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L^2-Solutions for Nonlinear Schrödinger Equations and Nonlinear Groups

By

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§1. Introduction and main results

We consider the unique global existence of solutions in a weaker class than $H^1(\boldsymbol{R}^n)$ and some properties of the solution operator for the following nonlinear Schrödinger equation:

$$i\frac{\partial u}{\partial t}=-\Delta u+\lambda|u|^{p-1}u,\quad t\in\boldsymbol{R},\quad x\in\boldsymbol{R}^n, \tag{1.1}$$

$$u(t_0,x)=u_0(x), \tag{1.2}$$

where $t_0\in\boldsymbol{R}$ and $\lambda\in\boldsymbol{R}$. By $\alpha(n)$ we denote ∞ if $n=1$ or $n=2$ and $(n+2)/(n-2)$ if $n\geqq 3$. There are many papers concerning the global existence of solutions for Problem (1.1)–(1.2) (see, e.g., [1]–[2], [4]–[7] and [9]). In [1] Baillon, Cazenave and Figueira show that if $1\leqq n\leqq 3$, $1<p<\alpha(n)$ and $\lambda>0$, Problem (1.1)–(1.2) has a unique global strong solution $u(t)\in C(\boldsymbol{R};H^2(\boldsymbol{R}^n))\cap C^1(\boldsymbol{R};L^2(\boldsymbol{R}^n))$ for any $u_0\in H^2(\boldsymbol{R}^n)$. In [2] Ginibre and Velo show that if $1<p<\alpha(n)$ and $\lambda>0$ or if $1<p<1+4/n$ and $\lambda<0$, Problem (1.1)–(1.2) has a unique global weak solution $u(t)\in C(\boldsymbol{R};H^1(\boldsymbol{R}^n))$ for any $u_0\in H^1(\boldsymbol{R}^n)$. In [6] Strauss shows that if $\lambda>0$ and $p>1$, Problem (1.1)–(1.2) has at least one global weak solution $u(t)\in L^\infty(\boldsymbol{R};H^1(\boldsymbol{R}^n)\cap L^{p+1}(\boldsymbol{R}^n))$ for any $u_0\in H^1(\boldsymbol{R}^n)\cap L^{p+1}(\boldsymbol{R}^n)$ (see also [5]). In [10] M. Tsutsumi and N. Hayashi discuss the unique global existence of classical solutions for (1.1)–(1.2) with $\lambda>0$ (see also Pecher and von Wahl [4]). Furthermore, in [9] M. Tsutsumi discusses the unique global solution in $\mathscr{S}(\boldsymbol{R}^n)$ or in the weighted Sobolev space for (1.1)–(1.2) with $\lambda>0$. In almost all of previous papers the solution of (1.1)–(1.2) has been constructed in a space not larger than the energy space, that is, $H^1(\boldsymbol{R}^n)$, because the proofs in almost all of previous papers are based on the energy conservation law. However, in [7] Strauss constructs the wave operators from $L^2(\boldsymbol{R}^n)$ to $L^2(\boldsymbol{R}^n)$ for the equation (1.1) with $p=1+4/n$ (see [7, Theorem 5]). His results are almost equivalent to the construction in $L^2(\boldsymbol{R}^n)$ of unique local solutions for (1.1)–(1.2) with $p=1+4/n$. In this paper we prove that when $1<p<1+4/n$, we can construct the unique global solution of (1.1)–(1.2) for u_0 in $L^2(\boldsymbol{R}^n)$ (but possibly not in $H^1(\boldsymbol{R}^n)$). Such a solution is called an "L^2-solution". Furthermore, we show that when $1<p<1+4/n$, the solution operator of the evolution equation (1.1) constitutes a strongly continuous nonlinear operator group in $L^2(\boldsymbol{R}^n)$. Our proof is based on the L^2-norm conservation law and the dispersive effect of solutions (see, e.g., Lemma 2.2).

Our main theorem in this paper is the following.

