Kobe University Repository : Kernel

PDF issue: 2026-11-13

On a Functional Equation on C∗-Algebras

Thaheem, A. B.

(Citation)
Funkcialaj Ekvacioj,31(3):411-413

(Issue Date)
1988-12

(Resource Type)
journal article

(Version)
Version of Record

(JaLCDOI)
https://doi.org/10.24546/0100499539

(URL)
https://hdl.handle.net/20.500.14094/0100499539

KOBE UNIVERSITY

Funkcialaj Ekvacioj, 31 (1988) 411–413

On a Functional Equation on C*-Algebras

By

A. B. THAHEEM

(Quaid-i-Azam University, Pakistan)

1. Introduction

Watatani [7] has proved that if α and β are $*$-automorphisms of the algebra $B(H)$ of all bounded linear operators on a Hilbert space H satisfying the equation

$$\alpha(x) + \alpha^{-1}(x) = \beta(x) + \beta^{-1}(x) \tag{1}$$

for all x in $B(H)$, then they commute. The purpose of this note is to consider this problem in a more general setting of C^*-algebras and provide a simpler proof without using Kadison's technical result about extreme points as in [7] and get Watatani's result as an easy corollary to our result (Theorem 2.2).

We may mention in brief that equation (1) has played a fundamental role in the geometric interpretation of the Tomita-Takesaki theory [1] and also in the generalization of the Tomita-Takesaki theory for Jordan algebras [2]. Also in recent years this equation has been solved (in various forms) by central projections. For instance it has been shown in [5] that if α and β are commuting $*$-automorphisms of a von Neumann algebra M satisfying equation (1) then there exists a central projection p in M such that $\alpha=\beta$ on Mp and $\alpha=\beta^{-1}$ on $M(1-p)$. For more details about this type of decomposition we refer to [5] and [6].

2. Results

Our main result depends on the following lemma of [4] which has been proved for von Neumann algebras. Essentially the same argument can work for C^*-algebras as well. We include it here for the sake of completeness and immediate reference.

Lemma 2.1 ([4]). *Let α be a $*$-automorphism of a C^*-algebra M. If x in M satisfies that $\alpha(x)+\alpha^{-1}(x)=2x$, then $\alpha(x)=x$.*

Proof. By assumption, we have $\alpha^2(x)-2\alpha(x)+x=0$. This implies that $(\alpha-1)(\alpha-1)(x)=0$. Put $(\alpha-1)(x)=y$. Then $\alpha(y)=y$ or one gets $\alpha^n(y)=y$ for all positive integers n. But $\alpha(x)=x+y$, so $\alpha^n(x)=x+ny$. This means

$$n\|y\| = \|\alpha^n(x)-x\| \leq \|\alpha^n(x)\| + \|x\| = 2\|x\|\,.$$

Therefore, $n\|y\| \leq 2\|x\|$ for all n. This shows that $\|y\| \leq 2\|x\|/n$ and as $n \to \infty$, we get that $y=0$. This shows that $y=0$ or $\alpha(x)=x$.

We now prove the commutativity result which partially answers the problem proposed in [5].

Theorem 2.2. *Let α and β be $*$-automorphisms of a C*-algebra M satisfying $\alpha(x)+\alpha^{-1}(x)=\beta(x)+\beta^{-1}(x)$ for all x in M. Suppose α (or β) is inner then α and β commute.*

Proof. Suppose α is inner. Then $\alpha(x)=uxu^{-1}$, $(x \in M)$ for an invertible u in M. It follows that

$$\beta(x) + \beta^{-1}(x) = uxu^{-1} + u^{-1}xu.$$

In particular when $x=u$, we get $\beta(u)+\beta^{-1}(u)=2u$. By Lemma 2.1, $\beta(u)=u$. Therefore, for any x in M,

$$\begin{aligned}(\alpha\beta)(x) = \alpha(\beta(x)) = u\beta(x)u^{-1} &= \beta(u)\beta(x)\beta(u^{-1}) \\ &= \beta(uxu^{-1}) = (\beta\alpha)(x).\end{aligned}$$

This shows that α and β commute.

Remark that for $M=B(H)$, we obtain Watatani's result as an immediate corollary to the above theorem.

Since every $*$-automorphism of a type I von Neumann algebra leaving the center pointwise fixed is inner (Sakai [3, Corollary 2.9.32]), therefore we have the following

Corollary 2.3. *Let M be a type I von Neumann algebra and α, β be $*$-automorphisms satisfying $\alpha(x)+\alpha^{-1}(x)=\beta(x)+\beta^{-1}(x)$ for all x in M. If α (or β) leaves the center pointwise fixed, then α and β commute.*

References

[1] Haagerup, U. and Skau, F., Geometric aspects of the Tomita-Takesaki theory II, Math. Scand., **48** (1981), 241–252.

[2] Haagerup, U. and Hanche-Olsen, Tomita-Takesaki theory for Jordan algebras, J. Operator Theory, **11** (1984), 343–364.

[3] Sakai, S., *C*-algebras and W*-algebras*, Springer-Verlag, Berlin, Heidelberg, New York, 1971.

[4] Thaheem, A. B., Decomposition of a von Neumann algebra relative to a $*$-automorphism, Proc. Edinburgh Math. Soc. (2), **22** (1979), 9–10.

[5] Thaheem, A. B., On a decomposition of a von Neumann algebra, Ren. Sem. Mat. Univ. Padova, **65** (1981), 1–7.

[6] Thaheem, A. B., A Van Daele and Vanheeswijck, L., A result on two one-parameter groups of automorphisms, Math. Scand., **51** (1982), 261–274.
[7] Watatani, Y., On commuting automorphisms, Amer. Math. Monthly, **88** (1981), 449.

nuna adreso:
Department of Mathematics
Quaid-i-Azam University
Islamabad, Pakistan

(Ricevita la 12-an de decembro, 1986)