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Existence and Uniqueness of Solutions of Neutral Delay-Differential Equations with State Dependent Delays*

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1. Introduction

Let there be given real numbers $\gamma \leq a < b$, a function $f: [a, b] \times \mathbf{R}^3 \rightarrow \mathbf{R}$, initial function $g: [\gamma, a] \rightarrow \mathbf{R}$, and delay functions $\alpha, \beta: [a, b] \times \mathbf{R} \rightarrow \mathbf{R}$ such that $\gamma \leq \alpha(t, y) \leq t$, $\gamma \leq \beta(t, y) \leq t$. Here, $\mathbf{R}$ denotes the set of real numbers. We consider the initial-value problem for delay-differential equations of neutral type

$$(1) \quad \begin{aligned} y'(t) &= F(t, y, y'), & t \in [a, b], \\ y(t) &= g(t), & t \in [\gamma, a], \end{aligned}$$

where for any functions $y, z: [\gamma, b] \rightarrow \mathbf{R}$, F is defined by

$$F(t, y, z) := f(t, y(t), y(\alpha(t, y(t))), z(\beta(t, y(t)))).$$

Denote by $\text{Lip}_1[t_1, t_2]$ the space of real-valued Lipschitz continuous functions on $[t_1, t_2]$ and by $C^{1,1}[t_1, t_2]$ the space of functions whose first derivative belongs to $\text{Lip}_1[t_1, t_2]$. We impose the following conditions on the functions f, g, α , and β which define problem (1):

- (i) $g \in C^{1,1}[\gamma, a]$;
- (ii) $F(a, g, g') = g'_-(a)$ ($g'_-(a)$ denotes the left hand derivative of g at a);
- (iii) $|f(t_1, y_1, u_1, z_1) - f(t_2, y_2, u_2, z_2)| \leq L_1(|t_1 - t_2| + |y_1 - y_2| + |u_1 - u_2|) + L_2|z_1 - z_2|$, $L_1, L_2 \geq 0$, $t_1, t_2 \in [a, b]$, $y_1, y_2, u_1, u_2, z_1, z_2 \in \mathbf{R}$;
- (iv) $|\alpha(t_1, y_1) - \alpha(t_2, y_2)| \leq A_1|t_1 - t_2| + A_2|y_1 - y_2|$, $A_1, A_2 \geq 0$, $t_1, t_2 \in [a, b]$, $y_1, y_2 \in \mathbf{R}$.
- (v) $|\beta(t_1, y_1) - \beta(t_2, y_2)| \leq B_1|t_1 - t_2| + B_2|y_1 - y_2|$, $B_1, B_2 \geq 0$, $t_1, t_2 \in [a, b]$, $y_1, y_2 \in \mathbf{R}$.

Additional conditions on some Lipschitz constants appearing above will be given in the formulation of our theorems.

Equations of type (1) arise as a model for a two-body problem of classical electrodynamics and were studied extensively by Driver [2-4]. He proved the

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existence and uniqueness results for the case where $\gamma \leq \beta(t, y) < t$ (see [2, 3]). The case where f is linear in the last argument was studied by Hale and Cruz [6]. Grimm [5] proved an existence result for (1) with $\gamma = a$ assuming that f is bounded by M , $L_2 < 1$, and $B_1 + B_2 M \leq 1$ by use of the Schauder fixed point theorem. He also proved a uniqueness result in case β is independent of y by the Banach contraction principle. In this paper we relaxed this very restrictive condition. We proved an existence and uniqueness result for (1) with β depending on both arguments t and y under the condition $L_2(1 + B_1 + B_2 G) < 1$, where $G = G(f, g)$. It is also shown that the last inequality can be replaced by $L_2(B_1 + B_2 G^*) < 1$, $G^* = G^*(f, g)$, if $\gamma \leq \beta(t, y) \leq t - \delta$ for some $\delta > 0$.

