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Oscillation Properties of the Solutions of a Class of Hyperbolic Equations of Neutral Type

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§ 1. Introduction

In recent years several results have been published concerning oscillation properties and the asymptotic behaviour of the solutions of certain classes of hyperbolic equations. We mention here the works [1]–[6] and their references. In the paper of K. Kreith, T. Kusano and N. Yoshida [6] sufficient conditions were found for oscillation of the solutions of nonlinear hyperbolic equations of the form

$$u_{tt}-\Delta u+c(x, t, u)=f(x, t)$$

considered in a cylindrical domain, where $\partial n/\partial u$ is prescribed on the surface.

In the present work sufficient conditions are obtained for oscillation of the solutions of the hyperbolic equation of neutral type

$$u_{tt}(x, t)+\lambda u_{tt}(x, t-\tau)-\Delta u(x, t)+c(x, t, u)=f(x, t), \tag{1}$$

where $\lambda=\text{const}>0$ and $\tau=\text{const}>0$.

§ 2. Sufficient conditions for oscillation of the solutions

Let G be a cylinder in the space (x, t), $x=(x_1, x_2, \cdots, x_n)$, defined by the conditions $x \in \mathscr{D}$, $t>0$, where $\mathscr{D}$ is a bounded domain in $\boldsymbol{R}^n$ with smooth boundary $\partial\mathscr{D}$. Let σ be the surface of G, i.e. $\sigma=\partial\mathscr{D}\times(0, \infty)$ and n be the vector of the exterior normal to σ. We denote by G_α, $\alpha \in \boldsymbol{R}$, the infinite cylinder $\mathscr{D}\times(\alpha, \infty)$.

Consider the following problem

$$\begin{cases} u_{tt}+\lambda u_{tt}(x, t-\tau)-\Delta u+c(x, t, u)=f(x, t), & (x, t) \in G \\ \dfrac{\partial u}{\partial n}=g(x, t), & (x, t) \in \sigma \end{cases} \tag{2}$$

where λ and τ are positive constants. Suppose that the following conditions (A) hold:

A1. $c(x, t, u) \in C(G\times\boldsymbol{R}; \boldsymbol{R})$,

$f(x, t) \in C(G; \boldsymbol{R})$,

$g(x, t) \in C(\sigma; \boldsymbol{R})$,

A2. $c(x, t, \xi) \geqq p(t)\varphi(\xi)$ for $(x, t, \xi) \in G \times (0, \infty)$ where $p(t)$ is a continuous and positive function in the segment $(0, \infty)$ and $\varphi(\xi)$ is a continuous, positive and convex function in the same segment $(0, \infty)$.

A3. $c(x, t, -\xi) = -c(x, t, \xi)$ for $(x, t, \xi) \in G \times (0, \infty)$.

Corresponding to each solution $u(x, t)$ of the problem (1), we also consider the function

$$U(t) = \frac{1}{|\mathscr{D}|}\int_{\mathscr{D}} u(x, t)dx, \qquad t \in (-\tau, \infty)$$

where $|\mathscr{D}| = \int_{\mathscr{D}} dx$. The following lemma holds:

Lemma 1. *Let the conditions* A1 *and* A2 *be satisfied. If $u(x, t)$ is a solution of the problem* (2) *for which $u(x, t) > 0$ in G_α, $\alpha \geqq 0$, then the function $U(t)$ satisfies the differential inequality of neutral type*

$$U''(t) + \lambda U''(t-\tau) + p(t)\varphi(U) \leqq \frac{1}{|\mathscr{D}|}\int_{\partial\mathscr{D}} g(x, t)d\sigma + \frac{1}{|\mathscr{D}|}\int_{\mathscr{D}} f(x, t)dx, \tag{3}$$

where $t > \alpha$.

Proof. Let $t > \alpha$. Using the formula for differentiating under the integral, we get

$$\begin{aligned} U''(t) + \lambda U''(t-\tau) &= \frac{1}{|\mathscr{D}|}\int_{\mathscr{D}} [u_{tt}(x, t) + \lambda u_{tt}(x, t-\tau)]dx \\ &= \frac{1}{|\mathscr{D}|}\int_{\mathscr{D}} [\Delta u - c(x, t, u) + f(x, t)]dx. \end{aligned} \tag{4}$$

Green's formula yields

$$\int_{\mathscr{D}} \Delta u dx = \int_{\partial\mathscr{D}} \frac{\partial u}{\partial n} d\sigma = \int_{\partial\mathscr{D}} g(x, t)d\sigma. \tag{5}$$

