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Boundary Value Problems for Systems of Second Order Ordinary Differential Equations

By

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Dedicated to Professor Taro Ura on his sixtieth birthday

§ 1. Introduction

This paper is concerned with the existence of solutions of the boundary value problem

$$(1) \quad x'' = f(t, x, x'),$$

$$(2) \quad x(0) = a, \quad x(1) = b,$$

where x is an n -vector and the prime denotes differentiation with respect to t .

Scorza-Dragoni [17] first proved that the equation (1) has a solution satisfying (2) for arbitrary a and b whenever $f: I \times R^n \times R^n \rightarrow R^n$ is bounded and continuous, where and through this paper $I = [0, 1]$. Hartman [5] (or refer to [6]) obtained an existence result for the problem (1)–(2) by imposing growth conditions on f which yield a priori bounds of $x'(t)$ in terms of a bound of $x(t)$. Such an idea had been found out by Nagumo [13] in 1937 for scalar second order equations. Hartman's result has been developed by Bernfeld, Ladde and Lakshmikantham [1], Lasota and Yorke [10], Schmitt and Thompson [16] and so on.

In the case where $n = 1$, Nagumo [14] has obtained a beautiful existence theorem for the problem (1)–(2). We have extended his result in [8, 9] by using topological properties of solution curves. In this paper, we consider a further extension of the result to vector equations. Our main theorem is Theorem 3 in Section 4, and the proof is based on the degree theory for a certain class of set-valued mappings which will be called *regular* mappings.

§ 2. Regular mapping and the degree

For a set $A \subset R^m$, we use the following notations; $\bar{A}$ is the closure, ∂A is the boundary and $\text{co } A$ is the convex hull. Furthermore, we set

$$\text{Comp}(R^m) = \{A: A \text{ is a nonempty compact set in } R^m\},$$

$$\text{Conv}(R^m) = \{A \in \text{Comp}(R^m): A \text{ is convex}\}.$$

By Caratheodory's lemma (refer to [4, p. 28]), $\text{co } A$ is written as

$$\text{co } A = \left\{ \sum_{i=0}^m r_i p_i : \sum_{i=0}^m r_i = 1, r_i \geq 0, p_i \in A, i=0, 1, \dots, m \right\}.$$

Therefore, if A is compact, then so is $\text{co } A$. Let M be a metric space, and let $C(M, R^m)$ be the set of all continuous mappings from M into R^m . For a given mapping $\Phi: M \rightarrow \text{Comp}(R^m)$, $\Phi^*: M \rightarrow \text{Conv}(R^m)$ denotes the mapping defined by $\Phi^*(x) = \text{co } \Phi(x)$ for $x \in M$, while $A(\Phi)$ denotes the set of all sequences $\{\Phi_k\}$ in $C(M, R^m)$ satisfying the following condition:

(R) If $\{\Phi_{k_j}\}$ is a subsequence of $\{\Phi_k\}$ and if $\{x_j\}$ is a sequence in M converging to an x in M , then the sequence $\{\Phi_{k_j}(x_j)\}$ contains a subsequence which converges to some point in $\Phi(x)$.

The element $\{\Phi_k\}$ of $A(\Phi)$ will be called an *approximate sequence* of Φ . A mapping $\Phi: M \rightarrow \text{Comp}(R^m)$ is called *upper semicontinuous* if, for every $x \in M$ and an open set V containing $\Phi(x)$, there exists a neighborhood W of x such that $\Phi(W) \subset V$, where $\Phi(W) = \bigcup \{\Phi(y) : y \in W\}$. It is clear that if Φ is upper semicontinuous, then so is Φ^* .

Cellina [2] has proved the following lemma (see also [3, Lemma 1]).

Lemma 1. *If $\Phi: M \rightarrow \text{Conv}(R^m)$ is upper semicontinuous and if M is compact, then $A(\Phi)$ is nonempty.*

Let D be a bounded and open set in R^m , and let $\Phi: \bar{D} \rightarrow \text{Conv}(R^m)$ be upper semicontinuous. Applying Lemma 1, Cellina and Lasota [3] defined the degree of Φ by

$$(3) \quad d(\Phi, D, p) = \lim_{k \rightarrow \infty} d(\Phi_k, D, p)$$

for a point $p \in R^m \setminus \Phi(\partial D)$ and a $\{\Phi_k\} \in A(\Phi)$, where $d(\Phi_k, D, p)$ is the degree of the continuous mapping Φ_k , refer to [11] or [15]. This statement involves that

(i) the limit in the right hand side of (3) exists for every $\{\Phi_k\} \in A(\Phi)$ and it is independent of the choice of $\{\Phi_k\}$,

(ii) $d(\Phi, D, p) \neq 0$ implies $p \in \Phi(D)$,

(iii) if $\Phi(\cdot, \cdot): I \times \bar{D} \rightarrow \text{Conv}(R^m)$ is upper semicontinuous and if $p \in R^m \setminus \Phi(I, \partial D)$, then $d(\Phi(0, \cdot), D, p) = d(\Phi(1, \cdot), D, p)$,

(iv) for Φ satisfying $\Phi(x) = \{x\}$ on $\bar{D}$, $d(\Phi, D, p) = 1$ if and only if $p \in D$.

