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Meromorphic Solutions of Some Difference Equations. II

By

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§ 1. Introduction

Let $y(x)$ be a meromorphic solution of the non-linear difference equation

$$y(x+1)=R(y(x)), \tag{1.1}$$

where $R(x)$ is a rational function of degree $p\geqq 2$. In this note, we will study some value distribution behaviors of $y(x)$.

Kimura [1], [2] investigated, in connection with the iteration of analytic functions, the difference equation

$$y(x+1)=y(x)+1+\frac{\lambda}{y(x)} \qquad (\lambda\neq 0) \tag{1.2}$$

and obtained a meromorphic solution $\phi(x)$ such that

$$\begin{cases} \phi(x) \text{ is holomorphic in the domain } D_l(B,\varepsilon)=\{|x|>B,\ |\arg x-\pi|<\pi/2-\varepsilon\}\cup \\ \{\mathrm{Im}\,[xe^{-i\varepsilon}]>B\}\cup\{\mathrm{Im}\,[xe^{i\varepsilon}]<-B\} \text{ and has an asymptotic expansion} \\ \qquad \phi(x)\sim x+\sum_{j+k\geq 1} p_{jk}x^{-j+1}\left(\frac{\log x}{x}\right)^k \qquad \text{in } D_l(B,\varepsilon), \\ \text{where } p_{10}=c \text{ is arbitrarily prescribed, } \varepsilon \text{ is an arbitrary positive number, and} \\ B \text{ is a sufficiently large number depending on } c \text{ and } \varepsilon. \end{cases} \tag{1.2'}$$

We proved [6] that $\phi(x)$ is of the order ∞ (in the sense of Nevanlinna), and there is a line $L_{\eta_0}=\{x;\ \mathrm{Im}\,x=\eta_0\}$ such that, for any $\delta>0$, $\phi(x)$ takes every value a, $|a|\leqq\infty$, infinitely often in

$$H(\eta_0,\delta)=\{x;\ \mathrm{Re}\,x>0,\ |\mathrm{Im}\,x-\eta_0|\leqq\delta\}, \tag{1.3}$$

if $\lambda\neq 1$. If $\lambda=1$, then $\phi(x)$ takes every value $a\neq -1$ infinitely often in $H(\eta_0,\delta)$.

We will show some corresponding results for solutions of the more general equation (1.1).

§ 2. Preliminaries. Statements of results

We proved in [7] that

(2.1) any non-constant meromorphic solution $y(x)$ of (1.1) is of the order ∞.

For a meromorphic solution $y(x)$, we showed in [7] that

(2.2) there is a number λ such that $\lambda=R(\lambda)$, $|R'(\lambda)|\geqq 1$, and $y(\xi+i\eta)\to\lambda$ for any η, as $\xi\to-\infty$.

We say that the solution $y(x)$ is *the one crresponding to* λ.

On the other hand we know, by a theorem of Julia [3, p. 158] that

(2.3) either there is a number λ such that $\lambda=R(\lambda)$ and $R'(\lambda)=1$

or there is a number λ such that

(2.4) $\lambda=R(\lambda)$ and $|R'(\lambda)|>1$.

If λ is a number for which (2.3) holds, then putting

(2.3–1) $$y(x)=\lambda+1/w(x),$$

we obtain

(2.3–2) $$w(x+1)=w(x)\left[1-\frac{R^{(m+1)}(\lambda)}{(m+1)!}w(x)^{-m}+\cdots\right] \qquad (m\geqq 1)$$
$$=R_1(w(x)) \qquad (R_1(x) \text{ is a rational function}).$$

