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(Citation)

Funkcialaj Ekvacioj, 28(1):83–92

(Issue Date)

1985-04

(Resource Type)

journal article

(Version)

Version of Record

(JaLCD0I)

<https://doi.org/10.24546/0100499396>

(URL)

<https://hdl.handle.net/20.500.14094/0100499396>



Morrey-Sobolev Type Imbedding Theorems in the $\mathcal{L}^{(p,\lambda)}$ Spaces of Strong Type

By

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Introduction

The theory of the $\mathcal{L}^{(p,\lambda)}$ spaces was first studied by C.B. Morrey and later was established by various authors including S. Campanato, F. John-L. Nirenberg, G.N. Meyers and G. Stampacchia (see [13], [14] and [16] for bibliography).

On the other hand, G. Stampacchia introduced the $\mathcal{L}^{(p,\lambda)}$ spaces of strong type, the structures of which are more general and complicated than those of the $\mathcal{L}^{(p,\lambda)}$ spaces in the usual sense, and the greater part of the spaces has been characterized by him, L.C. Piccinini, Y. Furusho, the author and the others (see [4], [8]–[10], [12], [14] and [15]). Moreover, he applied the theory to prove Morrey-Sobolev type imbedding theorems [15], which have played important roles in the study of partial differential equations (see [2], [3] and [7]). Afterwards, the author has generalized the Stampacchia's theorems to the $\mathcal{L}^{(p,\lambda)}$ spaces of strong type r ($=\mathcal{L}_r^{(p,\lambda)}$, where $1 < p < \infty$, $0 < \lambda < n$ and $1 < r < \infty$) in [9] and [10] except the case: $1 < p < \lambda < n$ and $(n/\lambda)\tilde{p} < r$ ($1/\tilde{p} = 1/p - 1/\lambda$). These theorems also have proved to be particularly useful in the study of partial differential equations (see [11]).

In this paper, the author will prove the theorem for this exceptional case which is closely analogous to those of Stampacchia and the author.

This article is organized as follows:

In § 1, the relevant definitions and main theorem are stated.

Proof of the theorem is given in § 2. Main tools for the proof are theorems and techniques due to Stampacchia [15] and the author [8], [10].

In § 3, we state in the unified form the theorems obtained in [9], [10] and this article. Finally, additional remarks are given.

§ 1. Preliminaries

We always consider subfamilies of integrable functions defined on a fixed bounded cube Q_0 in the n -Euclidean space E^n and call subcube Q of Q_0 if it is a

^{*)} This research is partially supported by Grant-in-Aid for Research of Sciences under the Ministry of Education of Japan Government.

subcube of Q_0 with parallel sides to those of Q_0 . In addition, we denote the measure of Q by $|Q|$ and the mean value of function $u(x)=u(x_1, \dots, x_n)$ over Q by $u_Q: u_Q=|Q|^{-1} \int_Q u(x)dx$.

Definition 1. A function u is said to belong to the space $\mathcal{L}_r^{(p,\lambda)} = \mathcal{L}_r^{(p,\lambda)}(Q_0)$ (the $\mathcal{L}^{(p,\lambda)}$ space of strong type r), where $1 \leq p < \infty$, $-\infty < \lambda < \infty$ and $1 \leq r < \infty$, if for any system of subcubes Q_j of finite number $S = \{Q_j: \bigcup_j Q_j \subset Q_0\}$, no two of which have interior point in common, the relation

$$(1.1) \quad [u]_{\mathcal{L}_r^{(p,\lambda)}(Q_j)} = \sup_{Q \subset Q_j} \left\{ |Q|^{\lambda/n-1} \int_Q |u(x) - u_Q|^p dx \right\}^{1/p} = K(Q_j) < \infty$$

holds and, furthermore, there exists a constant $L=L(u)$ such that

$$(1.2) \quad \sup_{\{Q_j\}=\bar{S} \in \bar{S}} \left[\sum_j |K(Q_j)|^r \right]^{1/r} = L$$

where $\bar{S}$ denotes the family of all systems of subcubes considered above. We denote L by $[u]_{\mathcal{L}_r^{(p,\lambda)}(Q_0)}$, and define a norm of the space $\mathcal{L}_r^{(p,\lambda)}(Q_0)$ by $[u]_{\mathcal{L}_r^{(p,\lambda)}(Q_0)} + \|u\|_{L^p(Q_0)}$. Endowed with this norm, the space $\mathcal{L}_r^{(p,\lambda)}(Q_0)$ is a Banach space.

