\|\leqslant e(s)+b\|x\|^{p/q}$ for $s\in\Omega$ and $x\in E$, where $e\in L^q(\Omega, R)$, $b\geqslant 0$ and $q>1$.

3° K is a strongly measurable function from Q into $B(F, E)$ such that

$$\iint_Q \|K(t, s)\|^m dtds<\infty, \text{ where } m=\max(p, r) \text{ and } r=q/(q-1).$$

(For convenience all norms of E, F and $B(F, E)$ are denoted by $\|\cdot\|$.)

Moreover, let μ denote the Lebesgue measure in R.

In what follows we shall need the following lemmas.

Lemma 1. *For any $u\in L^q(\Omega, F)$ the functions $t\to v(t)=\int_0^a K(t, s)u(s)ds$ $(0<a\leqslant\omega)$ and $t\to w(t)=\int_0^t K(t, s)u(s)ds$ are integrable on Ω.*

Proof. By 3° and the Hölder inequality for any $u\in L^q(\Omega, F)$ there exist the integrals $\iint_Q K(t, s)u(s)dtds$ and $\iint_Q K(t, s)u(s)\chi_{[0, t]}(s)dtds$, where $\chi_{[0, t]}$ is the characteristic function on $[0, t]$. In view of the Fubini theorem for vector functions (cf. [3], Ch. V. 8, Th. 1) this implies the integrability of v and w.

Lemma 2. *Suppose that u, v, w are nonnegative functions belonging to $L^p(\Omega, R)$, $\|v\|_p\leqslant d$, $d\geqslant 1$ and*

$$u(t)\leqslant v(t)+w(t)\left(\int_0^t u^p(s)ds\right)^{1/q} \qquad \text{for } t\in\Omega. \tag{2}$$

Then

$$(3)\qquad \int_0^t u^p(s)ds \leqslant U^{-1}\Big(U(2^{p-1}d^p)+2^{p-1}\int_0^t w^p(d)ds\Big) \qquad \text{for } t \in \Omega \cap [0, \tau),$$

where

$$U(y)=\int_1^y \frac{ds}{s^{p/q}} \qquad \text{for } y \geqslant 1$$

and τ *is the supremum of all* $t \in \Omega$ *for which* $U(2^{p-1}d^p)+2^{p-1}\int_0^t w^p(s)ds \in U([1, \infty])$. *Moreover, if* $q \geqslant p$, *then* (3) *holds for all* $t \in \Omega$.

Proof. Putting $x(t)=\int_0^t u^p(s)ds$, we see that (2) implies

$$x'(t) \leqslant (v(t)+w(t)x^{1/q}(t))^p \leqslant 2^{p-1}(v^p(t)+w^p(t)x^{p/q}(t))$$

for almost every $t \in \Omega$. Integrating this inequality between 0 and t we get $x(t) \leqslant 2^{p-1}\Big(\int_0^t v^p(s)ds+\int_0^t w^p(s)x^{p/q}(s)ds\Big) \leqslant 2^{p-1}d^p+2^{p-1}\int_0^t w^p(s)x^{p/q}(s)ds$ for $t \in \Omega$. Hence, applying Bihari's theorem (cf. [2], p. 83), we obtain

$$x(t) \leqslant U^{-1}\Big(U(2^{p-1}d^p)+2^{p-1}\int_0^t w^p(s)ds\Big) \qquad \text{for } t \in \Omega \cap [0, \tau$$

which proves (3).

Let k be the function defined by $k(t)=\Big(\int_0^t \|K(t, s)\|^r ds\Big)^{1/r}$ for $t \in \Omega$. Since $k(t) \leqslant t^{1/r-1/m}\Big(\int_0^t \|K(t, s)\|^m ds\Big)^{1/m}$, from 3° it follows that $k \in L^m(\Omega, R) \subset L^p(\Omega, R)$. Denote by τ the supremum of all $t \in \Omega$ for which $U(2^{p-1}(\|f\|_p+\|k\|_p\|e\|_q+1)^p)+2^{p-1}b\int_0^t k^p(s)ds$ belongs to the domain of definition of U^{-1}, where U is the function from Lemma 2. We choose a positive number a in such a way that $a<\tau$ if $q<p$, and $a=\omega$ if $q \geqslant p$. Let $J=[0, a]$ and $L^p=L^p(J, E)$. Without loss of generality we shall always assume that all functions from L^p are extended to R by putting $u(t)=0$ outside J.

Consider now a mapping G defined by

$$G(x)(t)=f(t)+\int_0^t K(t, s)g(s, x(s))ds \qquad \text{for } x \in L^p \text{ and } t \in J.$$

By Lemma 1, the Hölder inequality and the assumptions 1°–3° for any $x \in L^p$ the function $G(x)$ is strongly measurable on J and

$$(4)\qquad \|G(x)(t)\| \leqslant \|f(t)\|+k(t)\|e\|_q+bk(t)\Big(\int_0^t \|x(s)\|^p ds\Big)^{1/q} \qquad \text{for } t \in J,$$

so that $G(x) \in L^p$.