Further, if we put

(2.3–3) $$\omega(x)=w(x)^m/A^m, \qquad A=\left[\frac{-m}{(m+1)!}R^{(m+1)}(\lambda)\right]^{1/m},$$

then we get

(2.3–4) $$\omega(x+1)=F(\omega(x)),$$

where

(2.3–5) $$F(x)=x+1+\sum_{j\geq m+1} b_j x^{1-j/m}.$$

The equation (2.3–4) is studied by Kimura [1]. He obtained a local solution $\phi_\lambda(x)$, which is holomorphic in $D_l(B,\varepsilon)$ (see (1.2′)), and has asymptotic expansion

$$\phi_\lambda(x)\sim x\left[1+\sum_{j+k\geq 1}\alpha_{jk}x^{-j/m}\left(\frac{\log x}{x}\right)^k\right] \quad \text{in } D_l(B,\varepsilon). \tag{2.3–6}$$

Then we obtain a meromorphic solution $\tilde{\phi}(x)$ of (2.3–2) which is holomorphic in $D_l(B,\varepsilon)$ and has an asymptotic expansion

$$\begin{aligned}\tilde{\phi}_\lambda(x)&\sim Ax^{1/m}\left[1+\sum_{j+k\geq 1}\alpha_{jk}x^{-j/m}\left(\frac{\log x}{x}\right)^k\right]^{1/m}\\ &=Ax^{1/m}\left[1+\sum_{j+k\geq 1}\tilde{\alpha}_{jk}x^{-j/m}\left(\frac{\log x}{x}\right)^k\right] \quad \text{in } D_l(B,\varepsilon)\end{aligned} \tag{2.3–7}$$

and is continued analytically to $|x|<\infty$, by the equation (2.3–2). Further, a meromorphic solution $\psi_\lambda(x)$ of (2.3) is obtained by (2.3–1):

$$\psi_\lambda(x)=1+/\tilde{\phi}_\lambda(x). \tag{2.3–8}$$

If λ is a number for which (2.4) holds, then we obtain a meromorphic solution which is holomorphic in $D(B)=\{x;\,|e^{ax}|\leqq 1/B\}$ and

$$y(x)=\sigma_\lambda(x)=\lambda+\sum_{j=1}^{\infty}\kappa_j e^{jax} \quad \text{in } D(B), \tag{2.4–1}$$

where $e^a=R'(\lambda)$, and is continued analytically by the equation (1.1). $\sigma_\lambda(x)$ is a function of e^{ax}, and we can write

$$\sigma_\lambda(x)=f_\lambda(e^{ax}), \tag{2.4–2}$$

where $f_\lambda(t)$ is meromorphic in $|t|<\infty$ and

$$f_\lambda(t)=\lambda+\sum_{j=1}^{\infty}\kappa_j t^j \quad \text{in } |t|\leqq 1/B. \tag{2.4–3}$$

We showed in [7] that

(i) if a meromorphic solution $y(x)$ corresponds to λ for which (2.3) holds, then there is an entire periodic function $\pi(x)$ with the period 1 ($\pi(x+1)=\pi(x)$) such that

$$y(x)=\psi_\lambda(x+\pi(x)) \quad \text{(with } \psi_\lambda(x) \text{ in (2.3–8))}, \tag{2.3–9}$$

(ii) if $y(x)$ corresponds to λ for which (2.4) holds, then there is an entire periodic function $\pi(x)$ with the period 1 such that

$$y(x)=\sigma_\lambda\left(x+\frac{1}{a}\log\pi(x)\right)=f_\lambda(\pi(x)e^{ax}) \tag{2.4–4}$$

with $\sigma_\lambda(x)$ and $f_\lambda(t)$ in (2.4–1) and (2.4–3), respectively ($e^a=R'(\lambda)$),

(iii) when λ is a number such that $\lambda=R(\lambda)$, $R'(\lambda)=e^{i\alpha\pi}$, $0<\alpha<2$, and when

(2.5) either α is a rational number,
or α satisfies $\log\left|\sin\frac{n\alpha\pi}{2}\right|=O(\log n)$ as $n\to\infty$,

then there is no meromorphic solution $y(x)$ which corresponds to such λ.