From A2 and Jensen's inequality we obtain

$$\begin{aligned} \frac{1}{|\mathscr{D}|}\int_{\mathscr{D}} c(x, t, u)dx &\geqq \frac{1}{|\mathscr{D}|}\int_{\mathscr{D}} p(t)\varphi(u)dx \\ &\geqq p(t)\varphi\left(\frac{1}{|\mathscr{D}|}\int_{\mathscr{D}} u(x, t)dx\right) = p(t)\varphi(U). \end{aligned} \tag{6}$$

Then from (4), (5) and (6) it follows that for $t > \alpha$

$$U''(t)+\lambda U''(t-\tau)\leqq\frac{1}{|\mathscr{D}|}\int_{\partial\mathscr{D}} g(x,t)d\sigma-p(t)\varphi(U)+\frac{1}{|\mathscr{D}|}\int_{\mathscr{D}} f(x,t)dx.$$

This proves the lemma.

Definition 1. The solution $u(x, t)$ of the problem (2) is called oscillatory in G if $u(x, t)$ has zero in G_α for each $\alpha\geqq 0$.

Definition 2. (cf. 7). The inequality (3) is called oscillatory at $t=\infty$ if it does not possess a solution which is positive in the interval $[\alpha, \infty)$ for every $\alpha\geqq 0$.

Using Lemma 1 we prove the following theorem:

Theorem 1. *Suppose that the conditions* (A) *hold and the differential inequalities of neutral type*

$$U''(t)+\lambda U''(t-\tau)+p(t)\mathscr{C}(U)\leqq G(t)+F(t) \tag{7}$$

$$U''(t)+\lambda U''(t-\tau)+p(t)\mathscr{C}(U)\leqq -G(t)-F(t) \tag{8}$$

are oscillatory at $t=\infty$, *where*

$$G(t)=\frac{1}{|\mathscr{D}|}\int_{\partial\mathscr{D}} g(x,t)d\sigma \quad \text{and} \quad F(t)=\frac{1}{|\mathscr{D}|}\int_{\mathscr{D}} f(x,t)dx. \tag{9}$$

Then, every solution $u(x, t)$ *of the problem* (2) *is oscillatory in* G.

Proof. Assume to the contrary that there exists a solution $u(x, t)$ of the problem (2) which has no zero in G_α. If $u(x, t)>0$ in G_α, then from Lemma 1 it follows that $U(t)$ is a positive solution of the inequality (7) for $t>\alpha$, which contradicts the condition of the theorem. If $u(x, t)<0$ in G_α, then $v(x, t)=-u(x, t)$ is a positive solution of the problem

$$v_{tt}+\lambda v_{tt}(x, t-\tau)-\Delta v+c(x, t, v)=-f(x, t), \quad (x, t)\in G$$
$$\frac{\partial v}{\partial n}=-g(x, t), \quad (x, t)\in\sigma.$$

From Lemma 1 it follows that

$$V(t)=\frac{1}{|\mathscr{D}|}\int_{\mathscr{D}} v(x, t)dx$$

is a positive solution of the inequality (8) for $t>\alpha$ which contradicts the conditions of the theorem as well. This completes the proof of Theorem 1.

The above arguments imply that the oscillation properties of the solutions of the

problem (2) can be transformed to investigation of the oscillation properties of the solution of differential inequalities of neutral type

$$y''(t)+\lambda y''(t-\tau)+f(t, y)\leqq q(t) \tag{10}$$

where $\lambda=\text{const}>0$ and $\tau=\text{const}>0$.

Suppose the following conditions (B) hold:

B1. $f(t, y)\in C([a, \infty)\times(0, \infty); \boldsymbol{R})$, $q(t)\in C([a,\infty); \boldsymbol{R})$, where $a\in \boldsymbol{R}$.

B2. $f(t, y)>0$ for $(t, y)\in[a, \infty)\times(0, \infty)$.

Then, the following lemma can be proved:

Lemma 2. *Let the conditions* (B) *be satisfied and*

$$\lim_{t\to\infty}\inf\frac{1}{t-t_0}\int_{t_0}^{t}(t-s)q(s)ds=-\infty \tag{11}$$

for every $t_0>a$. *Then, the inequality* (10) *is oscillatory at* $t=\infty$.

