Kobe University Repository : Kernel

PDF issue: 2026-11-22

On Stabilization of Partial Differential Equations of Parabolic Type: Boundary Observation and Feedback

Nambu, Takao

(Citation)
Funkcialaj Ekvacioj,28(3):267-298

(Issue Date)
1985-12

(Resource Type)
journal article

(Version)
Version of Record

(JaLCDOI)
https://doi.org/10.24546/0100499417

(URL)
https://hdl.handle.net/20.500.14094/0100499417

KOBE UNIVERSITY

Funkcialaj Ekvacioj, **28** (1985) 267-293

On Stabilization of Partial Differential Equations of Parabolic Type: Boundary Observation and Feedback†

By

Takao NAMBU

(Kumamoto University, Japan)

§ 1. Introduction

We study in this paper the feedback stabilization of partial differential equations of parabolic type by using a functional observer. The same problem has been considered in our previous paper [7] when the controlled object has a simple structure; precisely, when the coefficient is a self-adjoint operator and has a compact resolvent. The observer is desirable in the sense that it is finite dimensional, while we have to suppose some gap and separation assumptions between the spectrums of the operators appearing in the control system. One may have a question if these assumptions would cause any difficulty in designing the observer, or expect another stabilizing scheme which could be applied to a more general class of systems. It is the purpose of the paper to remove the above assumptions and to improve the previous result by introducing a new type of observer, so that the stabilizing feedback scheme is applicable to systems with more complicated structures.

The setting of the problem is the same as before. The controlled object is, for simplicity, a parabolic differential equation, the coefficient L of which is a uniformly elliptic operator of order 2 in a bounded domain of $\boldsymbol{R}^m$. The equation has a state $u(t, x)$, outputs $(u(t, \cdot), w_k)_\Gamma$, $1 \leq k \leq N$, and inputs $f_k(t)$ through boundary controllers $h_k(\xi)$, $1 \leq k \leq M$. It is assumed that the equation has a finite number of unstable modes in the right half-plane. Our functional observer is, in the first place, a conceptional and an infinite-dimensional one to accomplish the stabilization of the evolution $u(t, \cdot)$ in a relatively easy manner. It is described by a differential equation of order 1 with a densely defined coefficient B in a separable Hilbert space H, and has a state $v(t)$, outputs $(v(t), \rho_k)_H$, $1 \leq k \leq M$, and inputs $g_k(t)$, $1 \leq k \leq M+N$. By setting $f_k(t)=(v(t), \rho_k)_H$, and $g_k(t)=(u(t, \cdot), w_k)_\Gamma$, $1 \leq k \leq N$, $=(v(t), \rho_{k-N})_H$, $N<k \leq M+N$, the whole system becomes a closed loop system. The problem is to design a suitable observer in order to stabilize the evolution $u(t, \cdot)$. The newly designed observer requires no gap condition on the operator L. The gap condition

† The major part of this paper was presented at the meeting "Mathematical Theory of Control and Systems" which was held in Kyoto University from January 25 to 27 in 1983.

between the spectrums $\sigma(-L)$ and $\sigma(B)$ at infinity is automatically fulfilled. The key to the success consists in the choice of the operator B. Our observer is in a sense an "intermediate" one since it stands between a parabolic obserber [7] and a hyperbolic one [6]. The next step is to reduce the observer to a finite-dimensional equation. This will be carried out in almost the same manner as in [7]. The spectrum growth assumption on the operator B is improved. The eigenvalues $\sqrt{\mu_n}\omega^{\pm}$ of B as is stated later may grow faster than, for example, n as $n\to\infty$. This seems an important remark. For, our result assumes algebraic conditions associated with B, which will be easily broken if these eigenvalues are mutually very close.

As is easily seen, we can restate the result here within a more abstract framework. It means that the new stabilization scheme is applied to a fairly wide class of partial differential equations. When L is a self-adjoint operator and has a compact resolvent, another construction of a stabilizing observer has been recently proposed [8, 11]. We will show in §5 that the same observer is also obtained by our approach, a completely different one.

§ 2. Formulation of the problem and preliminaries

In this section we are concerned with the formulation of our control problem. Some necessary results of the elliptic theory will be developed. Throughout the paper, Ω will denote a connected bounded domain in an m-dimensional Euclidean space $\boldsymbol{R}^m$, and Γ the boundary of Ω, consisting of a finite number of $(m-1)$-dimensional C^∞ hypersurfaces. $\mathscr{L}$ will denote a uniformly elliptic differential operator of order 2 in Ω defined by

$$\mathscr{L}u=-\sum_{i,j=1}^{m}\frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right)+\sum_{i=1}^{m}b_i(x)\frac{\partial u}{\partial x_i}+c(x)u,$$

where $a_{ij}(x)=a_{ji}(x)$, $1\leq i, j\leq m$, and for some positive δ

$$\sum_{i,j=1}^{m}a_{ij}(x)\xi_i\xi_j\geq\delta|\xi|^2,\quad \xi=(\xi_1,\cdots,\xi_m),\quad x\in\Omega.$$

Let $\nu=(\nu_1(\xi),\cdots,\nu_m(\xi))$ denote the outward normal vector at $\xi\in\Gamma$. A boundary operator associated with $\mathscr{L}$ is given by

$$\frac{\partial u}{\partial\nu_L}+\sigma(\xi)u,\qquad \xi\in\Gamma,$$

where

$$\frac{\partial u}{\partial\nu_L}=\sum_{i,j=1}^{m}a_{ij}(\xi)\nu_j(\xi)\frac{\partial u}{\partial x_i}(\xi).$$

All the coefficients appearing in these operators are assumed to be real valued and sufficiently smooth.

Our control system is a parabolic equation described by

$$\text{(2.1)}\qquad \begin{cases} \dfrac{\partial u}{\partial t} = -\mathscr{L}u, \\ \dfrac{\partial u}{\partial \nu_L} + \sigma(\xi)u = \sum_{k=1}^{M} f_k(t)h_k(\xi), \\ u(0, x) = u_0(x). \end{cases}$$

Here, $f_k(t)$ denote the inputs which will be specified later, and $h_k(\xi)$ the real-valued controllers in $C^2(\Gamma)$. The outputs are a finite number of observations on Γ given by

$$\text{(2.2)}\qquad (u(t, \cdot), w_k)_\Gamma, \qquad 1 \leq k \leq N,$$

where w_k indicate the real-valued weighting functions in $C^2(\Gamma)$, and $(\ ,\ )_\Gamma$ the inner product in $L^2(\Gamma)$. Our goal here is the same as before; given h_k and w_k, it is to design $f_k(t)$ as a suitable feedback of the outputs (2.2) in order to stabilize the evolution $u(t, \cdot)$ in (2.1). The feedback will be implemented via a functional observer as an auxiliary system.

Let us introduce now a functional observer. Let H_0 denote a separable complex Hilbert space, and A a positive-definite self-adjoint operator in H_0 with a dense domain $\mathscr{D}(A)$. It is assumed that there is a set of eigenpairs $\{\mu_i, \zeta_{ij}\}$ of the operator A satisfying the following conditions:

(i) $0 < \mu_1 < \mu_2 < \cdots < \mu_i < \cdots \to \infty$.

(ii) $A\zeta_{ij} = \mu_i \zeta_{ij}$, $i \geq 1$, $1 \leq j \leq n_i$ $(< \infty)$.

(iii) The set $\{\zeta_{ij};\ i \geq 1,\ 1 \leq j \leq n_i\}$ forms a complete orthonormal system in H_0.

Let a be $0 < a < 1$, and define an operator B in the product space $H = \mathscr{D}(\sqrt{A}) \times H_0$ by

$$B = \begin{bmatrix} 0 & 1 \\ -A & -2a\sqrt{A} \end{bmatrix}, \qquad \mathscr{D}(B) = \mathscr{D}(A) \times \mathscr{D}(\sqrt{A}).$$

Details of the operator B will be stated later in this section. Our conceptional observer is a differential equation in H given by

$$\text{(2.3)}\qquad \frac{d}{dt} v(t) = Bv(t) + \sum_{k=1}^{N} (u(t, \cdot), w_k)_\Gamma \xi_k + \sum_{k=1}^{M} f_k(t)\alpha_k, \qquad v(0) = v_0.$$

This equation will be reduced to a finite-dimensional one (see §§4 and 5). $f_k(t)$ in (2.1) and (2.3) are given by

$$\text{(2.4)}\qquad f_k(t) = (v(t), \rho_k)_H, \qquad 1 \leq k \leq M.$$

Here, $(\ ,\ )_H$ indicates the inner product in H. Thus, the set of the equations (2.1), (2.3) and (2.4) forms a closed loop control system.

We will recall some well-known results of the elliptic theory. The inner product and the norm in $L^2(\Omega)$ are hereafter denoted by $(\ ,\ )$ and $\|\ \|$ respectively. In the first place, let us define an operator L in $L^2(\Omega)$ by

$$Lu=\mathscr{L}u, \qquad u\in\mathscr{D}(L),$$

$$\mathscr{D}(L)=\left\{u\in H^2(\Omega);\ \frac{\partial u}{\partial\nu_L}+\sigma(\xi)u=0,\ \xi\in\Gamma\right\}.$$

As is well known [1], the operator L is closed, and has a compact resolvent $(\lambda-L)^{-1}$. Therefore, the spectrum $\sigma(L)$ consists only of eigenvalues λ_i, $i\geq 1$; ∞ is the only accumulation point. $\sigma(L)$ lies symmetrically in the interior of a parabola; $\sigma=a\tau^2-b$ $(\lambda=\sigma+i\tau,\ a>0)$. As usual, set

$$P_{\lambda_i}=\frac{1}{2\pi i}\int_{C_i}(\lambda-L)^{-1}d\lambda,$$

where C_i is a small counterclockwise circle with the center λ_i. P_{λ_i} is the projection operator mapping onto the generalized eigenspace for λ_i. The dimension of $P_{\lambda_i}L^2(\Omega)$ is finite and is denoted by m_i. Then, $\dim P_{\lambda_i}L^2(\Omega)=\dim P_{\bar{\lambda}_i}L^2(\Omega)=m_i$. Let $\{\phi_{i1},\cdots,\phi_{im_i}\}$ be a basis of $P_{\lambda_i}L^2(\Omega)$. Then, $\{\bar{\phi}_{i1},\cdots,\bar{\phi}_{im_i}\}$ forms a basis of $P_{\bar{\lambda}_i}L^2(\Omega)$. The adjoint operator L^* of L in $L^2(\Omega)$ corresponds to the formal adjoint $\mathscr{L}^*$ of $\mathscr{L}$, and is

$$L^*u=\mathscr{L}^*u, \qquad u\in\mathscr{D}(L^*),$$

$$\mathscr{D}(L^*)=\left\{u\in H^2(\Omega);\ \frac{\partial u}{\partial\nu_L}+\tilde{\sigma}(\xi)u=0,\ \xi\in\Gamma\right\},$$

where $\tilde{\sigma}(\xi)=\sigma(\xi)+\sum_{i=1}^{m}b_i(\xi)\nu_i(\xi)$. $P^*_{\lambda_i}$ is the projection operator mapping onto the generalized eigenspace of L^* corresponding to its eigenvalue $\bar{\lambda}_i$, and especially $\dim P^*_{\lambda_i}L^2(\Omega)=\dim P_{\lambda_i}L^2(\Omega)=m_i$. Let $\{\psi_{i1},\cdots,\psi_{im_i}\}$ be a basis of $P^*_{\lambda_i}L^2(\Omega)$. These relations are summarized as a diagram;

$$\begin{array}{ll}
\lambda_i & \nearrow P_{\lambda_i}L^2(\Omega);\ \text{spanned by}\ \{\phi_{i1},\cdots,\phi_{im_i}\}\ (L)\\
 & \searrow P^*_{\lambda_i}L^2(\Omega);\ \text{spanned by}\ \{\bar{\psi}_{i1},\cdots,\bar{\psi}_{im_i}\}\ (L^*)\\
\bar{\lambda}_i & \nearrow P^*_{\lambda_i}L^2(\Omega);\ \text{spanned by}\ \{\psi_{i1},\cdots,\psi_{im_i}\}\ (L^*)\\
 & \searrow P_{\bar{\lambda}_i}L^2(\Omega);\ \text{spanned by}\ \{\bar{\phi}_{i1},\cdots,\bar{\phi}_{im_i}\}\ (L)
\end{array}$$

Note that, when λ_i is real valued, we can choose ϕ_{ij} and ψ_{ij} as real-valued functions.