Further, we define as follows:

a value μ is said to be *maximally fixed value* (in short: *mf-value*) if the equation $\mu=R(x)$ has the only solution $x=\mu$;

a pair of values (μ_1, μ_2) $(\mu_1\neq\mu_2)$ is said to be *maximally fixed pair* (in short: *mf-pair*) if the equation $\mu_1=R(x)$ has the only solution $x=\mu_2$, and the equation $\mu_2=R(x)$ has the only solution $x=\mu_1$.

It is easy to see that, for a rational function $R(x)$, there are at most two mf-values, and at most one mf-pair [8].

We denote by $H(\eta_0, \delta)$ the half-strip in (1.3).

For a number a, $|a|\leqq\infty$, we denote by $\{x_n(a, \eta_0, \delta)\}$ the set of zero points of $y(x)-a$ in $H(\eta_0, \delta)$.

By these preliminaries, we can state our results.

Theorem 1. *If $y(x)$ is a meromorphic solution of* (1.1), *corresponding to λ, for which either* (2.3) *or* (2.4) *holds. Then*

(i) *if $R(x)$ has neither mf-values nor mf-pair, there is a line $L_{\eta_0}=\{x;\ \mathrm{Im}\ x=\eta_0\}$ such that $y(x)$ takes every value a, $|a|\leqq\infty$, infinitely often in $H(\eta_0, \delta)$ for any $\delta>0$, i.e.,*

$$\sum_n |x_n(a, \eta_0, \delta)|^{-\rho}=\infty \qquad \text{for any } \rho>0, \tag{2.6}$$

(ii) *if $R(x)$ has only one mf-value μ, then there is a line L_{η_0} such that* (2.6) *holds for any $a\neq\mu$,*

(iii) *if $R(x)$ has either two mf-values μ_1, μ_2 or a mf-pair (μ_1, μ_2) then there is a line L_{η_0} such that* (2.6) *holds for any $a\neq\mu_1$, μ_2.*

In (ii) and (iii) of this theorem, μ and (μ_1, μ_2) are really exceptional values for $y(x)$ (see [7]).

By theorem 1, we have

Theorem 2. *Suppose the rational function $R(x)$ has no values λ such that*

$$\lambda=R(\lambda) \quad \text{and} \quad R'(\lambda)=e^{i\alpha\pi},$$
$$\text{where } \alpha \text{ is irrational and } \log\left|\sin\frac{n\alpha\pi}{2}\right|\neq O(\log n). \tag{2.7}$$

Then, the statements of theorem 1 *are valid for any meromorphic solution $y(x)$ of* (1.1).

When $R(x)$ has a value λ for which (2.7) holds, we do not know whether there is a meromorphic solution $y(x)$ corresponding to such λ.

§ 3. An analogue of the second fundamental theorem for the half-strip domains

Tsuji [5, p. 272] defined the characteristic for meromorphic functions in an angular domain, and deduced an analogue of the second fundamental theorem of Nevanlinna. We will adapt his method for a half-strip [6].

Let Δ be a plane domain, and write $\Delta(r)=\Delta\cap\{|x|\leqq r\}$. For a meromorphic function $f(x)$ in Δ, we put

$$S(f;\Delta)=\frac{1}{\pi}\iint_{\Delta}\left(\frac{|f'(x)|}{1+|f(x)|^2}\right)^2 d\Sigma_x \qquad (d\Sigma_x \text{ is the area element}). \tag{3.1}$$

We write $S(f;\Delta(r))$ as $S(f,r;\Delta)$. Put

$$T(f,r;\Delta)=\int_0^r \frac{S(f,t;\Delta)}{t}dt. \tag{3.2}$$

When $\Delta=C=\{|x|<\infty\}$, we write $T(f,r;\Delta)$ etc. simply as $T(r,f)$ etc.. $T(r,f)$ is the characteristic of Nevanlinna in the usual sense [4].