We shall show that G is a continuous mapping $L^p \to L^p$. Let $x_n, x_0 \in L^p$ and $\lim_{n\to\infty}\|x_n - x_0\|_p = 0$. Suppose that $\|G(x_n) - G(x_0)\|_p$ does not converge to 0 as $n\to\infty$. Then there are $\varepsilon > 0$ and a subsequence (x_{n_j}) such that

$$\|G(x_{n_j}) - G(x_0)\|_p > \varepsilon \qquad \text{for } j = 1, 2, \cdots \tag{5}$$

and $\lim_{j\to\infty} x_{n_j}(t) = x_0(t)$ for almost every $t \in J$. By 2° (ii) we have $\lim_{j\to\infty}\|g(t, x_{n_j}(t)) - g(t, x_0(t))\| = 0$ for almost every $t \in J$. Moreover, as $\lim_{n\to\infty}\|x_n - x_0\|_p = 0$ implies that the sequence (x_n) has equi-absolutely continuous norms in L^p, from 2° (iii) it follows that the functions $\|g(\cdot, x_n) - g(\cdot, x_0)\|^q$ $(n = 1, 2, \cdots)$ are equi-integrable on J. Hence, by the Vitali convergence theorem, $\lim_{j\to\infty}\|g(\cdot, x_{n_j}) - g(\cdot, x_0)\|_q = 0$. Since $\|G(x_{n_j})(t) - G(x_0)(t)\| \leqslant k(t)\|g(\cdot, x_{n_j}) - g(\cdot, x_0)\|_q$ and $k \in L^p(J, R)$, this implies that $\lim_{j\to\infty}\|G(x_{n_j}) - G(x_0)\|_p = 0$, in contradiction with (5).

Let $c = \left(U^{-1}\left(U(2^{p-1}(\|f\|_p + \|k\|_p\|e\|_q + 1)^p) + 2^{p-1}b\int_0^a k^p(s)ds\right)\right)^{1/p}$ and $H = \{u \in L^p\colon \|u\|_p \leqslant c\}$, where U is the function from Lemma 2. Then, by (4),

$$\|G(x)(t)\| \leqslant M(t) \qquad \text{for } x \in H \text{ and } t \in J, \tag{6}$$

where $M(t) = \|f(t)\| + k(t)\|e\|_q + bc^{p/q}k(t)$.

Furthermore, by 1°–3° and the Hölder inequality, we have

$$\|G(x)(t+h) - G(x)(t)\| \leqslant d(t, h) \qquad \text{for each } x \in H, \tag{7}$$

where

$$d(t, h) = \begin{cases} M(t) & \text{if } t \in J \text{ and } t+h \notin J \\ \|f(t+h) - f(t)\| + (\|e\|_q + bc^{p/q})(\beta(t, h) + \varepsilon(t, h)) & \text{if } t, t+h \in J \end{cases}$$

and

$$\beta(t, h) = \left|\int_t^{t+h} \|K(t+h, s)\|^r ds\right|^{1/r}$$

$$\varepsilon(t, h) = \left(\int_0^a \|K(t+h, s) - K(t, s)\|^r ds\right)^{1/r}.$$

It follows from the assumptions 1°, 3° and the corresponding properties of the Lebesgue integral that

$$\lim_{h\to 0}\int_0^a \|f(t+h) - f(t)\|^p dt = 0, \quad \lim_{h\to 0}\int_0^a \left|\int_t^{t+h} \|K(t+h, s)\|^m ds\right| dt = 0$$

and

$$\lim_{h\to 0}\int_0^a \left(\int_0^a \|K(t+h, s) - K(t, s)\|^m ds\right) dt = \lim_{h\to 0}\int_0^a \left(\int_0^a \|K(t+h, s) - K(t, s)\|^m dt\right) ds = 0.$$

As

$$\beta(t, h) \leqslant |h|^{1/r-1/m} \left| \int_t^{t+h} \|K(t+h, s)\|^m ds \right|^{1/m},$$

$$\varepsilon(t, h) \leqslant a^{1/r-1/m} \left(\int_0^a \|K(t+h, s) - K(t, s)\|^m ds \right)^{1/m}$$

and $p \leqslant m$, from this we deduce that

$$\lim_{h \to 0} \int_0^a d^p(t, h) dt = 0. \tag{8}$$

Note also that $\iint_{Q_h} d^p(t, s) dt ds < \infty$, where $Q_h = \{(t, s): s \in [-h, h],\ t \in J\}$.

§ 2. The measure of noncompactness in L^p.

Let α and α_p be the Kuratowski measures of noncompactness in E and L^p, respectively. For any set V of functions from J into E denote by v the function defined by $v(t) = \alpha(V(t))$ for $t \in J$ (under the convention that $\alpha(A) = \infty$ if A is unbounded), where $V(t) = \{u(t): u \in V\}$. Moreover, for any subset T of J set $V(T) = \{u(t): u \in V,\ t \in T\}$.

Lemma 3. *Let V be a uniformly bounded and equicontinuous set of functions from J into E. Then*

$$\alpha_p(V) \leqslant \left(\int_0^a v^p(s) ds \right)^{1/p}.$$

Proof. Since the set V is equicontinuous and uniformly bounded, the function v is continuous on J. Fix $\varepsilon > 0$ and choose $\eta > 0$ such that $|v(t) - v(s)| \leqslant \varepsilon$ and $\|x(t) - x(s)\| \leqslant \varepsilon$ for each $x \in V$ and $t, s \in J$ such that $|t - s| \leqslant \eta$. Divide the interval J into n parts

$$0 = t_0 < t_1 < \cdots < t_n = a$$

in such a way that $t_i - t_{i-1} < \eta$ for $i = 1, \cdots, n$.