Let c be a positive number and set $L_c=L+c$ in order that $\sigma(L_c)$ is contained in the right half-plane. Fractional powers L_c^{α} and $L_c^{*\alpha}$, $\alpha \in \boldsymbol{R}^1$, are defined in a familiar way. Then, we easily find that

$$L_c^{\alpha}{}^*=L_c^{*\alpha}, \qquad \alpha \in \boldsymbol{R}^1.$$

It is well known [2] that

$$\mathscr{D}(L_c^{3/4-\varepsilon})=\mathscr{D}(L_c^{*3/2-2\varepsilon})=H^{3/2-2\varepsilon}(\Omega), \qquad 0<\varepsilon\leq 3/4.$$

The generalized eigenfunctions ϕ_{ij} satisfy the relations [1]:

$$L\phi_{ij}=\lambda_i\phi_{ij}+\sum_{k<j}\alpha_{kj}^i\phi_{ik}, \qquad 1\leq j\leq m_i.$$

For later use, we give an easily proved generalization of the above relations;

$$L_c^{\beta}\phi_{ij}=(\lambda_i+c)^{\beta}\phi_{ij}+\sum_{k<j}\alpha_{kj}^i(\beta)\phi_{ik}, \qquad \beta \in \boldsymbol{R}^1. \tag{2.5a}$$

Similar expressions hold for $L_c^{*\beta}$;

$$L_c^{*\beta}\psi_{ij}=(\bar{\lambda}_i+c)^{\beta}\psi_{ij}+\sum_{k<j}\tilde{\alpha}_{kj}^i(\beta)\psi_{ik}, \qquad \beta \in \boldsymbol{R}^1. \tag{2.5b}$$

Let us consider the Laurent expansion of the resolvent $(\lambda-L)^{-1}$ in a neighborhood of λ_i. It is based on that of $(\lambda-L_c^{-1})^{-1}$ in a neighborhood of $(\lambda_i+c)^{-1}$. $(\lambda-L)^{-1}$ is expanded in the following way [1, 9]:

$$\begin{aligned}(\lambda-L)^{-1}&=((\lambda+c)L_c^{-1}-1)^{-1}L_c^{-1}\\&=\sum_{j=1}^{m_i}\frac{-1}{\lambda+c}\left(\frac{(\lambda+c)(\lambda_i+c)}{\lambda_i-\lambda}\right)^j A_jL_c^{-1}+\sum_{j=0}^{\infty}\frac{-1}{\lambda+c}\left(\frac{\lambda_i-\lambda}{(\lambda+c)(\lambda_i+c)}\right)^j A_{-j}L_c^{-1},\end{aligned}$$

where

$$A_j=\frac{1}{2\pi i}\oint\left(\lambda-\frac{1}{\lambda_i+c}\right)^{j-1}(\lambda-L_c^{-1})^{-1}d\lambda, \qquad j=0, \pm 1, \pm 2, \cdots,$$

and the integrals are taken along a small counterclockwise circle with the center $(\lambda_i+c)^{-1}$. Thus, the above expression readily implies the Laurent expansion of $(\lambda-L)^{-1}$;

$$(\lambda-L)^{-1}=\sum_{j=1}^{m_i}(\lambda-\lambda_i)^{-j}B_j+f(\lambda),$$

where

$$B_j=\frac{1}{2\pi i}\int_{\sigma_i}(\lambda-\lambda_i)^{j-1}(\lambda-L)^{-1}d\lambda,$$

C_i being a small circle with the center λ_i, and $f(\lambda)$ is analytic in a neighborhood of $\lambda=\lambda_i$. Note that $B_1=P_{\lambda_i}$. We can also derive the relation $P_{\lambda_i}=A_1$[1].

Now, we return to the original problem. For each h_k, $1\leq k\leq M$, we can find a unique $\psi_k \in H^2(\Omega)$ such that

$$(\mathscr{L}+c)\psi_k=0, \qquad \frac{\partial}{\partial\nu_L}\psi_k+\sigma(\xi)\psi_k=h_k.$$

Similarly, for each w_k, $1\leq k\leq N$, we can find a unique $g_k \in H^2(\Omega)$ satisfying

$$(\mathscr{L}^*+c)g_k=0, \qquad \frac{\partial}{\partial\nu_L}g_k+\bar{\sigma}(\xi)g_k=w_k.$$

Using g_k and Green's formula, we find that[2]

$$(u, w_k)_\Gamma=(L_c^{1/4+\varepsilon}u, L_c^{*3/4-\varepsilon}g_k), \qquad u \in \mathscr{D}(L_c^{1/4+\varepsilon}). \tag{2.6}$$

By setting $\alpha=2(1/4+\varepsilon)$ and $x(t)=L_c^{-\alpha/2}u(t, \cdot)$, equation (2.1) is transformed into an equation in $L^2(\Omega)$;

$$\begin{aligned} &\frac{d}{dt}x(t)=-Lx(t)+\sum_{k=1}^{M}f_k(t)L_c^{3/4-\varepsilon}\psi_k, \\ &x(0)=x_0=L_c^{-\alpha/2}u_0. \end{aligned} \tag{2.7}$$

The outputs (2.2) inserted into the observer (2.3) are rewritten as

$$(u(t, \cdot), w_k)_\Gamma=(L_c^{\alpha}x(t), L_c^{*3/4-\varepsilon}g_k), \qquad 1\leq k\leq N.$$

Thus, the closed loop system is rewritten as a set of equations (2.7), (2.3) and (2.4) in $L^2(\Omega)\times H$.

We shall close this section with some remarks on the operator B. It is easily

---

1)
$$\begin{aligned} P_{\lambda_i}&=\frac{1}{2\pi i}\int_{C_i}\sum_{j=1}^{m_i}\frac{-1}{\lambda+c}\left(\frac{(\lambda+c)(\lambda_i+c)}{\lambda_i-\lambda}\right)^j A_j L_c^{-1}d\lambda \\ &=\sum_{j=1}^{m_i}(-1)^{1-j}(\lambda_i+c)^j A_j L_c^{-1} \\ &\qquad \text{by the use of relations } A_j=\left(L_c^{-1}-\frac{1}{\lambda_i+c}\right)^{j-1}A_1, \\ &\qquad\qquad\qquad\qquad\qquad\qquad j\geq 1 \\ &=\sum_{j=1}^{m_i}(1-(\lambda_i+c)L_c^{-1})^{j-1}A_1 L_c^{-1}(\lambda_i+c) \\ &=A_1-(\lambda_i+c)^{m_i}\left(\frac{1}{\lambda_i+c}-L_c^{-1}\right)^{m_i}A_1 \\ &=A_1-(\lambda_i+c)^{m_i}(-1)^{m_i}A_{m_i+1}=A_1. \end{aligned}$$

2) Green's formula implies that $(u, w_k)_\Gamma=(L_c u, g_k)$, $u \in \mathscr{D}(L)$. In the case where $u \in \mathscr{D}(L_c^{1/4+\varepsilon})$, passage to the limit with respect to u gives the desired relation.

seen that $\sigma(B)=\{\sqrt{\mu_i}\omega^{\pm};\ i\geq 1\}$, where $\omega^{\pm}=-a\pm\sqrt{1-a^2}i$. The eigenvectors corresponding to $\sqrt{\mu_i}\,\omega^{\pm}$ are written as

$$\eta_{ij}^{\pm}=\frac{1}{\sqrt{2\mu_i}}\begin{bmatrix}\zeta_{ij}\\ \sqrt{\mu_i}\,\omega^{\pm}\zeta_{ij}\end{bmatrix},\qquad B\eta_{ij}^{\pm}=\sqrt{\mu_i}\,\omega^{\pm}\eta_{ij}^{\pm}.$$

The sets $\{\eta_{ij}^{+};\ i\geq 1,\ 1\leq j\leq n_i\}$ and $\{\eta_{ij}^{-}\}$ form orthonormal systems in H respectively. Each $\begin{bmatrix}f\\ g\end{bmatrix}\in H$ has a unique expression

$$\begin{bmatrix}f\\ g\end{bmatrix}=\sum_{i,j}a_{ij}\eta_{ij}^{+}+\sum_{i,j}b_{ij}\eta_{ij}^{-},$$

where

$$a_{ij}=\frac{\sqrt{2}\,(g_{ij}-\sqrt{\mu_i}\,\omega^{-}f_{ij})}{\omega^{+}-\omega^{-}},\qquad b_{ij}=\frac{\sqrt{2}\,(\sqrt{\mu_i}\,\omega^{+}f_{ij}-g_{ij})}{\omega^{+}-\omega^{-}},$$

and

$$f_{ij}=(f,\zeta_{ij})_{H_0},\qquad g_{ij}=(g,\zeta_{ij})_{H_0}.$$

Thus, the set $\{\eta_{ij}^{\pm}\}$ forms a basis of H. The resolvent $(\lambda-B)^{-1}$ is expressed as

$$(\lambda-B)^{-1}\begin{bmatrix}f\\ g\end{bmatrix}=\sum_{i,j}\frac{a_{ij}}{\lambda-\sqrt{\mu_i}\,\omega^{+}}\eta_{ij}^{+}+\sum_{i,j}\frac{b_{ij}}{\lambda-\sqrt{\mu_i}\,\omega^{-}}\eta_{ij}^{-},\qquad \lambda\in\rho(B).$$

The semigroup e^{tB} is analytic in $t>0$, and has an estimate

$$\|e^{tB}\|_{\mathcal{L}(H)}\leq\left(\frac{2}{1-a}\right)^{1/2}e^{-a\sqrt{\mu_1}t},\qquad t\geq 0.$$

This is easily derived from the expression of e^{tB};

$$e^{tB}\begin{bmatrix}f\\ g\end{bmatrix}=\sum_{i,j}e^{t\sqrt{\mu_i}\omega^{+}}a_{ij}\eta_{ij}^{+}+\sum_{i,j}e^{t\sqrt{\mu_i}\omega^{-}}b_{ij}\eta_{ij}^{-},\qquad t\geq 0.$$

§ 3. Stabilization by static output feedback

This section deals with the behavior of solutions of the equation

$$\frac{d}{dt}x(t)=-Lx(t)+\sum_{k=1}^{M}(x(t),y_k)L_c^{3/4-\varepsilon}\psi_k,\qquad x(0)=x_0. \tag{3.1}$$

The second term in (3.1) means, of course, a feedback term. We seek suitable y_k, $1\leq k\leq M$, which stabilize the evolution $x(t)$. To begin with, let $L_R^2(\Omega)$ denote a sub-

space of $L^2(\Omega)$ consisting of real-valued functions. $L^2_R(\Omega)$ is clearly a real Hilbert space. We give some more remarks on the projection operators P_{λ_i}. $P_{\lambda_i}u$ is uniquely expressed by a linear combination of ϕ_{ij};

$$P_{\lambda_i}u=\sum_{j=1}^{m_i}u_{ij}\phi_{ij},$$

which leads to

$$(P_{\lambda_i}u,\ \psi_{il})=\sum_{j=1}^{m_i}u_{ij}(\phi_{ij},\ \psi_{il}),\qquad 1\leq l\leq m_i.$$

Set

$$\Pi_{\lambda_i}=\begin{bmatrix}(\phi_{ij},\ \psi_{il}); & j\to 1,\ \cdots,\ m_i\\ & l\downarrow 1,\ \cdots,\ m_i\end{bmatrix}.$$

Lemma 3.1. *For each i, we have that* $\det \Pi_{\lambda_i}\neq 0$.

Proof. Consider a linear combination of the column vectors of Π_{λ_i} satisfying

$$\sum_{j=1}^{m_i}a_j(\phi_{ij},\ \psi_{il})=\left(\sum_{j=1}^{m_i}a_j\phi_{ij},\ \psi_{il}\right)=0,\qquad 1\leq l\leq m_i.$$

This means that $\sum_{j=1}^{m_i}a_j\phi_{ij}\perp P^*_{\lambda_i}L^2(\Omega)$. Thus, we have for any $v\in L^2(\Omega)$

$$0=\left(\sum_{j=1}^{m_i}a_j\phi_{ij},\ P^*_{\lambda_i}v\right)=\left(P_{\lambda_i}\sum_{j=1}^{m_i}a_j\phi_{ij},\ v\right)=\left(\sum_{j=1}^{m_i}a_j\phi_{ij},\ v\right),$$

from which we can conclude that $a_1=\cdots=a_{m_i}=0$. Q.E.D.

According to the lemma, the coefficients u_{ij} are expressed by

$$\begin{bmatrix}u_{i1}\\ \vdots\\ u_{im_i}\end{bmatrix}=\Pi^{-1}_{\lambda_i}\begin{bmatrix}(P_{\lambda_i}u,\ \psi_{i1})\\ \vdots\\ (P_{\lambda_i}u,\ \psi_{im_i})\end{bmatrix}=\Pi^{-1}_{\lambda_i}\begin{bmatrix}(u,\ \psi_{i1})\\ \vdots\\ (u,\ \psi_{im_i})\end{bmatrix}.\tag{3.2}$$

Let u be in $L^2_R(\Omega)$ and set

$$P_{\bar\lambda_i}u=\sum_{j=1}^{m_i}v_{ij}\bar\phi_{ij}.$$

It will be clear that $\bar\Pi_{\lambda_i}=\Pi_{\bar\lambda_i}$. Therefore, we have

$$\begin{bmatrix}\bar u_{i1}\\ \vdots\\ \bar u_{im_i}\end{bmatrix}=\bar\Pi^{-1}_{\lambda_i}\begin{bmatrix}\overline{(u,\ \psi_{i1})}\\ \vdots\\ \overline{(u,\ \psi_{im_i})}\end{bmatrix}=\Pi^{-1}_{\bar\lambda_i}\begin{bmatrix}(u,\ \bar\psi_{i1})\\ \vdots\\ (u,\ \bar\psi_{im_i})\end{bmatrix}=\begin{bmatrix}v_{i1}\\ \vdots\\ v_{im_i}\end{bmatrix}.$$

This shows that $(P_{\lambda_i}+P_{\bar{\lambda}_i})u \in L^2_R(\Omega)$. When λ_i is a real, both ϕ_{ij} and ψ_{ij} belong to $L^2_R(\Omega)$. Thus, we have proved

Lemma 3.2. *$P_{\lambda_i}+P_{\bar{\lambda}_i} \in \mathscr{L}(L^2_R(\Omega))$ when λ_i is a complex, and $P_{\lambda_i} \in \mathscr{L}(L^2_R(\Omega))$ when λ_i is a real.*

Let $-\lambda_i$, $1 \leq i \leq K$, be the members of $\sigma(-L)$ such that $\mathscr{R}e-\lambda_i > -\beta$ for a given $\beta>0$, and let P_K be a projection operator defined by

$$P_K = P_{\lambda_1}+\cdots+P_{\lambda_K}.$$

According to the symmetry of $\sigma(-L)$ and Lemma 3.2, P_K belongs to $\mathscr{L}(L^2_R(\Omega))$. As for a relationship between L^α_c and $L^2_R(\Omega)$, we have the following result. Since the proof is easy, we omit it.