Take a number a, $|a|\leqq\infty$. Let $n(f,r,a;\Delta)$ be the number of zero-points of $f(x)-a$, contained in $\Delta(r)$, multiple zeros being counted only once. Put

$$N(f,r,a;\Delta)=\int_1^r \frac{n(f,t,a;\Delta)}{t}dt. \tag{3.3}$$

When Δ is a half-strip domain $H(\eta_0,\delta)$ (see (1.3)), then we write $S(f,r;H(\eta_0,\delta))$ etc. simply as $S(f,r;\eta_0,\delta)$ etc..

We proved in [6, p. 100] the following lemma 3:

Lemma 3. *Let $\delta_0>\delta$, and a_1, a_2, a_3 be three distinct values. Then we have*

$$S(f,r;\eta_0,\delta)\leqq 3\sum_{j=3}^{3} n(f,r+r_0,a_j;\eta_0,\delta_0)+O(r), \tag{3.4}$$

$$T(f,r;\eta_0,\delta)\leqq 3\sum_{j=1}^{3} N(f,r+r_0,a_j;\eta,\delta_0)+O(r), \tag{3.5}$$

where $r_0=2+|\eta_0|+\delta_0$.

The following lemma is easily proved [4, p. 217]:

Lemma 4. *For $\rho>0$, the three expressions*

$$\sum|x_n(f,a;\eta_0,\delta)|^{-\rho},\quad \int^{\infty}\frac{n(f,t,a;\eta_0,\delta)}{t^{\rho+1}}dt,\quad \int^{\infty}\frac{N(f,t,a;\eta_0,\delta)}{t^{\rho+1}}dt$$

are simultaneously finite or infinite.

Further, we need

Lemma 5. *Suppose a value a is neither an mf-value nor a member of some mf-pair for $R(x)$. Let $a'_j, j=1, \cdots, p$ ($p=$the degree of $R(x)$) be numbers such that $R(a'_j)=a$, and let $a''_k, k=1, \cdots, p^2$, be numbers such that $R^2(a''_k)=a$. Then, the set*

$$\{a\} \cup \{a'_j, j=1, \cdots, p\} \cup \{a''_j, k=1, \cdots, p^2\}$$

contains at least three distinct values.

Proof. When $R(a)=a$. Then there is $a'_1 \neq a$, $R(a'_1)=a$, since a is not an mf-value. Let $R(a''_1)=a'_1$. Then obviously $a''_1 \neq a'_1$, $a''_1 \neq a$.

When $R(a) \neq a$. Then there is $a'_1 \neq a$, $R(a'_1)=a$. If there is another $a'_2 \neq a'_1$, $R(a'_2)=a$, our lemma is proved. If the only solution of $R(x)=a$ is $x=a'_1$, then there is $a''_1 \neq a$, $R(a''_1)=a'_1$, since a is not a member of mf-pair. Obviously $a''_1 \neq a'_1$, and our proof is completed. Q.E.D.

§ 4. Proof of theorem 1, when $y(x)=\psi_\lambda(x)$

It suffices to show the assertion when $y(x)=\tilde{\phi}_\lambda(x)$ in (2.3–7). In the sequel, we write $\tilde{\phi}_\lambda(x)$ simply as $\tilde{\phi}(x)$. Then

$$\phi(x)=(\tilde{\phi}(x)/A)^m \qquad \text{(the constant } A \text{ in (2.3–3))} \tag{4.1}$$

is also a meromorphic function which satisfies (2.3–4) and (2.3–6).