Theorem 1.1. *Assume that* $1<p<1+4/n$. *Then, for any* $u_0 \in L^2(\boldsymbol{R}^n)$ *and any* $t_0 \in \boldsymbol{R}$ *there exists a unique global solution* $u(t)$ *of* (1.1)–(1.2) *such that*

$$u(t) \in C(\boldsymbol{R};\, L^2(\boldsymbol{R}^n)) \cap L^r_{\mathrm{loc}}(\boldsymbol{R};\, L^{p+1}(\boldsymbol{R}^n)), \tag{1.3}$$

$$u(t) = U(t-t_0)u_0 - i\int_{t_0}^{t} U(t-\tau)f(u(\tau))d\tau, \quad t \in \boldsymbol{R}, \tag{1.4}$$

$$\|u(t)\|_{L^2(\boldsymbol{R}^n)} = \|u_0\|_{L^2(\boldsymbol{R}^n)}, \quad t \in \boldsymbol{R}, \tag{1.5}$$

where $r = \dfrac{4(p+1)}{n(p-1)}$, $U(t)=e^{i\Delta t}$, $f(z)=\lambda|z|^{p-1}z$ $(z \in \boldsymbol{C})$ *and the integral in* (1.4) *is the Bochner integral in* $H^{-1}(\boldsymbol{R}^n)$. *Furthermore, let* u_{0j}, $j=1, 2,\cdots$, *and* u_0 *be such that* $u_{0j}, u_0 \in L^2(\boldsymbol{R}^n)$ *and* $u_{0j} \to u_0$ *in* $L^2(\boldsymbol{R}^n)$ $(j\to\infty)$. *Let* $u_j(t)$ *and* $u(t)$ *be the solutions of* (1.1) *with* $u_j(t_0)=u_{0j}$ *and* $u(t_0)=u_0$, *respectively. Then, for each* $t \in \boldsymbol{R}$

$$u_j(t) \longrightarrow u(t) \quad \text{in} \quad L^2(\boldsymbol{R}^n) \quad (j\to\infty). \tag{1.6}$$

By Theorem 1.1 we can define the solution operator of the evolution equation (1.1) as a mapping from $L^2(\boldsymbol{R}^n)$ to $L^2(\boldsymbol{R}^n)$, when $1<p<1+4/n$. We denote it by $S(t)$. The following result is an immediate consequence of Theorem 1.1.

Corollary 1.2. *Assume that* $1<p<1+4/n$. *Then,* $\{S(t); -\infty<t<+\infty\}$ *is a strongly continuous nonlinear operator group in* $L^2(\boldsymbol{R}^n)$. *That is,* $S(t)$ *is a homeomorphism from* $L^2(\boldsymbol{R}^n)$ *to* $L^2(\boldsymbol{R}^n)$ *for each* $t \in \boldsymbol{R}$, *and*

$$S(t+s) = S(t)S(s), \quad t, s \in \boldsymbol{R}, \tag{1.7}$$

$$S(0) = I, \tag{1.8}$$

$$S(h)v \longrightarrow v \quad \text{in} \quad L^2(\boldsymbol{R}^n) \quad (h\to 0), \quad v \in L^2(\boldsymbol{R}^n), \tag{1.9}$$

where I *is the identity operator from* $L^2(\boldsymbol{R}^n)$ *to* $L^2(\boldsymbol{R}^n)$.

Our plan in this paper is as follows. In Section 2 we summarize several lemmas needed for the proof of Theorem 1.1. In Section 3 we give a proof of Theorem 1.1.

We conclude this section with several notations given. We abbreviate $L^q(\boldsymbol{R}^n)$ and $H^m(\boldsymbol{R}^n)$ to L^q and H^m, respectively. $(\cdot,\cdot)$ denotes the scalar product in L^2. For a closed interval I in $\boldsymbol{R}$ and a Hilbert space H we denote the set of all weakly continuous functions from I to H by $C_w(I; H)$. Let $U(t)=e^{i\Delta t}$ be the evolution operator of the free Schrödinger equation. We put $f(z)=\lambda|z|^{p-1}z$ for $z\in \boldsymbol{C}$. Let $h(x)$ be an even and positive function in $C_0^\infty(\boldsymbol{R}^n)$ with $\|h\|_{L^1}=1$. We put $h_j(x)=j^n h(jx)$ for each positive integer j. $*$ denotes the convolution with respect to spatial variables. In the course of calculations below various constants will be simply denoted by C. In particular, $C=C(*,\cdots,*)$ will denote a constant depending only on the quantities appearing in parentheses.

§ 2. Lemmas.

In this section we summarize several results needed for the proof of Theorem 1.1.

For $U(t)$ we have the following two lemmas.

