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Precompactness and Convergence in Norm of Positive Orbits in a Certain Fading Memory Space

By

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1. Introduction

In various studies concerning qualitative properties of solutions of differential equations (ordinary, functional and partial), it is useful to establish that bounded positive orbits are precompact in the underlying solution space. (A set is said to be precompact if its closure is compact.) For example, an advantage in knowing that a positive orbit is precompact is that it must approach its omega (positive) limit set as $t\to\infty$. One can then set out to locate the omega limit sets of orbits in order to gain information regarding the behavior of solutions at infinity.

For ordinary differential equations (ODEs) a bounded positive orbit is automatically precompact since the underlying space, R^n, is finite dimensional. Also, it often is straightforward to prove precompactness of bounded positive orbits for retarded functional differential equations (FDEs) with finite delay. However, for FDEs with infinite delay and for partial differential equations (PDEs), the situation is more difficult, and it often requires additional work to obtain the desired precompactness property. For various PDE cases we refer to Walker [17] and references therein. For FDEs with infinite delay which have "exponential-type" weight norms, we refer to Coleman and Mizel [2] and Hale and Kato [8], to name a couple of sources.

The purpose of this paper is to give sufficient conditions for a bounded positive orbit to be precompact in certain infinite delay FDE fading memory spaces. A summary of the main precompactness results is given in Remark 3.1. As a side issue we provide information concerning convergence in norm at $t=\infty$ of precompact orbits. These matters will be explained in more detail throughout the remainder of the paper.

2. Preliminaries

Before examining the specific FDE spaces to which we alluded in the previous

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section, we give a brief discussion of some of the general properties of spaces that often are considered.

During the past 15 years or so, several papers have been devoted to the study of properties of spaces for FDEs. Along these lines, we refer to Hale and Kato [8], Hino [9], Kaminogo [10], Kappel and Schappacher [11], Kato [12], Naito [14], and Sawano [15, 16]. For a more complete list of references, see Corduneanu and Lakshmikantham [4] and Haddock [5]. The above investigators (as well many others) have examined fundamental properties of FDE spaces in a variety of forms. In Definition 2.1 below we list but one set of properties (sometimes referred to as axioms) that has been used.

Let $|\cdot|$ denote an $\boldsymbol{R}^n$-norm. Throughout the remainder of the paper, we assume:

(A1) B is a real vector space either (i) of continuous functions that map $(-\infty, 0]$ into $\boldsymbol{R}^n$ with $\phi=\psi$ in B if $\phi(s)=\psi(s)$ on $(-\infty, 0]$ or (ii) of measurable functions that map $(-\infty, 0]$ into $\boldsymbol{R}^n$ with elements $\phi=\psi$ (or ϕ is equivalent to ψ) in B if $\phi(s)=\psi(s)$ almost everywhere on $(-\infty, 0]$ and $\phi(0)=\psi(0)$.

(A2) B is endowed with a norm, denoted by $|\cdot|_B$.

(A3) B is complete with respect to $|\cdot|_B$.

Thus, B, equipped with norm $|\cdot|_B$, is a Banach space. We denote this space by $(B, |\cdot|_B)$, or simply B, whenever no confusion can result.

If $x: (-\infty, A)\to\boldsymbol{R}^n$, $0<A\leq\infty$, then, for any t in $[0, A)$, define $x_t: (-\infty, 0]\to\boldsymbol{R}^n$ by

$$x_t(s) = x(t+s), \quad s \leq 0.$$

For a given $t\geq 0$, x_t is the translation of x on $(-\infty, t]$ back to $(-\infty, 0]$, and x_0 is merely the restriction, $x|_{(-\infty,0]}$, of x to $(-\infty, 0]$.

Definition 2.1. A space B (as defined above) is said to be *admissible* whenever there exist constants K, $J>0$ and a continuous function $M: [0, \infty)\to[0, \infty)$ such that the following conditions hold:

Let $0\leq a<A\leq\infty$. If $x: (-\infty, A)\to\boldsymbol{R}^n$ is defined on $(-\infty, A)$ and continuous on $[a, A)$ with x_a in B, then, for all t in $[a, A)$,

(B1) $x_t\in B$;

(B2) x_t is continuous in t with respect to $|\cdot|_B$;

(B3) $|x_t|_B\leq K \max_{a\leq s\leq t}|x(s)| + M(t-a)|x_a|_B$; and

(B4) $|\phi(0)|\leq J|\phi|_B$ for all ϕ in B.