Hukuhara [7] and Ma [12] have also introduced the same degree as in the above by different approaches from that in [3].

A mapping $\Phi: M \rightarrow \text{Comp}(R^m)$ is said to be *regular* if $A(\Phi)$ is nonempty. Lemma 1 shows that an upper semicontinuous mapping from M into $\text{Conv}(R^m)$ is regular when M is compact.

Lemma 2. Consider a mapping $\Phi: M \rightarrow \text{Comp}(R^m)$. If there is a $\phi \in C(M, R^m)$ such that $\phi(x) \in \Phi(x)$ for all $x \in M$, then Φ is regular. Conversely, if Φ is regular and single-valued, $\Phi(x) = \{\phi(x)\}$, then $\phi \in C(M, R^m)$.

Proof. The first assertion of the lemma is clear since the sequence $\{\Phi_k\}$ in $C(M, R^m)$ defined by $\Phi_k = \phi$ satisfies (R). We prove the last assertion. Suppose that ϕ is not continuous. Then there exist $\varepsilon > 0$, $x \in M$ and a sequence $\{x_j\}$ in M converging to x such that $|\phi(x) - \phi(x_j)| \geq \varepsilon$ for all j , where $|\cdot|$ is any norm in R^m . Let $\{\Phi_k\} \in A(\Phi)$ be fixed. By (R), the sequence $\{\Phi_k\}$ converges to ϕ at every point in M , and hence there exists a subsequence $\{\Phi_{k_j}\}$ of $\{\Phi_k\}$ satisfying $|\Phi_{k_j}(x_j) - \phi(x_j)| < 1/j$ for all j . Therefore we have $\varepsilon \leq |\phi(x) - \phi(x_j)| < |\phi(x) - \Phi_{k_j}(x_j)| + 1/j$. On the other hand, by using (R), we have $|\phi(x) - \Phi_{k_j}(x_j)| \rightarrow 0$ as $j \rightarrow \infty$ by taking a subsequence if necessary, a contradiction. q.e.d.

The following lemma is trivial.

Lemma 3. (a) For mappings $\Phi, \Psi: M \rightarrow \text{Comp}(R^m)$, we have $A(\Phi) \subset A(\Psi)$ if $\Phi(x) \subset \Psi(x)$ for $x \in M$. Especially, if $\Phi: M \rightarrow \text{Comp}(R^m)$ is regular, then $\Phi^*: M \rightarrow \text{Conv}(R^m)$ is regular.

(b) If $\Phi(\cdot, \cdot): I \times M \rightarrow \text{Comp}(R^m)$ is regular, then $\Phi(t, \cdot): M \rightarrow \text{Comp}(R^m)$ is regular for every $t \in I$.

(c) Let M_1 and M_2 be metric spaces. Suppose that $g_1: M_1 \rightarrow M_2$ and $g_2: R^m \rightarrow R^n$ are continuous and that $\Phi: M_2 \rightarrow \text{Comp}(R^m)$ is regular. Then the composite mapping $g_2 \circ \Phi \circ g_1: M_1 \rightarrow \text{Comp}(R^n)$ is regular.

Examining the arguments in Cellina and Lasota [3], we can generalize the concept of the degree for a regular mapping by (3) if $p \in R^m \setminus \Phi^*(\partial D)$, and we can obtain the following theorem.

Theorem 1. (a) $d(\Phi, D, p) \neq 0$ implies $p \in \Phi(D)$ whenever $\Phi: \bar{D} \rightarrow \text{Comp}(R^m)$ is regular.

(b) If $\Phi(\cdot, \cdot): I \times \bar{D} \rightarrow \text{Comp}(R^m)$ is regular and if $p \in R^m \setminus \Phi^*(I, \partial D)$, then $d(\Phi(0, \cdot), D, p) = d(\Phi(1, \cdot), D, p)$.

Remark 1. Clearly, we have

$$(4) \quad d(\Phi, D, p) = d(\Phi^*, D, p)$$

for a regular mapping Φ . Even if Φ is not regular, Φ^* may be regular, which suffices to define $d(\Phi, D, p)$ by (4). In this case, we cannot conclude that $p \in \Phi(D)$ under $d(\Phi, D, p) \neq 0$, though $d(\Phi, D, p) \neq 0$ implies $p \in \Phi^*(D)$. This is seen in the following example.

Example. Let $D = (-1, 1)$, and let $\Phi: \bar{D} \rightarrow \text{Comp}(R)$ be the mapping defined by $\Phi(x) = \{x/|x|\}$ for $x \neq 0$ and $\Phi(0) = \{-1, 1\}$. Then Φ is upper semicontinuous, and hence Φ^* is regular. Since $\Phi^*(\partial D)$ does not contain 0, $d(\Phi^*, D, 0)$ is defined. Clearly, the sequence $\{\Phi_k\}$ in $C(\bar{D}, R)$ defined by

$$\Phi_k(x) = \begin{cases} 1, & 1/k \leq x \leq 1, \\ kx, & -1/k < x < 1/k, \\ -1, & -1 \leq x \leq -1/k \end{cases}$$

is an approximate sequence of Φ^* . Therefore we have $d(\Phi^*, D, 0) = 1$ and $0 \in \Phi^*(D)$, while $0 \notin \Phi(D)$.