Lemma 2.1. *Let q and r be positive numbers such that $1/q+1/r=1$ and $2\leqq q\leqq\infty$. For any $t\neq 0$, $U(t)$ is a bounded operator from L^r to L^q satisfying*

$$\|U(t)v\|_{L^q} \leqq (4\pi|t|)^{n/q-n/2}\|v\|_{L^r}, \quad v\in L^r, \quad t\neq 0, \tag{2.1}$$

and for any $t\neq 0$, the map $t\to U(t)$ is strongly continuous. For $q=2$, $U(t)$ is unitary and strongly continuous for all $t\in \boldsymbol{R}$.

Lemma 2.2. *Let q and r be positive numbers such that $1\leqq q-1<\alpha(n)$ and $(n/2-n/q)r=2$. Then,*

$$\|U(\cdot)v\|_{L^r(\boldsymbol{R};L^q)} \leqq C\|v\|_{L^2}, \tag{2.2}$$

where $C=C(n, q)$.

Lemma 2.1 is well known (see, e.g., [2, Lemma 1.2]). For Lemma 2.2, see Strichartz [8, Corollary 1 in §3] and Ginibre and Velo [3, Proposition 7].

Furthermore, we need the following two lemmas.

Lemma 2.3. *Let I be an open interval in $\boldsymbol{R}$. Let $1<q$, $r<\infty$ and a, $b>0$. We put*

$$\begin{aligned} M = \{&v(t)\in L^\infty(I; L^2)\cap L^r(I; L^q); \\ &\|v(t)\|_{L^2}\leqq a, \qquad a.e.\ t\in I, \\ &\|v\|_{L^r(I;L^q)}\leqq b\}. \end{aligned}$$

Then M is a closed subset in $L^r(I; L^q)$.

Proof. Let $\{v_j(t)\}\subset M$ and $v(t)\in L^r(I; L^q)$ be such that $\|v_j-v\|_{L^r(I;L^q)}\to 0$ $(j\to\infty)$. Then, since $\{v_j(t)\}$ is bounded in $L^\infty(I; L^2)$, we can choose a subsequence $\{v_{j'}(t)\}\subset\{v_j(t)\}$ and $w(t)\in L^\infty(I; L^2)$ such that

(2.3) $$v_{j'}(t) \longrightarrow w(t) \quad *\text{-weakly} \quad \text{in} \quad L^\infty(I; L^2).$$

Since by the definitions of $\{v_j(t)\}$ and $v(t)$ and (2.3) we have

(2.4) $$v_{j'}(t) \longrightarrow v(t) \quad \text{in} \quad \mathscr{D}'(I \times \boldsymbol{R}^n) \quad (j' \to \infty),$$

(2.5) $$v_{j'}(t) \longrightarrow w(t) \quad \text{in} \quad \mathscr{D}'(I \times \boldsymbol{R}^n) \quad (j' \to \infty),$$

we conclude by the uniqueness of limit that $v(t) = w(t) \in L^\infty(I; L^2) \cap L^r(I; L^q)$. Moreover, we have by (2.3) and the definitions of $\{v_j(t)\}$ and $v(t)$

(2.6) $$\|v\|_{L^\infty(I;L^2)} \leqq \liminf_{j' \to \infty} \|v_{j'}\|_{L^\infty(I;L^2)} \leqq a,$$

(2.7) $$\|v\|_{L^r(I;L^q)} = \lim_{j \to \infty} \|v_j\|_{L^r(I;L^q)} \leqq b,$$

which imply that $v(t) \in M$. This completes the proof of Lemma 2.3. Q. E. D.

Lemma 2.4. *Let T_1 and T_2 be constants with $T_1 < T_2$. Assume that $v(t) \in C([T_1, T_2]; H^{-1})$ and for some $K > 0$*

(2.8) $$\|v(t)\|_{L^2} \leqq K, \quad a.e.\ t \in [T_1, T_2].$$

Then, $v(t) \in C_w([T_1, T_2]; L^2)$ and (2.8) holds for all $t \in [T_1, T_2]$.

Proof. We first prove that (2.8) holds for all $t \in [T_1, T_2]$. For that purpose, we assume otherwise and derive the contradiction. We assume that (2.8) does not hold for some $t_0 \in [T_1, T_2]$. Then, by (2.8) we can choose $\{t_n\} \subset [T_1, T_2]$ such that $t_n \to t_0$ $(n \to \infty)$ and (2.8) holds for all t_n. Therefore, we can choose a subsequence $\{t_{n'}\} \subset \{t_n\}$ and $w \in L^2$ such that

(2.9) $$v(t_{n'}) \longrightarrow w \quad \text{weakly in} \quad L^2 \quad (n' \to \infty).$$