Definition 2.2. Let B be admissible and suppose $x: (-\infty, \infty)\to\boldsymbol{R}^n$ is such that $x_0\in B$ and x is continuous on $[0, \infty)$, then the set

$$\{x_t: t\geq 0\} \tag{2.1}$$

is called the *positive orbit* of x *in* B.

Note that (B1) assures that the set in (2.1) is contained in B and (B2) implies that the mapping $t \to x_t$ is continuous ($t \geq 0$). The significance of working with an admissible space for FDEs is that the standard existence-uniqueness-continuous dependence-continuation type results are available (cf., e.g., [8].). In the following example we outline the particular class of admissible spaces of continuous functions that we will be considering.

Example 2.1. Let

($g1$) $g: (-\infty, 0] \to [1, \infty)$ be continuous and nonincreasing with $g(0)=1$.

Further, let $C := C((-\infty, 0], \boldsymbol{R}^n)$ denote the vector space of continuous functions that map $(-\infty, 0]$ into $\boldsymbol{R}^n$, and define

$$C_g := \{\phi \text{ in } C: \sup_{s \leq 0} |\phi(s)|/g(s) < \infty\}.$$

For ϕ in C_g, define the norm of ϕ by

$$|\phi|_g := |\phi|_{C_g} := \sup_{s \leq 0} |\phi(s)|/g(s).$$

Then $(C_g, |\ |_g)$ is a Banach space (cf. [3]) which we denote by C_g. However, it is well-known that C_g is not admissible without additional restrictions (see [5]). To obtain an admissible space related to C_g, define

$$UC_g := \{\phi \text{ in } C_g: \phi/g \text{ is uniformly continuous on } (-\infty, 0]\}.$$

It is proven in [7] that UC_g is admissible if, in addition to ($g1$), g satisfies:

($g2$) $$[g(s+u)/g(s)] \to 1 \text{ uniformly on } (-\infty, 0] \text{ as } u \to 0-.$$

As was mentioned at the outset, we are interested in obtaining sufficient conditions for bounded positive orbits to be precompact in certain admissible spaces. The spaces under investigation will be C_g spaces. Before discussing results for these particular spaces, though, we give sufficient conditions (in Theorem 2.1 below) for precompactness of bounded positive orbits in general admissible spaces. As we shall see, the precompactness results that we seek will rely on certain "fading memory" properties.

Definition 2.3. Let B be an admissible space. B is said to be a *strong fading memory space* if, in (B3),

(2.2) $$M(t) \longrightarrow 0 \quad \text{as} \quad t \longrightarrow \infty.$$

Another condition that will be employed to prove the general result in

Theorem 2.1 is the following:

(2.3) if ϕ is bounded and continuous on $(-\infty, 0]$, then $\phi \in B$ and there exists a constant $L>0$ such that $|\phi_{-t}|B \leq L$ for all $t \geq 0$, where $\phi_{-t}(s) := \phi(-t+s)$ $(t \geq 0,\ s \leq 0)$.

(2.3) simply states that B contains all bounded, continuous functions and that right translations of these functions from $(-\infty, -t]$ to $(-\infty, 0]$ are in B and are bounded there.

Theorem 2.1. *Let B be a strong fading memory space such that* (2.3) *holds, and suppose $x: (-\infty, \infty) \to \boldsymbol{R}^n$ is bounded and continuous on $[0, \infty)$ with x_0 in B. If, in addition,*

(2.4) *$x_{t_n}(s) \to \phi(s)$ uniformly on compact subsets of $(-\infty, 0]$ as $t_n \to \infty$ for some sequence $\{t_n\}$,*

then $\phi \in B$ and $x_{t_n} \to \phi$ in B (i.e., $|x_{t_n} - \phi|_B \to 0$) *as $t_n \to \infty$.*

Proof. Since $x_{t_n}(s) \to \phi(s)$ uniformly on compact subsets of $(-\infty, 0]$, ϕ is continuous and bounded on $(-\infty, 0]$. So $\phi \in B$. Let L serve as a bound for $|x(t)|$, $M(t)$ and $|\phi_{-t}|_B$ on $[0, \infty)$. To show that $|x_{t_n} - \phi|_B \to 0$ as $t_n \to \infty$, let $\varepsilon > 0$ be given and let $T > 0$ be chosen sufficiently large that

$$M(T) < \varepsilon / 2L[K + |x_0|_B + 1].$$

Using admissible property (B3) together with (2.3), we can deduce

$$\begin{aligned} |x_{t_n} - \phi|_B &\leq K \max_{-T \leq s \leq 0} |x_{t_n}(s) - \phi(s)| + M(T)|x_{t_n - T} - \phi_{-T}|_B \\ &\leq K \max_{-T \leq s \leq 0} |x(t_n + s) - \phi(s)| + M(T)\,[|x_{t_n - T}|_B + |\phi_{-T}|_B]. \end{aligned} \tag{2.5}$$