2. Properties of modified euler sequences

Let $J = \{h : h = (b - a)/n, n \geq n_0\}$, where n_0 is a positive integer, and for $h \in J$ put $t_i = a + ih$, $i = 0, 1, \dots, n$. Define the modified Euler sequences for (1) by

$$(2) \quad \begin{aligned} y_h(t_i + rh) &= y_h(t_i) + rh z_h(t_i), & r \in (0, 1], \\ z_h(t_i + rh) &= (1 - r)z_h(t_i) + rz_h(t_{i+1}), & r \in (0, 1], \\ z_h(t_{i+1}) &= F(t_{i+1}, y_h, z_h), \end{aligned}$$

$i = 0, 1, \dots, n - 1$, where $y_h(t) = g(t)$ and $z_h(t) = g'(t)$ for $t \in [\gamma, a]$. Note that (2) differs from the Euler method considered by Castleton and Grimm [1], where the approximation to the derivative of the solution is a piecewise constant function rather than piecewise linear. Let $C[t_1, t_2]$ denote the space of continuous functions from $[t_1, t_2]$ into $\mathbf{R}$. For any $\phi \in C[\gamma, b]$ and $[c, d] \subset [\gamma, b]$ put $\|\phi\|_{[c, d]} := \sup \{|\phi(t)| : t \in [c, d]\}$. Let L_g and $L_{g'}$ be constants such that

$$\begin{aligned} |g(t) - g(\tau)| &\leq L_g |t - \tau|, \\ |g'(t) - g'(\tau)| &\leq L_{g'} |t - \tau|, \end{aligned}$$

for $t, \tau \in [\gamma, a]$. The existence of such constants follows from (i). Define

$$\begin{aligned} M &:= \sup \{|F(t, 0, 0)| : t \in [a, b]\}, \\ C_1 &:= (\|g'\|_{[\gamma, a]} + M)/(1 - L_2), \\ C_2 &:= 2L_1/(1 - L_2), \\ Y &:= (\|g\|_{[\gamma, a]} + C_1/C_2) \exp((b - a)C_2), \\ Z &:= C_1 + C_2 Y, \\ G &:= \max \{L_g, Z\}. \end{aligned}$$

We have the following:

Lemma 1. Assume that (i)–(v) hold and that $L_2(1 + B_1 + B_2G) < 1$. Then the sequences $\{y_h\}_{h \in J}$ and $\{z_h\}_{h \in J}$ are relatively compact in $C[\gamma, b]$.

Proof. It is clear that y_h and z_h are continuous. We will show that $\{y_h\}_{h \in J}$ and $\{z_h\}_{h \in J}$ are uniformly bounded and uniformly Lipschitz-continuous. From (2) it follows that

$$(3) \quad \|y_h\|_{[\gamma, t_{i+1}]} \leq \|y_h\|_{[\gamma, t_i]} + h\|z_h\|_{[\gamma, t_i]},$$

$i=0, 1, \dots, n-1$. We also have

$$|z_h(t_i)| \leq |F(t_i, y_h, z_h) - F(t_i, 0, 0)| + |F(t_i, 0, 0)|$$

(for $i=0$ this follows from (ii)). Hence, in view of (iii) we obtain

$$\|z_h\|_{[\gamma, t_i]} \leq \|z_h\|_{[\gamma, a]} + M + 2L_1\|y_h\|_{[\gamma, t_i]} + L_2\|z_h\|_{[\gamma, t_i]}.$$

Noting that $L_2 < 1$ we get

$$(4) \quad \|z_h\|_{[\gamma, t_i]} \leq C_1 + C_2\|y_h\|_{[\gamma, t_i]},$$

$i=0, 1, \dots, n$, where C_1 and C_2 are defined above. Substitution of (4) into (3) yields

$$\|y_h\|_{[\gamma, t_{i+1}]} \leq (1 + hC_2)\|y_h\|_{[\gamma, t_i]} + hC_1,$$

and by induction it follows that

$$\|y_h\|_{[\gamma, t_i]} \leq (1 + hC_2)^i\|y_h\|_{[\gamma, a]} + \frac{((1 + hC_2)^i - 1)C_1}{C_2},$$

$i=0, 1, \dots, n$. Consequently, $\|y_h\|_{[\gamma, b]} \leq Y$ and by (4) $\|z_h\|_{[\gamma, b]} \leq Z$, which proves that $\{y_h\}_{h \in J}$ and $\{z_h\}_{h \in J}$ are bounded uniformly in h . It is easy to check that y_h are uniformly Lipschitz-continuous on $[\gamma, b]$ with a constant G . Denote by D a constant such that $D \geq L_{g'}$ and

$$(5) \quad L_1(1 + G(1 + A_1 + A_2G)) + L_2D(B_1 + B_2G) \leq D(1 - L_2).$$

Note that existence of this constant is guaranteed by the condition $L_2(1 + B_1 + B_2G) < 1$. We will prove that the z_h , $h \in J$, are uniformly Lipschitz-continuous on $[\gamma, b]$ with the constant D , i.e.