§ 3. Solution mapping

Consider a differential equation

$$(E) \quad x' = h(t, x),$$

where $h: I \times R^m \rightarrow R^m$ is continuous, and assume that

$$(C) \quad \text{every solution of (E) is continuable over } I.$$

A subset S of $I \times R^m$ will be called a *positively invariant set* of (E) if every solution curve of (E) starting from a point in S remains in S on its right maximal interval of existence, that is, if every solution x of (E) satisfies $(t, x(t)) \in S$ for $\tau \leq t \leq 1$ whenever $(\tau, x(\tau)) \in S$ for some $\tau \in I$. Clearly, the intersection and the union of positively invariant sets are positively invariant. Similarly, the concept of the *negatively invariant set* of (E) is defined.

For $(\tau, \xi) \in I \times R^m$, we put

$$(5) \quad \Phi(\tau, \xi) = \{x(1) : x \text{ is a solution of (E) satisfying } x(\tau) = \xi\}.$$

Then, by well-known Kamke's theorem (see [6, Theorem 3.2, p.p. 14–15]) and the assumption (C), $\Phi(\tau, \xi)$ is a compact set in R^m , and we have the following theorem.

Theorem 2. *The solution mapping $\Phi: I \times R^m \rightarrow \text{Comp}(R^m)$ defined by (5) is regular, and it satisfies that $\Phi(1, \xi) = \{\xi\}$ for $\xi \in R^m$.*

Proof. It is well-known that the mapping h in the equation (E) admits a sequence $\{h_k\}$ in $C(I \times R^m, R^m)$ with the following properties:

- (A1) $\{h_k\}$ converges to h uniformly on every compact set in $I \times R^m$.
- (A2) Every solution of

$$(E_k) \quad x' = h_k(t, x)$$

is uniquely determined by initial data and it is continuable over I for $k=1, 2, \dots$.

Let $\Phi_k: I \times R^m \rightarrow R^m$ be the mapping defined by $\Phi_k(\tau, \xi) = x_k(1)$ for each k , where x_k is the solution of (E_k) satisfying $x_k(\tau) = \xi$. Then Φ_k are continuous. It follows from (A1) and Kamke's theorem that the sequence $\{\Phi_k\}$ satisfies (R) if $M = I \times R^m$ in (R). Therefore Φ is regular and $\{\Phi_k\} \in A(\Phi)$.

The last assertion of the theorem is clear.

q.e.d.

§ 4. Boundary value problems

In this section, we give an existence theorem for the problem (1)–(2). The equation (1) is equivalent to the system

$$(6) \quad x' = y, \quad y' = f(t, x, y).$$

Let $I = [0, 1]$ and $J = \{1, 2, \dots, n\}$. For two vectors x and y in R^n , we write $x \leq y$ when $x_i \leq y_i$ holds for each $i \in J$, where and hereafter the suffix i denotes the i -th component of a vector, and hence i runs over the set J . The scalar product of x and y will be denoted by $\langle x, y \rangle$, that is, $\langle x, y \rangle = \sum_{i \in J} x_i y_i$.

For given twice continuously differentiable functions $\alpha, \beta: I \rightarrow R^n$ satisfying $\alpha(t) \leq \beta(t)$ on I , set

$$\omega = \{(t, x) \in I \times R^n: \alpha(t) \leq x \leq \beta(t)\}.$$

Let Ω be the compact set defined by

$$\Omega = \{(t, x, y) \in \omega \times R^n: \phi(t, x) \leq y \leq \psi(t, x)\},$$

where ϕ and ψ are continuously differentiable functions from ω into R^n satisfying $\phi(t, x) \leq \psi(t, x)$ on ω . We assume that the function f in the equation (6) is defined and continuous on Ω .

Theorem 3. Suppose that the following inequalities hold on ω or on Ω for each $i \in J$;

$$(7) \quad \alpha'_i(t) \geq \phi_i(t, x) \quad \text{if } x_i = \alpha_i(t),$$

$$(8) \quad \beta'_i(t) \leq \psi_i(t, x) \quad \text{if } x_i = \beta_i(t),$$

$$(9) \quad \alpha''_i(t) \geq f_i(t, x, y) \quad \text{if } x_i = \alpha_i(t), \quad y_i = \alpha'_i(t),$$

$$(10) \quad \beta''_i(t) \leq f_i(t, x, y) \quad \text{if } x_i = \beta_i(t), \quad y_i = \beta'_i(t),$$

$$(11) \quad f_i(t, x, y) \geq \frac{\partial}{\partial t} \phi_i(t, x) + \left\langle \frac{\partial}{\partial x} \phi_i(t, x), y \right\rangle \quad \text{if } y_i = \phi_i(t, x),$$

$$(12) \quad f_i(t, x, y) \leq \frac{\partial}{\partial t} \psi_i(t, x) + \left\langle \frac{\partial}{\partial x} \psi_i(t, x), y \right\rangle \quad \text{if } y_i = \psi_i(t, x).$$

Then, for a given $b \in R^n$ with $\alpha(1) \leq b \leq \beta(1)$, the equation (6) has at least one solution (x, y) defined on I which satisfies $x(1) = b$ and

$$(13) \quad (t, x(t), y(t)) \in \Omega \quad \text{for } t \in I.$$

In particular, if $\alpha(0) = a = \beta(0)$ holds, then the solution satisfies (2).