Proof. Assume to the contrary that $y(t)$ is a positive solution of the inequality (10) defined in the interval $[t_0, \infty)$. Then, using the condition B2, we obtain

$$y''(t)+\lambda y''(t-\tau)\leqq q(t)-f(t, y)\leqq q(t)$$

i.e.

$$\frac{d^2}{dt^2}[y(t)+\lambda y(t-\tau)]\leqq q(t).$$

Integerating the above inequality twice in the segment $[t_0, t]$, we get

$$y(t)+\lambda y(t-\tau)\leqq C_1+C_2(t-t_0)+\int_{t_0}^{t}\left(\int_{t_0}^{w}q(s)ds\right)dw,$$

where C_1 and C_2 are constants. Since

$$\int_{t_0}^{t}\left(\int_{t_0}^{w}q(s)ds\right)dw=\int_{t_0}^{t}(t-s)q(s)ds$$

dividing both sides of the last inequality by $t-t_0$, $t>t_0$ we obtain

$$\frac{y(t)+\lambda y(t-\tau)}{t-t_0}\leqq\frac{C_1}{t-t_0}+C_2+\frac{1}{t-t_0}\int_{t_0}^{t}(t-s)q(s)ds. \tag{12}$$

Then, for $t\to\infty$, from (11) and (12) it follows that

$$\lim_{t\to\infty}\inf\frac{y(t)+\lambda y(t-\tau)}{t-t_0}=-\infty$$

which contradicts the assumption that $y(t)$ is positive for $t\geqq t_0$. This proves Lemma 2.

Using this result, we obtain the following theorem:

Theorem 2. *Suppose that the conditions* (A) *hold and*

$$\lim_{t\to\infty} \inf \int_{t_0}^{t} \left(1-\frac{s}{t}\right)(G(s)+F(s))ds = -\infty \tag{13}$$

$$\lim_{t\to\infty} \sup \int_{t_0}^{t} \left(1-\frac{s}{t}\right)(G(s)+F(s))ds = +\infty \tag{14}$$

for every sufficiently large number t_0. *Then, every solution* $u(x, t)$ *of the problem* (2) *is oscillatory in* G.

Proof. One can easily check that from the conditions (13) and (14) it follows respectively that

$$\lim_{t\to\infty} \inf \frac{1}{t-t_0} \int_{t_0}^{t} (t-s)(G(s)+F(s))ds = -\infty$$

$$\lim_{t\to\infty} \inf \frac{1}{t-t_0} \int_{t_0}^{t} (t-s)(-G(s)-F(s))ds = -\infty.$$

Then, according to Lemma 2 we obtain that the inequalities (7) and (8) are oscillatory at $t=\infty$. Hence, from Theorem 1 every solution $u(x, t)$ of the problem (2) is oscillatory in G.

Example. Consider the problem

$$\begin{aligned} &u_{tt}+u_{tt}(x, t-\pi)-u_{xx}+u=2e^t \cos x(\sin t+\cos t-e^{-\pi}\cos t), \\ &\qquad\qquad (x, t) \in G=\left(0, \frac{\pi}{2}\right)\times(0, \infty), \\ &u_x(0, t)=0, \qquad t \in (0, \infty), \\ &u_x\left(\frac{\pi}{2}, t\right)=-e^t \sin t, \qquad t \in (0, \infty). \end{aligned} \tag{15}$$

One can easily and immediately check that the functions

$$\begin{aligned} &c(x, t, u)=u, \qquad (x, t, u) \in G\times R, \\ &f(x, t)=2e^t \cos x(\sin t+\cos t-e^{-\pi}\cos t), \qquad (x, t) \in G, \\ &g(0, t)=0, \qquad t \in (0, \infty), \\ &g\left(\frac{\pi}{2}, t\right)=-e^t \sin t, \qquad t \in (0, \infty), \end{aligned}$$

satisfy the conditions (A). Moreover, taking into account (9), we obtain

$$G(t)=-\frac{2}{\pi}e^t \sin t, \qquad t\in(0,\infty),$$

$$F(t)=\frac{4}{\pi}e^t(\sin t+\cos t-e^{-\pi}\cos t), \qquad t\in(0,\infty),$$

which implies that

$$I(t)=\int_{t_0}^{t}\left(1-\frac{s}{t}\right)(G(s)+F(s))ds=\frac{e^t}{t\pi}(2\sin t-2e^{-\pi}\sin t-\cos t)+C,$$

where C does not depend on t. Hence

$$\liminf_{t\to\infty} I(t)=-\infty, \qquad \limsup_{t\to\infty} I(t)=+\infty,$$

i.e. the conditions (13) and (14) of Theorem 2 hold. Then, from Theorem 2 it follows that every solution of the problem (15) is oscillatory in the cylinder $G=(0,\pi/2)\times(0,\infty)$. For example,

$$u(x,t)=e^t \sin t\cos x$$

is one such solution.

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