Kobe University Repository : Kernel

PDF issue: 2026-11-19

A Fixed Point Theorem for Sequentially Continuous Mappings with Application to Ordinary Differential Equations

Arino, O.

Gautier, S.

Penot, J. P.

(Citation)
Funkcialaj Ekvacioj,27(3):273-279

(Issue Date)
1984-12

(Resource Type)
journal article

(Version)
Version of Record

(JaLCDOI)
https://doi.org/10.24546/0100499373

(URL)
https://hdl.handle.net/20.500.14094/0100499373

KOBE UNIVERSITY

Funkcialaj Ekvacioj, **27** (1984), 273–279

A Fixed Point Theorem for Sequentially Continuous Mappings with Application to Ordinary Differential Equations

By

O. ARINO, S. GAUTIER, J. P. PENOT

(Université de Pau et des Pays de l'Adour, France)

As everybody knows, the road to Hell is paved with good intentions... and faults. The story of existence results for ordinary differential equations is an illustration of this saying. Not only the common belief that continuous vector fields have unique integral curves issued from a point is shown to be false in any elementary course on ordinary differential equations but, moreover, existence fails in infinite dimensional Banach spaces, as Dieudonné [12] and Yorke [28] have shown (see also Cellina [7], Godunov [18]). This failure of the deterministic point of view principle is aggravated by Pianigiani [24] but is tempered by Lasota and Yorke [22], Vidossich [27], De Blasi et Myjak [10], [11] who deal with the generic point of view.

One of the first existence results for solutions of differential equations with continuous right hand side in infinite dimension due to F.E. Browder [5] was shown to be incorrect by W.J. Knight in [20], [21]. Here we point out a flaw in the proof of the results of [19] and we mend it using a fixed point theorem for sequentially weakly continuous mappings which has its own interest. We leave to the reader the task of setting another step in this inductive process!

§ 1. A fixed point theorem for weakly sequentially continuous maps

While it is not always possible to show that a given mapping between functional Banach spaces is weakly continuous, quite often its weak sequential continuity offers no problem. This follows from the fact that Lebesgue's dominated convergence theorem is valid for sequences but not for nets (we recall the following counter example taken from Bourbaki [4]: if A denotes the family of finite subsets of [0, 1], ordered by inclusion, and if x_α is the characteristic function of $\alpha \in A$, then $(x_\alpha) \to 1$ but $\int x_\alpha = 0$ for each $\alpha \in A$). An erroneous application of the dominated convergence theorem in [20] motivated the present paper.

As long as the Banach space involved is separable and reflexive, and this is usually the case in analysis, weak sequential continuity suffices (for recent nice applications of fixed points technics under these assumptions, see for instance G. Aubert - R. Tahraoui [1], J. Mossino [23], J. P. Zolesio [29]). The relationship be-

tween weak and sequential weak continuity for mappings between Banach spaces is studied in detail in J. M. Ball [2].

The following fixed point theorem for weakly sequentially continuous mappings does not require any restrictive assumption on the space as reflexivity or separability. However, metrizability is needed for the proof and we do not know if one can get rid of this restriction which is not important for our purpose.

Theorem 1. *Let E be a metrizable locally convex topological vector space and let C be a weakly compact convex subset of E. Then any weakly sequentially continuous map f: $C \to C$ has a fixed point.*

Proof. It suffices to prove that f is weakly continuous, so that the Schauder-Tychonoff fixed point theorem applies. Now for each weakly closed subset F of E, $f^{-1}(F)$ is sequentially closed in C, hence weakly compact by the Eberlein-Šmulian theorem (Edwards [14], Theorem 8.12.4), and $f^{-1}(F)$ is weakly closed. Hence f is weakly continuous. □

Remark 1. Using the Ky Fan's fixed point theorem [3] instead of the Schauder-Tychonoff theorem one sees that the same argument is valid if f is replaced by a closed convex valued multifunction F from C into C which is sequentially upper semicontinuous with respect to the weak topology. This means that if (x_n) converges weakly to x and if G is a weakly open subset of E containing $F(x)$, then $F(x_n)$ is contained in G for n large enough. Equivalently, for any weakly closed subset B of C, $F^{-1}(B)$ is sequentially closed for the weak topology on C.