But

$$|x_{t_n - T}|_B \leq K \max_{0 \leq s \leq t_n - T} |x(s)| + M(T - t_n)\,|x_0|_B \leq L[K + |x_0|_B]$$

as long as $t_n > T$. From (2.5) we have

$$\begin{aligned} |x_{t_n} - \phi|_B &\leq K \max_{-T \leq s \leq 0} |x_{t_n}(s) - \phi(s)| + M(T) L[K + |x_0|_B + 1] \\ &\leq K \max_{-T \leq s \leq 0} |x_{t_n}(s) - \phi(s)| + \varepsilon/2 \end{aligned} \tag{2.6}$$

Now, from (2.4) we choose N sufficiently large that

$$|x_{t_n}(s) - \phi(s)| < \varepsilon / 2K$$

for all $t_n \geq N$ and s in $[-T, 0]$. Thus, from (2.6)

$$|x_{t_n-T}-\phi|_B < \varepsilon \quad \text{if} \quad t_n > N.$$

It follows that $x_{t_n}\to\phi$ in B, which completes the proof.

Corollary 2.1. *In addition to the conditions of Theorem* 2.1, *suppose* $x_0 \in B$ *and* $|x'(t)|\leq P$ *for some* $P>0$ *and all* $t\geq 0$. *Then the positive orbit* $\{x_t\colon t\geq 0\}$ *is precompact in* B.

Proof. To establish precompactness, let $\{x_{t_n}\}$ denote any sequence of $\{x_t\colon t\geq 0\}$. If $\{t_n\}$ is bounded, then it has a subsequence – again denoted by $\{t_n\}$ – such that $t_n\to t^*$ as $n\to\infty$ for some $t^*\geq 0$. From the continuity property (B2), it follows that $x_{t_n}\to x_{t^*}\in B$. Now, assume that $\{t_n\}$ is unbounded. Then, by extracting a subsequence if necessary, we have that $t_n\to\infty$ as $n\to\infty$. The Ascoli-Arzelà theorem guarantees that (2.4) holds for some ϕ whenever $x(t)$ and $x'(t)$ are bounded on $[0, \infty)$. By Theorem 2.1, $\phi\in B$. Thus, we have shown that any sequence $\{x_{t_n}\}$ contains a subsequence which converges to an element in B. The conclusion follows.

Remark 2.1. Several comments are in order.

(i) We note that, for the most part, the above theorem and corollary represent a special case (but, we hope, a simplification) of Theorem 3.1 and Corollary 3.2 of [8].

(ii) It is straightforward to prove that a C_g space is a strong fading memory space whenever g has the form:

$$g(s) := \exp(-\gamma s) \quad \text{for some} \quad \gamma > 0.$$

This case is investigated in more detail in Theorem 3.2.

(iii) Some investigators (e.g., Kato [12] and Lakshmikantham and Leela [13]) have used the phrase "fading memory space" rather than "strong fading memory space" whenever (2.2) holds. However, we feel that there are important spaces which satisfy a fading memory type property for which (2.2) does not hold necessarily. A case in point is any C_g space for which

$$(g3) \qquad g(s)\longrightarrow\infty \quad \text{as} \quad s\longrightarrow-\infty.$$

As we shall see in the next section, $(g3)$ plays a prominent role regarding precompactness of orbits.

(iv) It is illustrated in Atkinson and Haddock [1] that $(g2)$ and $(g3)$ are not overly restrictive and often can be included in a natural way in studies involving FDEs with infinite delay.

(v) Many comments of this section have been stated without proof. However, we have made every attempt to give references whenever appropriate.

3. Precompactness of positive orbits in C_g spaces

If g is exponential and satisfies $(g3)$ (i.e., $g(s)\equiv\exp(-\gamma s)$ for some $\gamma>0$) and if $x_0\in C_g$ with x and x' bounded on $[0,\infty)$, then, from Corollary 2.1 and (ii) of Remark 2.1, we see that the positive orbit of x is precompact in C_g. However, as is shown in [1], there are C_g spaces that often arise in a natural way in studies involving FDEs with infinite delay, but for which the function g is not necessarily exponential.