For a number $B>0$, we put

$$D^B=\{|x|>B, \text{ Re } x<0\} \cup \{|\text{Im } x|>B\} \tag{4.2}$$

and

$$\tilde{D}^B=\{x; \text{ Re } x>0, |\text{Im } x| \leqq B\}. \tag{4.3}$$

Lemma 6. *Suppose B is sufficiently large. Then we have*

$$\phi(x)/x \to 1 \qquad \text{as } x \to \infty \text{ in } D^B. \tag{4.4}$$

Proof. By [1, p. 222, Theorem 9.1], we have

$$F^n(z)=z+n+\chi_n(z)+\zeta_n(z), \qquad \text{for } z \in \{|z|>B, \text{ Re } z>0\},$$

where

$$\begin{cases} \chi_n(z)=\chi_n^{(1)}(z)+\cdots+\chi_n^{(m)}(z), \\ \chi_n^{(j)}(z)=\sum_{\nu=0}^{n-1} b_{j+m}(z+\nu)^{-j/m}, \qquad j=1, \cdots, m \end{cases} \tag{4.5}$$

and

$$|\zeta_n(z)| \leqq K \sum_{\nu=0}^{n-1} \frac{|\chi_{\nu+1}^{(1)}(z)| + \cdots + |\chi_{\nu+1}^{(m)}(z)| + 1}{|z+\nu|^{1+1/m}}, \tag{4.6}$$

(K is a positive constant), provided B is sufficiently large. Thus

$$\phi(x+n) = \phi(x) + n + \chi_n(\phi(x)) + \zeta_n(\phi(x)),$$

and by easy estimations of (4.5) and (4.6), we have

$$\phi(x+n)/(x+n) \to 1 \qquad \text{as } n \to \infty \tag{4.7}$$

if $x \in D_l(B, \varepsilon) \cap \{\mathrm{Re}\, x > 0\}$. By asymptotic expansion (2.3–6) and (4.7) we obtain (4.4). Q.E.D.

Lemma 7. *$\phi'(x)$ is bounded in D^B, supposed B is so large that* (4.4) *holds.*

Proof. By (2.3–4) and (2.3–5), we obtain

$$\phi'(x+1) = \phi'(x)\left[1 + \sum_{j \geq m+1} b_j \left(1 - \frac{j}{m}\right) \phi(x)^{-j/m}\right].$$

Thus

$$\begin{aligned}
|\phi'(x+1)| &\leqq |\phi'(x)|\,[1 + K_1/|\phi(x)|^{1+1/m}], \\
|\phi'(x+n)| &\leqq |\phi'(x)| \prod_{\nu=0}^{n-1} [1 + K_1/|\phi(x+\nu)|^{1+1/m}] \\
&\leqq |\phi'(x)| \prod_{\nu=0}^{n-1} [1 + K_2/|x+\nu|^{1+1/m}] \\
&\leqq |\phi'(x)| \prod_{\nu=1}^{\infty} [1 + K_3/\nu^{1+1/m}] \leqq M,
\end{aligned}$$

where K_1, K_2, K_3 and M are suitable constants, since $\phi'(x)$ is bounded in $D_l(B, \varepsilon)$, as seen by differentiation of the asymptotic expansion (2.3–6). Q.E.D.

By (4.4), we have

$$\tilde{\phi}(x)/x^{1/m} \to A \qquad \text{as } x \to \infty \text{ in } D^B, \tag{4.8}$$

and

$$|\tilde{\phi}'(x)| = \left|\frac{A}{m} \phi(x)^{-1+1/m} \phi'(x)\right| \leqq M_1/|x|^{1-1/m} \qquad \text{in } D^B. \tag{4.9}$$

By (4.8) and (4.9), we get

$$S(\tilde{\phi}, r; D^B) = O(1),$$

hence

$$T(\tilde{\phi}, r; D^B)=O(\log r). \tag{4.10}$$

By (2.1) and (4.10), we have

$$\int^{\infty} \frac{T(\tilde{\phi}, t; \tilde{D}^B)}{t^{\rho+1}}dt=\infty \qquad \text{for any } \rho>0. \tag{4.11}$$