Remark 2. The condition (9) will be meaningless when the set $\{(t, x, y) \in \Omega : x_i = \alpha_i(t), y_i = \alpha'_i(t)\}$ is empty. This is similar for the condition (10).

Proof. The proof is fairly complicated and lengthy, so we proceed in the seven steps.

Step 1. We shall construct a bounded and continuous extension $F: I \times R^n \times R^n \rightarrow R^n$ of f so that the following inequalities hold for each $i \in J$;

$$(14) \quad \alpha''_i(t) > F_i(t, x, y) \quad \text{if } x_i < \alpha_i(t), \quad y_i = \alpha'_i(t),$$

$$(15) \quad \beta''_i(t) < F_i(t, x, y) \quad \text{if } x_i > \beta_i(t), \quad y_i = \beta'_i(t),$$

$$(16) \quad F_i(t, x, y) \geq f_i(t, x, \hat{y}) \quad \text{for } (t, x, y) \in \omega \times R^n, \quad y_i \leq \phi_i(t, x),$$

$$(17) \quad F_i(t, x, y) \leq f_i(t, x, \hat{y}) \quad \text{for } (t, x, y) \in \omega \times R^n, \quad y_i \geq \psi_i(t, x),$$

where $\hat{y}$ is the vector with the j -th component, $j \in J$, defined by

$$(18) \quad \hat{y}_j = \begin{cases} \phi_j(t, x), & y_j < \phi_j(t, x), \\ y_j, & \phi_j(t, x) \leq y_j \leq \psi_j(t, x), \\ \psi_j(t, x), & y_j > \psi_j(t, x) \end{cases}$$

which depends on t, x and y .

First of all, for each $i \in J$, we shall define F_i on the domain

$$V_i = \{(t, x, y) \in \omega \times R^n : y_i \in R, \quad \phi_j(t, x) \leq y_j \leq \psi_j(t, x) \quad \text{for } j \in J \setminus \{i\}\}$$

so as to satisfy

$$(19) \quad \alpha''_i(t) \geq F_i(t, x, y) \quad \text{on } A_i,$$

$$(20) \quad \beta''_i(t) \leq F_i(t, x, y) \quad \text{on } B_i,$$

$$(21) \quad F_i(t, x, y) \geq f_i(t, x, \hat{y}) \quad \text{on } V_i^-$$

and

$$(22) \quad F_i(t, x, y) \leq f_i(t, x, \hat{y}) \quad \text{on } V_i^+,$$

where

$$\begin{aligned}
A_i &= \{(t, x, y) \in V_i : x_i = \alpha_i(t), y_i = \alpha'_i(t)\}, \\
B_i &= \{(t, x, y) \in V_i : x_i = \beta_i(t), y_i = \beta'_i(t)\}, \\
V_i^- &= \{(t, x, y) \in V_i : y_i < \phi_i(t, x)\}, \\
V_i^+ &= \{(t, x, y) \in V_i : y_i > \psi_i(t, x)\}.
\end{aligned}$$

Here, notice that $V_i = V_i^- \cup \Omega \cup V_i^+$ and that the inequalities (19) and (20) are already satisfied on $A_i \cap \Omega$ and on $B_i \cap \Omega$, respectively, by (9) and (10). Therefore, since $A_i \subset \Omega \cup V_i^+$ and $B_i \subset \Omega \cup V_i^-$ by (7) and (8), it is not difficult to obtain a bounded and continuous extension F_i of f_i on the domain V_i which satisfies (19) through (22).

For an arbitrary $(t, x, y) \in \omega \times R^n$, define $F_i(t, x, y)$ by

$$F_i(t, x, y) = F_i(t, x, \hat{y}^*),$$

where $\hat{y}^* = (\hat{y}_1^*, \dots, \hat{y}_n^*)$ with $\hat{y}_i^* = y_i$ and $\hat{y}_j^* = \hat{y}_j$ given by (18) for $j \in J \setminus \{i\}$ which depends on i, t, x and y . Here, we note that $(t, x, \hat{y}^*)$ belongs to V_i . Finally, for an arbitrary $(t, x, y) \in I \times R^n \times R^n$ and $i \in J$, we set

$$F_i(t, x, y) = \begin{cases} F_i(t, \bar{x}, y) + \frac{x_i - \beta_i(t)}{1 + x_i - \beta_i(t)}, & x_i > \beta_i(t), \\ F_i(t, \bar{x}, y), & \alpha_i(t) \leq x_i \leq \beta_i(t), \\ F_i(t, \bar{x}, y) - \frac{\alpha_i(t) - x_i}{1 + \alpha_i(t) - x_i}, & x_i < \alpha_i(t), \end{cases}$$

where $\bar{x}$ is the vector with the j -th component, $j \in J$, defined by

$$(23) \quad \bar{x}_j = \begin{cases} \beta_j(t), & x_j > \beta_j(t), \\ x_j, & \alpha_j(t) \leq x_j \leq \beta_j(t), \\ \alpha_j(t), & x_j < \alpha_j(t) \end{cases}$$

which depends on t and x . Here, we note that $(t, \bar{x}, y)$ belongs to $\omega \times R^n$. It is easy to see that $F(t, x, y)$ satisfies all required conditions.

