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Some Results Concerning Injectivity of $\mathcal{D}$ -Modules

Dedicated to Professor Taro Ura on the occasion of his retirement

By

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§ 1. Introduction

Let us denote by $C[[x]]$ the ring of all formal power series in x with coefficients in C (the field of complex numbers), and by $C\{x\}$ the sub-ring of $C[[x]]$ which consists of all convergent power series in x with coefficients in C . We also denote by $\mathcal{D}$ the ring of all ordinary differential operators with coefficients in $C\{x\}$, i.e.

$$(1.1) \quad \mathcal{D} = \left\{ \sum_{j=0}^m a_j D^j; a_j \in C\{x\}, m \in N \right\},$$

where N is the ring of non-negative integers and $D = d/dx$. The ring $\mathcal{D}$ is non-commutative.

Definition 1.1. A $\mathcal{D}$ -module M is said to be *divisible* if

- (i) $x: M \rightarrow M$ is injective (i.e. if $a \in M$ and $xa = 0$, then $a = 0$), and
- (ii) for every $a \in M$ and every $P \in \mathcal{D} - \{0\}$, there exists $b \in M$ such that $Pb = a$.

Definition 1.2. A $\mathcal{D}$ -module M is said to be *injective* if for every diagram

$$\begin{array}{ccc} 0 & \longrightarrow & M_1 \xrightarrow{i} M_2 \\ & & \downarrow \lambda \\ & & M \end{array}$$

of homomorphisms of $\mathcal{D}$ -modules, where the horizontal sequence is exact, there exists $\mu \in \text{Hom}_{\mathcal{D}}(M_2, M)$ such that $\lambda = \mu \circ i$, i.e. the diagram

$$\begin{array}{ccc} 0 & \longrightarrow & M_1 \xrightarrow{i} M_2 \\ & & \downarrow \lambda \swarrow \mu \\ & & M \end{array}$$

is commutative.

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Let θ be a given direction in the complex x -plane. We shall denote by $\mathcal{A}(\theta)$ the set of all germs of functions $f(x)$ of x such that

- (i) f is holomorphic in a sector

$$V = \{x \in \mathbb{C}; \alpha < \arg x < \beta, 0 < |x| < r\}$$

which contains the direction θ , i.e. $\alpha < \theta < \beta$;

- (ii) f admits an asymptotic expansion

$$J(f) = \hat{f} = \sum_{m=0}^{\infty} c_m x^m \in C[[x]]$$

as x tends to 0 in V , i.e. there exist non-negative numbers K_N such that

$$\left| f(x) - \sum_{m=0}^N c_m x^m \right| \leq K_N |x|^{N+1} \quad (N \in \mathbb{N}) \quad \text{for } x \in V.$$

(The sector V depends on each f .) We shall also denote by $\mathcal{A}_0(\theta)$ the sub-set of $\mathcal{A}(\theta)$ which consists of germs of functions f such that $J(f) = 0$. It is easily seen that $\mathcal{A}(\theta)$, $\mathcal{A}_0(\theta)$, $C[[x]]$ and $C[[x]]/C\{x\}$ are $\mathcal{D}$ -modules.

Furthermore, $\mathcal{A}_0(\theta)$ and $C[[x]]/C\{x\}$ are divisible in the sense of Definition 1.1. In fact, $\mathcal{A}_0(\theta)$ is shown to be divisible by utilizing the fundamental existence theorem of asymptotic solutions (cf. M. Hukuhara [2]). The divisibility of $C[[x]]/C\{x\}$ can be verified by straight-forward computations (cf. B. Malgrange [3]). In this paper we shall first prove the following result:

Theorem 1.3. *If a $\mathcal{D}$ -module M is divisible in the sense of Definition 1.1, then M is injective in the sense of Definition 1.2.*

A proof of this result is found on page 6 of R. Godement [1]. However, we shall give a similar proof in Section 2.

It is known that the sequence

$$0 \longrightarrow \mathcal{A}_0(\theta) \hookrightarrow \mathcal{A}(\theta) \xrightarrow{J} C[[x]] \longrightarrow 0$$

of homomorphisms of $\mathcal{D}$ -modules is exact (cf. W. Wasow [5]). Now the injectivity of $\mathcal{A}_0(\theta)$ implies the following result:

Theorem 1.4. *The exact sequence*

$$(1.2) \quad 0 \longrightarrow \mathcal{A}_0(\theta) \hookrightarrow \mathcal{A}(\theta) \xrightarrow{J} C[[x]] \longrightarrow 0$$

of homomorphisms of $\mathcal{D}$ -modules splits, i.e. there exists $\varphi \in \text{Hom}_{\mathcal{D}}(C[[x]], \mathcal{A}(\theta))$ such that

$$(1.3) \quad J \circ \varphi = \text{id}_{C[[x]]}.$$

This result is purely algebraic and does not involve any topologies.