For any positive integer h, we divide $\tilde{D}^B$ into 2^h equal half-strips $H(\eta_j, \delta_h)$, $j=1, \cdots, 2^h$, $\delta_h=2B/2^h$. Then, by (4.11), there exists a sequence of half-strips $H_h=H(\eta_{j_h}, \delta_h)$ such that $H_1 \supset H_2 \supset \cdots$ and

$$\int^{\infty} T(\tilde{\phi}, t; \eta_{j_h}, \delta_h)t^{-\rho-1}dt=\infty \qquad \text{for any } \rho>0.$$

H_h converges to a line L_{η_0}. Thus, for any $\delta>0$, we have

$$\int^{\infty} T(\tilde{\phi}, t; \eta_0, \delta)t^{-\rho-1}dt=\infty \qquad \text{for any } \rho>0. \tag{4.12}$$

Take $\delta_0=2\delta$ and apply the lemma 3, then (supposing $\rho>2$)

$$\begin{aligned}&\int^{\infty} T(\tilde{\phi}, t; \eta_0, \delta)t^{-\rho-1}dt \\ &\leqq 3\sum_{j=1}^{3}\int^{\infty} N(\tilde{\phi}, t+r_0, a_j; \eta_0, \delta_0)t^{-\rho-1}dt+\text{const.}\end{aligned} \tag{4.13}$$

($r_0=2+|\eta_0|+\delta_0$). Put

$$G(a, \rho)=\int^{\infty} N(\tilde{\phi}, t, a; \eta_0, \delta_0)t^{-\rho-1}dt. \tag{4.14}$$

Then, by (4.13), if ρ (>2) is fixed,

(4.15) $G(a, \rho)$ can not be finite for three distinct values of a.

Suppose a value a is such that

(4.16) a is neither an mf-value nor a member of some mf-pair for $R(x)$.

Then, by lemma 5, there are three distinct values a, a', a'' such that $R(a'')=a'$, $R(a')=a$. Thus, it is easily seen that, if $G(a, \rho)$ is finite, then $G(a', \rho)$ as well as $G(a'', \rho)$ are also finite, which contradicts (4.15). Hence by lemma 4, (2.6) holds for any a which satisfies (4.16), and the theorem is proved when $y(x)=\tilde{\phi}(x)$, hence when $y(x)=\psi_\lambda(x)$.

§ 5. Proof of theorem 1, when $y(x)=\sigma_\lambda(x)$

We note that $\sigma_\lambda(x)$ is a function of e^{ax}, $e^a=R'(\lambda)$. We write $\sigma_\lambda(x)$ simply as $\sigma(x)$.

For a number $r>0$, we denote by P^r the parallelogram with verices

$$\pm r\pm\frac{\pi i}{a},$$

and by $\tilde{P}^r$ the rhomboid with vertices

$$\pm r\left(1\pm\frac{\pi i}{a}\Big/\left|\frac{\pi i}{a}\right|\right).$$

$\tilde{P}^r$ consists of $r/|\pi i/a|$ parallelogram P^r (see Fig. 1). Further, put

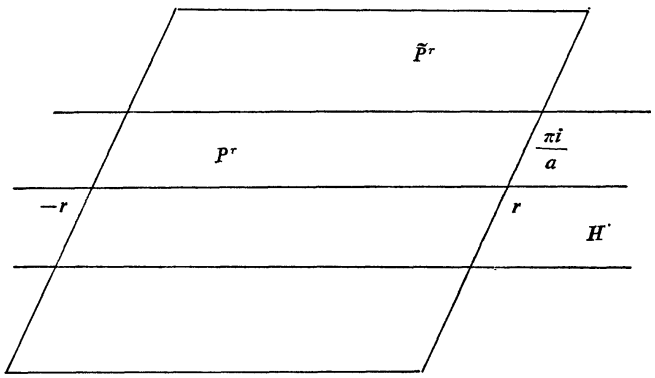


Fig. 1

$$\text{(5.1)}\qquad \begin{cases} H=\left\{x;\ \mathrm{Re}\,x>0,\ |\mathrm{Im}\,x|\leqq\left|\mathrm{Im}\left[\frac{\pi i}{a}\right]\right|\right\} \\ q=\mathrm{Im}\left[\frac{\pi i}{a}\right]\Big/\left|\frac{\pi i}{a}\right|. \end{cases}$$