The main purpose of this section is to determine which bounded positive orbits are precompact for C_g spaces in general (cf. Theorem 3.1). The exponential case discussed above is included in Theorem 3.2.

We begin our discussion in this section by equipping C (defined in Section 1) with the compact-open topology, and, for the sake of convenience, let $\hat{C}_g$ denote the C_g functions regarded as a subspace of C. It is instructive to compare the topologies of C_g and $\hat{C}_g$. It suffices to concentrate on neighborhood bases at zero. Let N denote the set of natural numbers. A neighborhood base at zero for C_g is given by

$$B_g := \{B_g(1/n)\colon n\in N\},$$

where, for any $\varepsilon>0$,

$$B_g(\varepsilon) := \{\phi \text{ in } C_g\colon |\phi|_g<\varepsilon\}.$$

A neighborhood base at zero for $\hat{C}_g$ is given by

$$\hat{B}_g := \{\hat{B}_g(1/n, n)\colon n\in N\},$$

where, for any $\varepsilon>0$ and n in N,

$$\hat{B}_g(\varepsilon, n) := \{\phi \text{ in } C_g\colon \max_{-n\le s\le 0}|\phi(s)|<\varepsilon\}.$$

Clearly, $B_g(\varepsilon/g(-n))\subseteq\hat{B}_g(\varepsilon, n)$. Let v in $\boldsymbol{R}^n$ be given with $|v|=1$. For n in N, define ϕ_n by

$$\phi_n(s)=\begin{cases} v/2n;\ -n\le s\le 0\\ \{(s+n)[(1/2n^2)-(g(-2n)/n)]+1/2n\}v;\ -2n\le s\le -n\\ g(s)v;\ s<-2n.\end{cases}$$

Now, $\phi_n\in\hat{B}_g(1/n, n)$, but $\phi_n\notin B_g(\delta)$ if $\delta<1$, and we see that the topology of C_g is strictly finer than that of $\hat{C}_g$.

In the following let $I_n := \{(t, s)\colon t\ge n,\ s\le -n\}$ for each n in N.

Theorem 3.1. *Suppose g satisfies $(g1)$ and $x\colon(-\infty,\infty)\to\boldsymbol{R}^n$ with*

(3.1) $x_0 \in C_g$ *and* x *is bounded and uniformly continuous on* $[0, \infty)$.

Further, suppose the mapping $t \to x_t$ *is continuous with respect to* $|\cdot|_g$ *on* $[0, \infty)$. *If*

$$\lim_{n\to\infty} \sup_{(t,s)\in I_n} |x(t+s)|/g(s) = 0, \tag{3.2}$$

then the positive orbit $\{x_t: t \geq 0\}$ *is precompact in* C_g.

Proof. Let $\varepsilon > 0$ be given. By (3.2), there exists $n_\varepsilon > 0$ such that

$$\sup_{(t,s)\in I_n} |x(t+s)|/g(s) < \varepsilon/3$$

whenever $n \geq n_\varepsilon$. By (3.1) above and the Ascoli-Arzelà Theorem, $\{x_t: t \geq n_\varepsilon\}$ is precompact in $\hat{C}_g$. Hence, there exist $t_1, t_2, \ldots, t_k \geq n_\varepsilon$ such that, for any $t \geq n_\varepsilon$, we have

$$\sup_{-n_\varepsilon \leq s \leq 0} |x_t(s) - x_{t_i}(s)| < \varepsilon/3$$

for some t_i. Now,

$$|x_t - x_{t_i}|_g \leq \sup_{-n_\varepsilon \leq s \leq 0} |x_t(s) - x_{t_i}(s)|/g(s) + \sup_{s \leq -n_\varepsilon} |x_t(s) - x_{t_i}(s)|/g(s) < \varepsilon.$$

By continuity of $t \to x_t$, the set $\{x_t: t \in [0, n_\varepsilon]\}$ is compact. By the above, $\{x_t: t \geq n_\varepsilon\}$ is totally bounded and its closure (in C_g) is complete. Therefore, $\{x_t: t \geq 0\}$ is precompact in C_g, and the proof is complete.