On the other hand, since $v(t) \in C([T_1, T_2]; H^{-1})$, we have

(2.10) $$v(t_{n'}) \longrightarrow v(t_0) \quad \text{in} \quad H^{-1} \quad (n' \to \infty).$$

(2.9) and (2.10) give us

(2.11) $$v(t_{n'}) \longrightarrow w \quad \text{in} \quad \mathscr{D}'(\boldsymbol{R}^n) \quad (n' \to \infty),$$

(2.12) $$v(t_{n'}) \longrightarrow v(t_0) \quad \text{in} \quad \mathscr{D}'(\boldsymbol{R}^n) \quad (n' \to \infty).$$

By (2.11–12) and the uniqueness of limit we obtain $v(t_0) = w \in L^2$. Moreover,

(2.13) $$\|v(t_0)\|_{L^2} \leqq \liminf_{n' \to \infty} \|v(t_{n'})\|_{L^2} \leqq K,$$

which contradicts the assumption of t_0. Accordingly, we conclude that (2.8) holds for all $t \in [T_1, T_2]$.

We next prove that $v(t) \in C_w([T_1, T_2]; L^2)$. For any $t_0 \in [T_1, T_2]$, let $\{t_n\}$

be an arbitrary sequence $\subset[T_1, T_2]$ such that $t_n \to t_0$ $(n\to\infty)$. Since $v(t)\in C([T_1, T_2]; H^{-1})$, we have

$$(v(t_n), \psi) \longrightarrow (v(t_0), \psi) \quad (n\to\infty) \tag{2.14}$$

for $\psi\in H^1$. By the boundedness in L^2 of $\{v(t_n)\}$ and (2.14) we obtain

$$v(t_n) \longrightarrow v(t_0) \quad \text{weakly in} \quad L^2 \quad (n\to\infty), \tag{2.15}$$

which implies the weak continuity in L^2 of $v(t)$ at $t=t_0$. Since t_0 is an arbitrary point in $[T_1, T_2]$, we conclude that $v(t)\in C_w([T_1, T_2]; L^2)$. Q. E. D.

Finally we describe the results of Ginibre and Velo [2] concerning the unique global existence of weak solutions for (1.1)–(1.2). We first formulate this problem precisely. We consider the integral equation

$$u(t) = U(t-t_0)u_0 - i\int_{t_0}^{t} U(t-\tau)f(u(\tau))d\tau, \tag{2.16}$$

as the integral version of the initial value problem (1.1)–(1.2). For (2.16) we have the following proposition (see, e.g., [2, Theorem 3.1]).

Proposition 2.5. *Assume that* $1<p<1+4/n$. *Then, for any* $t_0\in \boldsymbol{R}$ *and any* $u_0\in H^1$ *there exists a unique solution* $u(t)$ *of* (2.16) *such that*

$$u(t)\in C(\boldsymbol{R}; H^1), \tag{2.17}$$

$$\|u(t)\|_{L^2} = \|u_0\|_{L^2}, \quad t\in \boldsymbol{R}. \tag{2.18}$$

§ 3. Proof of Theorem 1.1.

In this section we give a proof of Theorem 1.1. By I_t and $\bar{I}_t$ we denote an open interval (t_0-t, t_0+t) and a closed interval $[t_0-t, t_0+t]$, respectively, for $t\geqq 0$. Let $r=\dfrac{4(p+1)}{n(p-1)}$ throughout this section.

We first prove the following result concerning the unique local existence of L^2-solutions for (1.1)–(1.2).

Lemma 3.1. *Assume that* $1<p<1+4/n$. *Then, for any* $t_0\in\boldsymbol{R}$ *and any* $u_0\in L^2$ *there exists a* $T=T(p, n, \lambda, \|u_0\|_{L^2})>0$ *such that Problem* (1.1)–(1.2) *has a unique local solution* $u(t)$:

$$u(t)\in C(\bar{I}_T; L^2) \cap L^r(I_T; L^{p+1}), \tag{3.1}$$

$$u(t) = U(t-t_0)u_0 - i\int_{t_0}^{t} U(t-\tau)f(u(\tau))d\tau, \quad t\in\bar{I}_T, \tag{3.2}$$

where the integral in (3.2) *is the Bochner integral in* H^{-1}. *Furthermore, the solution* $u(t)$ *satisfies*

$$\|u(t)\|_{L^2} = \|u_0\|_{L^2}, \quad t \in \bar{I}_T. \tag{3.3}$$

Proof. We consider the following integral equation:

$$u_j(t) = U(t-t_0)h_j * u_0 - i\int_{t_0}^{t} U(t-\tau)f(u_j(\tau))d\tau, \quad j = 1, 2, \cdots. \tag{3.4}$$