§ 2. Application to ordinary differential equations

We apply the preceding theorem to a Caratheodory type existence result for ordinary differential equations. Existence results of Peano type were given by Chow and Schur [8], [9], Fitzgibbon [15], Gautier-Penot [16], [17]. For Caratheodory type results in l.c.s. with a separability condition see Castaing-Valadier [6]; for a global existence result see Valadier [26]. For the reader's convenience we follow the terminology and the notations of [20] but we introduce a new concept of solution.

Let us recall that a function g from a measurable space $(T, \mathscr{C})$ to a Banach space X is said to be scalarly measurable if for every x^* in the topological dual space X^* the function $x^* \circ g$ is measurable on $(T, \mathscr{C})$. By a result of Edgar [13], this is equivalent to the measurability of g as a map from $(T, \mathscr{C})$ to $(X, \mathscr{A})$ where $\mathscr{A}$ is the Baire σ-algebra of X weak, i.e. the smallest σ-algebra with respect to which all weakly continuous real-valued functions are measurable. Here, and in all the following, we take for $(T, \mathscr{C})$ an interval of $\boldsymbol{R}$ endowed with the Lebesgue σ-algebra $\mathscr{C}$ and the Lebesgue measure $\lambda = ds$. A function g: $T \to X$ is said to be (strongly)

measurable if g is the a.e. pointwise limit of a sequence of measurable simple mappings; equivalently, g is (strongly) measurable iff g is scalarly measurable and there is a separable subspace X_0 of X such that $g^{-1}(X\backslash X_0)$ is negligible.

Two scalarly-measurable functions $g, h: T\to X$ are said to be *weakly equivalent* iff for all $x^* \in X^*$, $x^* \circ g = x^* \circ h$ a.e..

Let B be a closed ball with center x_0 and radius $r>0$ in the Banach space X, let $I=[0, b]$ and let $f: I\times B\to X$.

A *solution* to the differential equation

$$\dot{x}(t)=f(t, x(t)) \qquad x(0)=x_0 \tag{e}$$

is an absolutely continuous function $x: J\to B$, where J is a subinterval of $J=[0, c]$ of I with $x(t)=x_0+\int_0^t \dot{x}(s)ds$ where $\dot{x}$ is a (strongly) integrable mapping weakly equivalent to $f(\cdot, x(\cdot))$. Then $\dot{x}$ is the a.e. derivative of x and x is a *pseudo-solution* in the sense of [20], i.e. x is an absolutely continuous function and for every x^* in X^* there is a negligible subset N depending on x^* such that $x^* \circ x$ has $x^*(f(t, x(t)))$ as a derivative at each $t \in J\backslash N$. A *strong solution* to (e) is an absolutely continuous mapping x with a strong derivative $\dot{x}$ satisfying (e) a.e. on J. Equivalently, x is a strong solution iff $f(\cdot, x(\cdot))$ is (strongly) integrable and

$$x(t)=x_0+\int_0^t f(s, x(s))ds \quad t \in J \tag{ẽ}$$

whereas x is a pseudo-solution iff $f(\cdot, x(\cdot))$ is scalarly integrable and (ẽ) is valid as a Pettis integral (or scalar integral).

Theorem 2. *Let X be a Banach space, $I=[0, b]$, let B be the closed ball with center $x_0 \in X$ and radius r and let $f: I\times B\to X$. If f satisfies the following assumptions, then equation* (e) *has a solution on a subinterval $J=[0, c]$ of I, $c>0$.*

a) *For each $t \in I\backslash N$, where N is a negligible subset of I, the map $f_t=f(t, \cdot)$ is continuous with respect to the weak topology on B and X.*

b) *For each $x \in B$, $f(\cdot, x)$ is scalarly measurable on I.*

c) *$f((I\backslash N)\times B)$ is weaky relatively compact.*

In all the following we modify f on $N\times B$ (taking a constant map for instance) so that f is weakly continuous in x and scalarly measurable in t with weakly relatively compact range. We shall make use of the following lemmas.