Theorem 3.2. *Suppose* $g: (-\infty, \infty) \to [1, \infty)$ *satisfies* $(g1)$ *and* $(g2)$ *with* $g(s) \equiv 1$ *on* $[0, \infty)$. *Then the following conditions are equivalent*:

$$\lim_{n\to\infty} \sup_{(t,s)\in I_n} g(t+s)/g(s) = 0. \tag{3.3}$$

(3.4) UC_g *is a strong fading memory space.*

(3.5) *If* $x: (-\infty, \infty) \to \boldsymbol{R}^n$ *with* x *bounded and umiformly continuous on* $[0, \infty)$ *with* $x_0 \in UC_g$, *then* $\{x_t: t \geq 0\}$ *is precompact in* C_g.

(3.6) *There exists a decreasing, uniformly continuous function* $h: (-\infty, \infty) \to [0, \infty)$ *with* $h(t) \equiv 0$ *on* $[0, \infty)$ *such that* $\lim_{n\to\infty} \inf_{(t,s)\in I_n} [h(s) - h(s+t)] = \infty$ *and* $g(s) \equiv e^{h(s)}$ *on* $(-\infty, 0]$.

Moreover, each of these conditions implies:

(3.7) *There exist* $\gamma, \delta > 0$ *such that*

$$e^{-\gamma s} \leq g(s) \leq e^{-\delta s}$$

for all sufficiently large $|s|$ *with* $s \leq 0$.

Proof. [(3.3) *implies* (3.4)]. First note that $(g1)$ and $(g2)$ ensure admissibility of UC_g. Now, suppose x is continuous on $(-\infty, T)$, where $0<T\leq\infty$, and let $t\in[0, T)$. If $x_0 \in C_g$, we have

$$\begin{aligned}|x_t|_g &= \sup_{s\leq 0} |x(t+s)|/g(s)\\ &\leq \sup_{-t\leq s\leq 0} |x(t+s)|/g(s) + \sup_{s\leq -t} |x(t+s)|/g(s)\\ &\leq \sup_{0\leq s\leq t} |x(s)| + \sup_{s\leq -t} [g(t+s)/g(s)]\,[|x(t+s)|/g(t+s)]\\ &\leq \sup_{0\leq s\leq t} |x(s)| + M(t)|x_0|_g,\end{aligned}$$

where $M(t):= \sup_{s\leq -t} [g(t+s)/g(s)]\to 0$ as $t\to\infty$ by assumption. Hence, UC_g is a strong fading memory space.

[(3.4) *implies* (3.5)]. Let $x: (-\infty, \infty)\to \boldsymbol{R}^n$ be bounded and uniformly continuous on $[0, \infty)$ with $x_0 \in UC_g$. By the Ascoli-Arzelà Theorem, $\{x_t: t\geq 0\}$ is precompact in $\hat{C}_g$. Since $g(s)\geq 1$ for $s\leq 0$, condition (2.3) holds and precompactness of $\{x_t: t\geq 0\}$ in C_g follows from Theorem 2.1.

[(3.5) *implies* (3.3)]. Suppose (3.5) holds but (3.3) does not. Then there exist $\varepsilon>0$ and sequences $\{t_k\}\uparrow\infty$ and $\{s_k\}$ contained in $(-\infty, 0]$ such that

$$g(t_k+s_k)/g(s_k) > \varepsilon.$$

Define $x: (-\infty, \infty)\to \boldsymbol{R}^n$ so that x is continuous on $(-\infty, \infty)$, $x(t)\equiv 0$ on $[1, \infty]$, and $x_0 = gv$, where v is a unit vector in $\boldsymbol{R}^n$. Obviously, the zero function is the only possible limit point of the sequence $\{x_{t_k}\}$. However, we have $|x_{t_k}|_g>\varepsilon$ for all k. It follows that $\{x_t: t\geq 0\}$ is not precompact. That is (3.5) does not hold, which is a contradiction.

[(3.3) *implies* (3.6)]. For the sake of convenience, we set $g(s)\equiv 1$ on $[0, \infty)$. Then $(g2)$ is equivalent to the condition

$$g(s+u)/g(s)\to 1 \text{ as } u\to 0 \text{ uniformly for } s\leq 0.$$

Clearly, the function h defined by $h(s):= \log[g(s)]$, $s\leq 0$, is nonincreasing and uniformly continuous on $(-\infty, 0]$. Note that, for any $\varepsilon>0$,

(3.8) $\quad 0 < g(t+s)/g(s) < \varepsilon$ if and only if $h(s) - h(t+s) > -\log\varepsilon$.