It follows from the definition of α that for any i, $1 \leqslant i \leqslant n$, there exists a finite family $(A_{ij})_{j=1,\ldots,m_i}$ such that diam $A_{ij} \leqslant \alpha(V(t_i)) + \varepsilon$ and $V(t_i) \subset \bigcup_{j=1}^{m_i} A_{ij}$. Let $l = \max(m_1, \cdots, m_n)$. Denote by ϕ the set of all mappings φ of $\{1, \cdots, n\}$ into $\{1, \cdots, l\}$ such that $\varphi(i) \leqslant m_i$ for $i = 1, \cdots, n$. For any $\varphi \in \phi$ let Z_φ be the set of all $x \in V$ such that $x(t_i) \in A_{i\varphi(i)}$ for $i = 1, \cdots, n$. It is clear that the family $(Z_\varphi)_{\varphi \in \phi}$ is a finite cover of V. Moreover, for any $x, y \in Z_\varphi$ we have

$$\|x - y\|_p = \left(\int_0^a \|x(t) - y(t)\|^p dt \right)^{1/p}$$

$$\leqslant \left(\sum_{i=1}^n \int_{t_{i-1}}^{t_i} (\|x(t) - x(t_i)\| + \|x(t_i) - y(t_i)\| + \|y(t_i) - y(t)\|)^p dt \right)^{1/p}$$

$$\leqslant\left(\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}(3\varepsilon+v(t_i))^p\,dt\right)^{1/p}\leqslant\left(\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}(4\varepsilon+v(t))^p dt\right)^{1/p}$$
$$=\left(\int_0^a(4\varepsilon+v(t))^p dt\right)^{1/p}\leqslant 4\varepsilon a^{1/p}+\left(\int_0^a v^p(t)\,dt\right)^{1/p}.$$

Therefore $\operatorname{diam}_p Z_\varphi\leqslant 4\varepsilon a^{1/p}+\left(\int_0^a v^p(t)\,dt\right)^{1/p}$, and hence

$$\alpha_p(V)\leqslant 4\varepsilon a^{1/p}+\left(\int_0^a v^p(t)\,dt\right)^{1/p}.$$

Since the last inequality holds for every $\varepsilon>0$, we obtain

$$\alpha_p(V)\leqslant\left(\int_0^a v^p(t)dt\right)^{1/p}.$$

Lemma 4. *Let V be a subset of L^p such that $\|x(t)\|\leqslant M(t)$ and $\|x(t+h)-x(t)\|\leqslant d(t,h)$ for each $x\in V$, $t\in J$ and small $|h|$, where M and d are functions from* (6) *and* (7). *Then*

(a) $v\in L^p(J,R)$

(b) $\alpha_p(V)\leqslant\left(\int_0^a v^p(t)dt\right)^{1/p}$

(c) *for any $\varepsilon>0$ there exists a closed subset J_ε of J such that $\mu(J\setminus J_\varepsilon)<\varepsilon$ and for each compact subset T of J_ε there exists $\tau\in T$ such that $\alpha(V(T))=v(\tau)$.*

Proof Fix $h>0$. For any $x\in V$ denote by x_h the mean function defined by

$$x_h(t)=\frac{1}{2h}\int_{t-h}^{t+h}x(s)ds\qquad\text{for } t\in J.$$

Let $V_h=\{x_h\colon x\in V\}$. Since

$$\|x_h(t)\|\leqslant\left(\frac{1}{2h}\right)^{1/p}\|M\|_p\quad\text{for } x\in V\quad\text{and}\quad t\in J,$$

the set V_h is uniformly bounded Moreover, for any $x\in V$

$$\|x_h(t+\tau)-x_h(t)\|=\frac{1}{2h}\left\|\int_{t+\tau-h}^{t+\tau+h}x(s)ds-\int_{t-h}^{t+h}x(s)ds\right\|\leqslant\frac{1}{2h}\int_{t-h}^{t+h}\|x(s+\tau)-x(s)\|ds$$
$$\leqslant\frac{1}{2h}\int_{t-h}^{t+h}d(s,\tau)ds\leqslant\left(\frac{1}{2h}\int_0^a d^p(s,\tau)\,ds\right)^{1/p}.$$

In view of (8), this implies that the set V_h is equicontinuous. Hence the function $t\to\alpha(V_h(t))$ is continuous on J. Furthermore

$$\|x(t)-x_h(t)\| \leqslant \frac{1}{2h}\int_{t-h}^{t+h} \|x(t)-x(s)\|\,ds$$

$$= \frac{1}{2h}\int_{-h}^{h} \|x(t)-x(t+s)\|ds \leqslant \frac{1}{2h}\int_{-h}^{h} d(t, s)ds,$$

so that

(9) $$\|x(t)-x_h(t)\| \leqslant \eta(t, h) \qquad \text{for } x \in V \text{ and } t \in J,$$

where $\eta(t, h)=\left(\frac{1}{2h}\int_{-h}^{h} d^p(t, s)\,ds\right)^{1/p}$. From (8) and the equality

$$\int_0^a \eta^p(t, h)dt = \frac{1}{2h}\int_{-h}^{h}\left(\int_0^a d^p(t, s)dt\right)ds$$

it follows that

(10) $$\eta(\cdot, h) \in L^p(J, R) \quad \text{and} \quad \lim_{h\to 0+}\int_0^a \eta^p(t, h)dt=0.$$

Now, by (9),

$$|\alpha(V(t))-\alpha(V_h(t))| \leqslant 2\eta(t, h) \qquad \text{for } t \in J \text{ and small } h>0$$

As the function $t\to\alpha(V_h(t))$ is continuous on J, from this and (10) we deduce that the function $v=\alpha(V(\cdot))$ is measurable on J, $v \in L^p(J, R)$ and

(11) $$\lim_{h\to 0+} \|v-\alpha(V_h(\cdot))\|_p=0.$$

On the other hand, (9) implies that $\|x-x_h\|_p \leqslant \|\eta(\cdot, h)\|_p$ for each $x \in V$. Hence $|\alpha_p(V)-\alpha_p(V_h)| \leqslant 2\|\eta(\cdot, h)\|_p$, and therefore, by (10),

$$\alpha_p(V)=\lim_{h\to 0+} \alpha_p(V_h).$$

Thus, by (11) and Lemma 3, we obtain

$$\alpha_p(V)=\lim_{h\to 0+} \alpha_p(V) \leqslant \lim_{h\to 0+} \|\alpha(V_h(\cdot))\|_p=\|v\|_p.$$