By Proposition 2.5 we have the unique global solution $u_j(t)$ in $C(\boldsymbol{R}; H^1)$ of (3.4) for each j. Let $\rho = \|u_0\|_{L^2}$. By δ we denote the constant appearing in (2.2) with $q = p+1$ and $r = \dfrac{4(p+1)}{n(p-1)}$. We note that δ depends only on n and p. We put

$$\begin{aligned} M = \{v(t) \in L^\infty(I_T; L^2) \cap L^r(I_T; L^{p+1});& \\ \|v\|_{L^\infty(I_T;L^2)} \leqq \rho,\ \|v\|_{L^r(I_T;L^{p+1})} \leqq 2\delta\rho\},& \end{aligned} \tag{3.5}$$

where T is a small positive constant to be determined later. We note that by Lemma 2.3 M is closed in $L^r(I_T; L^{p+1})$.

We first show that if T is sufficiently small, then

$$u_j(t) \in M \quad \text{for all} \quad j. \tag{3.6}$$

By (2.18) we have

$$\|u_j(t)\|_{L^2} = \|h_j * u_0\|_{L^2} \leqq \|u_0\|_{L^2}, \quad t \in \bar{I}_T, \tag{3.7}$$

for all j. We define $\tilde{u}_j^s(t)$ by

$$\tilde{u}_j^s(t) = \begin{cases} u_j(t), & t \in I_s, \\ 0, & t \notin I_s, \end{cases} \tag{3.8}$$

for $s \geqq 0$. By (3.4) and (2.1–2) we have for s with $0 \leqq s \leqq T$

$$\begin{aligned} \|u\|_{L^r(I_s;L^{p+1})} &\leqq \delta\rho \\ &+ C\Big\|\int_{t_0}^{t} |t-\tau|^{-n/2+n/(p+1)} \|f(u_j(\tau))\|_{L^{1+1/p}} d\tau\Big\|_{L^r(I_s)} \\ &\leqq \delta\rho + C\Big\|\int_{-\infty}^{+\infty} |t-\tau|^{-n/2+n/(p+1)} \|\tilde{u}_j^s(\tau)\|_{L^{p+1}}^p d\tau\Big\|_{L^r(\boldsymbol{R})} \\ &\leqq \delta\rho + C\|\tilde{u}_j^s\|_{L^{q_1}(\boldsymbol{R};L^{p+1})}^p \\ &\leqq \delta\rho + C\|u_j\|_{L^{q_1}(I_s;L^{p+1})}^p, \quad j = 1, 2, \cdots, \end{aligned} \tag{3.9}$$

where $q_1 = \dfrac{4p(p+1)}{n+4-(n-4)p}$. Here we have used the generalized Young inequality (see [12, §4 in Chapter IX]) at the third inequality. We note that $1<q_1<r$ for $1<p<1+4/n$. On the other hand, we have by Hölder's inequality

$$(3.10)\quad \|u_j\|_{L^{q_1}(I_s;L^{p+1})} \leqq \left(\int_{t_0-s}^{t_0+s} d\tau\right)^{1/q_2} \|u_j\|_{L^r(I_s;L^{p+1})}$$
$$\leqq CT^{1/q_2}\|u_j\|_{L^r(I_s;L^{p+1})},\quad 0\leqq s\leqq T,\quad j=1,2,\cdots,$$

where $q_2=\dfrac{4p}{n+4-np}$ and $C=C(n,p)$. (3.9) and (3.10) give us

$$(3.11)\quad \|u_j\|_{L^r(I_s;L^{p+1})} \leqq \delta\rho + C_0T^{p/q_2}\|u_j\|^p_{L^r(I_s;L^{p+1})},$$
$$0\leqq s\leqq T,\quad j=1,2,\cdots,$$

where $C_0=C_0(n,p,\lambda)$. Now we choose $T>0$ so small that there exists a positive number y satisfying $C_0T^{p/q_2}y^p+\delta\rho-y<0$ and $0<y\leqq 2\delta\rho$. For that purpose, it is sufficient to choose $T>0$ so that

$$(3.12)\qquad T<(2C_0(2\delta\rho)^{p-1})^{-q_2/p}.$$

Then we put

$$(3.13)\qquad y_0=\min\{2\delta\rho\geqq y>0;\ C_0T^{p/q_2}y^p+\delta\rho-y=0\}.$$