But, by (3.3), for each $\varepsilon>0$ there exists n_ε such that the first–hence, the second–inequality in (3.8) holds whenever $(t, s)\in I_n$, $n\geq n_\varepsilon$, and (3.6) is satisfied.

[(3.6) *implies* (3.3)]. This follows since the liminf condition in (3.6) implies that (3.3) holds.

[3.6) *implies* (3.7)]. By uniform continuity of h, there is $\delta=\delta(1)>0$ such that $|h(s_1)-h(s_2)|<1$ whenever $|s_1-s_2|<\delta$, $s_1, s_2\leq 0$. For $s<0$, set $s=-q\delta-r_1$,

where q is a nonnegative integer and $0 \leq r_1 < \delta$. Then

$$h(s) = [h(s) - h(-q\delta)] + h(-q\delta) < 1 + q \leq 1 - s/\delta.$$

Hence, $g(s) < e^{-2s/\delta}$ for $s < -\delta$.

By the liminf condition on h, there exists $N = N(1)$ such that, for any $n \geq N$, we have $h(s) - h(t+s) > 1$ whenever $s \leq -n$, $t \geq n$. For $s < -N$, write $s = -kN - r_2$, where k is a nonnegative integer and $0 \leq r_2 < N$. Then

$$h(s) \geq h(-kN) \geq k > -s/N - 1,$$

and, thus, $g(s) > e^{-s/2N}$ whenever $s < -2N$. This completes the proof of the theorem.

Rrmark 3.1. Theorems 3.1 and 3.2 give us a fairly complete picture of precompactness of bounded, positive orbits in C_g spaces. For instance, suppose $x: (-\infty, \infty) \to \boldsymbol{R}^n$ is continuous with $x_0 \in C_g$.

(i) It is easy to check that (3.2) holds whenever x is bounded on $[0, \infty)$, g satisfies $(g3)$ and, for example,

$$|x_0(s)|/g(s) \longrightarrow 0 \quad \text{as} \quad s \longrightarrow -\infty. \tag{3.9}$$

If x is bounded on $(-\infty, \infty)$, then (3.9) holds for any g which satisfies $(g3)$. On the other hand, (3.9) certainly does not require x_0 to be bounded on $(-\infty, 0]$. For example, if $(g3)$ holds and, for the scalar case, $x_0(s) := [g(s)]^{1/2}$, then (3.9) holds. Hence, if x is bounded and uniformly continuous on $[0, \infty)$, then $\{x_t: t \geq 0\}$ is precompact in C_g.

(ii) If (3.3) holds (that is, if g is exponential in the sense of (3.7)), then the restriction (3.9) can be dropped. In particular, Theorem 3.2 assures that $\{x_t: t \geq 0\}$ is precompact in C_g whenever (3.3) and the assumption in (3.5) hold. On the other hand, if (3.3) does not hold, then, in the proof of Theorem 3.2 that (3.5) implies (3.3), it is illustrated that we cannot necessarily conclude the desired precompactness property. In particular, it is always possible to find x such that (3.1) holds and $\{x_t: t \geq 0\}$ is not precompact.

Next, we present a theorem concerning convergence in norm of a precompact orbit in C_g. This result provides a supplement to related convergence results in Haddock, Krisztin and Terjéki [6].

Theorem 3.3 *Let $r \geq 0$ be given and suppose $g: (-\infty, 0] \to [1, \infty)$ satisfies $(g1)$ with $g(s) \equiv 1$ on $[-r, 0]$ and $g(s) > 1$ for $s < -r$. Further, suppose $x: (-\infty, \infty) \to \boldsymbol{R}^n$ is such that $x_0 \in C_g$ and $\{x_t: t \geq 0\}$ is precompact in C_g. Then*

$$|x_t|_g \longrightarrow c (\leq \infty) \quad \text{as} \quad t \longrightarrow \infty \tag{3.10}$$

if and only if

$$\sup_{-r\le s\le 0}|x(t+s)| \longrightarrow c \quad as \quad t\longrightarrow\infty. \tag{3.11}$$

Proof. Clearly,

$$\limsup_{t\to\infty}|x_t|_g \ge \limsup_{t\to\infty}\sup_{-r\le s\le 0}|x_t(s)| = \limsup_{t\to\infty}|x(t)|$$

and

$$\liminf_{t\to\infty}|x_t|_g \ge \liminf_{t\to\infty}\sup_{-r\le s\le 0}|x_t(s)| \ge \liminf_{t\to\infty}|x(t)|.$$