$$(6) \quad |z_h(t) - z_h(\tau)| \leq D|t - \tau|,$$

for $t, \tau \in [\gamma, b]$. By (i) this condition is satisfied for $t, \tau \in [\gamma, a]$. Assume that (6) holds for $t, \tau \in [\gamma, t_i]$. Then

$$|z_h(t_{i+1}) - z_h(t_i)| \leq L_1[h + |y_h(t_{i+1}) - y_h(t_i)|]$$

$$\begin{aligned}
& + |y_h(\alpha(t_{i+1}, y_h(t_{i+1}))) - y_h(\alpha(t_i, y_h(t_{i+1})))| \\
& + |y_h(\alpha(t_i, y_h(t_{i+1}))) - y_1(\alpha(t_i, y_h(t_i)))|] \\
& + L_2[|z_h(\beta(t_{i+1}, y_h(t_{i+1}))) - z_h(\beta(t_i, y_h(t_{i+1})))| \\
& + |z_h(\beta(t_i, y_h(t_{i+1}))) - z_h(\beta(t_i, y_h(t_i)))|] \\
& \leq L_1(1 + G(1 + A_1 + A_2G))h + L_2DB_2Gh \\
& + L_2|z_h(\beta(t_{i+1}, y_h(t_{i+1}))) - z_h(\beta(t_i, y_h(t_{i+1})))|.
\end{aligned}$$

Two cases are possible: either $\beta(t_{i+1}, y_h(t_{i+1})) \leq t_i$ or $\beta(t_{i+1}, y_h(t_{i+1})) \in (t_i, t_{i+1}]$. In the first case

$$|z_h(\beta(t_{i+1}, y_h(t_{i+1}))) - z_h(\beta(t_i, y_h(t_{i+1})))| \leq DB_1h$$

and

$$\begin{aligned}
|z_h(t_{i+1}) - z_h(t_i)| & \leq L_1(1 + G(1 + A_1 + A_2G))h + L_2D(B_1 + B_2G)h \\
& \leq D(1 - L_2)h \leq Dh
\end{aligned}$$

in view of (5). In the second case

$$z_h(\beta(t_{i+1}, y_h(t_{i+1}))) = (1 - r)z_h(t_i) + rz_h(t_{i+1}),$$

where $r = (\beta(t_{i+1}, y_h(t_{i+1})) - t_i)/h$, and

$$\begin{aligned}
& |z_h(\beta(t_{i+1}, y_h(t_{i+1}))) - z_h(\beta(t_i, y_h(t_{i+1})))| \\
& \leq |z_h(t_i) - z_h(\beta(t_i, y_h(t_{i+1})))| + r|z_h(t_{i+1}) - z_h(t_i)| \\
& \leq D|\beta(t_{i+1}, y_h(t_{i+1})) - \beta(t_i, y_h(t_{i+1}))| + |z_h(t_{i+1}) - z_h(t_i)| \\
& \leq DB_1h + |z_h(t_{i+1}) - z_h(t_i)|.
\end{aligned}$$

Hence,

$$\begin{aligned}
(1 - L_2)|z_h(t_{i+1}) - z_h(t_i)| & \leq L_1(1 + G(1 + A_1 + A_2G))h \\
& + L_2D(B_1 + B_2G)h \leq (1 - L_2)D.
\end{aligned}$$

The last inequality follows from (5). Because z_h is piecewise continuous on $[a, b]$, this proves (6) for $t, \tau \in [\gamma, t_{i+1}]$. Thus $\{y_h\}_{h \in J}$ and $\{z_h\}_{h \in J}$ are uniformly bounded and uniformly Lipschitz-continuous. In view of the Ascoli-Arzelà theorem they are relatively compact in $C[\gamma, b]$, which completes the proof of Lemma 1.

Since $L_2(1 + B_1 + B_2G) < 1$, L_2 is necessarily less than 1. This condition can be relaxed if $\gamma \leq \beta(t, y) \leq t - \delta$ for some $\delta > 0$. Denote by K the smallest integer such that $K \geq 2(b - a)/\delta$ and put $S(\mu) = \sum_{j=0}^{\mu} L_2^j$. Define

$$\begin{aligned}
C_1^* &:= S(K) \|g'\|_{[\gamma, a]} + S(K-1)M, \\
C_2^* &:= 2L_1 S(K-1), \\
Y^* &:= (\|g\|_{[\gamma, a]} + C_1^*/C_2^*) \exp((b-a)C_2^*), \\
Z^* &:= C_1^* + C_2^* Y^*, \\
G^* &:= \max \{L_g, Z^*\}.
\end{aligned}$$

We have the following:

Lemma 2. Assume that (i)–(v) hold, $\gamma \leq \beta(t, y) \leq t - \delta$, $t \in [a, b]$, for some $\delta > 0$ and that $L_2(B_1 + B_2 G^*) < 1$. Then the sequences $\{y_h\}_{h \in J}$ and $\{z_h\}_{h \in J}$ are relatively compact in $C[\gamma, b]$.