Lemma 3.3. *If $u \in \mathscr{D}(L^\alpha_c) \cap L^2_R(\Omega)$, then $L^\alpha_c u \in L^2_R(\Omega)$. Similarly, if $u \in \mathscr{D}(L^{*\alpha}_c) \cap L^2_R(\Omega)$, then $L^{*\alpha}_c u \in L^2_R(\Omega)$. Here, $\alpha \in \boldsymbol{R}^1$.*

Returning to the control system (3.1), set $x_1(t)=P_K x(t)$, and $x_2(t)=Q_K x(t)$, where $Q_K=1-P_K$. The subspaces $P_K L^2(\Omega)$ and $Q_K L^2(\Omega)$ are invariant relative to the operator L. We are seeking y_k in the finite-dimensional subspace $P^*_K L^2_R(\Omega)$. Assuming such y_k, we note that every solution $x(t)$ belongs to $L^2_R(\Omega)$ if the same is true for x_0. Equation (3.1) is divided into two equations;

$$\frac{d}{dt}x_1 = -Lx_1+\sum_{k=1}^{M}(x_1, y_k)P_K L_c^{3/4-\varepsilon}\psi_k, \qquad x_1(0)=P_K x_0, \tag{3.3a}$$

$$\frac{d}{dt}x_2 = -Lx_2+\sum_{k=1}^{M}(x_1, y_k)Q_K L_c^{3/4-\varepsilon}\psi_k, \qquad x_2(0)=Q_K x_0. \tag{3.3b}$$

Equation (3.3a) is a finite-dimensional one, and we are going to write it in matrix form. Note that the operator L acting on $P_K L^2(\Omega)$ is equivalent to an upper triangular matrix, the diagonal elements of which are, of course, consisting of the eigenvalues of L. Let $P_K L_c^{3/4-\varepsilon}\psi_k$ be expressed by

$$P_K L_c^{3/4-\varepsilon}\psi_k = \sum_{i,j\,(i\leq K)} \psi^k_{ij}\phi_{ij}.$$

In view of (3.2), we have

$$\begin{bmatrix} \psi^k_{i1} \\ \vdots \\ \psi^k_{im_i} \end{bmatrix} = \Pi^{-1}_{\lambda_i} \begin{bmatrix} (L_c^{3/4-\varepsilon}\psi_k, \psi_{i1}) \\ \vdots \\ (L_c^{3/4-\varepsilon}\psi_k, \psi_{im_i}) \end{bmatrix} = \Pi^{-1}_{\lambda_i} \begin{bmatrix} (\psi_k, L_c^{*3/4-\varepsilon}\psi_{i1}) \\ \vdots \\ (\psi_k, L_c^{*3/4-\varepsilon}\psi_{im_i}) \end{bmatrix} = \tilde{\Pi}_i \begin{bmatrix} (\psi_k, \psi_{i1}) \\ \vdots \\ (\psi_k, \psi_{im_i}) \end{bmatrix},$$

$$1 \leq i \leq K, \qquad 1 \leq k \leq M.$$

Here, we have used (2.5b), and $\tilde{\Pi}_i$ indicates a non-singular matrix. Green's formula applied to ψ_k and ψ_{ij} gives a relation $(\psi_k, L_c^*\psi_{ij})=(h_k, \psi_{ij})_\Gamma$. Thus, using (2.5b) again, we find a non-singular matrix Θ_i such that

$$\begin{bmatrix} \psi_{i1}^k \\ \psi_{im_i}^k \end{bmatrix} = \Theta_i \begin{bmatrix} h_{i1}^k \\ h_{im_i}^k \end{bmatrix}, \qquad h_{ij}^k = (h_k, \psi_{ij})_\Gamma.$$

Thus, setting $\Theta = \mathrm{diag}\,[\Theta_1\, \Theta_2 \cdots \Theta_K]$, we obtain

$$\Psi = \begin{bmatrix} & k \to 1 \leq k \leq M \\ \psi_{ij}^k; & \\ & i,j \downarrow\ 1 \leq i \leq K,\ 1 \leq j \leq m_i \end{bmatrix} = \Theta \begin{bmatrix} & k \to \\ h_{ij}^k; & \\ & i,j \downarrow \end{bmatrix} = \Theta H. \tag{3.4}$$

For our convenience, let the eigenvalues $-\lambda_1, \cdots, -\lambda_K$ be relabelled as

$$-\lambda_1, \cdots, -\lambda_{K_1}, \quad -\bar{\lambda}_1, \cdots, -\bar{\lambda}_{K_1}; \quad \text{complex conjugates},$$
$$-\lambda_{2K_1+1}, \cdots, -\lambda_K; \quad \text{reals}.$$

We are interested in solutions of (3.1) which belong to $L_R^2(\Omega)$. According to this ordering, we can write

$$x_1(t) \longleftrightarrow \begin{bmatrix} u(t) \\ \bar{u}(t) \\ v(t) \end{bmatrix}; \quad K \times 1, \quad v(t); \quad \text{real valued},$$

$$L \text{ acting on } P_K L^2(\Omega) \longleftrightarrow \mathrm{diag}\,[\Lambda\, \bar{\Lambda}\, \Sigma], \quad \Sigma; \text{ a real matrix}.$$

Vectors y_k are designed in $P_K^* L_R^2(\Omega)$, and thus written as

$$y_k = \sum_{i,j(i\leq K_1)} y_{ij}^k \psi_{ij} + \sum_{i,j(i\leq K_1)} \bar{y}_{ij}^k \bar{\psi}_{ij} + \sum_{i,j(2K_1<i\leq K)} y_{ij}^k \psi_{ij}.$$

Set

$$Y_i = \begin{bmatrix} & k \downarrow 1, \cdots, M \\ y_{ij}^k; & \\ & j \to 1, \cdots, m_i \end{bmatrix},$$

and

$$Y = \left[\bar{Y}_1 \Pi_{\lambda_1} \vdots \cdots \vdots \bar{Y}_{K_1} \Pi_{\lambda_{K_1}} \right], \qquad Z = \left[Y_{2K_1+1} \Pi_{\lambda_{2K_1+1}} \vdots \cdots \vdots Y_K \Pi_{\lambda_K} \right].$$

By writing Ψ as $\begin{bmatrix} U \\ \bar{U} \\ V \end{bmatrix}$, (3.3a) is transformed into an equivalent equation

$$\frac{d}{dt}\begin{bmatrix} u \\ \bar{u} \\ v \end{bmatrix} = \left(-\begin{bmatrix} \Lambda & 0 & 0 \\ 0 & \bar{\Lambda} & 0 \\ 0 & 0 & \Sigma \end{bmatrix} + \begin{bmatrix} U \\ \bar{U} \\ V \end{bmatrix} [Y \ \bar{Y} \ Z] \right) \begin{bmatrix} u \\ \bar{u} \\ v \end{bmatrix}. \tag{3.5}$$

Writing now $u=p+iq$, $\Lambda=\Lambda_R+i\Lambda_I$, etc., we find that (3.5) is equivalent to

$$\frac{d}{dt}\begin{bmatrix} p \\ q \\ v \end{bmatrix} = \left(-\begin{bmatrix} \Lambda_R & -\Lambda_I & 0 \\ \Lambda_I & \Lambda_R & 0 \\ 0 & 0 & \Sigma \end{bmatrix} + \begin{bmatrix} U_R \\ U_I \\ V \end{bmatrix} [2Y_R \ -2Y_I \ Z] \right) \begin{bmatrix} p \\ q \\ v \end{bmatrix}. \tag{3.6}$$

We remark that $\begin{bmatrix} U_R \\ U_I \\ V \end{bmatrix} = \tilde{\Theta} H$, where $\tilde{\Theta}$ is a non-singular matrix. The well-known pole assignment argument can be applied to (3.6). Then, we have

Theorem 3.4. *Suppose that* $\left(-\begin{bmatrix} \Lambda_R & -\Lambda_I & 0 \\ \Lambda_I & \Lambda_R & 0 \\ 0 & 0 & \Sigma \end{bmatrix}, \begin{bmatrix} U_R \\ U_I \\ V \end{bmatrix} \right)$ *is a controllable pair. Then, we can find a set of* $y_k \in P_K^* L_R^2(\Omega)$, $1 \leq k \leq M$, *such that*

$$\left\| \exp t \left\{ -L + \sum_{k=1}^{M} (\cdot, y_k) L_c^{3/4-\varepsilon} \psi_k \right\} \right\|_{\mathcal{L}(L^2(\Omega))} \leq \text{const.} \ (1+t^m) e^{-\kappa t}, \qquad t \geq 0, \tag{3.7}$$

where $\kappa = \min_{i>K} \mathcal{R}e \lambda_i \geq \beta$, *and* $m \geq 0$ *is an integer.*

Remark. The semigroup in (3.7) is clearly analytic in $t>0$ since the perturbation to the operator $-L$ is a bounded operator.

Proof. Firstly, we consider solutions of (3.1) with $x_0 \in L_R^2(\Omega)$. Choose any $\kappa' > \kappa$. In view of the assumption, we can find a matrix $[2Y_R \ -2Y_I \ Z]$ such that every solution of (3.6) is evaluated as

$$\left\| \begin{bmatrix} p(t) \\ q(t) \\ v(t) \end{bmatrix} \right\| \leq \text{const.} \ e^{-\kappa' t} \left\| \begin{bmatrix} p(0) \\ q(0) \\ v(0) \end{bmatrix} \right\|, \qquad t \geq 0.$$

In other words, we can find $y_1, \cdots, y_M \in P_K^* L_R^2(\Omega)$ such that every solution of (3.3a) in $L_R^2(\Omega)$ has an estimate

$$\|x_1(t)\| \leq \text{const.} \ e^{-\kappa' t} \|x_1(0)\|, \qquad t \geq 0.$$

Let us turn to an estimate for $x_2(t)$. The restriction of L into the subspace $Q_K L^2(\Omega)$ is denoted by L_2. Let $-\lambda_{K+1}, \cdots, -\lambda_J$ be the eigenvalues of $-L$ lying on the vertical line $\mathcal{R}e \ \lambda = -\kappa$. Setting $m = \max(m_{K+1}, \cdots, m_J) - 1$, we easily derive an estimate

$$\|e^{-tL_2}\|_{\mathcal{L}(Q_K L^2(\Omega))} \leq \text{const.} \ (1+t^m) e^{-\kappa t}, \qquad t \geq 0.$$

Writing (3.3b) as an integral form, we have

$$x_2(t)=e^{-tL_2}x_2(0)+\int_0^t e^{-(t-s)L_2}\sum_{k=1}^{M}(x_1(s), y_k)Q_K L_c^{3/4-\varepsilon}\psi_k ds,$$

and using the above estimates, we easily obtain

$$\|x_2(t)\|\leq \text{const.}\,(1+t^m)e^{-\kappa t}\|x_0\|, \qquad t\geq 0.$$

Thus, we have established the estimate (3.7) with the norm replaced by $\|\ \|_{\mathscr{L}(L^2_R(\Omega))}$, from which we easily obtain the final estimate (3.7) with the same constant. Q.E.D.

§ 4. Stabilization by a functional observer

The purpose of this section is to design the stabilizing observer (2.3) under suitable conditions. After being designed, the observer will be reduced to a finite-dimensional equation. We begin with the definition of a subset $\hat{H}$ of H. It is defined by

$$\hat{H}=\left\{h=\sum_{i,j}a_{ij}\eta^+_{ij}+\sum_{i,j}b_{ij}\eta^-_{ij};\ \ \overline{a_{ij}}=b_{ij} \text{ for any } i \text{ and } j\right\}.$$

$\hat{H}$ is clearly a real Hilbert space with the norm and the inner product induced by those in H. We remark that H is uniquely decomposed as

$$H=\hat{H}\oplus i\hat{H}.$$

In fact, every $v=\sum_{i,j}a_{ij}\eta^+_{ij}+\sum_{i,j}b_{ij}\eta^-_{ij}\in H$ is uniquely expressed as $v=v_R+iv_I$, $v_R, v_I\in\hat{H}$, where

$$v_R=\sum_{i,j}\frac{1}{2}(a_{ij}+\overline{b_{ij}})\eta^+_{ij}+\sum_{i,j}\frac{1}{2}(\overline{a_{ij}}+b_{ij})\eta^-_{ij},$$

$$v_I=\sum_{i,j}\frac{1}{2i}(a_{ij}-\overline{b_{ij}})\eta^+_{ij}+\sum_{i,j}\frac{1}{-2i}(\overline{a_{ij}}-b_{ij})\eta^-_{ij}.$$

It is not difficult to show the following properties:

Lemma 4.1. (i) $\|v_R\|_{\hat{H}}, \|v_I\|_{\hat{H}}\leq(2/(1-a))^{1/2}\|v\|_H$[3].

---

3) In fact, let $v=\begin{bmatrix} f \\ g \end{bmatrix}$ as in §2. Then,

$$\|v_R\|^2_{\hat{H}}\leq 4\sum_{i,j}\left|\frac{1}{2}(a_{ij}+\overline{b_{ij}})\right|^2\leq 2\sum_{i,j}(|a_{ij}|^2+|b_{ij}|^2)$$

$$\leq 2\sum_{i,j}\frac{1}{2(1-a^2)}\{|g_{ij}-\sqrt{\mu_i}\omega^- f_{ij}|^2+|\sqrt{\mu_i}\omega^+ f_{ij}-g_{ij}|^2\}$$

$$\leq\frac{2}{1-a}\{\|g\|^2_{H_0}+\|\sqrt{A}f\|^2_{H_0}\}\leq\frac{2}{1-a}\|v\|^2_H.$$

(ii) B; $\mathscr{D}(B)\cap\hat{H}\xrightarrow{onto}\hat{H}$.
(iii) $v\in\mathscr{D}(B)\longleftrightarrow v_R$ *and* $v_I\in\mathscr{D}(B)$.