Suppose

$$\lim_{t\to\infty}|x_t|_g = c \quad \text{and} \quad \liminf_{t\to\infty}\sup_{-r\le s\le 0}|x_t(s)| = d < c.$$

Then there exists a sequence $\{t_n\}\uparrow\infty$ with $\lim_{n\to\infty}\sup_{-r\le s\le 0}|x_{t_n}(s)|=d$. By precompactness of $\{x_t\colon t\ge 0\}$, there exists a sequence $\{n(k)\}\uparrow\infty$ with $x_{t_{n(k)}}\to\phi$ in C_g. Now,

$$d = \lim_{n\to\infty}\sup_{-r\le s\le 0}|x_{t_n}(s)| = \sup_{-r\le s\le 0}|\phi(s)|,$$

but $|\phi|_g=c$. Since g is nonincreasing and $g(s)>1$ for $s<-r$, there exists $s_0<-r$ with $|\phi(s_0)|>c$, and, hence,

$$c < |\phi(s_0)| = \lim_{k\to\infty}|x(s_0+t_{n(k)})| \le \limsup_{t\to\infty}|x(t)| \le c,$$

a contradiction. Next, suppose $\lim_{t\to\infty}\sup_{-r\le s\le 0}|x_t(s)|=c$. Then $\liminf_{t\to\infty}|x_t|_g\ge c$, and the assumption that $\limsup_{t\to\infty}|x_t|_g>c$ leads to a contradiction as above. This proves the theorem.

We conclude the paper with a rather lengthy but informative example. The main objective of the example is to tie Theorems 3.1, 3.2 and 3.3 together.

Example 3.1. Consider the scalar Volterra integrodifferential equation

$$\begin{aligned} x'(t) = [&-A(t)x(t)+B(t)x(t-r) \\ &+ C(t)\int_{-\infty}^{t} D(t-s)x(s)ds]G(x_t), \end{aligned} \tag{3.12}$$

where $r\ge 0$ is a constant. Let C_r denote the space $C([-r, 0], \boldsymbol{R})$ of continuous functions that map $[-r, 0]$ into $\boldsymbol{R}$ with sup (max) norm. We assume:

$A\colon \boldsymbol{R}^+\longrightarrow\boldsymbol{R}^+$, $B, C, D\colon \boldsymbol{R}^+\longrightarrow\boldsymbol{R}$ and $G\colon C_r\longrightarrow\boldsymbol{R}^+$ are continuous and

$$\int_{-\infty}^{0}|D(-s)|ds < 1. \tag{3.13}$$

Further, suppose

$$A(t) - |B(t)| - |C(t)| \geq 0 \qquad \text{for all} \quad t \geq 0. \tag{3.14}$$

Now, from Theorem 3.1 of [1], we can choose a function g which satisfies $(g1)$, $(g2)$ and $(g3)$ such that $g(s) \equiv 1$ on $[-r, 0]$ and

$$\int_{-\infty}^{0} |D(-s)|g(s)ds < 1. \tag{3.15}$$

Claim 1. For any ϕ in C_g each solution x with $x_0 = \phi$ satisfies

$$\sup_{-r \leq s \leq 0} |x(t+s)| \longrightarrow \text{constant} \quad \text{as } t \longrightarrow \infty.$$

To prove this, we define an auxiliary (Liapunov-Razumikhin) function $V: \boldsymbol{R} \to \boldsymbol{R}^+$ by $V(x) = x^2/2$. After some straightforward calculations, we see that

$$dV(t)/dt \leq 0 \text{ for all } t \geq 0 \text{ for which } |x(t)| = |x_t|_g. \tag{3.16}$$

It follows from Corollary 1 (Section 2) of [6] that (3.10) holds for some $c < \infty$. However, we cannot yet conclude that (3.11) holds since we have not determined that $\{x_t: t \to 0\}$ is precompact in C_g. On the other hand we can generate a new C_g space that will help. In particular, choose g^* which satisfies $(g1)$, $(g2)$ and $(g3)$ with $g^*(s) \equiv 1$ on $[-r, 0]$ and

$$\int_{-\infty}^{0} |D(-s)|g(s)g^*(s)ds < 1. \tag{3.17}$$

Again, [1, Theorem 3.1] permits us to do this. Next, set $g^{**} = gg^*$. Then, g^{**} satisfies the same conditions above as g and g^*. Also, $\phi \in C_{g^{**}}$ and it is easily seen that (3.2) holds with g replaced by g^{**}. Thus, from Theorem 3.1, $\{x_t: t \geq 0\}$ is precompact in $C_{g^{**}}$ for a solution x with x_0 in C_g. As in the argument above using $V(x) = x^2/2$ and the fact that (3.16) holds with g replaced by g^{**}, we have that

$$|x_t|_{g^{**}} \longrightarrow c \quad \text{as } t \longrightarrow \infty \quad \text{for some} \quad c < \infty.$$

From Theorem 3.3 we can now conclude that (3.11) holds, and this proves Claim 1.