This completes the proof of (a) and (b). Now we are going to prove (c). It follows from (10) that there exists a sequence (h_n) such that $h_n>0$, $\lim_{n\to\infty} h_n=0$ and for any $\varepsilon>0$ there exists a closed subset J_ε of J such that $\mu(J\setminus J_\varepsilon)<\varepsilon$ and $\lim_{n\to\infty} \eta(t, h_n)=0$ uniformly on J_ε. Fix $\varepsilon>0$ and put $r_n=\sup\{\eta(t, h_n): t \in J_\varepsilon\}$. Then $\lim_{n\to\infty} r_n=0$ and $\|x(t)-x_{h_n}(t)\| \leqslant r_n$ for each $x \in V$, $t \in J_\varepsilon$ and $n=1, 2, \cdots$. For a given $\gamma>0$ we choose a positive integer n such that $r_n \leqslant \gamma/4$. As the set V_{h_n} is equicontinuous,

there exists $\delta>0$ such that $\|x_{h_n}(t)-x_{h_n}(s)\|\leqslant\gamma/2$ for all $x\in V$ and $t, s\in J$ such that $|t-s|\leqslant\delta$. Hence

$$\|x(t)-x(s)\|\leqslant\|x(t)-x_{h_n}(t)\|+\|x_{h_n}(t)-x_{h_n}(s)\|+\|x_{h_n}(s)-x(s)\|\leqslant 2r_n+\gamma/2\leqslant\gamma$$

for each $x\in V$ and $t, s\in J_\varepsilon$ such that $|t-s|\leqslant\delta$. This shows that the set V is equicontinuous on J_ε. Moreover, V is uniformly bounded on J_ε, because the set V_{h_n} is uniformly bounded and

$$\|x(t)\|\leqslant r_n+\|x_{h_n}(t)\| \quad \text{for } x\in V,\ t\in J_\varepsilon \quad \text{and } n=1, 2, \cdots.$$

Applying now Ambrosetti's lemma [1; Th. 2.2], we conclude that for any compact subset T of J_ε the function $t\to v(t)=\alpha(V(t))$ is continuous on T and $\alpha(V(T))=\sup\{v(t)\colon t\in T\}=v(\tau)$ for some $\tau\in T$.

§ 3. An existence theorem.

In this section we give some sufficient conditions for the existence of L^p-solutions of the integral equation (1), where the functions f, g and K satisfy the assumptions 1°–3° from Section 1.

Theorem 1. *Assume that* $p\geqslant\min(q, 2)$, $1\leqslant l\leqslant\max(p-1, p/q)$ *and* $N\geqslant 0$. *If for any* $\varepsilon>0$ *and for any bounded subset* X *of* E *there exists a closed subset* J_ε *of* J *such that* $\mu(J\setminus J_\varepsilon)<\varepsilon$ *and*

$$\alpha(g(J_\varepsilon\times X))\leqslant N\alpha^1(X), \tag{12}$$

then the equation (1) *has a solution* $x\in L^p$.

Proof. For any positive integer n we define a function $u_n\colon J\to E$ by

$$u_n(t)=\begin{cases} f(t) & \text{for } 0\leqslant t\leqslant a_n \\ f(t)+\int_0^{t-a_n} K(t-a_n, s)g(s, u_n(s))ds & \text{for } a_n\leqslant t\leqslant a, \end{cases}$$

where $a_n=a/n$. Analogously as in Section 1 for $G(x)$, it can be shown that $u_n\in L^p$ and

$$\|u_n(t)\|\leqslant\|f(t)\|+k_n(t)\|e\|_q+bk_n(t)\Big(\int_0^t\|u_n(s)\|^p ds\Big)^{1/q} \qquad \text{for } t\in J,$$

where $k_n(t)=0$ for $0\leqslant t\leqslant a_n$ and $k_n(t)=k(t-a_n)$ for $a_n\leqslant t\leqslant a$. Hence, by Lemma 2,

$$\int_0^a\|u_n(t)\|^p dt\leqslant U^{-1}(U(2^{p-1}(\|f\|_p+\|k_n\|_p\|e\|_q+1)^p)+2^{p-1}b\|k_n\|_p^p).$$

As $\|k_n\|_p \leqslant \|k\|_p$, this proves that $\|u_n\|_p \leqslant c$, i.e. $u_n \in H$. It follows from (6) and (7) that

$$\|u_n(t)-G(u_n)(t)\| \leqslant \|f(t)\|+M(t) \qquad \text{for } 0 \leqslant t \leqslant a_n$$

and

$$\|u_n(t)-G(u_n)(t)\| \leqslant \|f(t)-f(t-a_n)\|+d(t, a_n) \qquad \text{for } a_n \leqslant t \leqslant a.$$

By (8) this implies that

$$\lim_{n\to\infty} \|u_n-G(u_n)\|_p=0. \tag{13}$$

Without loss of generality (by passing to a subsequence if necessary) we may assume that

$$\lim_{n\to\infty} (u_n(t)-G(u_n)(t))=0 \qquad \text{for almost every } t \in J. \tag{14}$$

Let $V=\{u_n: n=1, 2, \cdots\}$ and $W=G(V)$. Denote by I the identity mapping on L^p. By (13) and (14) we have

$$\alpha_p((I-G)(V))=0 \text{ and } \alpha((I-G)(V)(t))=0 \qquad \text{for almost every } t \in J.$$

As $V \subset (I-G)(V)+W$ and $W \subset -(I-G)(V)+V$, this implies that

$$\alpha_p(V)=\alpha_p(W) \quad \text{and} \quad \alpha(V(t))=\alpha(W(t)) \quad \text{for almost every } t \in J. \tag{15}$$