Putting $X_j(s)=\|u_j\|_{L^r(I_s;L^{p+1})}$, we have by (3.11)

$$(3.14)\qquad X_j(s)\leqq \delta\rho + C_0T^{p/q_2}X_j(s)^p,\quad 0\leqq s\leqq T,$$

$$(3.15)\qquad X_j(0)=0,$$

for all j. If T is chosen so small that (3.12) holds, then by (3.13) and (3.14–15) we obtain

$$(3.16)\qquad X_j(s)\leqq y_0\leqq 2\delta\rho,\quad 0\leqq s\leqq T,\quad j=1,2,\cdots.$$

From (3.16) and Fatou's lemma it follows that

$$(3.17)\qquad \|u_j\|_{L^r(I_T;L^{p+1})}\leqq 2\delta\rho,\quad j=1,2,\cdots.$$

By (3.7) and (3.17) we obtain (3.6), if T is chosen so small that (3.12) holds. From now on we assume that T is chosen so small that (3.12) holds.

For any j and k, in the same way as (3.9–11) we have by (3.4), Lemmas 2.1, 2.2 and the generalized Young inequality

$$(3.18)\quad \|u_j-u_k\|_{L^r(I_T;L^{p+1})}\leqq K(j,k)$$
$$+\ C\left\|\int_{t_0}^{t}|t-\tau|^{-n/2+n/(p+1)}\|f(u_j(\tau))-f(u_k(\tau))\|_{L^{1+1/p}}d\tau\right\|_{L^r(I_T)}$$

$$\leqq K(j,\ k)+C\|\int_{t_0}^{t}|t-\tau|^{-n/2+n/(p+1)}$$

$$\times(\|u_j(\tau)\|_{L^{p+1}}^{p-1}+\|u_k(\tau)\|_{L^{p+1}}^{p-1})\|u_j(\tau)-u_k(\tau)\|_{L^{p+1}}d\tau\|_{L^r(I_T)}$$

$$\leqq K(j,\ k)+C(\|u_j\|_{L^{q_1}(I_T;L^{p+1})}^{p-1}+\|u_k\|_{L^{q_1}(I_T;L^{p+1})}^{p-1})$$

$$\times\|u_j-u_k\|_{L^{q_1}(I_T;L^{p+1})}$$

$$\leqq K(j,\ k)+\bar{C}_0T^{p/q_2}(\|u_j\|_{L^r(I_T;L^{p+1})}^{p-1}+\|u_k\|_{L^r(I_T;L^{p+1})}^{p-1})$$

$$\times\|u_j-u_k\|_{L^r(I_T;L^{p+1})}$$

$$\leqq K(j,\ k)+\bar{C}_0T^{p/q_2}\cdot 2(2\delta\rho)^{p-1}\|u_j-u_k\|_{L^r(I_T;L^{p+1})},$$

where $K(j,\ k)=\delta\|h_j*u_0-h_k*u_0\|_{L^2}$, $q_1=\dfrac{4p(p+1)}{n+4-(n-4)p}$, $q_2=\dfrac{4p}{n+4-np}$ and $\bar{C}_0=\bar{C}_0(n,\ p,\ \lambda)$. We have used the generalized Young inequality at the third inequality and have used Hölder's inequality with $1/q_2+q_1/r=1$ at the last inequality but one. If we choose T so small in (3.18) that

$$\bar{C}_0T^{p/q_2}\cdot 2(2\delta\rho)^{p-1}\leqq\frac{1}{2},\tag{3.19}$$

then we have by (3.18)

$$\|u_j-u_k\|_{L^r(I_T;L^{p+1})}\leqq 2K(j,\ k)\tag{3.20}$$

for all j and k. Since $K(j,\ k)\to 0$ $(j,\ k\to\infty)$, we have by (3.20)

$$\|u_j-u_k\|_{L^r(I_T;L^{p+1})}\longrightarrow 0\quad(j,\ k\to\infty),\tag{3.21}$$

if T is chosen so small that (3.19) holds. Furthermore, we have by (3.4), (3.21), the Sobolev imbedding theorem and Hölder's inequality