With a restriction slightly stronger than (3.14), we can sometimes show that each solution of (3.12) actually tends to a constant as $t \to \infty$. To illustrate this, define

$$G(\phi) := [\phi(0) - \phi(-r)]^2$$

so that (3.12) becomes

$$\begin{aligned} x'(t) = [&-A(t)x(t) + B(t)x(t-r) \\ &+ C(t)\int_{-\infty}^{t} D(t-s)x(s)ds][x(t) - x(t-r)]^2. \end{aligned} \tag{3.18}$$

Instead of (3.14), we impose:

$$A(t)\theta - |B(t)| - |C(t)| \geq 0 \quad \text{for all} \quad t \geq 0 \tag{3.19}$$

and some constant θ with $0 \leq \theta < 1$.

Again, let g and g^* be chosen such that (3.15) and (3.17) hold and each satisfies $(g1)$, $(g2)$ and $(g3)$. This time, though, we choose g and g^* so that g is strictly decreasing on $(-\infty, 0]$ and with the additional restrictions:

(i) $g(-r) = 1/\theta$ and $g^*(s) \equiv 1$ on $[-r, 0]$;

(ii) $\int_{-\infty}^{0} |D(-s)|g(s)ds < 1/\theta$

(iii) $\int_{-\infty}^{0} |D(-s)|g(s)g^*(s)ds < 1/\theta.$

Using $V = x^2/2$ as above, we have that

$$dV[x(t)]/dt \leq |x(t)|\,[-A(t)|x(t)| + |B(t)|\,|x(t-r)| + |C(t)|\,|x_t|_{g^{**}} \int_{-\infty}^{0} |D(-s)|g(s)g^*(s)ds]\,[x(t)-x(t-r)]^2, \tag{3.20}$$

where $g^{**} = gg^*$. We have that g^{**} is strictly decreasing on $(-\infty, 0]$ and, from (i), $g^{**}(-r) = 1/\theta$. By applying (3.19) to (3.20), we see that $dV[x(t)]/dt \leq 0$ whenever $|x(t)| = |x_t|_{g^{**}}$. As was previously mentioned, this implies that $|x_t|_{g^{**}}$ is nonincreasing $\{x_t : t \geq 0\}$ is precompact in $C_{g^{**}}$, so Theorem 3.1 and the fact that g^{**} is strictly decreasing assures that $|x(t)|$ tends to a finite limit as $t \to \infty$. Since (3.18) is a scalar equation, it follows that each solution with initial value in C_g tends to a constant as $t \to \infty$. Of course, we can apply the same ideas by starting with g^{**} and draw the same conclusion for initial values in $C_{g^{**}}$. Note that each constant function is a solution of (3.18), therefore we cannot conclude asymptotic stability.

Remark 3.1. Several comments are in order regarding the previous example.

(a) The requirement that g and g^* above satisfy $(g1)$ and $(g2)$ guarantees that g^{**} will also satisfies these two properties, thus ensuring that $UC_{g^{**}}$ is admissible. The fact that g^* satisfies $(g3)$, in turn, implies that $\phi \in UC_{g^{**}}$ for any ϕ in C_g. One can then argue that standard existence results hold for ϕ in C_g (cf. [8]).

(b) It is interesting to note that to obtain a desired precompactness property for a given positive orbit in a C_g space, we must pass to another such space $C_{g^{**}}$, where ϕ in C_g implies $|\phi(s)|/g^{**}(s) \to 0$ as $s \to -\infty$. Thus, in dealing with a given Volterra equation, we can sometimes find it beneficial to consider a family of underlying phase spaces. Additional details along these lines are given in [1].

(c) The technique of using V and dV/dt in Example 3.1 often is referred to as a method of Razumikhin. More information regarding this method as applied to C_g spaces as well as additional references can be found in [6].

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