Let us remark that the set W satisfies the assumptions of Lemma 4 and therefore the function $t \to w(t)=\alpha(W(t))$ belongs to $L^p(J, R)$. Fix $\varepsilon>0$ and $t \in J$ such that $\int_0^t \|K(t, s)\|^r ds<\infty$. By the Luzin theorem (cf. [3], Ch. IV.5, Prop. 1), the Jegorov theorem and Lemma 4 there exists a closed subset Z of $[0, t]$ such that

$$\mu([0, t]\setminus Z)<\varepsilon; \qquad \left(\int_{[0, t]\setminus Z} \|K(t, s)\|^r ds\right)^{1/r} (\|e\|_q+bc^{p/q})<\varepsilon;$$

the functions $s \to K(t, s)$ and $s \to w(s)$ are continuous on Z; for each compact subset T of Z there exists $\tau \in T$ such that $\alpha(W(T))=w(\tau)$; $\lim_{n\to\infty} (u_n(s)-G(u_n)(s))=0$ uniformly for $s \in Z$. Ambrosetti's lemma [1; Th. 2.2] and this last fact imply that $\alpha((I-G)(V)(T))=0$ for each compact subset T of Z. Since $V(T) \subset (I-G)(V)(T)+W(T)$ and $W(T) \subset -(I-G)(V)(T)+V(T)$, we infer that

$$\alpha(V(T))=\alpha(W(T)) \qquad \text{for each compact subset } T \text{ of } Z. \tag{16}$$

Let $\int_Y K(t, s)g(s, V(s))ds$ denote the set $\left\{\int_Y K(t, s)g(s, u(s))ds: u \in V\right\}$. Then

$$W(t)\subset f(t)+\int_Z K(t,s)g(s,V(s))ds+\int_{Z'} K(t,s)g(s,V(s))ds,$$

and consequently

$$\alpha(W(t))\leqslant\alpha\Big(\int_Z K(t,s)g(s,V(s))ds\Big)+\alpha\Big(\int_{Z'} K(t,s)g(s,V(s))ds\Big), \tag{17}$$

where $Z'=[0,t]\setminus Z$.

Since Z is compact and the functions $s\to w^1(s)$ and $s\to\|K(t,s)\|$ are continuous on Z, there exists a positive number β such that

$$\|K(t,s)\|\leqslant\beta \quad\text{and}\quad Nw^1(s)\leqslant\beta \quad\text{for each } s\in Z.$$

Choose a positive number δ such that

$$\|K(t,s)-K(t,s')\|\leqslant\varepsilon/2\beta \text{ and } N|w^1(s)-w^1(s')|\leqslant\varepsilon/2\beta \tag{18}$$

for each $s, s'\in Z$ such that $|s-s'|\leqslant\delta$, and divide the interval $[0,t]$ into n parts $0=t_0<t_1<\cdots<t_n=t$ in such a way that $t_i-t_{i-1}<\delta$ for $i=1,\cdots,n$. Let $T_i=[t_{i-1},t_i]\cap Z$. Then for any i, $1\leqslant i\leqslant n$, there exist numbers $s_i, \tau_i\in T_i$ such that

$$\alpha(W(T_i))=w(\tau_i) \quad\text{and}\quad \max_{s\in T_i}\|K(t,s)\|=\|K(t,s_i)\|. \tag{19}$$

Furthermore, by (12) there exists a closed subset J_ε of J such that

$$\mu(J\setminus J_\varepsilon)<\varepsilon, \qquad \Big(\int_{[0,t]\setminus J_\varepsilon}\|K(t,s)\|^r ds\Big)^{1/r}(\|e\|_q+bc^{p/q})<\varepsilon$$

and

$$\alpha(g(J_\varepsilon\times V(T_i)))\leqslant N\alpha^1(V(T_i)) \qquad \text{for } i=1,\cdots,n.$$

Let $D=Z\cap J_\varepsilon$, $D_i=T_i\cap J_\varepsilon$ and $P_i=\{K(t,s)g(s,x)\colon s\in D_i, x\in V(T_i)\}$. Since, by the continuity of $K(t,\cdot)$ on Z, the set $\{K(t,s)\colon s\in D_i\}$ is compact, it is clear that $\alpha(P_i)\leqslant\max_{s\in D_i}\|K(t,s)\|N\alpha^1(V(T_i))$, and hence, by (16) and (19),

$$\alpha(P_i)\leqslant N\|K(t,s_i)\|w^1(\tau_i). \tag{20}$$

By the mean value theorem we have

$$\int_D K(t,s)g(s,u(s))ds=\sum_{i=1}^n\int_{D_i} K(t,s)g(s,u(s))ds\in\sum_{i=1}^n\mu(D_i)\,\overline{\operatorname{conv}}\,P_i$$

for each $u\in V$. Therefore, by (20) and the corresponding properties of α,

$$\alpha\Big(\int_D K(t,s)g(s,V(s))ds\Big)\leqslant\sum_{i=1}^n\mu(D_i)\alpha\,(\overline{\operatorname{conv}}\,P_i)$$
$$=\sum_{i=1}^n\mu(D_i)\alpha(P_i)\leqslant\sum_{i=1}^n\mu(D_i)N\|K(t,s_i)\|w^1(\tau_i).$$

Moreover, it follows from (18) that

$$N\mu(D_i)\|K(t, s_i)\|w^1(\tau_i)\leqslant N\int_{D_i}\|K(t, s)\|w^1(s)ds+\varepsilon\mu(D_i).$$

Hence

$$\begin{aligned}\alpha\left(\int_D K(t, s)g(s, V(s))ds\right)&\leqslant\sum_{i=1}^{n}N\int_{D_i}\|K(t, s)\|w^1(s)ds+\sum_{i=1}^{n}\varepsilon\mu(D_i)\\&=N\int_D\|K(t, s)\|w^1(s)ds+\varepsilon\mu(D),\end{aligned}$$

so that

$$\alpha\left(\int_D K(t, s)g(s, V(s))ds\right)\leqslant N\int_0^t\|K(t, s)\|w^1(s)ds+\varepsilon t. \tag{21}$$