(3.22)
$$|(u_j(t)-u_k(t),\ \psi)|$$

$$\leqq\|h_j*u_0-h_k*u_0\|_{L^2}\|\psi\|_{L^2}$$

$$+\left|\int_{t_0}^{t}(U(t-\tau)\{f(u_j(\tau))-f(u_k(\tau))\},\ \psi)d\tau\right|$$

$$\leqq\|h_j*u_0-h_k*u_0\|_{L^2}\|\psi\|_{L^2}$$

$$+\int_{t_0-T}^{t_0+T}\|f(u_j(\tau))-f(u_k(\tau))\|_{L^{1+1/p}}\|U(\tau-t)\psi\|_{L^{p+1}}d\tau$$

$$\leqq\|h_j*u_0-h_k*u_0\|_{L^2}\|\psi\|_{L^2}$$

$$+C\|\psi\|_{H^1}\int_{t_0-T}^{t_0+T}(\|u_j(\tau)\|_{L^{p+1}}^{p-1}+\|u_k(\tau)\|_{L^{p+1}}^{p-1})$$

$$\times\|u_j(\tau)-u_k(\tau)\|_{L^{p+1}}d\tau$$

$$\leqq \|h_j * u_0 - h_k * u_0\|_{L^2}\|\psi\|_{L^2}$$
$$+ CT^{q_3}\|\psi\|_{H^1}\cdot 2(2\delta\rho)^{p-1}\|u_j - u_k\|_{L^r(I_T;^{p+1})}$$
$$\longrightarrow 0 \quad (j,\ k\to\infty) \quad \text{uniformly in} \quad t\in \bar{I}_T,$$

for $\psi\in H^1$, where $q_3=\dfrac{4+(n+4)p-np^2}{4(p+1)}$. We note that $q_3>0$ for $1<p<1+4/n$. Since by Proposition 2.5 $u_j(t)\in C(\bar{I}_T;\ H^{-1})$ for any j, (3.22) implies that $\{u_j(t)\}$ is the Cauchy sequence in $C(\bar{I}_T;\ H^{-1})$.

Therefore, by (3.6), (3.21), (3.22) and Lemma 2.3 we obtain the solution $u(t)$ of (1.1)–(1.2) such that

$$u(t)\in L^\infty(I_T;\ L^2)\cap L^r(I_T;\ L^{p+1})\cap C(\bar{I}_T;\ H^{-1}), \tag{3.23}$$

$$u(t) = U(t-t_0)u_0 - i\int_{t_0}^t U(t-\tau)f(u(\tau))d\tau, \quad t\in \bar{I}_T, \tag{3.24}$$

$$\|u(t)\|_{L^2}\leqq \|u_0\|_{L^2}, \quad \text{a.e.} \quad t\in I_T \tag{3.25}$$

$$u_j(t)\longrightarrow u(t) \quad \text{in} \quad L^r(I_T;\ L^{p+1}) \quad \text{and} \quad \text{in} \quad C(\bar{I}_T;\ H^{-1}) \quad (j\to\infty), \tag{3.26}$$

where T is a positive constant with (3.12) and (3.19) and the integral in (3.24) is the Bochner integral in H^{-1}. (3.23), (3.25) and Lemma 2.4 imply that

$$u(t)\in C_w(\bar{I}_T;\ L^2) \tag{3.27}$$

and that for all $t\in \bar{I}_T$ (3.25) holds.

We next show that the solution satisfying (3.23) and (3.24) is unique. Let $u(t)$ and $v(t)$ be two solutions satisfying (3.23) and (3.24) with the same initial datum. We put $t_1=\sup\{t\in[0,\ T];\ u(s)=v(s) \text{ on } \bar{I}_t\}$. If $t_1=T$, then $u(t)=v(t)$ on $\bar{I}_T$, which is the desired result. If $t_1<T$, in the same way as (3.18) we have by (3.24) and the assumption of t_1

$$\begin{aligned}\|u-v\|_{L^r(I_{t_1,t_2};L^{p+1})} &= \|u-v\|_{L^r(I_{t_2};L^{p+1})}\\ &\leqq C(t_1-t_2)^{p/q_2}(\|u\|^{p-1}_{L^r(I_{t_1,t_2};L^{p+1})}+\|v\|^{p-1}_{L^r(I_{t_1,t_2};L^{p+1})})\\ &\quad\times \|u-v\|_{L^r(I_{t_1,t_2};L^{p+1})}, \quad t_2\in(t_1,\ T],\end{aligned} \tag{3.28}$$

where $q_2=\dfrac{4p}{n+4-np}$, $C=C(n,\ q,\ p)$ and $I_{t_1,t_2}=(-t_2,\ -t_1)\cup(t_1,\ t_2)$. We can choose t_2 such that $t_2>t_1$ and the coefficient of $\|u-v\|_{L^r(I_{t_1,t_2};L^{p+1})}$ is smaller than 1/2 in the right hand side of (3.28). Then,

$$\|u-v\|_{L^r(I_{t_1,t_2};L^{p+1})}\leqq 0, \tag{3.29}$$

which implies that $u(t)=v(t)$ on $\bar{I}_{t_1,t_2}$. This contradicts the assumption of t_1.