Step 2. Instead of the equation (6), we consider the equation

$$(24) \quad x' = y, \quad y' = F(t, x, y).$$

The boundedness of F assures that every solution of (24) is continuable over I . Suppose that (x, y) is a solution of (24) satisfying $x_i(t) < \alpha_i(t)$ and $y_i(t) = \alpha'_i(t)$ for some $i \in J$ and $t \in I$. Then, by (14), we have

$$y'_i(t) = F_i(t, x(t), y(t)) < \alpha''_i(t).$$

Therefore we can easily observe that

$$N_i = \{(t, x, y) \in I \times R^n \times R^n : x_i < \alpha_i(t), y_i > \alpha'_i(t)\}$$

is a negatively invariant set of (24) for each $i \in J$, while

$$P_i = \{(t, x, y) \in I \times R^n \times R^n : x_i < \alpha_i(t), y_i \leq \alpha'_i(t)\}$$

is a positively invariant set of (24). Similarly, by (15),

$$M_i = \{(t, x, y) \in I \times R^n \times R^n : x_i > \beta_i(t), y_i < \beta'_i(t)\}$$

and

$$Q_i = \{(t, x, y) \in I \times R^n \times R^n : x_i > \beta_i(t), y_i \geq \beta'_i(t)\}$$

are, respectively, a negatively invariant set and a positively invariant set of (24).

Let $X_i, i \in J$, be the set defined by

$$X_i = \{(t, x, y) \in I \times R^n \times R^n : \alpha_i(t) \leq x_i \leq \beta_i(t)\}.$$

Then the outside of X_i is the disjoint union of P_i, Q_i, N_i and M_i . Since $M_i \cup N_i$ is negatively invariant, the set $Y_i = P_i \cup X_i \cup Q_i$ is positively invariant. Therefore the intersection $Y = \bigcap \{Y_i : i \in J\}$ is positively invariant.

Step 3. Let ε and δ be arbitrary fixed numbers such that $0 < \varepsilon < \delta < 1$. Then we shall obtain a continuous mapping $\rho : [\varepsilon, \delta] \times R^n \rightarrow R^n$ which satisfies

$$(25) \quad \phi(t, x) \leq \rho(t, x) \leq \psi(t, x) \quad \text{for } (t, x) \in \omega, \quad \varepsilon \leq t \leq \delta$$

and

$$(26) \quad (t, x, \rho(t, x)) \in Y \quad \text{for } (t, x) \in [\varepsilon, \delta] \times R^n.$$

Let $C : \{(t, x) \in \omega : \varepsilon \leq t \leq \delta\} \rightarrow R^n$ be the mapping such that the i -th component $C_i(t, x)$ is linear in x_i and satisfies $C_i(t, x) = \min \{\alpha'_i(t), \psi_i(t, x)\}$ at $x_i = \alpha_i(t)$, $= \max \{\beta'_i(t), \phi_i(t, x)\}$ at $x_i = \beta_i(t)$. Since $\alpha'_i(t) = \beta'_i(t)$ when $\alpha_i(t) = \beta_i(t)$ for some $t \in [\varepsilon, \delta]$, $C_i(t, x)$ turns out to be continuous by (7) and (8) if we set $C_i(t, x) = \alpha'_i(t)$ when $\alpha_i(t) = \beta_i(t)$. Define $\rho_i : [\varepsilon, \delta] \times R^n \rightarrow R, i \in J$, by

$$\rho_i(t, x) = \begin{cases} \phi_i(t, \bar{x}), & C_i(t, \bar{x}) < \phi_i(t, \bar{x}), \\ C_i(t, \bar{x}), & \phi_i(t, \bar{x}) \leq C_i(t, \bar{x}) \leq \psi_i(t, \bar{x}), \\ \psi_i(t, \bar{x}), & \psi_i(t, \bar{x}) < C_i(t, \bar{x}), \end{cases}$$

where $\bar{x}$ is the vector given by (23). Then we can easily see that ρ is continuous and satisfies (25). When $x_i < \alpha_i(t)$, we have $C_i(t, \bar{x}) \leq \psi_i(t, \bar{x})$ since $\bar{x}_i = \alpha_i(t)$, and hence $\rho_i(t, x) = \max \{\phi_i(t, \bar{x}), C_i(t, \bar{x})\} \leq \alpha'_i(t)$ by (7). Consequently, we have $(t, x, \rho(t, x)) \in P_i$ if $x_i < \alpha_i(t)$. Similarly, we have $(t, x, \rho(t, x)) \in Q_i$ if $x_i > \beta_i(t)$. Thus, $(t, x,$

$\rho(t, x)$ belongs to Y_i for each $i \in J$, namely, (26) holds.