Here, we make the following important:

Remark 1. For an arbitrary function $u \in \mathcal{L}_r^{(p,\lambda)}(Q_0)$ we can extend u to a function belonging to the space $\mathcal{L}_r^{(p,\lambda)}(E^n)$ with a support contained in the concentric and parallel cube to Q_0 with side length $(\sqrt{n}+2)|Q_0|^{1/n}$ by the method of symmetric transformation and truncation (see [10] for the detail). Furthermore, we have

$$(1.3) \quad [u]_{\mathcal{L}_r^{(p,\lambda)}(E^n)} \leq C \|u\|_{\mathcal{L}_r^{(p,\lambda)}(Q_0)}$$

Therefore, we can identify the function $u \in \mathcal{L}_r^{(p,\lambda)}(Q_0)$ to the redefined function $u \in \mathcal{L}_r^{(p,\lambda)}(E^n)$ and from now on we always consider the functions belonging to the space $\mathcal{L}_r^{(p,\lambda)}(E^n)$, unless otherwise stated.

Moreover, we need in § 3 the following:

Definition 2. For an arbitrary constant $1 < p < \infty$, the space $\mathcal{L}_r^{(p,0)}$ is isomorphic to the space $\mathcal{L}_r^{(1,0)}$ with their corresponding norms by John-Nirenberg's lemma [5] and therefore we call the space $\mathcal{L}_r^{(1,0)}$ the John-Nirenberg space of strong type r .

Definition 3. For an arbitrary constant $0 < \alpha < 1$ the seminorm $[u]_{\mathcal{L}_r^{(p,-p\alpha)}}$ is equivalent to the following seminorm:

$$(1.4) \quad [u]_{\mathcal{L}_r^{(p,-p\alpha)}}^\alpha = \sup_{\{Q_j\} \in \bar{S}} \left\{ \sum_j [u]_{C^\alpha(Q_j)}^r \right\}^{1/r}$$

¹⁾ Throughout this paper, C denotes always a constant independent of functions.

by Campanato-Meyers' theorem [1], [6] and therefore we call the space $\mathcal{H}_r^\alpha$ the space of Hölder continuous functions with exponent α of strong type r .

Now, our main result reads as follows:

Theorem. *Let u be a function such that the derivatives u_x belong to the space $\mathcal{L}_r^{(p,\lambda)}(Q_0)$, where $1 < p < \lambda < n$ and $(n/\lambda)\tilde{p} \leq r$ ($1/\tilde{p} = 1/p - 1/\lambda$).²⁾ Then, in fact after the extension as in Remark 1, u belongs to the space $\mathcal{L}_r^{(\tilde{q},\mu)}(E^n)$ and the following estimate holds for u :*

$$(1.5) \quad [u]_{\mathcal{L}_r^{(\tilde{q},\mu)}(E^n)} \leq C \{ \|u_x\|_{\mathcal{L}_r^{(p,\lambda)}(Q_0)} + \|u\|_{L^p(Q_0)} \}$$

where q is an arbitrary constant satisfying $1 < q < p$, $1/\tilde{q} = 1/q - 1/\mu$ and $\mu/q = \lambda/p$.³⁾

§ 2. Proof of Theorem

For the proof we need the following:

Lemma 1 ([15] Stampacchia). *Let u be a function such that the derivatives u_x belong to the space $\mathcal{L}^{(p,\lambda)}(Q_0)$ ($= \mathcal{L}_\infty^{(p,\lambda)}(Q_0)$), where $1 < p < \lambda < n$.*

Then u belongs to the space $\mathcal{L}^{(\tilde{q},\mu)}(Q_0)$ and the following estimate holds for u :

$$(2.1) \quad [u]_{\mathcal{L}^{(\tilde{q},\mu)}(Q_0)} \leq C \|u_x\|_{\mathcal{L}^{(p,\lambda)}(Q_0)}$$

where q and μ are as in the Theorem.

Next, we make the following:

Definition 4. We denote the subfamily of $\bar{S}$ by $\bar{S}_N$ all of the systems of disjoint subcubes Q_j satisfying:

$$(2.2) \quad \forall \{Q_j\} \in \bar{S}_N \quad N^{-1}|Q_j| \leq |Q_i| \leq N|Q_j| \quad \forall i, \forall j$$

where N is a constant greater than unity.