Proof. As in the proof of Lemma 1, we obtain

$$\|z_h\|_{[\gamma, t_i]} \leq \|z_h\|_{[\gamma, a]} + M + 2L_1 \|y_h\|_{[\gamma, t_i]} + L_2 \|z_h\|_{[\gamma, t_i - \delta]},$$

$i=0, 1, \dots, n$. To estimate $\|z_h\|_{[\gamma, t_i]}$ by $\|y_h\|_{[\gamma, t_i]}$ we use arguments similar to those in [7, 8]. Put $i(0)=i$ and denote by $i(v+1)$ the smallest integer such that $t_{i(v)} - \delta \leq t_{i(v+1)}$, $v=0, 1, \dots$. Note that if $h \leq \delta$ then $i(v+1) < i(v)$, $i=0, 1, \dots, K_i-1$, and $i(v)=0$ for $v \geq K_i$, where K_i is the smallest integer such that $K_i[\delta/h] \geq i$. Here $[\delta/h]$ denotes the integer part of δ/h . For $h \leq \delta$ we have $K[\delta/h] \geq 2(b-a)[\delta/h]/\delta \geq (b-a)h \geq i$. Since K_i is the smallest integer with this property it follows that $K_i \leq K$, i.e. K_i can be bounded by a constant independent of h . As in [7, 8], after $v \leq K_i$ iterations we obtain

$$\|z_h\|_{[\gamma, t_i]} \leq S(v-1)(\|g'\|_{[\gamma, a]} + M) + 2S(v-1)L_1 \|y_h\|_{[\gamma, t_i]} + L_2^v \|z_h\|_{[\gamma, t_{i(v)}}.$$

Putting $v=K_i$ in this inequality gives

$$\begin{aligned}
\|z_h\|_{[\gamma, t_i]} &\leq S(K_i) \|g'\|_{[\gamma, a]} + S(K_i-1)M + 2S(K_i-1)L_1 \|y_h\|_{[\gamma, t_i]} \\
&\leq C_1^* + C_2^* \|y_h\|_{[\gamma, t_i]},
\end{aligned}$$

which is the desired estimate for $\|z_h\|_{[\gamma, t_i]}$. From this point, proceeding as in the proof of Lemma 1 we get $\|y_h\|_{[\gamma, b]} \leq Y^*$, $\|z_h\|_{[\gamma, b]} \leq Z^*$ and $|y_h(t) - y_h(\tau)| \leq G^*|t - \tau|$ for $t, \tau \in [\gamma, b]$. Denote by $D^* \geq L_g$, a constant such that

$$(7) \quad L_1(1 + G^*(1 + A_1 + A_2 G^*)) + L_2 D^*(B_1 + B_2 G^*) \leq D^*.$$

The existence of such a constant is guaranteed by the condition $L_2(B_1 + B_2 G^*) < 1$. Assume that

$$|z_h(t) - z_h(\tau)| \leq D^*|t - \tau|$$

for $t, \tau \in [\gamma, t_i]$. Then, if $h \leq \delta$, in view of (7) we have

$$|z_h(t_{i+1}) - z_h(t_i)| \leq L_1(1 + G^*(1 + A_1 + A_2 G^*))h + L_2 D^*(B_1 + B_2 G^*) \leq D^*.$$

This proves that z_h , $h \in J$, are uniformly Lipschitz-continuous, and the Lemma follows.

3. Existence and uniqueness

For any $\phi \in C[\gamma, b]$ and $h \in J$ define ϕ^h and $\bar{\phi}^h$ by

$$\phi^h(t) := \begin{cases} \phi(t), & t \in [\gamma, a), \\ \phi(t_i), & t \in [t_i, t_{i+1}); \end{cases}$$

and

$$\bar{\phi}^h(t) := \begin{cases} \phi(t), & t \in [\gamma, a), \\ \frac{t_{i+1} - t_i}{h} \phi(t_i) + \frac{t - t_i}{h} \phi(t_{i+1}), & t \in [t_i, t_{i+1}), \end{cases}$$

$i=0, 1, \dots, n$. We will use this notation in the proof of the following theorem.