On the other hand, for any $u\in V$ we have

$$\left\|\int_{Z'}K(t, s)g(s, u(s))ds\right\|\leqslant\left(\int_{Z'}\|K(t, s)\|^r ds\right)^{1/r}(\|e\|_q+bc^{p/q})<\varepsilon,$$

which implies

$$\alpha\left(\int_{Z'}K(t, s)g(s, V(s))ds\right)\leqslant 2\varepsilon.$$

Analogously

$$\alpha\left(\int_{Z\setminus D}K(t, s)g(s, V(s))ds\right)\leqslant 2\varepsilon,$$

because $Z\setminus D\subset[0, t]\setminus J_\varepsilon$. As

$$\int_Z K(t, s)g(s, V(s))ds\subset\int_D K(t, s)g(s, V(s))ds+\int_{Z\setminus D}K(t, s)g(s, V(s))ds,$$

the above inequalities, (21) and (17) imply that

$$w(t)=\alpha(W(t))\leqslant N\int_0^t\|K(t, s)\|w^1(s)ds+\varepsilon a+4\varepsilon.$$

Since this inequality holds for every $\varepsilon>0$, we get

$$w(t)\leqslant N\int_0^t\|K(t, s)\|w^1(s)ds. \tag{22}$$

Let $\rho=p/l$ and $\nu=\rho/(\rho-1)$. As $1\leqslant l\leqslant\max(p-1, p/q)$, we see that $p\geqslant\rho\geqslant\min(p/(p-1), q)$ and $\nu\leqslant\max(p, q/(p-1))=m$. Since (22) holds for almost every

$t \in J$, by the Hölder inequality we obtain

$$w(t) \leqslant h(t)\left(\int_0^t w^p(s)ds\right)^{1/\rho} \qquad \text{for almost every } t \in J,$$

where $h(t)=N\left(\int_0^t \|K(t,s)\|^\nu ds\right)^{1/\nu}$. From 3° and the inequality $h(t) \leqslant Nt^{1/\nu-1/m}$ $\left(\int_0^t \|K(t,s)\|^m ds\right)^{1/m}$ it follows that $h \in L^p(J,R)$. Putting $z(t)=\int_0^t w^p(s)ds$ we see that $z(0)=0$ and

$$z'(t) \leqslant h^p(t)z^{p/\rho}(t) \qquad \text{for almost every } t \in J.$$

As $p/\rho \geqslant 1$, the last inequality implies that $z(t)=0$ for $t \in J$. By (15) and Lemma 4 this proves that $\alpha_p(V)=\alpha_p(W)=0$. Therefore we can find a subsequence (u_{n_j}) of (u_n) which converges in L^p to a limit u. Consequently, from (13) and the continuity of G it follows that

$$\|u-G(u)\|_p=\lim_{j\to\infty}\|u_{n_j}-G(u_{n_j})\|_p=0,$$

which proves that u is a solution of (1).

§ 4. The set of solutions.

Denote by S the set of all L^p-solutions of (1), and denote by S_ε the set of all $u \in L^p$ such that $\|u-G(u)\|_p<\varepsilon$. First we shall prove the following Hukuhara-type theorem.

Theorem 2. *Under the assumptions* 1°–3° *from Section* 1, *for any* ε, $0<\varepsilon\leqslant 1$, *the set* S_ε *is nonempty and connected. Moreover,* $S_\varepsilon \subset H$.

Proof. For any positive integer n we define a mapping G_n by

$$G_n(x)(t)=\begin{cases} f(t) & \text{for } x \in L^p,\ 0 \leqslant t \leqslant a_n \\ f(t)+\int_0^{t-a_n} K(t-a_n,s)g(s,x(s))ds & \text{for } x \in L^p,\ a_n \leqslant t \leqslant a, \end{cases}$$

where $a_n=a/n$. Similarly as in Section 1, it can be shown that G_n is a continuous mapping $L^p \to L^p$ and

$$\|G_n(x)(t)\| \leqslant \|f(t)\|+k_n(t)\|e\|_q+bk_n(t)\left(\int_0^t \|x(s)\|^p ds\right)^{1/q} \tag{23}$$

for $x \in L^p$ and $t \in J$, where $k_n(t)=0$ for $0 \leqslant t \leqslant a_n$ and $k_n(t)=k(t-a_n)$ for $a_n \leqslant t \leqslant a$.

We shall prove that $I-G_n$ is a homeomorphism $L^p \to L^p$. Let $y \in L^p$. We

define a finite sequence of functions x_i, $i=1, \cdots, n$, by the formulas $x_1(t)=y(t)+f(t)$ for $0\leqslant t\leqslant a_n$

$$x_{i+1}(t)=\begin{cases}x_i(t) & \text{for } 0\leqslant t\leqslant ia_n\\ y(t)+f(t)+\int_0^{t-a_n}K(t-a_n, s)g(s, x_i(s))ds & \text{for } ia_n\leqslant t\leqslant (i+1)a_n.\end{cases}$$

We observe that

$$x_i(t)=y(t)+f(t) \qquad \text{for } 0\leqslant t\leqslant a_n$$

$$x_i(t)=y(t)+f(t)+\int_0^{t-a_n}K(t-a_n, s)g(s, x_i(s))ds \qquad \text{for } a_n\leqslant t\leqslant ia_n$$

$$x_{i+1}|[0, ia_n]=x_i \quad \text{and} \quad x_i\in L^p([0, ia_n], E).$$

Hence $x_n\in L^p$ and $x_n-G_n(x_n)=y$. Conversely, if $x-G_n(x)=y$ and $x\in L^p$, then $x|[0, ia_n]=x_i$ for $i=1, \cdots, n$, so that $x=x_n$. This shows that $I-G_n$ is a bijection $L^p\to L^p$.