Lemma 1. *Under the assumptions of theorem* 2, *for each continuous function $x: J\to B$, $J=[0, c]$, the function $g: J\to X$ given by $g(t)=f(t, x(t))$ is scalarly measurable.*

Proof. As x is continuous, x is the uniform limit of a sequence (x_n) of step

functions from J to B. It follows easily from assumption b) that for each n, $g_n(t)=f(t, x_n(t))$ defines a scalarly measurable function. As g is (a.e.) the weak limit of (g_n), g is scalarly measurable. □

Lemma 2. *Under the assumptions of theorem* 2, *every pseudo-solution to* (e) *is a solution.*

Proof. Let $x: J\to B$ be a pseudo-solution. Then $g: I\to X$ given by $g(t)=f(t, x(t))$ is scalarly measurable and its range is contained in a weakly compact subset of X. By a result of Philipps (see [2] or [13] for a proof) g is weakly equivalent to a strongly measurable mapping h. As $\|h\|$ is bounded, h is strongly integrable. If

$$y(t)=x_0+\int_0^t h(s)ds,$$

then for each $x^*\in X^*$, $x^*\circ x=x^*\circ y$, thus $y=x$, and x is a solution. □

Remark 2. a) If X^* contains a countable subset (x_n^*) which separates the points of X, in particular if X is separable, then any solution (hence any pseudo-solution) is a strong solution as then $\dot{x}(t)$ and $f(t, x(t))$ coincide outside the union of the negligible subsets N_n where $\langle x_n^*, \dot{x}(\cdot)\rangle$ and $\langle x_n^*, f(\cdot, x(\cdot))\rangle$ differ. This does not hold in general, even when X has the Radon-Nikodym property as W. J. Knight claims ([20] Corollary 6).

For instance, let $I=[0, 1]$ and let X be the Hilbert space $\ell_2(I)$. Let us define a scalarly measurable map $g: I\to X$ by $g(t)(s)=0$ if $s\neq t$, $g(t)(t)=1$. Then g is weakly equivalent to the zero map. The zero map $z: I\to X$ is a solution of the differential equation (e) with $x_0=0$ and $f(t, x)=g(t)$ for each $(t, x)\in I\times X$. But the derivative $\dot{z}(t)\neq f(t, z(t))$ for every $t\in I$ and z is not a strong solution.

b) However, if f is a continuous map from $I\times B$ strong to X weak (in particular if f is jointly continuous for the weak topology) then every solution x is a strong solution as then x is weakly differentiable everywhere in $(0, c)$ with weak derivative $f(\cdot, x(\cdot))$ (see [17] for details).

Proof of Theorem 2. Let K be the closed convex hull of the weak closure of the range $F=f(I\times B)$ of f. By the Krein theorem K is weakly compact, hence bounded in norm, say by M. Let $c=r/M$ and let $J=[0, c]$. For each t in J let $K_t=x_0+tK\subset B$. Let D be a countable dense subset of J and let $F=X^D$. Endowed with the product topology, F is a metrizable locally convex vector space. We consider the space $E=C(J, X)$ of continuous mappings from J into X as a vector subspace of F. Hence its weak topology is the topology induced by the weak topology $\sigma(F, F^*)$ of F. But $\sigma(F, F^*)$ is the product topology of the weak topology $\sigma(X, X^*)$ of X. Thus the subset $P=\prod_{t\in D} K_t$ of F is compact in $\sigma(F, F^*)$ by the

Tychonoff theorem. Let Q be the subset of E consisting of mappings $x: J \to X$ satisfying $\|x(t)-x(s)\| \leq M|t-s|$ for each $(t, s) \in J^2$. It is a closed convex subset of E and F. Thus Q is weakly closed in F and $C=P \cap Q$ is weakly compact in F, hence in E.