Then, we prepare the following:

Lemma 2. *There exists a constant N sufficiently large such that the following inequality holds:*

$$(2.3) \quad [u]_{\mathcal{L}_r^{(\tilde{q},\mu)}(Q_0)} \leq C [u]_{\mathcal{L}_{r,N}^{(\tilde{q},\mu)}(Q_0)}$$

where q , u and r are as in the Theorem and

$$(2.4) \quad [u]_{\mathcal{L}_{r,N}^{(\tilde{q},\mu)}(Q_0)} = \sup_{\{Q_j\} \in \bar{S}_N} \left\{ \sum_j \left(|Q_j|^{\mu/n-1} \int_{Q_j} |u(x) - u_{Q_j}|^{\tilde{q}} dx \right)^{r/\tilde{q}} \right\}^{1/r}$$

^{2),3)} From now on, $\tilde{p}$ and $\tilde{q}$ always denote the constants $(1/p - 1/\lambda)^{-1}$ and $(1/q - 1/\mu)^{-1}$ respectively.

Now, let $\{Q_j^m\} \in \bar{S}$ be a sequence of systems of subcubes of finite number such that the following inequality holds:

Next, if m tends to infinity, then we may assume without loss of generality that the sequence of the systems $\{Q_j^m\}$ tends to a system of countable subcubes $\{Q_{ij}\}$ having points x_i of finite number as condensation points. Here, we take an arbitrary large integer k and classify the system $\{Q_{ij}\}_{1 \leq j \leq k}$ into subsystems $\{Q_{ij}\}_{1 \leq j \leq j_1}, \dots, \{Q_{ij}\}_{j_{l-1} \leq j \leq k}$ belonging to $\bar{S}_N$ respectively. On the other hand, by the assumption on $\bar{S}_N$ we have for an arbitrary positive number ε ,

Next, we verify (2.5) for an arbitrary j_0 assuming that this is true for every $j < j_0$. Now, we have

$$\begin{aligned} & \left\{ |Q_{j_0}|^{\mu/n-1} \int_{Q_{j_0}} |u(x) - u_{Q_{j_0}}|^{\bar{q}} dx \right\}^{1/\bar{q}} \\ &= \left\{ |Q_{j_0}|^{\mu/n-1} \int_{Q_{j_0}} |v(x) - v_{Q_{j_0}} + \tau_h u(x) - (\tau_h u)_{Q_{j_0}}|^{\bar{q}} dx \right\}^{1/\bar{q}} \end{aligned}$$

applying Minkowski's inequality

$$\leq \left\{ |Q_{j_0}|^{\mu/n-1} \int_{Q_{j_0}} |v(x) - v_{Q_{j_0}}|^{\bar{q}} dx \right\}^{1/\bar{q}} + \left\{ |Q_{j_0}|^{\mu/n-1} \int_{Q_{j_0}} |\tau_h u(x) - (\tau_h u)_{Q_{j_0}}|^{\bar{q}} dx \right\}^{1/\bar{q}}$$

Hence, the conclusion is immediate, if $\tau_{-h}Q_{j_0}$ is a subcube of some cube Q_j ($j < j_0$). If there is no such a cube Q_j ($j < j_0$), we can select a subsystem of $\{Q_j\}_{j < j_0}$ such that $\tau_{-h}Q_{j_0} \subset \bigcup_j Q_j$ holds. Furthermore, we may assume by Lemma 2 that for any system of disjoint subcubes Q_j , there exists a constant N sufficiently large satisfying (2.2) and therefore the number of the subcubes Q_j appearing in the right hand side of the above inclusion relation depends only on n and N .

Hence, the proof of this lemma is complete.

Lemma 4 ([8] Ono). *For an arbitrary cube Q and function w in $L^p(Q)$ ($p \geq 1$), we have*

$$(2.6) \quad \left\{ \int_Q dx \left(\int_Q |w(x) - w(y)| dy \right)^p \right\}^{1/p} \leq 2 \left\{ \int_Q dx \left| \int_Q (w(x) - w(y)) dy \right|^p \right\}^{1/p}$$

Now, we are going to give the

Proof of Theorem. By Lemmas 1, 2 and 3, it is sufficient for us to prove that for any system $\{Q_j\} \in \bar{S}_N$, there exists a constant C such that the following estimate holds:

$$(2.7) \quad \left\{ \sum_j \left(|Q_j|^{\mu/n-1} \int_{Q_j} |v(x) - v_{Q_j}|^{\bar{q}} dx \right)^{r/\bar{q}} + \|v\|_{L^{\bar{q}}(Q_j)}^r \right\}^{1/r} \leq C \{ \|u_x\|_{\mathcal{X}_r^{(p, \lambda)}(Q_0)} + \|u\|_{L^p(Q_0)} \}$$