If two maps φ_1 and φ_2 in $\text{Hom}_{\mathcal{D}}(C[[x]], \mathcal{A}(\theta))$ satisfy Condition (1.3), i.e.

$$J \circ \varphi_1 = \text{id}_{C[[x]]} \quad \text{and} \quad J \circ \varphi_2 = \text{id}_{C[[x]]}.$$

Then,

$$\varphi_1 - \varphi_2 \in \text{Hom}_{\mathcal{D}}(C[[x]], \mathcal{A}_0(\theta)).$$

Furthermore, since Condition (1.3) implies that

$$(1.4) \quad \varphi|_{C\{x\}} = \text{id}_{C\{x\}} \quad (\text{regarding } C\{x\} \hookrightarrow \mathcal{A}(\theta)),$$

we get

$$(\varphi_1 - \varphi_2)|_{C\{x\}} = 0.$$

Thus we conclude that the arbitrariness involved in homomorphisms φ in $\text{Hom}_{\mathcal{D}}(C[[x]], \mathcal{A}(\theta))$ which satisfy Condition (1.3) is given by $\text{Hom}_{\mathcal{D}}(C[[x]]/C\{x\}, \mathcal{A}_0(\theta))$.

In order to study the structure of $\text{Hom}_{\mathcal{D}}(C[[x]]/C\{x\}, \mathcal{A}_0(\theta))$, we shall use the following result:

Theorem 1.5. *Let M be a divisible $\mathcal{D}$ -module, and set*

$$(1.5) \quad \tilde{M} = \{a \in M; Pa = 0 \text{ for some } P \in \mathcal{D} - \{0\}\}.$$

Then $\tilde{M}$ is also a divisible $\mathcal{D}$ -module, and there exists a sub-module $\tilde{\tilde{M}}$ of M such that

$$(1.6) \quad M = \tilde{M} \oplus \tilde{\tilde{M}} \quad (\text{a direct sum of } \mathcal{D}\text{-modules}).$$

Furthermore, for every $P \in \mathcal{D} - \{0\}$, the map

$$(1.7) \quad P: \tilde{\tilde{M}} \longrightarrow \tilde{\tilde{M}}$$

is a bijective C -linear map. Regarding $\mathcal{D} \subset \text{Hom}_C(\tilde{\tilde{M}}, \tilde{\tilde{M}})$, set

$$(1.8) \quad \mathcal{R} = \{P^{-1}Q; P \in \mathcal{D} - \{0\}, Q \in \mathcal{D}\}.$$

Then, $\mathcal{R}$ is a division ring and $\tilde{\tilde{M}}$ is a free $\mathcal{R}$ -module.

This theorem will be proved in Section 3. The structure of $\text{Hom}_{\mathcal{D}}(C[[x]]/C\{x\}, \mathcal{A}_0(\theta))$ will be studied in Section 4.

The injectivity of $\mathcal{A}_0(\theta)$ also implies the following two results:

(I) *For any exact sequence $0 \rightarrow M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow 0$ of homomorphisms of $\mathcal{D}$ -modules, we have an exact sequence*

$$(1.9) \quad 0 \longrightarrow \text{Hom}_{\mathcal{D}}(M_3, \mathcal{A}_0(\theta)) \longrightarrow \text{Hom}_{\mathcal{D}}(M_2, \mathcal{A}_0(\theta)) \longrightarrow \text{Hom}_{\mathcal{D}}(M_1, \mathcal{A}_0(\theta)) \longrightarrow 0$$

of C -linear maps;

(II) for any $\mathcal{D}$ -module M , we have an exact sequence

$$(1.10) \quad 0 \longrightarrow \text{Hom}_{\mathcal{D}}(M, \mathcal{A}_0(\theta)) \longrightarrow \text{Hom}_{\mathcal{D}}(M, \mathcal{A}(\theta)) \longrightarrow \text{Hom}_{\mathcal{D}}(M, \mathcal{C}[[x]]) \longrightarrow 0$$

of $\mathcal{C}$ -linear maps; furthermore this exact sequence splits.

Note that, if M is a differential system in the sense of the definition given on page 30 of F. Pham [4] (i.e. if a $\mathcal{D}$ -module M is of a finite presentation), then $\text{Hom}_{\mathcal{D}}(M, N)$ is the space of solutions of M which take values in the $\mathcal{D}$ -module N (cf. F. Pham [4]).

§ 2. Proof of Theorem 1.3.