Equations (2.7), (2.3), and (2.4) are written as an equation in $L^2(\Omega)\times H$;

$$\frac{d}{dt}\begin{bmatrix}x\\ v\end{bmatrix}=\mathscr{M}\begin{bmatrix}x\\ v\end{bmatrix},\qquad \begin{bmatrix}x(0)\\ v(0)\end{bmatrix}=\begin{bmatrix}x_0\\ v_0\end{bmatrix}. \tag{4.1}$$

The operator $\mathscr{M}$ is of the form

$$\begin{bmatrix}-L & 0\\ 0 & B\end{bmatrix}+\begin{bmatrix}0 & \sum_{k=1}^{M}(\cdot,\rho_k)_H L_c^{3/4-\varepsilon}\psi_k\\ \sum_{k=1}^{N}(L_c^{\alpha}\cdot, L_c^{*3/4-\varepsilon}g_k)\xi_k & \sum_{k=1}^{M}(\cdot,\rho_k)_H\alpha_k\end{bmatrix}, \tag{4.2}$$

the first term of which generates an analytic semigroup. Thus, in exactly the same way as in [7], we find that the perturbed operator $\mathscr{M}$ also generates an analytic semigroup $e^{t\mathscr{M}}$, $t>0$. Henceforth, we assume that ξ_k, α_k and ρ_k belong to $\hat{H}$. Then, Lemma 3.3 and 4.1 imply that, if $\begin{bmatrix}x_0\\ v_0\end{bmatrix}$ is in $L_R^2(\Omega)\times\hat{H}$, so is $\begin{bmatrix}x(t)\\ v(t)\end{bmatrix}$, $t>0$. In other words, $e^{t\mathscr{M}}\in\mathscr{L}(L_R^2(\Omega)\times\hat{H})$, $t\geq 0$.

We briefly outline the procedure to obtain the finally designed control system. A finite-dimensional observer which will be derived from (4.1) is an equation in $\boldsymbol{R}^S$, $S<\infty$, described by

$$\frac{d}{dt}\boldsymbol{y}(t)=\mathscr{F}\boldsymbol{y}(t)+\sum_{k=1}^{N}(L_c^{\alpha}x(t), L_c^{*3/4-\varepsilon}g_k)\boldsymbol{\xi}_k+\mathscr{G}\boldsymbol{y}(t),\qquad \boldsymbol{y}(0)=\boldsymbol{y}_0, \tag{4.3}$$

where the appearing matrices and vectors are real valued. Then, $f_k(t)$ will be, instead of (2.4), given by

$$f_k(t)=\boldsymbol{c}_k'\boldsymbol{y}(t),\qquad 1\leq k\leq M, \tag{4.4}$$

where $\boldsymbol{c}_k$ indicate real column vectors. Equations (2.7), (4.3) and (4.4) are summarized as an equation in $L_R^2(\Omega)\times\boldsymbol{R}^S$ (or in $L^2(\Omega)\times\boldsymbol{C}^S$)

$$\frac{d}{dt}\begin{bmatrix}x\\ \boldsymbol{y}\end{bmatrix}=\mathscr{N}\begin{bmatrix}x\\ \boldsymbol{y}\end{bmatrix},\qquad \begin{bmatrix}x(0)\\ \boldsymbol{y}(0)\end{bmatrix}=\begin{bmatrix}x_0\\ \boldsymbol{y}_0\end{bmatrix}. \tag{4.5}$$

It is easily seen that the operator $\mathscr{N}$ generates an analytic semigroup $e^{t\mathscr{N}}$, $t>0$. Putting $u(t,\cdot)=L_c^{\alpha/2}x(t)$ in (4.5), we will prove later that the pair $u(t,\cdot)$, $\boldsymbol{y}(t)$ is the solution of the control system described by

$$
(4.6)\qquad \begin{cases} \dfrac{d}{dt}u(t,\cdot)=-\mathscr{L}u(t,\cdot), \\ \dfrac{\partial}{\partial\nu_L}u(t,\xi)+\sigma(\xi)u(t,\xi)=\sum_{k=1}^{M}\boldsymbol{c}_k'\boldsymbol{y}(t)h_k(\xi), \\ \dfrac{d}{dt}\boldsymbol{y}(t)=\mathscr{F}\boldsymbol{y}(t)+\sum_{k=1}^{N}(u(t,\cdot),w_k)_\Gamma\boldsymbol{\xi}_k+\mathscr{G}\boldsymbol{y}(t), \\ u(0,\cdot)=u_0,\quad \boldsymbol{y}(0)=\boldsymbol{y}_0. \end{cases}
$$

Before stating the main result in this section, we need some preparations and assumptions. According to the former labelling of ϕ_{ij} or λ_i, let us define the matrices W_i by

$$
W_i=\begin{bmatrix} & k\downarrow 1,\cdots,N \\ (w_k,\phi_{ij})_\Gamma; & \\ & j\to 1,\cdots,m_i \end{bmatrix},\qquad i\geq 1.
$$

Expressing ξ_k as $\sum_{i,j}\xi_{ij}^k\eta_{ij}^+ + \sum_{i,j}\overline{\xi_{ij}^k}\eta_{ij}^-$, set

$$
\Xi_i=\begin{bmatrix} & k\downarrow 1,\cdots,N \\ \xi_{ij}^k; & \\ & j\to 1,\cdots,n_i \end{bmatrix},\qquad i\geq 1.
$$

Throughout the section, it is assumed that the eigenvalues of the operator A satisfy

$$
(4.7)\qquad {}^\exists c_0>0,\quad 0<{}^\exists\gamma<2;\quad \sqrt{\mu_n}\leq c_0 n^\gamma,\quad n\geq 1.
$$

In addition, adjusting A or a if necessary, we may assume that

$$
\sigma(-L)\cap\sigma(B)=\varnothing.
$$

In [7], the growth order γ was less than one. Thus, the present assumption (4.7) is more desirable than the previous one (see §1). Let κ in Theorem 3.4 be $>a\sqrt{\mu_1}$. The main result in this section is

Theorem 4.2. *Suppose that the controllers $h_1,\cdots,h_M$ are chosen in order to satisfy the assumption in Theorem* 3.4. *Suppose further that the matrices W_i and Ξ_i satisfy*

$$
(4.8)\qquad \begin{cases} \operatorname{rank} W_i=m_i, & 1\leq i\leq K, \\ \operatorname{rank}\Xi_i=N, & i\geq 1. \end{cases}
$$

Then, we can find suitable vectors α_k, $\rho_k\in\hat{H}$, $1\leq k\leq M$, and consecutively a constant $c_1>0$ which ensure an estimate

$$
(4.9)\qquad \|e^{t\mathscr{A}}\|_{\mathscr{L}(L^2(\Omega)\times H)}\leq c_1 e^{-a\sqrt{\mu_1}t},\qquad t\geq 0.
$$

Furthermore, (4.1) *is reduced to* (4.6), *and there is a constant* $c_2>0$ *such that an estimate*

$$\left\|\begin{bmatrix} u(t,\cdot) \\ y(t) \end{bmatrix}\right\|_{L^2(\Omega)\times R^S} \leq c_2 e^{-a\sqrt{\mu_1}t}\left\|\begin{bmatrix} u_0 \\ y_0 \end{bmatrix}\right\|_{L^2(\Omega)\times R^S}, \qquad t\geq 0 \tag{4.10}$$

holds for every solution of (4.6).

Proof of Theorem 4.2. The proof is divided into several steps, and its outline will be first drawn. To begin with, we define the operator C by

$$Cu=\sum_{k=1}^{N}(L_c^a u,\ L_c^{*3/4-\varepsilon}g_k)\xi_k, \qquad u\in\mathscr{D}(L_c^a),$$

and seek a solution of the operator equation $XL+BX=-C$ on $\mathscr{D}(L)$. According to the assumption $\sigma(-L)\cap\sigma(B)=\varnothing$, we can state

Proposition 4.3. *There exists the unique solution* $X\in\mathscr{L}(L^2(\Omega);H)\cap\mathscr{L}(L_R^2(\Omega);\hat{H})$ *of the operator equation* $XL+BX=-C$. *The solution* X *is explicitly written by*

$$Xu=-\sum_{i,j}\sum_{k=1}^{N}f_k(\sqrt{\mu_i}\,\omega^+;u)\xi_{ij}^k\eta_{ij}^+-\sum_{i,j}\sum_{k=1}^{N}f_k(\sqrt{\mu_i}\,\omega^-;u)\bar{\xi}_{ij}^k\bar{\eta}_{ij}^-, \tag{4.11}$$

where

$$f_k(\lambda;u)=(L_c^a(L+\lambda)^{-1}u,\ L_c^{*3/4-\varepsilon}g_k), \qquad 1\leq k\leq N.$$

In view of Proposition 4.3, it is clear that $X^*\in\mathscr{L}(H;L^2(\Omega))\cap\mathscr{L}(\hat{H};L_R^2(\Omega))$. Our main interest is in the spaces $L_R^2(\Omega)$ and $\hat{H}$ since we are seeking real-valued solutions of (4.6). As is stated in §3, we remark that P_K^* also belongs to $\mathscr{L}(L_R^2(\Omega))$. As in [7], the following result is most important:

Proposition 4.4. *Under the assumptions of Theorem* 4.2, *we have a relation*

$$\overline{X^*\hat{H}}\supset P_K^*L_R^2(\Omega). \tag{4.12}$$

The following proposition is a "perturbed" result of Theorem 3.4, and is proved with the help of Proposition 4.4. Since the proof is the same as that of [7, Lemma 4.4], it is omitted.

Proposition 4.5. *Let* D *be a bounded operator in* $L^2(\Omega)$ *(or* $L_R^2(\Omega)$*) defined by*

$$Du=\sum_{k=1}^{M}(u,\ X^*\rho_k)L_c^{3/4-\varepsilon}\psi_k, \qquad \rho_k\in\hat{H}.$$

Let any κ', $a\sqrt{\mu_1}<\kappa'<\kappa$, *be fixed. Under the assumptions of Theorem* 4.2, *we can find a set of weighting vectors* $\rho_1,\cdots,\rho_M\in\hat{H}$ *such that a region* $\Sigma\cup\{\lambda;\ \mathrm{Re}\,\lambda\geq-\kappa'\}$ *is*

contained in the resolvent set $\rho(-L+D)$, *where* Σ *denotes a sector* $\Sigma=\{\lambda=\sigma+i\tau;\ \sigma\geq -a|\tau|+b,\ \tau\in \boldsymbol{R}^1\}$ *for some* $a, b>0$.

The operator $-L+D$ *generates a semigroup* $e^{t(-L+D)}$ *analytic in* $t>0$.

The evaluation of the semigroup $e^{t\mathcal{A}}$ will be carried out in the same manner as in [7]. We choose a set of weighting vectors ρ_k as stated in Proposition 4.5, and set $\alpha_k=XL_c^{3/4-\varepsilon}\psi_k\in\hat{H}$, $1\leq k\leq M$. It is easily seen from (4.1) (or (2.7) and (2.3)) that the function $Xx(t)-v(t)$ satisfies

$$\frac{d}{dt}(Xx(t)-v(t))=B(Xx(t)-v(t)),\qquad Xx(0)-v(0)=Xx_0-v_0.$$

Thus, $Xx(t)-v(t)=e^{tB}(Xx_0-v_0)$. Substituting this relation into (2.7), we have

$$\frac{d}{dt}x(t)=(-L+D)x(t)+\sum_{k=1}^{M}(e^{tB}(v_0-Xx_0),\ \rho_k)_H L_c^{3/4-\varepsilon}\psi_k,$$

which leads to an expression of $x(t)$ in an integral form;

$$x(t)=e^{t(-L+D)}x_0+\int_0^t e^{(t-s)(-L+D)}\sum_{k=1}^{M}(e^{sB}(v_0-Xx_0),\ \rho_k)_H L_c^{3/4-\varepsilon}\psi_k ds.$$

According to Proposition 4.5, we easily derive an estimate

$$\|e^{t(-L+D)}\|_{\mathcal{L}(L^2(\Omega))}\leq \text{const. } e^{-\varepsilon' t},\qquad t\geq 0.$$

Furthermore, by recalling that $\|\exp tB\|_{\mathcal{L}(H)}\leq\sqrt{2/(1-a)}\exp(-a\sqrt{\mu_1}\,t)$, $t\geq 0$, the above expression of $x(t)$ readily implies that

$$\|x(t)\|\leq \text{const. } e^{-a\sqrt{\mu_1}t}(\|x_0\|+\|v_0\|_H),\qquad t\geq 0,$$

through which a similar estimate for $\|v(t)\|_H$ is immediately obtained. The estimate (4.9) has just been established.