Therefore, $u(t)=v(t)$ for $t\in\bar{I}_T$.

Thus, for any $t_1\in\bar{I}_T$ we can uniquely solve (1.1)–(1.2) in the time interval $[t_1-T,\ t_1+T]$ with the initial time t_0 and the initial datum u_0 replaced by t_1 and $u(t_1)$, respectively, where T is the same as in the case of the initial time t_0 and the initial datum u_0. Therefore, reversing the roles of 0 and t, we obtain the reverse inequality to (3.25) for all $t\in\bar{I}_T$. Accordingly, we obtain (3.3). (3.3) and (3.27) give us

$$u(t)\in C(\bar{I}_T;\ L^2). \tag{3.30}$$

This completes the proof of Lemma 3.1. Q.E.D.

We are now in a position to prove Theorem 1.1.

Proof of Theorem 1.1. The unique global existence of L^2-solutions for (1.1)–(1.2) follows directly from Lemma 3.1, which shows the unique local solvability in L^2 of (1.1)–(1.2) and the a priori bound of the L^2-norm of L^2-solutions.

It remains only to prove the continuous dependence of L^2-solutions on the initial data. Let u_{0j}, $j=1, 2,\cdots$, and u_0 be such that u_{0j}, $u_0\in L^2$ and $u_{0j}\to u_0$ in L^2 $(j\to\infty)$. Let $u_j(t)$ and $u(t)$ be the global L^2-solutions of (1.1) with $u_j(t_0)=u_{0j}$ and $u(t_0)=u_0$, respectively. We put $\rho=\sup\{\|u_0\|_{L^2}, \|u_{0j}\|_{L^2}, j=1, 2,\cdots\}$. For this ρ, let $T>0$ be defined as in (3.12) and (3.19). Then, by using the same argument as in the proof of Lemma 3.1 we have

$$u_j(t)\longrightarrow u(t) \quad \text{in} \quad C(\bar{I}_t;\ H^{-1}) \quad (j\to\infty) \tag{3.31}$$

(see, e.g., (3.22) and (3.26)). Particularly, we have

$$(u_j(t),\ \psi)\longrightarrow (u(t),\ \psi), \quad t\in\bar{I}_t \tag{3.32}$$

for $\psi\in H^1$. By (3.32) and the boundedness in L^2 of $\{u_j(t)\}$ we obtain

$$u_j(t)\longrightarrow u(t) \quad \text{weakly in} \quad L^2 \quad (j\to\infty), \quad t\in\bar{I}_T. \tag{3.33}$$

On the other hand, we have by (1.5)

$$\|u_j(t)\|_{L^2}\longrightarrow \|u(t)\|_{L^2} \quad (j\to\infty), \quad t\in\boldsymbol{R}. \tag{3.34}$$

Therefore, (3.33) and (3.34) give us

$$u_j(t)\longrightarrow u(t) \quad \text{in} \quad L^2 \quad (j\to\infty) \tag{3.35}$$

for each $t\in\bar{I}_T$. The length of T is determined only by n, p, λ and ρ (see (3.12) and (3.19)). By the L^2-norm conservation law (see (1.5) and (3.3)) we see that $\sup\{\|u(t)\|_{L^2}, \|u_j(t)\|_{L^2}, j=1, 2,\cdots\}$ is constant for $t\in\boldsymbol{R}$. Accordingly, we use the above argument with the initial time t_0 and the initial data u_0, u_{0j}, $j=1, 2,\cdots$, replaced by t_0+T and $u(t_0+T)$, $u_j(t_0+T)$, $j=1, 2,\cdots$, or by t_0-T and $u(t_0-T)$, $u_j(t_0-T)$, $j=1, 2,\cdots$, respectively, to obtain (3.35) for each $t\in \bar{I}_{2T}$. Repeating this procedure, we obtain (1.6) for each $t\in \boldsymbol{R}$. This completes the proof of Theorem 1.1. Q. E. D.

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