Suppose that u_j, $u_0\in L^p$ and

$$\lim_{j\to\infty}\|u_j-G_n(u_j)-u_0+G_n(u_0)\|_p=0. \tag{24}$$

Since $G_n(u_j)(t)=G_n(u_0)(t)=f(t)$ for $0\leqslant t\leqslant a_n$, (24) implies that

$$\lim_{j\to\infty}\|(u_j-u_0)\chi_{[0, a_n]}\|_p=0.$$

Further

$$\begin{aligned}u_j(t)&=u_j(t)-G_n(u_j)(t)+f(t)-f(t-a_n)+G(u_j)(t-a_n)\\ &=u_j(t)-G_n(u_j)(t)+f(t)-f(t-a_n)+G(u_j\chi_{[0, a_n]})(t-a_n)\end{aligned}$$

for $a_n\leqslant t\leqslant 2a_n$ and $j=0, 1, 2, \cdots$. In view of (24) and the continuity of G, from this we deduce that $\lim_{j\to\infty}\|(u_j-u_0)\chi_{[a_n, 2a_n]}\|_p=0$, and consequently $\lim_{j\to\infty}\|(u_j-u_0)\times\chi_{[0, 2a_n]}\|_p=0$. By repeating this argument we find $\lim_{j\to\infty}\|(u_j-u_0)\chi_{[0, ia_n]}\|_p=0$ for $i=1, \cdots, n$, and hence $\lim_{j\to\infty}\|u_j-u_0\|_p=0$. This proves the continuity of $(I-G_n)^{-1}$. Furthermore, by (6) and (7) we have

$$\|G_n(x)(t)-G(x)(t)\|\leqslant\|f(t)\|+M(t) \qquad \text{if } 0\leqslant t\leqslant a_n$$

and

$$\|G_n(x)(t)-G(x)(t)\|\leqslant\|f(t)-f(t-a_n)\|+d(t, a_n) \qquad \text{if } a_n\leqslant t\leqslant a$$

for each $x\in H$, and therefore, by (8),

$$\lim_{n\to\infty}\|G_n(x)-G(x)\|_p=0 \text{ uniformly for } x\in H. \tag{25}$$

Let us remark that

(26) $\quad (I-G_n)^{-1}(y) \in H \quad$ for each n and $y \in L^p$ such that $\|y\|_p \leqslant 1$.

Indeed, if $\|y\|_p \leqslant 1$ and $x=(I-G_n)^{-1}(y)$, then from Lemma 2 and the inequalities $\|x(t)\| \leqslant \|y(t)\|+\|G_n(x)(t)\|$ and (23) it follows that $\|x\|_p \leqslant c$.

Fix ε, $0<\varepsilon \leqslant 1$. The set S_ε is nonempty, because (25) and (26) imply that $(I-G_n)^{-1}(0) \in S_\varepsilon$ for sufficiently large n. Moreover, if $x \in S_\varepsilon$, then $\|x\|_p \leqslant c$, and hence $S_\varepsilon \subset H$. This follows from Lemma 2 and the inequalities $\|x(t)\| \leqslant \|x(t)-G(x)(t)\|+\|G(x)(t)\|$, (4) and $\|x-G(x)\|_p<\varepsilon \leqslant 1$.

Assume that $x_1, x_2 \in S_\varepsilon$. Let $\eta=\max(\|x_1-G(x_1)\|_p, \|x_2-G(x_2)\|_p)$ and $\delta=\varepsilon-\eta$. By (25) we can choose a positive integer n such that

$$\|G_n(x)-G(x)\|_p<\delta/2 \qquad \text{for } x \in H.$$

For any $\lambda \in [0, 1]$ put $v_\lambda=(I-G_n)^{-1}(h(\lambda))$, where

$$h(\lambda)=\lambda(x_1-G_n(x_1))+(1-\lambda)(x_2-G_n(x_2)).$$

Since

$$\|x_i-G_n(x_i)\|_p \leqslant \|x_i-G(x_i)\|_p+\|G(x_i)-G_n(x_i)\|_p<\eta+\delta/2 \qquad (i=1,2),$$

we see that $\|h(\lambda)\|_p<\eta+\delta/2$ for $0 \leqslant \lambda \leqslant 1$, and hence, by (26), $v_\lambda \in H$ for $0 \leqslant \lambda \leqslant 1$. This implies that

$$\begin{aligned}\|v_\lambda-G(v_\lambda)\|_p &= \|G_n(v_\lambda)+h(\lambda)-G(v_\lambda)\|_p \\ &\leqslant \|h(\lambda)\|_p+\|G_n(v_\lambda)-G(v_\lambda)\|_p<\eta+\delta=\varepsilon,\end{aligned}$$

i.e. $v_\lambda \in S_\varepsilon$ for every $\lambda \in [0, 1]$. As $v_0=x_2$ and $v_1=x_1$, this shows that for any $x_1, x_2 \in S_\varepsilon$ there exists a continuous curve in S_ε connecting x_1 and x_2. This proves that S_ε is arcwise connected.

The next result is a Kneser-type theorem for (1).