The next step is to design a finite-dimensional observer (4.3). We recall that both $\{\eta_{ij}^+;\ i\geq 1,\ 1\leq j\leq n_i\}$ and $\{\eta_{ij}^-\}$ form orthonormal systems in H. Thus, we may, with no loss of generality, that ρ_k are expressed by linear combinations of a finite number of $\eta_{ij}^{\pm}$, namely

$$\rho_k=\sum_{i,j\,(i\leq I)}\rho_{ij}^k\eta_{ij}^+ +\sum_{i,j\,(i\leq I)}\overline{\rho_{ij}^k}\eta_{ij}^-,\qquad 1\leq k\leq M.$$

Let $\tilde{P}_I$ be a projection operator in H defined by

$$\tilde{P}_I=\frac{1}{2\pi i}\oint(\lambda-B)^{-1}d\lambda,$$

where the integral is taken along a contour encircling the eigenvalues $\sqrt{\mu_i}\,\omega^{\pm}$, $1\leq i$

$\leq I$. By setting $v_1(t)=\tilde{P}_I v(t)$, (4.1) is reduced to an equation in $L^2(\Omega)\times\tilde{P}_I H$;

$$(4.13)\quad \begin{cases} \dfrac{d}{dt}x(t)=-Lx(t)+\sum_{k=1}^{M}(v_1(t),\ \rho_k)_H L_c^{3/4-\varepsilon}\psi_k, \\ \dfrac{d}{dt}v_1(t)=B_1 v_1(t)+\sum_{k=1}^{N}(L_c^{\alpha}x(t),\ L_c^{*3/4-\varepsilon}g_k)\tilde{P}_I\xi_k+\sum_{k=1}^{M}(v_1(t),\ \rho_k)_H\tilde{P}_I\alpha_k, \end{cases}$$

where B_1 indicates the restriction of B onto the invariant subspace $\tilde{P}_I\hat{H}$. Equation (4.13) is a closed loop system with respect to $x(t)$ and $v_1(t)$. The same consideration as that for (4.1) applies to (4.13); the coefficient operator appearing in (4.13) generates an analytic semigroup in $L^2(\Omega)\times\tilde{P}_I H$ or $L_R^2(\Omega)\times\tilde{P}_I\hat{H}$. It follows from (4.9) that every solution of (4.13) is estimated as

$$\left\|\begin{bmatrix}x(t)\\ v_1(t)\end{bmatrix}\right\|_{L^2(\Omega)\times\tilde{P}_I H}\leq c_3 e^{-a\sqrt{\mu_1}t}\left\|\begin{bmatrix}x(0)\\ v(0)\end{bmatrix}\right\|_{L^2(\Omega)\times H},\qquad t\geq 0.$$

The left-hand side of the above inequality is, however, independent of $(1-\tilde{P}_I)v_0$ since the solution of (4.13) is uniquely determined by $x(0)$ and $v_1(0)$. Therefore, setting $(1-\tilde{P}_I)v_0=0$, we obtain an estimate

$$(4.14)\qquad \left\|\begin{bmatrix}x(t)\\ v_1(t)\end{bmatrix}\right\|_{L^2(\Omega)\times\tilde{P}_I H}\leq c_3 e^{-a\sqrt{\mu_1}t}\left\|\begin{bmatrix}x(0)\\ v_1(0)\end{bmatrix}\right\|_{L^2(\Omega)\times\tilde{P}_I H},\qquad t\geq 0.$$

Considering (4.13) in $L_R^2(\Omega)\times\tilde{P}_I\hat{H}$, we express $v_1(t)$ as

$$v_1(t)=\sum_{i,j(i\leq I)} v_{ij}(t)\eta_{ij}^{+}+\sum_{i,j(i\leq I)}\overline{v_{ij}(t)}\eta_{ij}^{-},$$

$$v_{ij}(t)=p_{ij}(t)+\sqrt{-1}q_{ij}(t).$$

Set $\boldsymbol{y}=[p_{11}\ q_{11}p_{12}\ q_{12}\cdots p_{In_I}q_{In_I}]'$. A simple, but tedious calculation shows that (4.13) considered in $L_R^2(\Omega)\times\tilde{P}_I\hat{H}$ is equivalent to (4.5), where the matrices and vectors are suitably determined. Especially, $\mathscr{F}$ is written as

$$\mathscr{F}=\operatorname{diag}[\sqrt{\mu_1}F\cdots\sqrt{\mu_1}F\sqrt{\mu_2}F\cdots\sqrt{\mu_I}F\cdots\sqrt{\mu_I}F],$$

$$F=\begin{bmatrix}-a & -\sqrt{1-a^2}\\ \sqrt{1-a^2} & -a\end{bmatrix},$$

where $\sqrt{\mu_i}F$ are arranged according to the multiplicities n_i. In view of an equivalence relation $|\boldsymbol{y}(t)|\sim\|v_1(t)\|_{\hat{H}}$, we conclude that

$$(4.15)\qquad \|e^{t\mathscr{A}}\|_{\mathscr{L}(L_R^2(\Omega)\times\boldsymbol{R}^S)}\leq c_4 e^{-a\sqrt{\mu_1}t},\qquad t\geq 0.$$

Here, we may replace the subscript $\mathscr{L}(L_R^2(\Omega)\times\boldsymbol{R}^S)$ by $\mathscr{L}(L^2(\Omega)\times\boldsymbol{C}^S)$. It is not

difficult to prove that the operator $\mathcal{N}$ has a compact resolvent since the same is true for $\begin{bmatrix} -L & 0 \\ 0 & \mathcal{F} \end{bmatrix}$ and $\mathcal{N}$ is this operator plus a suitable perturbation. Thus, the spectrum $\sigma(\mathcal{N})$ consists only of eigenvalues, and each eigenvalue is a pole of the resolvent $(\lambda-\mathcal{N})^{-1}$. An easy conclusion from (4.15) is that $(\lambda-\mathcal{N})^{-1}$ exists for $\mathrm{Re}\,\lambda > -a\sqrt{\mu_1}$. More is true for our $\mathcal{N}$.

Lemma 4.6. *If an eigenvalue of $\mathcal{N}$ lies on the vertical line $\mathrm{Re}\,\lambda = -a\sqrt{\mu_1}$, the order of its pole cannot be more than one.*

Proposition 4.7. *Let k be a positive integer. Then, we have*

$$\|\mathcal{N}^k e^{t\mathcal{N}}\|_{\mathcal{L}(L^2(\Omega)\times C^S)} \leq \text{const.} \max(t^{-k}, 1) e^{-a\sqrt{\mu_1}t} \qquad t>0.$$

The final step is as follows: Setting $u(t)=L_c^{\alpha/2}x(t)$, we will evaluate the norm of the pair $(u(t), y(t))$ in (4.6). Noting that $\mathcal{D}(\mathcal{N})=\mathcal{D}(L)\times C^S$ (or $=(\mathcal{D}(L)\cap L_R^2(\Omega))\times R^S$), we first have a relation

$$\begin{bmatrix} u(t) \\ 0 \end{bmatrix} = \begin{bmatrix} L_c^{\alpha/2} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} L_c^{\alpha/2} & 0 \\ 0 & 0 \end{bmatrix} \mathcal{N}^{-1} \mathcal{N} e^{t\mathcal{N}} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}, \qquad t>0.$$

The factor $[\cdots]\mathcal{N}^{-1}$ in the last equation is a bounded operator. Applying Proposition 4.7 to the above relation, we find that

$$\|u(t)\| \leq \text{const. } e^{-a\sqrt{\mu_1}t}(\|u_0\|+|y_0|), \qquad t\geq 1.$$

On the other hand, the estimate (4.15) implies that

$$|y(t)| \leq c_4 e^{-a\sqrt{\mu_1}t}(\|u_0\|+|y_0|), \qquad t\geq 0.$$

The first equation of (4.5) gives an expression of $u(t)$ in an integral form;

$$u(t) = e^{-tL}u_0 + \int_0^t L_c^{\alpha/2} e^{-(t-s)L} \sum_{k=1}^{M} c_k' y(s) L_c^{5/4-\varepsilon} \psi_k ds.$$

Thus, $u(t)$ is evaluated on $0\leq t\leq 1$ as

$$\|u(t)\| \leq \text{const. } (\|u_0\|+|y_0|).$$

Here, we have used the moment inequality for $L_c^{\alpha/2}$. Combining these estimates altogether, we have reached the final conclusion (4.10).

Several results used above are left unproved. To complete the proof of Theorem 4.2, we will henceforth prove them. We first show

Proof of Proposition 4.3. By recalling the moment inequality for L_c^{α}, it will be

clear that the operator X defined by (4.11) belongs to $\mathscr{L}(L_R^2(\Omega); \hat{H})$ as well as $\mathscr{L}(L^2(\Omega); H)$. Let u be in $\mathscr{D}(L)$. Then, noting that

$$f_k(\lambda; Lu)=(L_c^a u, L_c^{*3/4-\varepsilon} g_k)-\lambda f_k(\lambda; u),$$

we calculate as

$$XLu=-\sum_{i,j}\sum_k\{(L_c^a u, L_c^{*3/4-\varepsilon} g_k)-\sqrt{\mu_i}\,\omega^+ f_k(\sqrt{\mu_i}\,\omega^+; u)\}\xi_{ij}^k\eta_{ij}^+$$
$$-\sum_{i,j}\sum_k\{(L_c^a u, L_c^{*3/4-\varepsilon} g_k)-\sqrt{\mu_i}\,\omega^- f_k(\sqrt{\mu_i}\,\omega^-; u)\}\overline{\xi_{ij}^k}\eta_{ij}^-.$$

The operator B is closed. Therefore, we can show that

$$BXu=-\sum_{i,j}\sum_k f_k(\sqrt{\mu_i}\,\omega^+; u)\xi_{ij}^k\sqrt{\mu_i}\,\omega^+\eta_{ij}^+-\sum_{i,j}\sum_k f_k(\sqrt{\mu_i}\,\omega^-; u)\overline{\xi_{ij}^k}\sqrt{\mu_i}\,\omega^-\eta_{ij}^-.$$

In fact, since u is in $\mathscr{D}(L)$, we find that $|f_k(\sqrt{\mu_i}\,\omega^\pm; u)|\leq \text{const.}/\sqrt{\mu_i}$, $i\geq 1$. Thus, the right-hand side of the above relation has the meaning. From these two equations, we see that

$$XLu+BXu=-\sum_k (L_c^a u, L_c^{*3/4-\varepsilon} g_k)\{\sum_{i,j}\xi_{ij}^k\eta_{ij}^+ + \sum_{i,j}\overline{\xi_{ij}^k}\eta_{ij}^-\}$$
$$=-Cu,\ u\in\mathscr{D}(L).$$

Uniqueness[4]. We adopt the approach used in [6] to suggest that our theory developed here has a substantial applicability to a more general operator L. Let Γ be a contour separating $\sigma(-L)$ from $\sigma(B)$. When $\mathscr{R}e\,\lambda$ goes to $-\infty$, Γ consists of two symmetrical lines; $\mathscr{I}m\,\lambda=\pm p\,\mathscr{R}e\,\lambda$, where $0<p<\tan(\sqrt{1-a^2}/a)$ (see Fig. 1). We have to prove that $C=0$ implies $X=0$. Now, $C=0$ implies that $(B-\lambda)^{-1}X=-X(L+\lambda)^{-1}$ for $\lambda\in\rho(-L)\cap\rho(B)$. Multiplying the both sides by $e^{t\lambda}$, $t>0$, and integrating them along Γ, we have

$$\frac{1}{2\pi i}\int_\Gamma e^{t\lambda}(B-\lambda)^{-1}Xd\lambda=\frac{-1}{2\pi i}\int_\Gamma e^{t\lambda}X(L+\lambda)^{-1}d\lambda.$$

It is easily seen that the left-hand side of the above equation is equal to zero. On the other hand, the right-hand side is equal to $-Xe^{-tL}$. Since e^{-tL} converges to 1

4) In our case, the set $\{\phi_{ij};\ i\geq 1,\ 1\leq j\leq m_i\}$ spans $L^2(\Omega)$ [1]. Based on this fact, another proof is possible. Since $XL=-BX$ on $\mathscr{D}(L)$, we find that

$$-BX\phi_{ij}=XL\phi_{ij}=X\{\lambda_i\phi_{ij}+\sum_{k<j}\alpha_{kj}^i\phi_{ik}\}.$$

Therefore

$$X\phi_{ij}+\sum_{k<j}\alpha_{kj}^i(B+\lambda_i)^{-1}X\phi_{ik}=0.$$

When $j=1$, we have that $X\phi_{i1}=0$, and thus inductively that $X\phi_{ij}=0$, $1\leq j\leq m_i$. Consequently, $Xu=0$, $u\in$ a dense subset. Passage to the limit with respect to u shows that $X=0$.

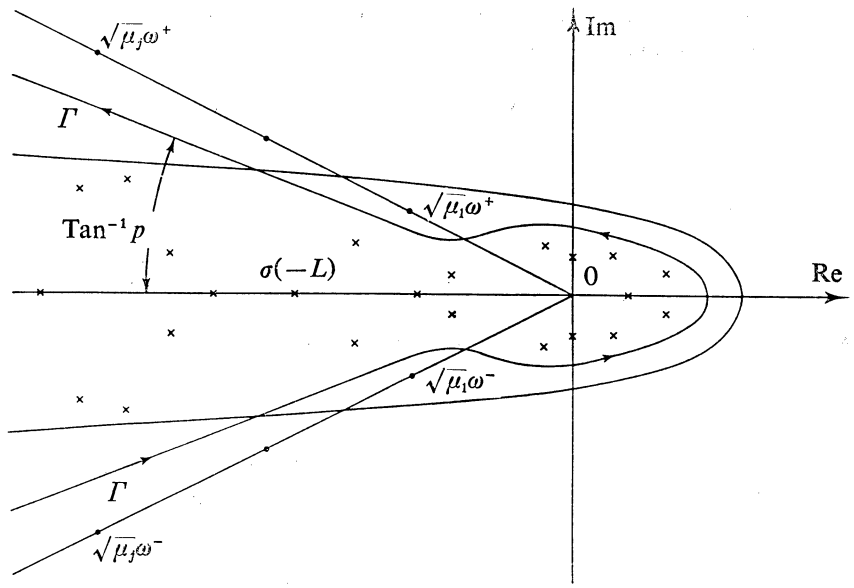


Fig. 1

strongly as $t \downarrow 0$, we conclude that $\lim_{t\downarrow 0} Xe^{-tL}=0$. Q.E.D.