By Theorem 2, the mapping $\Phi: [\varepsilon, \delta] \times R^n \times R^n \rightarrow \text{Comp}(R^n \times R^n)$ defined by

$$\Phi(\tau, \xi, \eta) = \{(x(\delta), y(\delta)) : (x, y) \text{ is a solution of (24) through } (\tau, \xi, \eta)\}$$

is regular. We define two continuous mappings $\gamma: [\varepsilon, \delta] \times R^n \rightarrow [\varepsilon, \delta] \times R^n \times R^n$ and $\pi: R^n \times R^n \rightarrow R^n$ by $\gamma(t, x) = (t, x, \rho(t, x))$ and $\pi(x, y) = x$. By Lemma 3 (c), the composite mapping $\Psi = \pi \circ \Phi \circ \gamma: [\varepsilon, \delta] \times R^n \rightarrow \text{Comp}(R^n)$ is regular.

Step 4. Let r be a number satisfying $r > \max \{|\alpha_i(t)|, |\beta_i(t)| : \varepsilon \leq t \leq \delta, i \in J\}$, and let D be the set defined by

$$D = \{x \in R^n : |x_i| < r \text{ for all } i \in J\}.$$

Then D is a bounded open set in R^n . The restriction of Ψ to $[\varepsilon, \delta] \times \bar{D}$, denoted by Ψ again, is regular. By the assumption, there exists a continuous mapping $p: I \rightarrow R^n$ satisfying $p(1) = b$ and

$$(27) \quad \alpha(t) \leq p(t) \leq \beta(t) \quad \text{on } I.$$

We want to prove

$$(28) \quad d(\Psi(\varepsilon, \cdot), D, p(\delta)) = 1.$$

The definition of r and (27) imply that $p(\delta) \in D$. Clearly, Ψ satisfies that $\Psi(\delta, x) = \{x\}$ for all x in $\bar{D}$, and hence we have $d(\Psi(\delta, \cdot), D, p(\delta)) = 1$.

Let (t, x) be an arbitrary point in $[\varepsilon, \delta] \times \partial D$. Then we have that $|x_i| = r$ for some $i \in J$. First, consider the case where $x_i = -r$. As was seen in Step 3, we have $\gamma(t, x) = (t, x, \rho(t, x)) \in P_i$ because $x_i = -r < \alpha_i(t)$. Since P_i is positively invariant, the set $\Psi(t, x)$ is contained in the convex set $\{x \in R^n : x_i < \alpha_i(\delta)\}$. Therefore we have $\Psi^*(t, x) \subset \{x \in R^n : x_i < \alpha_i(\delta)\}$. Similarly, in the case where $x_i = r$, we have $\Psi^*(t, x) \subset \{x \in R^n : x_i > \beta_i(\delta)\}$. It follows from (27) that $\Psi^*(t, x)$ does not contain $p(\delta)$ for all $(t, x) \in [\varepsilon, \delta] \times \partial D$, namely,

$$p(\delta) \in R^n \setminus \Psi^*([\varepsilon, \delta], \partial D).$$

By Theorem 1 (b), we obtain that $d(\Psi(\varepsilon, \cdot), D, p(\delta)) = d(\Psi(\delta, \cdot), D, p(\delta))$. Thus, (28) is proved.

Step 5. It follows from (28) and Theorem 1 (a) that there exists a $\xi \in D$ satisfying $p(\delta) \in \Psi(\varepsilon, \xi)$. In other words, the equation (24) has a solution (x, y) such that

$$(29) \quad x(\varepsilon) = \xi, \quad y(\varepsilon) = \rho(\varepsilon, \xi)$$

and that $x(\delta) = p(\delta)$. We show that the solution satisfies

$$(30) \quad \alpha(t) \leq x(t) \leq \beta(t) \quad \text{on } [\varepsilon, \delta].$$

By (26) and (29), we have $(\varepsilon, x(\varepsilon), y(\varepsilon)) \in Y$. If the solution does not satisfy (30), then the solution curve enters $P_i \cup Q_i$ for some $i \in J$ because Y is positively invariant. Namely, there exists a $t \in [\varepsilon, \delta]$ such that $(t, x(t), y(t)) \in P_i \cup Q_i$. Since $P_i \cup Q_i$ is positively invariant, either $x_i(\delta) < \alpha_i(\delta)$ or $x_i(\delta) > \beta_i(\delta)$ holds. This contradicts $x(\delta) = p(\delta)$ and (27), and hence we have (30).

Step 6. We shall prove that the solution satisfies

$$(31) \quad (t, x(t), y(t)) \in \Omega \quad \text{on } [\varepsilon, \delta].$$

Suppose that (31) is false. Then there exists a subinterval $[\sigma, \tau]$ of $[\varepsilon, \delta]$ such that

$$(32) \quad (\sigma, x(\sigma), y(\sigma)) \in \Omega, \quad (t, x(t), y(t)) \notin \Omega \quad \text{for } \sigma < t \leq \tau,$$

where we note that $(\varepsilon, x(\varepsilon), y(\varepsilon)) \in \Omega$ by (25), (29) and (30).