As for the first term, we have by Lemma 1

$$\begin{aligned} & \left\{ |Q_j|^{\mu/n-1} \int_{Q_j} |v(x) - v_{Q_j}|^{\bar{q}} dx \right\}^{1/\bar{q}} \leq C \sup_{Q \subset Q_j} \left(|Q|^{\lambda/n-1} \int_Q |v_x|^p dx \right)^{1/p} \\ & \leq C \left[\sup_{Q \subset Q_j} \left\{ |Q|^{\lambda/n-1} \int_Q |(u_x - \tau_h u_x)(x)|^p dx \right\}^{1/p} + \|u_x\|_{L^p(Q_j)} + \|u\|_{L^p(Q_j)} \right] \end{aligned}$$

For the first term of the right hand side, it suffices to estimate the following:

$$\left\{ |Q_j|^{\lambda/n-1} \int_{Q_j} |(u_x - \tau_h u_x)(x)|^p dx \right\}^{1/p}$$

as we may assume that an arbitrary system of subcubes $\{Q'_j: Q'_j \subset Q_j\}$ belongs to the family $\bar{S}_N$ for an appropriate N . Now,

$$\begin{aligned} & \left\{ |Q_j|^{\lambda/n-1} \int_{Q_j} |(u_x - \tau_h u_x)(x)|^p dx \right\}^{1/p} \\ &= \left\{ |Q_j|^{\lambda/n-1} \int_{Q_j} |(u_x(x) - (u_x)_{Q_j}) - (\tau_h u_x(x) - (\tau_h u_x)_{Q_j}) - (u_x - \tau_h u_x)_{Q_j}|^p dx \right\}^{1/p} \end{aligned}$$

applying Minkowski's inequality, we have

$$\begin{aligned} & \leq \left\{ |Q_j|^{\lambda/n-1} \int_{Q_j} |u_x(x) - (u_x)_{Q_j}|^p dx \right\}^{1/p} \\ & \quad + \left\{ |Q_j|^{\lambda/n-1} \int_{Q_j} |\tau_h u_x(x) - (\tau_h u_x)_{Q_j}|^p dx \right\}^{1/p} + |Q_j|^{\lambda/n p} |(u_x - \tau_h u_x)_{Q_j}| \end{aligned}$$

Obviously, the first and the second terms are

$$\left\{ |Q_j|^{\lambda/n-1} \int_{Q_j} |u_x(x) - (u_x)_{Q_j}|^p dx \right\}^{1/p} \leq [u_x]_{\mathcal{L}^{(p, \lambda)}(Q_j)}$$

and

$$\left\{ |Q_j|^{\lambda/n-1} \int_{Q_j} |\tau_h u_x(x) - (\tau_h u_x)_{Q_j}|^p dx \right\}^{1/p} \leq [u_x]_{\mathcal{L}^{(p, \lambda)}(\tau_{-h} Q_j)}$$

Finally, the third term is

$$\begin{aligned} & |Q_j|^{\lambda/n p} |(u_x - \tau_h u_x)_{Q_j}| \\ &= |Q_j|^{\lambda/n p} |Q_j|^{-1} \left| \int_{Q_j} (u_x(x) - u_x(x-h)) dx \right| \\ &= |Q_j|^{\lambda/n p - 1} \left| \int_{Q_j} u_x(x) dx - \int_{\tau_{-h} Q_j} u_y(y) dy \right| \\ &= |Q_j|^{\lambda/n p - 2} \left| \int_{Q_j} dx \int_{\tau_{-h} Q_j} (u_x(x) - u_y(y)) dy \right| \\ &\leq |Q_j|^{\lambda/n p - 2} \int_{Q_j} dx \int_{\tau_{-h} Q_j} |u_x(x) - u_y(y)| dy \end{aligned}$$