Then, $\{|x|<qr\}\subset\tilde{P}^r$, and we have, using the fact that $\sigma(x+2\pi i/a)=\sigma(x)$,

$$S(\sigma, qr)\ (=S(\sigma;\{|x|\leqq qr\}))\leqq S(\sigma;\tilde{P}^r)=\frac{r}{|\pi i/a|}S(\sigma;p^r)$$
$$\leqq\frac{r}{|\pi i/a|}\left[S\left(\sigma, r+\left|\frac{\pi i}{a}\right|;H\right)+O(1)\right].$$

Thus

$$\int^r\frac{S(\sigma,qt)}{qt}qdt\leqq\frac{r}{b}\int^r\frac{S(\sigma,t+b;H)}{t}dt+O(r)\qquad\left(b=\left|\frac{\pi i}{a}\right|\right)$$

i.e.

$$T(\sigma; qr) \leqq KrT(\sigma, r+b; H)+O(r),$$

where K is a constant. Thus, we have by (2.1),

$$\int^{r} \frac{T(\sigma, r; H)}{r^{\rho+1}} dr=\infty \qquad \text{for any } \rho>0. \tag{5.2}$$

For any integer h, we divide H into 2^h equal half-strips. Arguing as in § 4, we know that there is a line L_{η_0} such that, for any $\delta>0$,

$$\int^{\infty} \frac{T(\sigma, r; \eta_0, \delta)}{r^{\rho+1}} dr=\infty \qquad \text{for any } \rho>0,$$

from which we obtain our conclusion as in § 4, using lemmas 3, 5, and 6.

§ 6. Proof of theorem 1, when $y(x)=\psi_\lambda(x+\pi(x))$

For $B'>0$ and $B''>0$, we put

$$\mathscr{D}_0 = \text{the image of } \tilde{D}^{B'} \text{ by the function } z=x+\pi(x),$$
$$D^B_{B''} = \tilde{D}^B \cap \{\operatorname{Re} x > B''\} \qquad (\text{for } \tilde{D}^B, \text{ see (4.3)}).$$

Take B' and B'' so large that $\mathscr{D}_0$ contains $D^B_{B''}$. Then

$$\begin{aligned} S(y, r; \tilde{D}^{B'}) &= \iint_{\tilde{D}^{B'}(r)} \left(\frac{|y'(x)|}{1+|y(x)|^2}\right)^2 d\Sigma_x \\ &\geqq \iint_{\mathscr{D}_0(r')} \left(\frac{|\psi_\lambda'(z)|}{1+|\psi_\lambda(z)|^2}\right)^2 d\Sigma_z \\ &\geqq \iint_{D^B_{B''}(r')} \left(\frac{|\psi_\lambda'(x)|}{1+|\psi_\lambda(x)|^2}\right)^2 d\Sigma_x, \end{aligned} \tag{6.1}$$

where

$$r' = \min \{|x+\pi(x)|\,;\, x \in \{|x|=r\} \cap \tilde{D}^{B'}\}. \tag{6.2}$$

Obviously,

$$r'/r \to 1 \qquad \text{as } r \to \infty.$$

From (6.1), we obtain, using (4.11),

$$\int^{\infty} \frac{T(y, r; \tilde{D}^{B'})}{r^{\rho+1}} dr=\infty \qquad \text{for any } \rho>0. \tag{6.3}$$

Then, dividing $\tilde{D}^{B'}$ into 2^h equal half-strips, $h=1, 2, \cdots$, we get our conclusion as in § 4, by lemmas 3, 5, and 6.