Let $U: \{(t, x, y) \in \omega \times R^n: \sigma \leq t \leq \tau\} \rightarrow R$ be the mapping defined by $U(t, x, y) = \sum_{i \in J} \text{dist}(y_i, [\phi_i(t, x), \psi_i(t, x)])$, namely,

$$(33) \quad U(t, x, y) = \sum_{i \in G_0} [\phi_i(t, x) - y_i] + \sum_{i \in H_0} [y_i - \psi_i(t, x)],$$

where $G_0 = \{i \in J: y_i < \phi_i(t, x)\}$ and $H_0 = \{i \in J: y_i > \psi_i(t, x)\}$, which depend on t, x and y , and we understand that for empty G_0 or H_0 the corresponding sum makes zero. It is clear that $U(t, x, y) = 0$ if and only if $(t, x, y) \in \Omega$. Along the solution curve, put

$$u(t) = U(t, x(t), y(t)) \quad \text{for } \sigma \leq t \leq \tau,$$

where we note (30). Since clearly U is Lipschitz continuous in (t, x, y) , u is absolutely continuous. The relation (32) implies $u(\sigma) = 0$ and $u(t) > 0$ for $\sigma < t \leq \tau$.

We show that u satisfies

$$(34) \quad \log u(\tau) - \log u(s) \leq nK(\tau - s) \quad \text{for } \sigma < s < \tau,$$

where $K = \max \{ |(\partial/\partial x_j)\phi_i(t, x)|, |(\partial/\partial x_j)\psi_i(t, x)|: (t, x) \in \omega, i, j \in J \}$. We put

$$\begin{aligned} \Gamma(t, x, y) = & \sum_{i \in G_0} \left\{ \sum_{j \in G_0} \left(-\frac{\partial}{\partial x_j} \phi_i(t, x) \right) [\phi_j(t, x) - y_j] \right. \\ & \left. + \sum_{j \in H_0} \left(-\frac{\partial}{\partial x_j} \phi_i(t, x) \right) [\psi_j(t, x) - y_j] \right\} \\ & + \sum_{i \in H_0} \left\{ \sum_{j \in G_0} \left(-\frac{\partial}{\partial x_j} \psi_i(t, x) \right) [y_j - \phi_j(t, x)] \right. \\ & \left. + \sum_{j \in H_0} \left(-\frac{\partial}{\partial x_j} \psi_i(t, x) \right) [y_j - \psi_j(t, x)] \right\}, \end{aligned}$$

where G_0 and H_0 are those as in (33). Then Γ is continuous and satisfies $|\Gamma(t, x, y)| \leq nKu(t, x, y)$, and hence

$$(35) \quad |\Gamma(t, x(t), y(t))| \leq nKu(t) \quad \text{for } \sigma < t \leq \tau.$$

Let s be fixed with $\sigma < s < \tau$. Then we have $u(t) > 0$ on $[s, \tau]$ and the function

$$(36) \quad \mu(t) = \log u(t) + \int_s^t \Gamma(w, x(w), y(w)) dw / u(w)$$

is absolutely continuous on $[s, \tau]$. Differentiating (36), we have

$$(37) \quad u(t)\mu'(t) = u'(t) + \Gamma(t, x(t), y(t)) \quad \text{a.e. on } [s, \tau].$$

We want to show that $\mu'(t) \leq 0$ a.e. on $[s, \tau]$. Let $t \in [s, \tau]$ be fixed. Then there exist two subset G and H of J and a sequence $\{\nu_k\}$ of nonzero numbers such that

$$\lim_{k \rightarrow \infty} \nu_k = 0, \quad s \leq t + \nu_k \leq \tau \quad \text{for } k = 1, 2, \dots$$

and that $u(\theta)$ is expressed in

$$(38) \quad u(\theta) = \sum_{i \in G} [\phi_i(\theta, x(\theta)) - y_i(\theta)] + \sum_{i \in H} [y_i(\theta) - \psi_i(\theta, x(\theta))]$$

for $\theta = t, t + \nu_k, k = 1, 2, \dots$. Clearly, $G_0 \subset G \subset \{i \in J: y_i(t) \leq \phi_i(t, x(t))\}$ and $H_0 \subset H \subset \{i \in J: y_i(t) \geq \psi_i(t, x(t))\}$. Here, we emphasize that G and H do not depend on θ . Therefore $\Gamma(t, x(t), y(t))$ is expressed in

$$(39) \quad \begin{aligned} \Gamma(t, x(t), y(t)) = & \sum_{i \in G} \left\{ \sum_{j \in G} \left[\frac{\partial}{\partial x_j} \phi_i(t, x(t)) \right] [\phi_j(t, x(t)) - y_j(t)] \right. \\ & + \sum_{j \in H} \left[\frac{\partial}{\partial x_j} \phi_i(t, x(t)) \right] [\psi_j(t, x(t)) - y_j(t)] \Big\} \\ & + \sum_{i \in H} \left\{ \sum_{j \in G} \left[\frac{\partial}{\partial x_j} \psi_i(t, x(t)) \right] [y_j(t) - \phi_j(t, x(t))] \right. \\ & + \sum_{j \in H} \left[\frac{\partial}{\partial x_j} \psi_i(t, x(t)) \right] [y_j(t) - \psi_j(t, x(t))] \Big\}. \end{aligned}$$