Here, we take a subcube Q_i containing the cubes Q_j and $\tau_{-h} Q_j$ simultaneously and having sides with Q_j and $\tau_{-h} Q_j$ in common. Then, the right hand side of the above inequality is

$$\begin{aligned} & \leq |Q_j|^{\lambda/n p - 2} \int_{Q_i} dx \int_{Q_i} |u_x(x) - u_y(y)| dy \\ & \leq C |Q_i|^{\lambda/n p - 2} \int_{Q_i} dx \int_{Q_i} |u_x(x) - u_y(y)| dy \end{aligned}$$

and we have by Lemma 4

$$\begin{aligned} &\leq C |Q_i|^{\lambda/n p - 2} \int_{Q_i} dx \left| \int_{Q_i} (u_x(x) - u_y(y)) dy \right| \\ &= C |Q_i|^{\lambda/n p - 1} \int_{Q_i} |u_x(x) - (u_x)_{Q_i}| dx \end{aligned}$$

applying Hölder's inequality

$$\begin{aligned} &\leq C |Q_i|^{\lambda/n p - 1} \left(\int_{Q_i} |u_x(x) - (u_x)_{Q_i}|^p dx \right)^{1/p} |Q_i|^{1-1/p} \\ &\leq C [u_x]_{\mathcal{L}^{(p, \lambda)}(Q_i)} \end{aligned}$$

Hence, we have

$$\begin{aligned} &\sum_j \left(|Q_j|^{\lambda/n - 1} \int_{Q_j} |(u_x - \tau_h u_x)(x)|^p dx \right)^{r/p} \\ &\leq C \sum_j [u_x]_{\mathcal{L}^{(p, \lambda)}(Q_j)}^r \\ &\leq C [u_x]_{\mathcal{L}^{(p, \lambda)}(Q_0)}^r \end{aligned}$$

Next, for the second term of the left hand side of (2.7), we have

$$\begin{aligned} &\sum_j \|v\|_{L^{\tilde{q}}(Q_j)}^r \\ &= \sum_j \left(\int_{Q_j} |(u - \tau_h u)(x)|^{\tilde{q}} dx \right)^{r/\tilde{q}} \end{aligned}$$

as r is equal to or greater than $(n/\mu)\tilde{q}$ ($= (n/\lambda)\tilde{p}$), this is

$$\begin{aligned} &\leq \left(\sum_j \int_{Q_j} |(u - \tau_h u)(x)|^{\tilde{q}} dx \right)^{r/\tilde{q}} \\ &\leq \left(\int_{E^n} |(u - \tau_h u)(x)|^{\tilde{q}} dx \right)^{r/\tilde{q}} \end{aligned}$$

u has a support contained in a fixed bounded cube and therefore

$$= \left(\int_{E^n} |u(x) - u_0 - \tau_h u(x) + (\tau_h u)_0|^{\tilde{q}} dx \right)^{r/\tilde{q}}$$

where u_0 denotes the mean value of u over the support of u . Applying Minkowski's inequality, this is

$$\begin{aligned} &\leq C \left(\int_{E^n} |u(x) - u_0|^{\tilde{q}} dx \right)^{r/\tilde{q}} \\ &= C |Q_0|^{r/\tilde{q}(1-\mu/n)} \left(|Q_0|^{\mu/n-1} \int_{E^n} |u(x) - u_0|^{\tilde{q}} dx \right)^{r/\tilde{q}} \end{aligned}$$

by Definition 1 and Remark 1, we have

$$\leq C[u]_{\mathcal{L}^{(\tilde{q}, \mu)}(E^n)}$$

applying Remark 1 again and Lemma 1, we have finally

$$\leq C \left\{ \|u_x\|_{\mathcal{L}^{(p, \lambda)}(Q_0)} + \|u\|_{L^p(Q_0)} \right\}$$

Therefore, we have deduced (2.7).

This completes the proof of Theorem.

§ 3. Comments on Morrey-Sobolev type imbedding theorems

Combining theorems proved in [9], [10] and this paper, our theorems of this type read as follow:

Theorem 1. *Let u be a function such that the derivatives u_x belong to the space $\mathcal{L}_r^{(p, \lambda)}(Q_0)$, where $1 < p < \infty$, $0 < \lambda < n$, $1 < r < \infty$ and $\lambda/p \leq n/r < \lambda/p + 1$. Then, after the extension as in Remark 1, the following estimates hold for u :*

(1) $p < \lambda$, then u belongs to the space $\mathcal{L}_{r_1}^{((\tilde{p}/\lambda)\nu, \nu)} = \mathcal{L}_{r_1}^{((\tilde{p}/\lambda)\nu, \nu)}(E^n)$ and we have

$$(3.1) \quad [u]_{\mathcal{L}_{r_1}^{((\tilde{p}/\lambda)\nu, \nu)}} \leq C \{ \|u_x\|_{\mathcal{L}_r^{(p, \lambda)}(Q_0)} + \|u\|_{L^p(Q_0)} \}$$

where r_1 and ν are arbitrary constants satisfying $r_1 > (n/\lambda)p$ and $\lambda/\tilde{p} \leq \nu < n$.