Proof of Proposition 4.4. We consider X as an element of $\mathscr{L}(L^2_R(\Omega); \hat{H})$. The relation to be proved is equivalent to the relation; $\mathrm{Ker}\, X \subset (P^*_K L^2_R(\Omega))^\perp$. Let $u \in L^2_R(\Omega)$ be an arbitrary element of $\mathrm{Ker}\, X$. Since the system $\{\eta^\pm_{ij}\}$ forms a basis of H, we find from (4.11) that

$$\sum_{k=1}^{N} f_k(\sqrt{\mu_i}\,\omega^+; u)\xi^k_{ij}=0, \quad i\geq 1, \quad 1\leq j\leq n_i.$$

This, combined with our assumptions rank $\Xi_i=N$, $i\geq 1$, implies that

$$(4.16) \qquad f_k(\sqrt{\mu_i}\,\omega^+; u)=(L^a_c(L+\sqrt{\mu_i}\,\omega^+)^{-1}u,\ L_c^{*3/4-\varepsilon}g_k)=0, \quad 1\leq k\leq N, \quad i\geq 1.$$

We will see how (4.16) implies that $f_k(\lambda; u)\equiv 0$. Let $\eta>1$ be a constant satisfying $\gamma<\eta<2$, and δ a positive constant. Let us consider a map

$$m;\ \lambda \longrightarrow (\lambda+\delta i)^\eta e^{-i\zeta}$$

defined on the upper half-plane $\mathscr{I}m\,\lambda\geq 0$. Here, $(\cdot)^\eta$ is considered as single-valued in the plane with a cut along the negative imaginary axis. Therefore, $m(\cdot)$ is analytic in a domain containing $\{\lambda;\ \mathscr{I}m\,\lambda\geq 0\}$. Choose a constant ζ so that

$$(\eta-1)\pi<\zeta<\min\left\{\pi,\ (\eta-1)\pi+\mathrm{Tan}^{-1}\frac{\sqrt{1-a^2}}{a}\right\}.$$

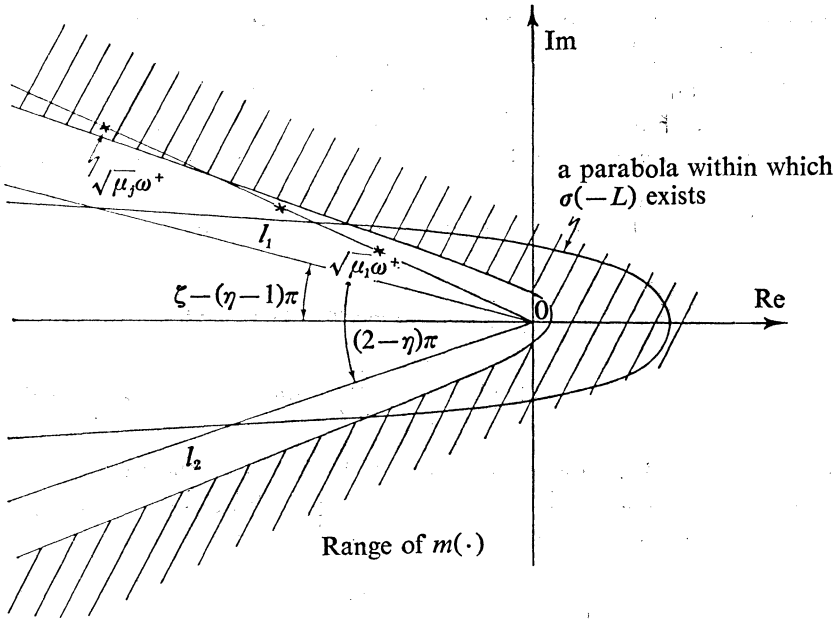


Fig. 2

Then, the range of $m(\cdot)$ is as shown in Fig. 2. When λ tends to $-\infty$ and $+\infty$ on the real axis, the asymptotes of the contour $m(\lambda)$ are consisting of rays l_1 and l_2 respectively, and they are the only asymptotes. The map $m(\cdot)$ is one-to-one, and hence there are only a finite number of λ such that $m(\lambda) \in \sigma(-L)$. If there is a real λ such that $m(\lambda) \in \sigma(-L)$, we can adjust a constant δ in order that $m(\lambda)$ belongs to $\rho(-L)$ for every real λ.

Let us consider a function $f_k(m(\lambda); u)$ instead of $f_k(\lambda; u)$. For simplicity, we may abbreviate u and the subscript k. It is not difficult to show that every singularity of $f(m(\lambda))$ on the upper half-plane is a pole. Thus, $f(m(\lambda))$ is a meromorphic function. It follows from (4.16) that

$$\sigma_j = \sqrt{\mu_j}^{1/\eta} e^{i(\pi-\beta+\zeta)/\eta} - i\delta = \sqrt{\mu_j}^{1/\eta}\omega_0 - i\delta, \qquad j \geq j_0$$

are zeros of $f(m(\lambda))$, where j_0 is an integer such that

$$\sqrt{\mu_{j_0}}^{1/\eta} \sin\frac{\pi-\beta+\zeta}{\eta} > \delta \quad \text{and} \quad \beta = \mathrm{Tan}^{-1}(\sqrt{1-a^2}/a).$$

We suppose that $f(m(\lambda)) \not\equiv 0$ in order to derive a contradiction. If necessary, we may further adjust δ so that $f(m(0)) \neq 0$. The following result is an extension of Carleman's theorem to a meromorphic function. Since the proof is the same as that of [7, Lemma 4.7], it is omitted.

Lemma 4.8. *Let $R>0$ be arbitrary, but be chosen large enough. Suppose that $f(m(\lambda))$ has the zeros $r_\nu e^{i\theta_\nu}$ of order m_ν, $1\leq\nu\leq p$, inside the closed contour C_R consisting of the semicircle $|\lambda|=R$, $0<\arg\lambda<\pi$, and the segment $|\lambda|\leq R$ on the real axis. Furthermore, let $\zeta_j=|\zeta_j|e^{i\kappa_j}$, $1\leq j\leq J$, be the poles of $f(m(\lambda))$ of order l_j on the upper half-plane, J being independent of R. Finally set $a=1/f(m(0))$. Then, we have a relation*

$$\sum_{\nu=1}^{p} m_\nu\left(\frac{1}{r_\nu}-\frac{r_\nu}{R^2}\right)\sin\theta_\nu=\frac{1}{\pi R}\int_0^\pi \log|af(m(Re^{i\theta}))|\cdot\sin\theta d\theta$$

$$+\frac{1}{2\pi}\int_0^R \log|af(m(-x))\,af(m(x))|\cdot\left(\frac{1}{x^2}-\frac{1}{R^2}\right)dx \tag{4.17}$$

$$+\sum_{j=1}^{J} l_j\left(\frac{1}{|\zeta_j|}-\frac{|\zeta_j|}{R^2}\right)\sin\kappa_j+\frac{1}{2}\mathscr{I}ma\frac{d}{d\lambda}f(m(0)).$$

We now proceed to evaluate each term of (4.17). First, the right-hand region with the boundaries l_1 and l_2 in the plane is denoted by $\tilde{\Sigma}$. When λ is in $\tilde{\Sigma}\cap\{\lambda;|\lambda|\geq C_1\}$ for some large $C_1>0$, we have an estimate

$$\|(L+\lambda)^{-1}\|_{\mathscr{L}(L^2(\Omega))}\leq\frac{\text{const.}}{1+|\lambda|}.$$

As for the first term of the right-hand side of (4.17), the integrand is calculated, with the help of the moment inequality, as

$$|f(m(Re^{i\theta}))|\leq\frac{\text{const.}}{\{1+|m(Re^{i\theta})|\}^{1-\alpha}}\leq\frac{\text{const.}}{\{1+(R-\delta)^\eta\}^{1-\alpha}}\leq\frac{\text{const.}}{R^{(1-\alpha)\eta}},$$
$$0\leq\theta\leq\pi.$$

Thus, the first term of the right-hand side of (4.17) is estimated from above by

$$\frac{1}{\pi R}\int_0^\pi\{\text{const.}-(1-\alpha)\eta\log R\}\sin\theta d\theta,$$

which clearly converges to zero as $R\to\infty$. Next, we proceed to the second term. When a real x is large enough, i.e., $x\geq C_2$ for some large C_2, $m(\pm x)$ belongs to $\tilde{\Sigma}\cap\{\lambda;|\lambda|\geq C_1\}$. The second term is decomposed as

$$\frac{1}{2\pi}\left\{\int_0^{C_2}+\int_{C_2}^{R}\right\}\log|af(m(-x))af(m(x))|\cdot\left(\frac{1}{x^2}-\frac{1}{R^2}\right)dx.$$

The above first term $\frac{1}{2\pi}\int_0^{C_2}\cdots dx$ is bounded as $R\to\infty$. As for the above second term, an estimate similar to the above holds; $|f(m(\pm x))|\leq\text{const.}\,x^{-(1-\alpha)\eta}$, $x\geq C_2$. Thus, the above second term is estimated from above by

$$\int_{C_2}^{R}\{\text{const.}-2\eta(1-\alpha)\log x\}\left(\frac{1}{x^2}-\frac{1}{R^2}\right)dx$$
$$\leq \text{const.}\int_{C_2}^{R}\left(\frac{1}{x^2}-\frac{1}{R^2}\right)dx=\text{const.}\left(-\frac{2}{R}+\frac{1}{C_2}+\frac{C_2}{R^2}\right),$$

which remains bounded as $R\to\infty$. The integer J in the third term of (4.17) is independent of R. Therefore, the third term remains bounded as $R\to\infty$. Consequently, we find that the right-hand side of (4.17) is bounded as $R\to\infty$.

We will estimate the left-hand side of (4.17) from below by a suitable increasing function of R. For each $x\geq 0$, let $N(x)$ denote the number of $\sqrt{\mu_n}^{1/\eta}<x$, that is, $N(x)=\#\{n;\ \sqrt{\mu_n}^{1/\eta}<x\}$. Then, we have the following lemma. Since the proof is easily carried out, it is omitted.

Lemma 4.9. *The increasing function $N(x)$ satisfies an estimate*

$$N(x)\geq\left(\frac{x}{c_0}\right)^{\eta/\gamma}-2, \qquad x\geq 0.$$

Since σ_j are zeros of $f(m(\lambda))$, we have an obvious inequality

$$\sum_{\substack{|\sigma_j|<R\\ j\geq j_0}}\left(\frac{1}{|\sigma_j|}-\frac{|\sigma_j|}{R^2}\right)\sin(\arg\sigma_j)\leq\sum_{\nu=1}^{p}m_\nu\left(\frac{1}{r_\nu}-\frac{r_\nu}{R^2}\right)\sin\theta_\nu. \tag{4.18}$$

According to the choice of j_0, σ_j are estimated as

$$|\sigma_j|^2=\sqrt{\mu_j}^{2/\eta}+\delta^2-2\sqrt{\mu_j}^{1/\eta}\delta\sin\frac{\pi-\beta+\zeta}{\eta}$$
$$<\sqrt{\mu_j}^{2/\eta}, \qquad j\geq j_0.$$

Thus, it follows that the number $\#\{j\geq j_0;\ |\sigma_j|<R\}$ is no less than $\#\{j\geq j_0;\ \sqrt{\mu_j}^{1/\eta}<R\}$. By noticing that $1/x-x/R^2$ is monotonically decreasing, the left-hand side of (4.18) is estimated from below by

$$\sum_{\sqrt{\mu_j}^{1/\eta}<R,\ j\geq j_0}\left(\frac{1}{\sqrt{\mu_j}^{1/\eta}}-\frac{\sqrt{\mu_j}^{1/\eta}}{R^2}\right)\sin(\arg\sigma_{j_0})$$
$$=\int_{\sqrt{\mu_{j_0}}^{1/\eta-\varepsilon}}^{R}\left(\frac{1}{x}-\frac{x}{R^2}\right)dN(x) \qquad \text{(ignoring } \sin(\arg\sigma_{j_0})\text{)}$$
$$=\left(\frac{1}{x}-\frac{x}{R^2}\right)N(x)\Big|_{\sqrt{\mu_{j_0}}^{1/\eta-\varepsilon}}^{R}+\int_{\sqrt{\mu_{j_0}}^{1/\eta-\varepsilon}}^{R}\left(\frac{1}{x^2}+\frac{1}{R^2}\right)N(x)dx \tag{4.19}$$
$$\geq\int_{\sqrt{\mu_{j_0}}^{1/\eta-\varepsilon}}^{R}\left(\frac{1}{x^2}+\frac{1}{R^2}\right)\left\{\left(\frac{x}{c_0}\right)^{\eta/\gamma}-2\right\}dx-\text{const.}$$
$$\geq\frac{2(\eta/\gamma)c_0^{-\eta/\gamma}}{(\eta/\gamma)^2-1}R^{\eta/\gamma-1}-\text{const.},$$

for sufficiently small positive ε. Recalling that $\eta/\gamma>1$, we find that the last term of (4.19) tends to infinity as $R\to\infty$, and so does the left-hand side of (4.18). This is a contradiction. We have finally proved that $f(m(\lambda))\equiv 0$, i.e., $f_k(m(\lambda); u)\equiv 0$. Thus, we see that

$$f_k(\lambda; u)\equiv 0, \qquad 1\leq k\leq N.$$

Integrating $f_k(\lambda; u)$ along a small circle centered at each $-\lambda_i$, and using the Laurent expansion of $(\lambda+L)^{-1}$, we find that

$$(L_c^{\alpha} P_{\lambda_i} u,\ L_c^{*3/4-\varepsilon} g_k)=0.$$

By expressing $P_{\lambda_i}u$ as $\sum_{j=1}^{m_i} u_{ij}\phi_{ij}$, these relations are rewritten as

$$\sum_{j=1}^{m_i} (L_c^{5/4+\varepsilon}\phi_{ij}, g_k)u_{ij}=0, \quad 1\leq k\leq N, \quad i\geq 1. \tag{4.20}$$

Let us recall (2.5a) with $\beta=5/4+\varepsilon$. If we set

$$G_i=\begin{bmatrix} & k\downarrow 1, \cdots, N \\ (\phi_{ij}, g_k); & \\ & j\to 1, \cdots, m_i \end{bmatrix}, \qquad i\geq 1,$$

we find from (4.20) that

$$G_i \Lambda_i \boldsymbol{u}_i=\boldsymbol{0}, \qquad \boldsymbol{u}_i=\begin{bmatrix} u_{i1} \\ u_{i2} \\ \vdots \\ u_{im_i} \end{bmatrix}, \qquad i\geq 1.$$

Here, Λ_i indicates an $m_i\times m_i$ upper triangular matrix with m_i diagonal elements $(\lambda_i+c)^{\beta}$. If we prove that rank $G_i=m_i$, $i\leq K$, it then follows that $\Lambda_i\boldsymbol{u}_i=\boldsymbol{0}$, and therefore $\boldsymbol{u}_i=\boldsymbol{0}$, $i\leq K$. We have proved that $P_K u=0$. It is clear that $\{u\in L_R^2(\Omega);\ P_K u=0\}=(P_K^* L_R^2(\Omega))^{\perp}$.