When u is differentiable at t , the equality $u'(t) = \lim_{k \rightarrow \infty} [u(t + \nu_k) - u(t)] / \nu_k$ and (38) imply

$$(40) \quad \begin{aligned} u'(t) = & \sum_{i \in G} \left\{ \frac{\partial}{\partial t} \phi_i(t, x(t)) + \left\langle \frac{\partial}{\partial x} \phi_i(t, x(t)), y(t) \right\rangle - F_i(t, x(t), y(t)) \right\} \\ & + \sum_{i \in H} \left\{ F_i(t, x(t), y(t)) - \frac{\partial}{\partial t} \psi_i(t, x(t)) - \left\langle \frac{\partial}{\partial x} \psi_i(t, x(t)), y(t) \right\rangle \right\}. \end{aligned}$$

Substitute (39) and (40) into (37). Then a direct calculation gives

$$\begin{aligned}
(41) \quad u(t)\mu'(t) = & \sum_{i \in G} \left\{ \frac{\partial}{\partial t} \phi_i(t, x(t)) + \left\langle \frac{\partial}{\partial x} \phi_i(t, x(t)), \hat{y}(t) \right\rangle - F_i(t, x(t), y(t)) \right\} \\
& + \sum_{i \in H} \left\{ F_i(t, x(t), y(t)) - \frac{\partial}{\partial t} \psi_i(t, x(t)) - \left\langle \frac{\partial}{\partial x} \psi_i(t, x(t)), \hat{y}(t) \right\rangle \right\},
\end{aligned}$$

where $\hat{y}(t)$ is the vector given by (18) with $x=x(t)$. By (16) and (17), we have

$$(42) \quad F_i(t, x(t), y(t)) \geq f_i(t, x(t), \hat{y}(t)), \quad \text{for } i \in G$$

and

$$(43) \quad F_i(t, x(t), y(t)) \leq f_i(t, x(t), \hat{y}(t)), \quad \text{for } i \in H.$$

It follows from (41) through (43) that

$$\begin{aligned}
u(t)\mu'(t) \leq & \sum_{i \in G} \left\{ \frac{\partial}{\partial t} \phi_i(t, x(t)) + \left\langle \frac{\partial}{\partial x} \phi_i(t, x(t)), \hat{y}(t) \right\rangle - f_i(t, x(t), \hat{y}(t)) \right\} \\
& + \sum_{i \in H} \left\{ f_i(t, x(t), \hat{y}(t)) - \frac{\partial}{\partial t} \psi_i(t, x(t)) - \left\langle \frac{\partial}{\partial x} \psi_i(t, x(t)), \hat{y}(t) \right\rangle \right\}.
\end{aligned}$$

Therefore, by the assumptions (11) and (12), we obtain that $u(t)\mu'(t) \leq 0$ because $\hat{y}_i(t) = \phi_i(t, x(t))$ for $i \in G$ and $\hat{y}_i(t) = \psi_i(t, x(t))$ for $i \in H$. This implies $\mu'(t) \leq 0$ a.e. on $[s, \tau]$. Thus, we have $\mu(\tau) - \mu(s) \leq 0$. On the other hand, it follows from (35) and (36) that

$$\begin{aligned}
\mu(\tau) - \mu(s) &= \log u(\tau) - \log u(s) + \int_s^\tau \Gamma(w, x(w), y(w)) dw / u(w) \\
&\geq \log u(\tau) - \log u(s) - nK \int_s^\tau dw,
\end{aligned}$$

and hence we obtain (34).

Making $s \rightarrow \sigma$ in (34), we arrive a contradiction since the left hand side of (34) tends to $+\infty$ (note $u(\sigma)=0$). Thus, we have (31).

Step 7. Let $\{\varepsilon_k\}$ and $\{\delta_k\}$ be two sequences in the open interval $(0, 1)$ such that $\lim_{k \rightarrow \infty} \varepsilon_k = 0$ and $\lim_{k \rightarrow \infty} \delta_k = 1$. As was seen in the above argument, for each integer k , the equation (24) has a solution (x^k, y^k) which satisfies $x^k(\delta_k) = p(\delta_k)$ and

$$(44) \quad (t, x^k(t), y^k(t)) \in \Omega \quad \text{on } [\varepsilon_k, \delta_k].$$

We may assume that (x^k, y^k) are defined on I because every solution of (24) is continuable over I . Since Ω is compact and F is bounded, the family $\{(x^k, y^k): k=1, 2, \dots\}$ is uniformly bounded and equicontinuous on I . By taking a subsequence if necessary, we may assume that $\{(x^k, y^k)\}$ converges to a solution (x, y) of (24) uniformly on I . Since $x^k(\delta_k) = p(\delta_k) \rightarrow b$ as $k \rightarrow \infty$, we have $x(1) = b$. Furthermore, we

can conclude that (13) holds by (44). At the same time, this shows that (x, y) is a solution of (6). This completes the proof.

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