Theorem 1. Assume that (i)–(v) hold and that $L_2(1 + B_1 + B_2 G) < 1$, where G is defined as above. Then the initial-value problem (1) has a solution $y \in C^{1,1}[\gamma, b]$. This solution is unique in the space of continuously differentiable functions $C^1[\gamma, b]$.

Proof. We will first show existence. It follows from Lemma 1 that $\{y_h\}_{h \in J}$ and $\{z_h\}_{h \in J}$ defined by (2) are relatively compact in $C[\gamma, b]$. Denote by $\{z_h\}_{h \in J'}$, $J' \subset J$, $\inf J' = 0$, any convergent subsequence of $\{z_h\}_{h \in J}$ with limit z , i.e. $\|z_h - z\|_{[\gamma, b]} \rightarrow 0$ as $h \rightarrow 0$, $h \in J'$. It is clear that $z \in \text{Lip}_1[\gamma, b]$. We will show that $\{y_h\}_{h \in J'}$ converges to the function y defined by

$$y(t) = \begin{cases} g(t), & t \in [\gamma, a), \\ g(a) + \int_a^t z(s) ds, & t \in [a, b], \end{cases}$$

and that y is a solution of our problem (1). Note that for $t \in [a, b]$

$$z_h(t) = \bar{z}_h^h(t) = \bar{z}^h(t) + o(1)$$

and

$$y_h(t) = g(a) + \int_a^t z_h^h(s) ds = g(a) + \int_a^t z^h(s) ds + o(1).$$

Define

$$\bar{y}_h(t) = g(a) + \int_a^t z_h(s) ds.$$

Clearly, $\bar{y}_h \rightarrow y$ as $h \rightarrow 0$, $h \in J'$. We have also

$$|y_h(t) - \bar{y}_h(t)| \leq \int_a^t |z^h(s) - \bar{z}^h(s)| ds + o(1).$$

Hence, $\|y_h - \bar{y}_h\|_{[\gamma, b]} \rightarrow 0$ as $h \rightarrow 0$, $h \in J'$. Consequently

$$\|y_h - y\|_{[\gamma, b]} \leq \|y_h - \bar{y}_h\|_{[\gamma, b]} + \|\bar{y}_h - y\|_{[\gamma, b]},$$

which proves that $y_h \rightarrow y$ as $h \rightarrow 0$, $h \in J'$. For $t \in [a, b]$ we have

$$\begin{aligned} (8) \quad & |y_h(t) - \int_a^t F(s, y, y') ds - g(a)| \\ & \leq \int_a^t |F^h(s, y_h, z_h) - F(s, y, y')| ds \\ & \leq \gamma_1(y, h) + \gamma_2(y, h), \end{aligned}$$

where

$$\begin{aligned} \gamma_1(y, h) &:= \int_a^t |F^h(s, y_h, z_h) - F^h(s, y, y')| ds, \\ \gamma_2(y, u) &:= \int_a^t |F^h(s, y, y') - F(s, y, y')| ds. \end{aligned}$$

To estimate $\gamma_1(y, h)$ note that

$$\begin{aligned} \gamma_1(y, h) &= \sum_{j=0}^{i-1} \int_{t_j}^{t_{j+1}} |F(t_j, y_h, z_h) - F(t_j, y, y')| ds \\ &\quad + \int_{t_i}^t |F(t_i, y_h, z_h) - F(t_i, y, y')| ds, \end{aligned}$$

for $t \in (t_i, t_{i+1}]$. As in the proof of Lemma 1 we obtain

$$\begin{aligned} |F(t_j, y_h, z_h) - F(t_j, y, y')| &\leq (L_1(2 + GA_2) + L_2DB_2) \|y_h - y\|_{[\gamma, b]} \\ &\quad + L_2 \|z_h - y'\|_{[\gamma, b]}. \end{aligned}$$

From this inequality it follows that $\gamma_1(y, h) \rightarrow 0$ as $h \rightarrow 0$, $h \in J'$. It is also clear that $\gamma_2(y, h) \rightarrow 0$ as $h \rightarrow 0$, $h \in J'$, because the function $s \rightarrow F(s, y, y')$ is continuous (in particular Riemann integrable). Thus, in view of (8), y is a solution of (1).