Let us prove that rank $G_i=m_i$, $i\leq K$. Applying a version of Green's formula (2.6) or its footnote in the case where $u=\phi_{ij}$, we obtain relations

$$\begin{aligned}(\phi_{ij}, w_k)_{\Gamma} &= (L_c\phi_{ij}, g_k) \\ &= (\lambda_i+c)(\phi_{ij}, g_k)+\sum_{l<j}\alpha_{lj}^{i}(\phi_{il}, g_k).\end{aligned}$$

These relations are rewritten in a matrix form as $\overline{W}_i=G_i\tilde{\Lambda}_i$, $i\geq 1$. Here, $\tilde{\Lambda}_i$ indicates again an $m_i\times m_i$ upper triangular matrix with m_i diagonal elements λ_i+c. Thus, it follows from these relations that rank $W_i=$ rank G_i, $i\geq 1$. Now, since rank $W_i=m_i$, $i\leq K$, we have the desired result. This completes the proof of Proposition 4.4.

Proof of Lemma 4.6. In the first place, we note that $e^{t\mathcal{N}}$ is expressed as

$$e^{t\mathcal{N}}=\frac{1}{2\pi i}\int_I e^{t\lambda}(\lambda-\mathcal{N})^{-1}d\lambda,$$

where I consists of two rays $\{\lambda=\sigma+i\tau;\ \sigma=-a|\tau|+b,\ \tau\in\boldsymbol{R}^1\}$ for some $a,\ b>0$, and is oriented according to increasing τ. We set

$$\max_{\lambda\in\sigma(\mathcal{N}),\ \mathcal{R}e\lambda\neq-a\sqrt{\mu_1}}\mathcal{R}e\,\lambda=-\nu_1<-a\sqrt{\mu_1},$$

and choose ν_2; $-\nu_1<-\nu_2<-a\sqrt{\mu_1}$. Let us modify the contour I into I_1, $I_1=\{\lambda=\sigma+i\tau;\ \sigma=-a|\tau|+b$ for $|\tau|\geq(\nu_2+b)/a$, and $\sigma=-\nu_2$ for $|\tau|\leq(\nu_2+b)/2\}$. Since there is no eigenvalue on the strip $-\nu_1<\mathcal{R}e\,\lambda<-a\sqrt{\mu_1}$, the above expression of $e^{t\mathcal{N}}$ is rewritten as (the eigenvalues on the vertical line $\mathcal{R}e\,\lambda=-a\sqrt{\mu_1}$ are denoted by $\zeta_1,\cdots,\zeta_J$)

$$e^{t\mathcal{N}}=\frac{1}{2\pi i}\int_{I_1}e^{t\lambda}(\lambda-\mathcal{N})^{-1}d\lambda+\sum_j\text{residue of } e^{t\lambda}(\lambda-\mathcal{N})^{-1}\text{ at }\lambda=\zeta_j. \tag{4.21}$$

The above first term has no problem. In fact, its norm is obviously estimated from above by

$$\text{const. } e^{-\nu_2 t}\leq\text{const. } e^{-a\sqrt{\mu_1}t},\quad t\geq 0.$$

Suppose that $(\lambda-\mathcal{N})^{-1}$ has a pole of order $m\,(\geq 2)$ at $\lambda=\zeta_j$. Set $N_i=\{\phi;\ (\zeta_j-\mathcal{N})^i\phi=0\}$. As is well known [9], we have a relation

$$N_1\subsetneqq N_2\subsetneqq\cdots\subsetneqq N_m=N_{m+1}=\cdots.$$

Thus, we find a $\phi\in N_m$ such that $(\zeta_j-\mathcal{N})^{m-1}\phi\neq 0$. For such a ϕ, we can inductively find $\phi_i\in N_{m-i}$, $1\leq i\leq m-1$, such that a relation

$$(\lambda-\mathcal{N})^{-1}\phi=\frac{1}{\lambda-\zeta_j}\phi+\sum_{i=2}^{m}\frac{1}{(\lambda-\zeta_j)^i}\phi_{i-1}$$

holds in a neighborhood of $\lambda=\zeta_j$. Thus, $(\lambda-\mathcal{N})^{-1}\phi$ is extended to an analytic function of $\lambda\neq\zeta_j$. Therefore, only the residue at the particular $\lambda=\zeta_j$ should be considered in (4.21) in order to calculate $e^{t\mathcal{N}}\phi$. Let the Laurent expansion of $(\lambda-\mathcal{N})^{-1}$ in a neighborhood of $\lambda=\zeta_j$ be expressed by

$$(\lambda-\mathcal{N})^{-1}=\sum_{l=1}^{m}\frac{C_l}{(\lambda-\zeta_j)^l}+g(\lambda),\qquad C_l=\frac{1}{2\pi i}\oint(\lambda-\zeta_j)^{l-1}(\lambda-\mathcal{N})^{-1}d\lambda,$$

where $g(\lambda)$ indicates an analytic function. Since

$$C_m\phi=(\mathcal{N}-\zeta_j)^{m-1}C_1\phi=(\mathcal{N}-\zeta_j)^{m-1}\phi\neq 0,$$

the order of the pole ζ_j of $e^{t\lambda}(\lambda-\mathcal{N})^{-1}\phi$ is m. Therefore, the residue of $e^{t\lambda}(\lambda-\mathcal{N})^{-1}\phi$

is calculated as

$$\frac{1}{(m-1)!}\frac{d^{m-1}}{d\lambda^{m-1}}\{(\lambda-\zeta_j)^m e^{t\lambda}(\lambda-\mathcal{N})^{-1}\phi\}\Big|_{\lambda=\zeta_j}$$
$$=\frac{1}{(m-1)!}\sum_{l=0}^{m-1}\sum_{\nu=0}^{m-1}\{e^{t\lambda}\}^{(m-1-\nu)}\{(\lambda-\zeta_j)^l\}^{(\nu)}\binom{m-1}{\nu}C_{m-l}\phi\Big|_{\lambda=\zeta_j}$$
$$=\sum_{l=0}^{m-1}\frac{1}{(m-l-1)!}t^{m-l-1}e^{t\zeta_j}C_{m-l}\phi.$$

Thus, multiplying the both sides of (4.21) by ϕ, we arrive at an estimate

$$\|e^{t\mathcal{N}}\phi\|_{L^2(\Omega)\times C^S}\geq \text{const. } t^{m-1}e^{-a\sqrt{\mu_1}t}-\text{const. } e^{-a\sqrt{\mu_1}t}$$
$$\geq \text{const. } t^{m-1}e^{-a\sqrt{\mu_1}t}, \qquad \text{as } t\to\infty.$$

This estimate contradicts the formerly obtained estimate (4.15). Q.E.D.

Proof of Proposition 4.7. We prove the inequality when $k=1$. Other cases are similar. Using the same notations as in Lemma 4.6, we calculate as

$$\mathcal{N}e^{t\mathcal{N}}=\frac{1}{2\pi i}\int_I \lambda e^{t\lambda}(\lambda-\mathcal{N})^{-1}d\lambda$$
$$=\frac{1}{2\pi i}\int_{I_1}\lambda e^{t\lambda}(\lambda-\mathcal{N})^{-1}d\lambda+\sum_j \text{residue of } \lambda e^{t\lambda}(\lambda-\mathcal{N})^{-1} \text{ at } \lambda=\zeta_j.$$

The norm of the first term is estimated from above by

$$\text{const. } t^{-1}e^{-\nu_2 t}, \qquad t>0.$$

According to Lemma 4.6, each residue of $\lambda e^{t\lambda}(\lambda-\mathcal{N})^{-1}$ at $\lambda=\zeta_j$ is calculated as

$$(\lambda-\zeta_j)\lambda e^{t\lambda}(\lambda-\mathcal{N})^{-1}|_{\lambda=\zeta_j}=\zeta_j e^{t\zeta_j}C_1, \qquad 1\leq j\leq J.$$

Here, the operator C_1 is, of course, dependent on j. Thus, the norm of the second term is estimated from above by

$$\text{const. } e^{-a\sqrt{\mu_1}t}, \qquad t\geq 0.$$

Combining these estimates, we have proved the desired estimate. This completes the proof of Proposition 4.7, and thereby the proof of Theorem 4.2.

In the rest of the section, we will study the existence and uniqueness of solutions of (4.6). Our result is

Theorem 4.10. *The control system* (4.6) *has the unique solution* $u(t, x)$ *and* $\boldsymbol{y}(t)$

for each $u_0 \in L^2_R(\Omega)$ *and* $\boldsymbol{y}_0 \in \boldsymbol{R}^S$. *Here, the solutions are considered within the following class of functions*:

$$u(t, \cdot) \in C(\overline{\boldsymbol{R}^1_+}; L^2_R(\Omega)) \cap C(\boldsymbol{R}^1_+; H^2(\Omega)) \cap C^1(\boldsymbol{R}^1_+; L^2_R(\Omega)),$$

and

$$\boldsymbol{y}(t) \in C(\overline{\boldsymbol{R}^1_+}; \boldsymbol{R}^S) \cap C^1(\boldsymbol{R}^1_+; \boldsymbol{R}^S).$$

Furthermore, $u(t, \cdot)$ *and* $\boldsymbol{y}(t)$ *are analytic functions of* $t>0$ *in* $L^2_R(\Omega)$ *and in* $\boldsymbol{R}^S$ *respectively.*

Proof. Uniqueness will be clear. In fact, the solutions $u(t, \cdot)$ and $\boldsymbol{y}(t)$ with the above properties satisfy (4.5) which generates an analytic semigroup. Analyticity of $u(t, \cdot)$ and $\boldsymbol{y}(t)$ follows from the relation

$$\begin{bmatrix} u(t, \cdot) \\ \boldsymbol{y}(t) \end{bmatrix} = \begin{bmatrix} L_c^{\alpha/2} & 0 \\ 0 & 1 \end{bmatrix} \mathcal{N}^{-1} \mathcal{N} e^{t\mathcal{N}} \begin{bmatrix} x_0 \\ \boldsymbol{y}_0 \end{bmatrix}, \qquad t>0.$$

We will show that $u(t, \cdot) = L_c^{\alpha/2} x(t)$ and $\boldsymbol{y}(t)$ satisfies (4.6). It follows from (2.7) that (by abbreviating the summation $\sum$ and the subscript k)

$$x(t) = e^{-tL} x_0 + \int_0^t e^{-(t-s)L} \boldsymbol{c}' \boldsymbol{y}(s) L_c^{3/4-\varepsilon} \psi \, ds. \tag{4.22}$$

Multiplying the both sides of (4.22) by L_c^{α}, we obtain an estimate

$$\| L_c^{\alpha} x(t) \| \leq \text{const. } t^{-\alpha/2}, \qquad t>0.$$

The function $L_c^{\alpha} x(t)$ is continuous. Thus, the second equation of (4.5) implies that

$$\left| \frac{d}{dt} \boldsymbol{y}(t) \right| \leq \text{const. } t^{-\alpha/2}, \qquad t>0.$$

Now, in exactly the same way as in [7, Theorem 3.1], (4.22) leads to an expression of $u(t, \cdot)$;

$$\begin{aligned} u(t, \cdot) &= e^{-tL} u_0 + \left\{ 1 - e^{-tL} + c \int_0^t e^{-\sigma L} d\sigma \right\} \boldsymbol{c}' \boldsymbol{y}_0 \psi \\ &\quad + \int_0^t \left\{ 1 - e^{-(t-s)L} + c \int_0^{t-s} e^{-\sigma L} d\sigma \right\} \boldsymbol{c}' \frac{d}{ds} \boldsymbol{y}(s) \psi \, ds \\ &= f_1(t) + f_2(t) + f_3(t). \end{aligned}$$

It is readily seen that

$$\frac{d}{dt} f_1(t) = -L f_1(t), \qquad f_1(0) = u_0$$

and

$$\begin{cases} \dfrac{d}{dt} f_2(t) = -\mathscr{L} f_2(t), \qquad f_2(0)=0, \\ \dfrac{\partial}{\partial \nu_L} f_2(t) + \sigma(\xi) f_2(t) = \boldsymbol{c}' \boldsymbol{y}_0 h(\xi). \end{cases}$$

The third term $f_3(t)$ is decomposed as

$$f_3(t) = \boldsymbol{c}'(\boldsymbol{y}(t)-\boldsymbol{y}_0)\psi - \int_0^t e^{-(t-s)L} \boldsymbol{c}' \frac{d}{ds} \boldsymbol{y}(s) \psi ds$$
$$- c\int_0^t e^{-\sigma L} d\sigma \, \boldsymbol{c}' \boldsymbol{y}_0 \psi + c\int_0^t e^{-(t-s)L} \boldsymbol{c}' \boldsymbol{y}(s) \psi ds.$$

The above first, third, and fourth terms have no problem. The above estimate of $d\boldsymbol{y}/ds$ as well as the fact that $\psi \in \mathscr{D}(L_c^{3/4-\varepsilon})$ implies that the second term belongs to $\mathscr{D}(L)$ and is differentiable in $t>0$. Therefore,

$$\begin{cases} \dfrac{d}{dt} f_3(t) = -\mathscr{L} f_3(t), \qquad f_3(0)=0, \\ \dfrac{\partial}{\partial \nu_L} f_3(t) + \sigma(\xi) f_3(t) = \boldsymbol{c}'(\boldsymbol{y}(t)-\boldsymbol{y}_0) h(\xi). \end{cases}$$

The equations on $f_1(t)$, $f_2(t)$ and $f_3(t)$ immediately give (4.6). It is clear that the solution $u(t, \cdot)$ and $\boldsymbol{y}(t)$ possesses the properties mentioned in the Theorem. Q.E.D.

As for further regularity, we can give the following remark: Suppose that

(i) $u_0 \in \mathscr{D}(L_c^r)$, $r>1/4$, and the spatial dimension m is ≤ 6,

or (ii) $u_0 \in \mathscr{D}(L_c^r)$, $r>5/4$, m is ≤ 10, and $\boldsymbol{y}_0=0$.