Theorem 3. *Under the assumptions of Theorem* 1, *the set S is a continuum in L^p.*

Proof. For any positive integer n denote by V_n the closure of $S_{1/n}$ in L^p. By Theorem 2 the set V_n is nonempty and connected, and $V_n \subset H$. We shall show that

(27) $$\lim_{n\to\infty} \alpha_p(V_n)=0.$$

Suppose the contrary. Then there exist $\varepsilon>0$ and a sequence (v_n) such that

(28) $$v_n \in V_n \quad \text{and} \quad \|v_n-v_i\|_p \geqslant \varepsilon \quad \text{for } i, n=1, 2, \cdots, i \neq n.$$

Let $V=\{v_n: n=1, 2, \cdots\}$. Since $V\subset H$ and $\lim_{n\to\infty}\|v_n-G(v_n)\|_p=0$, by repeating the same argument as in the proof of Theorem 1, we can prove that $\alpha_p(V)=0$. Thus we can find a subsequence (v_{n_i}) of (v_n) which converges in L^p. This contradicts (28) and, consequently, proves (27). Hence, applying the well-known Kuratowski theorem (cf. [6]).

"Let X be a complete metric space, and let α be the measure of noncompactness in X. If (C_n) is a decreasing sequence of nonempty, closed and connected subsets of X such that $\lim_{n\to\infty}\alpha(C_n)=0$, then the intersection $\bigcap_{n=1}^{\infty}C_n$ is nonempty, compact and connected.", we conclude that the set $S=\bigcap_{n=1}^{\infty}V_n$ is a continuum in L^p.

§ 5. An example.

We now want to illustrate Theorem 1 by a simple example.

Consider the integral equation (1) with f, g and K satisfying the assumptions 1°–3° of Section 1. Assume that $1\leqslant l\leqslant\max(p-1, p/q)$, $p\geqslant\min(q, 2)$, $N\geqslant 0$ and $g=g_1+g_2$, where g_1, g_2 are such that

(*) for any $\varepsilon>0$ there exists a closed subset J_ε of J such that $\mu(J\setminus J_\varepsilon)<\varepsilon$ and the set $g_1(J_\varepsilon\times X)$ is relatively compact for every bounded subset X of E;

(**) $\|g_2(t, x)-g_2(t, y)\|\leqslant N\|x-y\|^l$ for each $x, y\in E$ and almost every $t\in J$.

Then the set S of all L^p-solutions of (1) is nonempty, compact and connected in L^p.

Proof. By Theorems 1 and 3, it is sufficient to show that the function g satisfies (12).

Let X be a bounded subset of E. First we shall prove that for any η, $\delta>0$ there exists a closed subset $J(\eta, \delta)$ of J such that $\mu(J\setminus J(\eta, \delta))<\delta$ and $\alpha(g_2(J(\eta, \delta)\times X))\leqslant 2N(\alpha(X)+\eta)^l+\eta$. It follows from the definition of $\alpha(X)$ that for any $\eta>0$ there exists a finite cover $(X_i)_{1\leqslant i\leqslant n}$ of X such that diam $X_i\leqslant\alpha(X)+\eta$ $(i=1, \cdots, n)$. Choose $x_i\in X_i$ $(i=1, \cdots, n)$. By the Luzin theorem for any $\delta>0$ there exists a closed subset $J(\eta, \delta)$ of J such that $\mu(J\setminus J(\eta, \delta))<\delta$, the functions $t\to g_2(t, x_i)$ $(i=1, \cdots, n)$ are uniformly continuous on $J(\eta, \delta)$ and (**) holds for each $t\in J(\eta, \delta)$. Choose $\rho>0$ such that $\|g_2(t, x_i)-g_2(s, x_i)\|\leqslant\eta$ for each $t, s\in J(\eta, \delta)$ such that $|t-s|\leqslant\rho$ and $i=1, \cdots, n$. We divide J into k parts $0=t_0<t_1<\cdots<t_k=a$ in such a way that $t_j-t_{j-1}\leqslant\rho$ for $j=1, \cdots, k$. Putting $T_j=[t_{j-1}, t_j]\cap J(\eta, \delta)$ and $H_{ij}=g_2(T_j\times X_i)$ we observe that the family $(H_{ij})_{1\leqslant i\leqslant n, 1\leqslant j\leqslant k}$ is a finite cover of $g_2(J(\eta, \delta)\times X)$. Moreover, if $g_2(t, x)$, $g_2(s, y)\in H_{ij}$, then by (**)

$$\|g_2(t, x)-g_2(s, y)\|\leqslant\|g_2(t, x)-g_2(t, x_i)\|+\|g_2(t, x_i)-g_2(s, x_i)\|+\|g_2(s, x_i)-g_2(s, y)\|$$
$$\leqslant N\|x-x_i\|^l+N\|y-x_i\|^l+\eta\leqslant 2N(\alpha(X)+\eta)^l+\eta.$$

Hence

$$\alpha(g_2(J(\eta, \delta)\times X))\leqslant 2N(\alpha(X)+\eta)^l+\eta.$$

Now for any $\varepsilon > 0$ let $J_\varepsilon = \bigcap_{n=1}^{\infty} J(1/n, \varepsilon/2^n)$. Then

$$\mu(J \setminus J_\varepsilon) \leqslant \sum_{n=1}^{\infty} \mu(J \setminus J(1/n, \varepsilon/2^n)) \leqslant \sum_{n=1}^{\infty} \varepsilon/2^n = \varepsilon$$

and

$$\alpha(g_2(J_\varepsilon \times X)) \leqslant \alpha(g_2(J(1/n, \varepsilon/2^n) \times X)) \leqslant 2N(\alpha(X) + 1/n)^1 + 1/n \quad \text{for } n = 1, 2, \cdots.$$

Therefore

$$\alpha(g_2(J_\varepsilon \times X)) \leqslant 2N\alpha^1(X).$$

From this and (*) it is clear that the function $g = g_1 + g_2$ satisfies (12).

Remark. If the spaces E and F are finite-dimensional, then the condition (*) holds for any function g satisfying the Caratheodory conditions.

Acknowledgement. The author expresses his appreciation to the referee for his suggestions to amend an earlier version of the paper.

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(Ricevita la 8-an de marto, 1983)
(Reviziita la 22-an de novembro, 1983)