§ 7. Proof of theorem 1, when $y(x)=\sigma_\lambda(x+(1/a)\log\pi(x))$

For $B'>0$ and $B''>0$, we put

$$\mathscr{D}_0'=\text{the image of } \tilde{D}^{B'} \text{ by the function } z=x+(1/a)\log\pi(x),$$
$$H_{B''}=H\cap\{\mathrm{Re}\,x>B''\} \qquad (\text{with } \tilde{D}^B \text{ in (4.3) and } H \text{ in (5.1)}).$$

Take B' and B'' so large that $\mathscr{D}_0'$ contains $H_{B''}$. Further, we can suppose that $\pi(x)\neq 0$ on the lines $\{\mathrm{Im}\,x=\pm B'\}$.

Let $\{r_n\}, r_n\uparrow\infty$, be the sequence such that

$$\{|x|=r_n\}\cap D^{B'} \text{ is distant } \geqq\gamma_0 \text{ from zeros of } \pi(x). \tag{7.1}$$
(γ_0 is a suitably fixed constant.)

We can suppose that

$$r_{n+1}/r_n\to 1 \qquad \text{as } n\to\infty. \tag{7.2}$$

Then, if $r=r_n$, we have

$$\begin{aligned} S(y,r;\tilde{D}^{B'}) &= \iint_{\tilde{D}^{B'}(r)}\left(\frac{|y'(x)|}{1+|y(x)|^2}\right)^2 d\Sigma_x \\ &\geqq \iint_{\mathscr{D}_0'(r')}\left(\frac{|\sigma_\lambda'(z)|}{1+|\sigma_\lambda(z)|^2}\right)^2 d\Sigma_z \\ &\geqq \iint_{H_{B''}(r')}\left(\frac{|\sigma_\lambda'(x)|}{1+|\sigma_\lambda(x)|^2}\right)^2 d\Sigma_x \end{aligned} \tag{7.3}$$

where

$$r'=\min\{|x+(1/a)\log\pi(x)|;\ x\in\{|x|=r\}\cap\tilde{D}^{B'}\}. \tag{7.4}$$

By (7.1), we obtain

$$r_n'/r_n\to 1 \qquad \text{as } n\to\infty. \tag{7.5}$$

Further, by (7.2), we get

$$\text{if } r_n\leqq r\leqq r_{n+1}, \text{ then } r_n'/r\to 1 \text{ as } n\to\infty. \tag{7.6}$$

From (7.3) and (7.6), we have

$$\int^\infty \frac{T(y,r;\tilde{D}^{B'})}{r^{\rho+1}}dr=\infty \qquad \text{for any } \rho>0, \tag{7.7}$$

using (5.2). Dividing $\tilde{D}^{B'}$ into 2^h equal half-strips, $h=1,\cdots$, we obtain our conclusion as in § 5, using lemmas 3, 5, and 6.

Thus, our proof of theorem 1 is completed for all cases. Q.E.D.

We will call the line L_{γ_0} for which (2.6) holds as a *"Borel" line* for $y(x)$.

Example. The equation $y(x+1)=y(x)^p$ has a solution

$$y(x)=\exp[p^x]. \tag{7.8}$$

For any $a=r_0e^{i\alpha}$, the solutions of $y(x)=a$ are

$$x=\frac{1}{\log p}\left[\frac{1}{2}\log((\log r_0)^2+(\alpha+2n\pi)^2)+i\left(\tan^{-1}\left(\frac{\alpha+2n\pi}{\log r_0}\right)+2k\pi\right)\right]$$
$$n=0, \pm1, \pm2, \cdots, \qquad k=0, \pm1, \pm2, \cdots.$$

Thus, the lines $\{\operatorname{Im} x=(\pi/2+2k\pi)/\log p\}$, $k=0, \pm1, \pm2, \cdots$, are "Borel" lines for $y(x)$ in (7.8).

Further, $y(x)=\exp[p^{x+\pi i/2}]$ is also a solution, for which the "Borel" lines are

$$\{\operatorname{Im} x=(\pi+2k\pi)/\log p\}, \qquad k=0, \pm1, \pm2, \cdots.$$

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