We define a mapping T from C into C by

$$(Tx)(t)=x_0+\int_0^t f(s, x(s))ds,$$

where the integral is a Pettis integral (or equivalently, using the proof of lemmas 1 and 2) the Bochner integral of an integrable function weakly equivalent to $f(\cdot, x(\cdot))$. It is easy to see that a sequence (x_n) of C converges weakly to $x \in C$ iff $(x_n(t)) \to x(t)$ in $(X, \sigma(X, X^*))$ for each $t \in J$ (and not only $t \in D$). Using the Lebesgue dominated convergence theorem we deduce from this that T is weakly continuous. Theorem 1 implies that T has a fixed point, so that (e) has a solution. □

An existence result of Peano's type follows immediately from theorem 2 and remark 2.

Corollary 1. ([16]). *If $f: I \times B \to X$ is jointly continuous for the weak topology on B and X, and if $f(I \times B)$ is weakly relatively compact then* (e) *has a strong solution.*

In fact, in [8], [9] and [16] X is supposed to be reflexive so that $f(I \times B)$ is weakly compact as soon as f is jointly weakly continuous.

Let us remark that we can treat differential inclusions by the same method.

Theorem 3. *Let B be a closed ball in a Banach space X, let x_0 be an interior point of B, $I=[0, b]$ and let $F: I \times B \to X$ be a multivalued mapping satisfying the following assumptions*:

a) *for each $t \in I \backslash N$, where N is a negligible subset of I, the multifunction $F_t = F(t, \cdot)$ is upper-semi-continuous for the weak topology on B and X with closed convex values,*

b) *for each continuous map $y: I \to B$, there exists a scalarly measurable function $z: I \to X$ with $z(t) \in F(t, y(t))$ a.e. on I,*

c) *$F((I \backslash N) \times B)$ is weakly relatively compact.*

Then the following differential inclusion has a solution on a subinterval $J=[0, c]$ of I:

$$\dot{x}(t) \in F(t, x(t)). \tag{i}$$

Here x is a solution to (i) *if there exists a (strongly) integrable map $\dot{x}: J \to X$ such that for all $x^* \in X^*$*

$$x(t)=x_0+\int_0^t \dot{x}(s)ds, \qquad \langle x^*, \dot{x}(t)\rangle \leq \sup \langle x^*, F(t, x(t))\rangle$$

a.e. in J.

The proof is an easy modification of the proof of theorem 2. It consists in finding a fixed point to the multivalued mapping T: $P \cap Q \to P \cap Q$ given by

$$T(y)(t) = \left\{ x_0 + \int_0^t z(s)ds \mid z \in M_\sigma(I, X),\ z(\cdot) \in F(\cdot, y(\cdot)) \text{ a.e.} \right\}$$

where $M_\sigma(I, X)$ denotes the set of scalarly measurable functions from J to X. Then one applies the Kakutani-Ky Fan theorem.

Analytic conditions can be given, especially when X is separable, ensuring condition b) via the theory of measurable selections. A problem arising from plasmas physics which is solved by a direct method in J. P. Zolesio [29], chap. 7, can be treated as a direct application of theorem 3.

References

[1] Aubert, G. et Tahraoui, R., Théorèmes d'existence pour des problèmes du calcul des variations du type: inf $\int_0^L f(x, u'(x))dx$ et inf $\int_0^L f(x, u(x), u'(x))dx$, J. Differential Equations, **33** (1979), 1–15.

[2] Ball, J. M., Properties of mappings and semigroups, Proc. Royal Soc. Edinburgh Sect A, **72** (4) (1973/74), 275–280.

[3] Berge, C., *Espaces topologiques—fonctions multivoques,* Dunod, Paris, 1966.

[4] Bourbaki, N., Integration chap. IV §.1 n°3 rem. 1, Hermann, Paris, 1968, 2ème édition.

[5] Browder, F. E., Nonlinear equations of evolution, Ann. of Math., **80** (1964), 485–523, MR 30 #4167.