(2) $p = \lambda$, then u belongs to the space $\mathcal{L}_{r_1}^{(1, 0)}$ and we have

$$(3.2) \quad [u]_{\mathcal{L}_{r_1}^{(1, 0)}} \leq \text{the right hand side of (3.1)}$$

where r_1 is as in (1).

(3) $p > \lambda$, then u belongs to the space $\mathcal{H}_{r_1}^{1-\lambda/p}$ and we have

$$(3.3) \quad [u]_{\mathcal{H}_{r_1}^{1-\lambda/p}} \leq \text{the right hand side of (3.1)}$$

where r_1 is as in (1).

Theorem 2. *Among the conditions of Theorem 1, if we replace the condition “ $\lambda/p \leq n/r < \lambda/p + 1$ ” by “ $n/r < \lambda/p$, that is $(n/\lambda)p < r$ ”, we obtain the following estimates for u :*

(1) $p < \lambda$ and $(n/\lambda)p < r \leq (n/\lambda)\tilde{p}$, then u belongs to the space $\mathcal{L}_r^{((\tilde{p}/\lambda)\nu, \nu)}$ and we have

$$(3.4) \quad [u]_{\mathcal{L}_r^{((\tilde{p}/\lambda)\nu, \nu)}} \leq \text{the right hand side of (3.1)}$$

where ν is as in Theorem 1, (1).

(2) $p = \lambda$, then u belongs to the space $\mathcal{L}_r^{(1, 0)}$ and we have

$$(3.5) \quad [u]_{\mathcal{L}_r^{(1,0)}} \leq \text{the right hand side of (3.1)}$$

(3) $p > \lambda$, then u belongs to the space $\mathcal{H}_r^{1-\lambda/p}$ and we have

$$(3.6) \quad [u]_{\mathcal{H}_r^{1-\lambda/p}} \leq \text{the right hand side of (3.1)}$$

For the proof of (1) in Theorems 1 and 2 which are of more general forms than those stated in [9] and [10], we refer the following:

Lemma 5 ([12] Ono-Furusho). *Let p, λ and r be arbitrary constants satisfying $1 < p < \infty$, $0 < \lambda < n$, $1 < r < \infty$ and $\lambda/p \leq n/r < \lambda/p + 1$. Then, the space $\mathcal{L}_r^{(p,\lambda)}$ is isomorphic to the space $\mathcal{L}_r^{(1,\lambda/p)}$ and we have*

$$(3.7) \quad C^{-1}[u]_{\mathcal{L}_r^{(p,\lambda)}} \leq [u]_{\mathcal{L}_r^{(1,\lambda/p)}} \leq [u]_{\mathcal{L}_r^{(p,\lambda)}}$$

Theorem 3. *In Theorem 2, (1), if we replace the condition “ $(n/\lambda)p < r \leq (n/\lambda)\tilde{p}$ ” by “ $(n/\lambda)\tilde{p} < r$ ”, then u belongs to the space $\mathcal{L}_r^{(q,\mu)}$ and we have*

$$(3.8) \quad [u]_{\mathcal{L}_r^{(q,\mu)}} \leq \text{the right hand side of (3.1)}$$

where q is an arbitrary constant satisfying $1 < q < p$ and $\mu/q = \lambda/p$.

Finally, we make the following:

Remark 2. If r is equal to infinity, then Theorems 2 (2), (3) and 3 reduce to Stampacchia's imbedding theorems [15].

Remark 3. If we replace the spaces $\mathcal{L}_r^{(p,\lambda)}$ by the spaces $L_r^{(p,\lambda)}$ (the Morrey spaces of strong type $r > (n/\lambda)p$, which have been defined in [14] and $L_r^{(p,\lambda)} \subset \mathcal{L}_r^{(p,\lambda)}$ with their corresponding norms), then we can verify more easily that closely analogous estimates to Theorems 2 and 3 hold.

However, it is quite difficult to improve the results mentioned in Remarks 2 and 3. Hence, Theorems 1–3 will be certainly final results among imbedding theorems in the $\mathcal{L}_r^{(p,\lambda)}$ spaces of strong type.

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(Ricevita la 12-an de novembro, 1983)