Due to Zorn's Lemma, it suffices to consider a diagram:

$$\begin{array}{ccccc} 0 & \longrightarrow & M_1 & \xrightarrow{i} & M_2 \\ & & \downarrow \lambda & & \\ & & M & & \end{array}$$

where

$$(2.1) \quad M_2 = M_1 + \mathcal{D}f, \quad f \notin M_1, \quad M_1 \cap \mathcal{D}f \neq \{0\}.$$

Set

$$(2.2) \quad \mathcal{I} = \{P \in \mathcal{D} - \{0\}; Pf \in M_1\}.$$

Then (2.1) implies that

$$(2.3) \quad \mathcal{I} \neq \emptyset.$$

Let $P_0 \neq 0$ be an element of the smallest order in $\mathcal{I}$. It is easy to show that for every $P \in \mathcal{I}$ there exist $N \in \mathbb{N}$ and $Q \in \mathcal{D}$ such that $x^N P = QP_0$.

Since $P_0 f \in M_1$, we have $\lambda(P_0 f) \in M$, and hence there exists $b \in M$ such that $P_0 b = \lambda(P_0 f)$. For $P \in \mathcal{I}$, we have

$$(2.4) \quad x^N P b = Q P_0 b = Q \lambda(P_0 f) = \lambda(Q P_0 f) = \lambda(x^N P f) = x^N \lambda(P f).$$

Hence

$$(2.5) \quad P b = \lambda(P f) \quad \text{for every } P \in \mathcal{I}.$$

In particular, if $P f = 0$ for some $P \in \mathcal{D}$, then $P b = 0$. Therefore, if we set

$$(2.6) \quad \mu(P f) = P b \quad \text{for } P \in \mathcal{D},$$

the map

$$(2.7) \quad \mu: \mathcal{D}f \longrightarrow M$$

is a homomorphism of $\mathcal{D}$ -modules and $\mu(g) = \lambda(g)$ for $g \in M_1 \cap \mathcal{D}f$.

Set

$$\mu(f_1 + f_2) = \lambda(f_1) + \mu(f_2) \quad \text{for } f_1 \in M_1 \quad \text{and } f_2 \in \mathcal{D}f.$$

Then

$$\mu: M_2 \longrightarrow M$$

is a homomorphism of $\mathcal{D}$ -modules and $\mu(g) = \lambda(g)$ for $g \in M_1$.

Q.E.D.

Since $\mathcal{A}_0(\theta)$ is divisible, this is an injective $\mathcal{D}$ -module. Therefore, there exists $\mu \in \text{Hom}_{\mathcal{D}}(\mathcal{A}(\theta), \mathcal{A}_0(\theta))$ such that the diagram

$$\begin{array}{ccc} 0 & \longrightarrow & \mathcal{A}_0(\theta) \hookrightarrow \mathcal{A}(\theta) \\ & & \downarrow \text{id} \quad \swarrow \mu \\ & & \mathcal{A}_0(\theta) \end{array}$$

is commutative, i.e.

$$(2.8) \quad \mu|_{\mathcal{A}_0(\theta)} = \text{id}_{\mathcal{A}_0(\theta)}.$$

If $f \in \mathcal{A}(\theta)$, $g \in \mathcal{A}(\theta)$ and $f - g \in \mathcal{A}_0(\theta)$, then $\mu(f - g) = f - g$. This means that

$$(2.9) \quad f - \mu(f) = g - \mu(g) \quad \text{if } f - g \in \mathcal{A}_0(\theta).$$

For every $\hat{f} \in C[[x]]$ choose $f \in \mathcal{A}(\theta)$ such that $J(f) = \hat{f}$, and set

$$(2.10) \quad \varphi(\hat{f}) = f - \mu(f).$$

Then, $\varphi: C[[x]] \rightarrow \mathcal{A}(\theta)$ is a well-defined homomorphism of $\mathcal{D}$ -modules and

$$(2.11) \quad J \circ \varphi = \text{id}_{C[[x]]},$$

since $\mu(f) \in \mathcal{A}_0(\theta)$. This proved Theorem 1.4.

To prove (1.4) it suffices to show that

$$(2.12) \quad \varphi(1) = 1.$$

Note that $D1 = 0$, and hence $D\varphi(1) = 0$. Therefore $\varphi(1) \in C$. Further, we have $J(\varphi(1)) = 1$ (cf. (2.11)). Thus we get (2.12).

§ 3. Proof of Theorem 1.5.