[6] Castaing, Ch. et Valadier, M., Equations différentielles multivoques dans les espaces vectoriels localement convexes, R.I.R.O., **16** (1969), 3–16.

[7] Cellina, A., On the non existence of solutions of differential equations in non reflexive spaces, Bull. Amer. Math. Soc., **78** (6) (1972), 1069–1072.

[8] Chow, S. et Schuur J. D., An existence theorem for ordinary differential equations in Banach spaces, Bull. Amer. Math. Soc., **77** (1971), 1018–1020, MR 44 #4334.

[9] ——, Fundamental theory of contingent differential equations in Banach spaces, Trans. Amer. Math. Soc., **179** (1973), 133–144.

[10] De Blasi, F. S. et Myjak, J., La convergence des approximations successives pour les équations différentielles dans les espaces de Banach est une propriété générique, C. R. Acad. Sci. Paris, **286** (1) (1978), 29–32.

[11] ——, Generic properties of differential equations in Banach spaces, Bull. Acad. Polon. Sci. Sér. Math. Astronom. Phys., **26** (14) (1978), 287–292.

[12] Dieudonné, J., Deux exemples singuliers d'équations différentielles, Acta. Sci. Math. (Szeged), **12** (1950), 38–40.

[13] Edgar, G. A., Measurability in a Banach space, Indiana Univ. Math. J., **26** (4) (1977), 663–678.

[14] Edwards, R. E., *Functional analysis, theory and applications,* Holt, Rinehart and Winston, New York, 1965.

[15] Fitzgibbon, W. E., Weakly continuous accretive operators, Bull. Amer. Math. Soc., **79** (1973), 473–474.

[16] Gautier, S. et Penot, J. P., Existence des courbes intégrales pour les champs de vecteurs faiblement continus, Publications Mathématiques de pau, 1973.

[17] ——, Fermés invariants par un système dynamique, C. R. Acad. Sci. Paris, **276** (1973), 1457–1460.
[18] Godunov, A. N., Peano's theorem in Banach space (in Russian), Funktional Anal. i Prilożen, **9** (1975), 59–60.
[19] Knight, W. J., Existence of solutions of differential equations in Banach space, Bull. Amer. Math. Soc., **80** (I) (1974), 148–149.
[20] ——, Solutions of differential equations in Banach spaces, Duke Math. J., **41** (2) (1974), 437–442.
[21] ——, A counter example to a theorem on differential equations in Hilbert spaces, Proc. Amer. Math. Soc., **51** (2) (1975), 378–380.
[22] Lasota, A. and Yorke, J. A., The generic property of existence of solutions of differential equations in Banach spaces, J. Differential Equations, **13** (1973), 1–12.
[23] Mossino, J., Etude de quelques problèmes nonlinéaires d'un type nouveau apparaissant en physique des plasmas, Ph. D. Thesis, Univ. Paris XI, Orsay (1977) and J. Differential Equations, **34** (I) (1979), 114–138.
[24] Pianigiani, G., A density result for differential equations in Banach spaces, Bull. Acad. Polon. Sci., **26** (9–10) (1978), 791–793.
[25] Schaefer, H. H., *Topological vector spaces,* Springer Verlag, New York, 1971.
[26] Valadier, M., Complément sur les équations différentielles multivoques, Sem. Analyse Convexe, Montpellier, 1970.
[27] Vidossich, G., Most of the successive approximations do converge, J. Math. Anal. Appl., **45** (1974), 127–131.
[28] Yorke, J. A., A continuous differential equation in Hilbert Space without existence, Funkcial. Ekvac., **13** (1970), 19–21.
[29] Zolesio, J. P., Identification de domaines par déformation, Thesis, Univ. of Nice, 1979.

nuna adreso:
Faculté des Sciences Exactes
Département de Mathématiques
Université de Pau et des Pays
de l'Adour
64000 PAU
France

(Ricevita la 12-an de aprilo, 1983)
(Reviziita la 25-an de junio, 1984)