Now we will show uniqueness. Assume that there is another solution $\tilde{y} \in C^1[\gamma, b]$ of (1) and put $u := y - \tilde{y}$. Then $u \equiv 0$ on $[\gamma, a]$ and

$$\begin{aligned}
|u'(t)| &\leq |F(t, y, y') - F(t, \tilde{y}, \tilde{y}')| \leq L_1[|u(t)| \\
&\quad + |y(\alpha(t, y(t))) - y(\alpha(t, \tilde{y}(t)))| + |y(\alpha(t, \tilde{y}(t))) - \tilde{y}(\alpha(t, \tilde{y}(t)))|] \\
&\quad + L_2[|y'(\beta(t, y(t))) - y'(\beta(t, \tilde{y}(t)))| + |y'(\beta(t, \tilde{y}(t))) - \tilde{y}'(\beta(t, \tilde{y}(t)))|] \\
&\leq (L_1(2 + GA_2) + L_2DB_2) \|u\|_{[a,t]} + L_2 \|u'\|_{[a,t]}.
\end{aligned}$$

Hence,

$$\|u'\|_{[a,b]} \leq W \|u\|_{[a,b]},$$

where

$$W = (L_1(2 + GA_2) + L_2DB_2)/(1 - L_2).$$

We also have

$$\begin{aligned}
|u(t)| &\leq \int_a^t |F(s, y, y') - F(s, \tilde{y}, \tilde{y}')| ds \\
&\leq (L_1(2 + GA_2) + L_2DB_2) \int_a^t \|u\|_{[a,s]} ds + L_2 \int_a^t \|u'\|_{[a,s]} ds.
\end{aligned}$$

Hence,

$$\|u\|_{[a,t]} \leq E \int_a^t \|u\|_{[a,s]} ds,$$

where

$$E = (L_1(2 + GA_2) + L_2DB_2) + L_2W.$$

By induction it follows that

$$\|u\|_{[a,t]} \leq \frac{(E(t-a))^k}{k!} \|u\|_{[a,b]}$$

for any $t \in [a, b]$ and any integer $k \geq 0$ (compare also Lemma 2 in [9]). Consequently, $u \equiv 0$ and $y \equiv \tilde{y}$ which is our claim.

If $\gamma \leq \beta(t, y) \leq t - \delta$ for some $\delta > 0$ we can relax the condition that $L_2 < 1$, and we have the following:

Theorem 2. Assume that (i)–(v) hold, $\gamma \leq \beta(t, y) \leq t - \delta$, $\delta > 0$, for $t \in [a, b]$, and that $L_2(B_1 + B_2G^*) < 1$, where G^* is defined as above. Then problem (1) has a solution $y \in C^{1,1}[\gamma, b]$. This solution is unique in $C^1[\gamma, b]$.

Proof. It follows from Lemma 2 that the modified Euler sequences $\{y_h\}_{h \in J}$ and $\{z_h\}_{h \in J}$ are relatively compact in $C[\gamma, b]$. From this point the proof is similar to that of Theorem 1.

References

- [1] Castleton, R. N. and Grimm L. J., A first order method for differential equations of neutral type, *Math. Comp.*, **27** (1973), 571–577.
- [2] Driver, R. D., A two-body problem of classical electrodynamics: the one-dimensional case, *Ann. Physics*, **21** (1963), 122–142.
- [3] Driver, R. D., A functional-differential system of neutral type arising in a two-body problem of classical electrodynamics, In: *International Symposium on Nonlinear Differential Equations and Nonlinear Mechanics*, pp. 474–484, Academic Press, New York, 1963.
- [4] Driver, R. D., A neutral system with state dependent delay, *J. Differential Equations*, **54** (1984), 73–86.
- [5] Grimm, L. J., Existence and continuous dependence for a class of nonlinear equations, *Proc. Amer. Math. Soc.*, **29** (1971), 467–473.
- [6] Hale, J. K. and Cruz, M. A., Existence, uniqueness, and continuous dependence for hereditary systems, *Ann. Mat. Pura Appl.*, **85** (1970), 63–81.
- [7] Jackiewicz, Z., Adams methods for neutral functional differential equations, *Numer. Math.*, **39** (1982), 221–230.
- [8] Jackiewicz, Z., One step methods of any order for neutral functional differential equations, *SIAM J. Numer. Anal.*, **21** (1984), 486–511.
- [9] Tavernini, L., The approximate solution of Volterra differential systems with state-dependent time lags, *SIAM J. Numer. Anal.*, **15** (1978), 1039–1052.

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