We shall use notations and definitions of Theorem 1.5. First of all, remark that for $P \in \mathcal{D} - \{0\}$ and $Q \in \mathcal{D}$ there exist $R \in \mathcal{D}$ and $S \in \mathcal{D} - \{0\}$ such that $RP = SQ$. Hence if $a \in \tilde{M}$ and $Q \in \mathcal{D}$, we have $Qa \in \tilde{M}$. Furthermore, if $Pa = 0$ and $Qb = 0$ for

some $a \in M$, $b \in M$, $P \in \mathcal{D} - \{0\}$ and $Q \in \mathcal{D} - \{0\}$, then $SQ(a+b) = SQa = RPa = 0$. Hence $\tilde{M}$ is a $\mathcal{D}$ -module. For $a \in \tilde{M}$ and $Q \in \mathcal{D} - \{0\}$, there exists $b \in M$ such that $Qb = a$. Since $Pa = 0$ for some $P \in \mathcal{D} - \{0\}$, we have $PQb = 0$, and hence $b \in \tilde{M}$. This proves that $\tilde{M}$ is divisible. Therefore, by virtue of Theorem 1.3, $\tilde{M}$ is injective. The direct-sum decomposition (1.6) of M is a consequence of the injectivity of $\tilde{M}$.

Let us consider the map (1.7) for $P \in \mathcal{D} - \{0\}$;

1) *Surjectivity of P* : For any $a \in \tilde{M}$, there exists $b \in M$ such that $Pb = a$. Write $b = b_1 + b_2$, where $b_1 \in \tilde{M}$ and $b_2 \in \tilde{M}$. Then $Pb_1 \in \tilde{M}$, $Pb_2 \in \tilde{M}$ and $Pb_1 + Pb_2 = a$. Hence $Pb_1 = 0$ and $Pb_2 = a$.

2) *Injectivity of P* : If $Pa = 0$ for some $a \in \tilde{M}$, we have $a \in \tilde{M} \cap \tilde{M}$. Hence $a = 0$.

Note that

(i) $QP^{-1} = S^{-1}R$ for some $R \in \mathcal{D}$ and $S \in \mathcal{D} - \{0\}$;

(ii) $(P_1^{-1}Q_1)(P_2^{-1}Q_2) = P_1^{-1}(Q_1P_2^{-1})Q_2 = P_1^{-1}S^{-1}RQ_2$ for some $R \in \mathcal{D}$ and $S \in \mathcal{D} - \{0\}$;

(iii) $P_1^{-1}Q_1 + P_2^{-1}Q_2 = P_1^{-1}Q_1\{1 + R^{-1}S\} = P_1^{-1}Q_1R^{-1}\{R + S\}$ for some $R \in \mathcal{D}$ and $S \in \mathcal{D} - \{0\}$;

(iv) $(P^{-1}Q)^{-1} = Q^{-1}P$.

Therefore, $\mathcal{R}$ is a division ring. Hence, $\tilde{M}$ is a free $\mathcal{R}$ -module. Q.E.D.

§ 4. The structure of $\text{Hom}_{\mathcal{D}}(C[[x]]/C\{x\}, \mathcal{A}_0(\theta))$.

Set

$$(4.1) \quad \tilde{\mathcal{A}}_0(\theta) = \{f \in \mathcal{A}_0(\theta); Pf = 0 \text{ for some } P \in \mathcal{D} - \{0\}\},$$

and

$$(4.2) \quad \mathcal{C} = \{[\hat{f}] \in C[[x]]/C\{x\}; P[\hat{f}] = 0 \text{ for some } P \in \mathcal{D} - \{0\}\}.$$

Then, by virtue of Theorem 1.5, there exist a sub-module M_1 of $\mathcal{A}_0(\theta)$ and a sub-module M_2 of $C[[x]]/C\{x\}$ such that

$$(4.3) \quad \mathcal{A}_0(\theta) = \tilde{\mathcal{A}}_0(\theta) \oplus M_1 \quad \text{and} \quad C[[x]]/C\{x\} = \mathcal{C} \oplus M_2.$$

Regarding $\mathcal{D} \subset \text{Hom}_{\mathcal{C}}(M_1, M_1)$ and $\mathcal{D} \subset \text{Hom}_{\mathcal{C}}(M_2, M_2)$ respectively, we define two division rings

$$(4.4) \quad \mathcal{R}_j = \{P^{-1}Q; P \in \mathcal{D} - \{0\}, Q \in \mathcal{D}\} \quad (j=1, 2).$$

Then M_j is a free $\mathcal{R}_j$ -module ($j=1, 2$).

From these facts, we conclude that

$$(4.5) \quad \begin{aligned} \text{Hom}_{\mathcal{D}}(C[[x]]/C\{x\}, \mathcal{A}_0(\theta)) \\ = \text{Hom}_{\mathcal{D}}(\mathcal{C}, \tilde{\mathcal{A}}_0(\theta)) \oplus \text{Hom}_{\mathcal{D}}(M_2, \tilde{\mathcal{A}}_0(\theta)) \oplus \text{Hom}_{\mathcal{D}}(M_2, M_1). \end{aligned}$$

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