Then,

$$u \in C^1(\boldsymbol{R}_+^1; C(\bar{\Omega})) \cap C(\boldsymbol{R}_+^1; C^2(\bar{\Omega})).$$

The proof is based on the m-accretiveness of the operator L_c, Sobolev's lemma, and so on. However, it is tedious, and therefore we omit it.

§ 5. Another construction of a functional observer

In this section, we consider a case where L is a self-adjoint operator. In this case, the eigenvalues of L are lying on the real axis. They are labeled in increasing order:

$$\lambda_1 < \lambda_2 < \cdots < \lambda_i < \cdots \to \infty.$$

Each λ_i is a pole of $(\lambda - L)^{-1}$ of order one, and thus ϕ_{ij}, $1 \leq j \leq m_i$, are eigenfunctions

of L. Especially, $\{\phi_{ij};\ i\geq 1,\ 1\leq j\leq m_i\}$ forms a complete orthonormal system in $L^2(\Omega)$. The resolvent $(\lambda-L)^{-1}$ is expressed in terms of $\{\phi_{ij}\}$;

$$(\lambda-L)^{-1}u=\sum_{i=1}^{\infty}\sum_{j=1}^{m_i}\frac{1}{\lambda-\lambda_i}(u,\phi_{ij})\phi_{ij},\qquad \lambda\in\rho(L).$$

This situation allows us to construct another functional observer in a more concrete form. The observer shown below was recently obtained in [8]. Our approach here is, however, quite different from his. It is based on the arguments in §4. The first conceptional observer is given by (2.3) in $H=L^2(\Omega)$. The choice of the operator B is different from the previous one in §4. It is instead defined as

$$B=-L-C=-L-\sum_{k=1}^{N}(L_c^{\alpha}\cdot,\ L_c^{3/4-\varepsilon}g_k)\xi_k. \tag{5.1}$$

Thus, the operator equation $XL+BX=-C$ on $\mathcal{D}(L)$ has a solution $X=1$. The feedback terms $f_k(t)$ are given by (2.4). Before stating our result, let us set

$$H_i=\begin{bmatrix}k\to 1,\ \cdots,\ M\\ (h_k,\phi_{ij})_\Gamma;\\ j\downarrow 1,\ \cdots,\ m_i\end{bmatrix},\qquad i\geq 1.$$

Let J and K be integers such that λ_{J+1} and $\lambda_{K+1}>0$. Our main result in this section is

Theorem 5.1. *Suppose that the algebraic conditions*

$$\begin{cases}\text{rank } H_i=m_i, & 1\leq i\leq J,\\ \text{rank } W_i=m_i, & 1\leq i\leq K\end{cases} \tag{5.2}$$

are satisfied. Then, we can find suitable α_k, ρ_k, $\xi_k\in L_R^2(\Omega)$ *such that*

$$\|e^{t\mathcal{A}}\|_{\mathcal{L}(L^2(\Omega)\times L^2(\Omega))}\leq c_1e^{-\min(\lambda_{J+1},\lambda_{K+1})t},\qquad t\geq 0. \tag{5.3}$$

For any κ, $0<\kappa<\min(\lambda_{J+1},\lambda_{K+1})$, *we can find an integer* S *such that the equation* (4.1) *is reduced to* (4.6), *and an estimate*

$$\left\|\begin{bmatrix}u(t,\cdot)\\ y(t)\end{bmatrix}\right\|_{L^2(\Omega)\times R^S}\leq c_2e^{-\kappa t}\left\|\begin{bmatrix}u_0\\ y_0\end{bmatrix}\right\|_{L^2(\Omega)\times R^S},\qquad t\geq 0 \tag{5.4}$$

holds for every solution of (4.6).

Proof of Theorem 5.1. In the first place, let P_K be the projection operator in $L^2(\Omega)$ corresponding to $\{\lambda_1,\ \cdots,\ \lambda_K\}$. Then, we can show

Proposition 5.2. *Under the first condition of* (5.2), *there exist* $\rho_1,\ \cdots,\ \rho_M$ *belonging to a finite-dimensional subspace* $P_JL_R^2(\Omega)$ *such that*

$$\left\|\exp t\left\{-L+\sum_{k=1}^{M}(\cdot,\rho_k)L_c^{3/4-\varepsilon}\psi_k\right\}\right\|_{\mathscr{L}(L^2(\Omega))}\leq c_3e^{-\lambda_{J+1}t},\qquad t\geq 0.$$

Proposition 5.3. *Under the second condition of* (5.2), *there exist* $\xi_1,\cdots,\xi_N$ *belonging to* $P_KL_R^2(\Omega)$ *such that*

$$\|\exp tB\|_{\mathscr{L}(L^2(\Omega))}=\left\|\exp t\left\{-L-\sum_{k=1}^{N}(L_c^{\alpha}\cdot,L_c^{3/4-\varepsilon}g_k)\xi_k\right\}\right\|_{\mathscr{L}(L^2(\Omega))}$$
$$\leq c_4e^{-\lambda_{K+1}t},\qquad t\geq 0.$$

Let us fix $\{\rho_1,\cdots,\rho_M\}$ and $\{\xi_1,\cdots,\xi_N\}$ as stated in these propositions, and set $\alpha_k=L_c^{3/4-\varepsilon}\psi_k$, $1\leq k\leq M$. Then, we find that an equation

$$\frac{d}{dt}(x-v)=B(x-v),\qquad x(0)-v(0)=x_0-v_0$$

holds for any solution x and v of (4.1). The procedure to obtain (5.3) is the same as in Theorem 4.2. Setting $X=1$ in the proof of Theorem 4.2, we find that

$$\|x(t)\|\leq\text{const. } e^{-\min(\lambda_{J+1},\lambda_{K+1})t}(\|x_0\|+\|v_0\|),\qquad t\geq 0.$$

A corresponding estimate for $v(t)$ is similarly obtained. Thus, (5.3) is established.

We will add a perturbation to (4.1) in order to obtain a finite-dimensional equation (4.6). For this, we prepare

Lemma 5.4. *Let* n *be* $\geq\max(J,K)$. *Then, we have an estimate*

$$\|(BP_n-P_nB)u\|\leq\varepsilon_n\|L_c^{\alpha}u\|,\qquad u\in\mathscr{D}(L),$$

where ε_n *tends to* 0 *as* $n\to\infty$.

Let us define an operator $\mathscr{K}_n$ in $L^2(\Omega)\times L^2(\Omega)$ as $\mathscr{K}_n=\begin{bmatrix}0 & 0\\ 0 & BP_n-P_nB\end{bmatrix}$, and consider a perturbed equation

$$\frac{d}{dt}\begin{bmatrix}x\\ v\end{bmatrix}=(\mathscr{M}+\mathscr{K}_n)\begin{bmatrix}x\\ v\end{bmatrix},\qquad \begin{bmatrix}x(0)\\ v(0)\end{bmatrix}=\begin{bmatrix}x_0\\ v_0\end{bmatrix}.\tag{5.5}$$

To investigate the operator $\mathscr{M}+\mathscr{K}_n$, we note that

$$\mathscr{K}_n\mathscr{M}^{-1}=\begin{bmatrix}0 & 0\\ 0 & (BP_n-P_nB)L_c^{-1}\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & L_c\end{bmatrix}\mathscr{M}^{-1}.$$

Since $\mathscr{D}(\mathscr{M})=\mathscr{D}(L)\times\mathscr{D}(L)$, it will be clear that the second factor $[\cdots]\mathscr{M}^{-1}$ is a bounded operator. Therefore, it follows from Lemma 5.4 that $\mathscr{K}_n\mathscr{M}^{-1}$ is bounded and that

$$\|\mathscr{K}_n\mathscr{M}^{-1}\|_{\mathscr{L}(L^2(\Omega))}\leq\varepsilon_n'\to 0.$$

If $\|\mathscr{K}_n(\lambda-\mathscr{M})^{-1}\|_{\mathscr{L}(L^2(\Omega))}<1$ for some $\lambda\in\rho(\mathscr{M})$, then, λ belongs to $\rho(\mathscr{M}+\mathscr{K}_n)$ and

$$(\lambda-\mathscr{M}-\mathscr{K}_n)^{-1}=(\lambda-\mathscr{M})^{-1}[1-\mathscr{K}_n(\lambda-\mathscr{M})^{-1}]^{-1}.$$

Now, a region $\Sigma\cup\{\lambda;\ \mathscr{R}e\,\lambda>-\min(\lambda_{J+1},\lambda_{K+1})\}$ is contained in $\rho(\mathscr{M})$, where Σ indicates a sector $\{\lambda=\sigma+i\tau;\ \sigma\geq-a|\tau|+b,\ \tau\in\boldsymbol{R}^1\}$ for some $a,\ b>0$. Furthermore, an estimate

$$\|(\lambda-\mathscr{M})^{-1}\|_{\mathscr{L}(L^2(\Omega)\times L^2(\Omega))}\leq\frac{\text{const.}}{1+|\lambda|}$$

holds for $\lambda\in\Sigma\cup\{\lambda;\ \mathscr{R}e\,\lambda\geq-\kappa\}$. If n is large enough, we have

$$\begin{aligned}\|\mathscr{K}_n(\lambda-\mathscr{M})^{-1}\|_{\mathscr{L}(L^2(\Omega)\times L^2(\Omega))}&\leq\varepsilon_n'\|1+\lambda(\lambda-\mathscr{M})^{-1}\|_{\mathscr{L}(L^2(\Omega)\times L^2(\Omega))}\\&\leq\text{const.}\ \varepsilon_n'<1\end{aligned}$$

for any $\lambda\in\Sigma\cup\{\lambda;\ \mathscr{R}e\,\lambda\geq-\kappa\}$. Thus, $\mathscr{M}+\mathscr{K}_n$ generates an analytic semigroup $\exp t(\mathscr{M}+\mathscr{K}_n)$, and an estimate

$$\|\exp t(\mathscr{M}+\mathscr{K}_n)\|_{\mathscr{L}(L^2(\Omega)\times L^2(\Omega))}\leq\text{const.}\ e^{-\kappa t},\qquad t\geq0$$

holds. Set $v_1(t)=P_nv(t)$. Then, (5.5) is reduced to an equation in $L^2(\Omega)\times P_nL^2(\Omega)$;

$$\text{(5.6)}\qquad\begin{cases}\dfrac{d}{dt}x(t)=-Lx(t)+\displaystyle\sum_{k=1}^{M}(v_1(t),\rho_k)L_c^{3/4-\varepsilon}\psi_k,\\[2ex]\dfrac{d}{dt}v_1(t)=B_1v_1(t)+Cx(t)+\displaystyle\sum_{k=1}^{M}(v_1(t),\rho_k)P_nL_c^{3/4-\varepsilon}\psi_k,\end{cases}$$

where $B_1=P_nBP_n=BP_n$. By setting $u(t)=L_c^{\alpha/2}x(t)$, the rest of the proof is carried out in the same way as in Theorem 4.2.

Proof of Proposition 5.2. According to Green's formula, the $m_i\times M$ matrix whose (j,k)-component is $(L_c^{3/4-\varepsilon}\psi_k,\phi_{ij})$ is expressed as $(\lambda_i+c)^{-\alpha/2}H_i$. Thus, the rank of this matrix is m_i, $1\leq i\leq J$. Since $\rho_1,\cdots,\rho_M$ are freely designed in $P_JL_R^2(\Omega)$, a well-known pole assignment technique gives the desired result. Q.E.D.

Proposition 5.3 is almost a special case of [5, Theorem 3.1]. Thus, we omit the proof.

Proof of Lemma 5.4. According to the definition of B, we calculate as

$$(BP_n-P_nB)u=\sum_{k=1}^{N}(L_c^{\alpha}u,(1-P_n)L_c^{3/4-\varepsilon}g_k)\xi_k.$$

The set $\{\phi_{ij};\ i\geq1,\ 1\leq j\leq m_i\}$ in our case is a complete orthonormal system. Consequently,

$$\|(BP_n - P_nB)u\| \leq \|L_c^\alpha u\| \sum_{k=1}^{N} \|(1-P_n)L_c^{3/4-\varepsilon} g_k\| \|\xi_k\|$$
$$= \varepsilon_n \|L_c^\alpha u\|,$$

and ε_n converges to zero as $n \to \infty$. This completes the proof of Lemma 5.4, and therefore the proof of Theorem 5.1 is completed.

References

[1] Agmon, S., *Lectures on Elliptic Boundary Value Problems,* Princeton, Van Nostrand, 1965.

[2] Fujiwara, D., Concrete characterization of the domain of fractional powers of some elliptic differential operators of the second order, Proc. Japan Acad., **43** (1967), 82–86.

[3] Gilbarg, D. and Trudinger, N. S., *Elliptic Partial Differential Equations of Second Order,* Springer-Verlag, New York, 1977.

[4] Levinson, N., *Gap and Density Theorems,* Amer. Math. Soc. Colloquium Publications, New York, 1940.

[5] Nambu, T., Feedback stabilization for distributed parameter systems of parabolic type II, Arch. Rational Mech. Anal., **79** (1982), 241–259.

[6] ——, Feedback stabilization of diffusion equations by a functional observer, J. Differential Equations, **43** (1982), 257–280.

[7] ——, On the stabilization of diffusion equations: boundary observation and feedback, J. Differential Equations, **52** (1984), 204–233.

[8] Sakawa, Y., Feedback stabilization of linear diffusion systems, SIAM J. Control Optim., **21** (1983), 667–676.

[9] Taylor, A. E., *Introduction to Functional Analysis,* John Wiley & Sons, New York, 1958.

[10] Titchmarsh, E. C., *The Theory of Functions,* The Clarendon Press, Oxford, 1939.

[11] Curtain, R. F., Finite dimensional compensators for parabolic distributed systems with unbounded control and observation, SIAM J. Control Optim., **22** (1984), 255–276.

nuna adreso:
Department of Mathematics
Faculty of Engineering
Kumamoto University
Kumamoto 860
Japan

(Ricevita la 15-an de marto, 1984)
(Reviziita la 12-an de decembro, 1984)