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(Citation)

Funkcialaj Ekvacioj, 30(1):163-168

(Issue Date)

1987-04

(Resource Type)

journal article

(Version)

Version of Record

(JaLCD0I)

<https://doi.org/10.24546/0100499541>

(URL)

<https://hdl.handle.net/20.500.14094/0100499541>



The Cauchy Problem for Linear Distribution Differential Equations

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1. Introduction

In this note we extend the results in Persson [3] on the one-sided Cauchy problem for linear distribution differential equations. The equations here are of a more general type with measure differential equations of arbitrary order $n > 0$ as a special case. We also reformulate the definitions of the spaces $\mathcal{P}^j$ and $\mathcal{B}^j$ of [3]. Especially some conventions used implicitly in [3] are made explicit in the following definition.

Definition 1.1. If g is a complex Borel measure on $\mathbf{R}$ then g is said to be in $\mathcal{P}^0$. If g is a complex valued Borel measurable function on $\mathbf{R}$ then g is said to be in $\mathcal{B}^0$. We agree that two functions in $\mathcal{B}^0$ are different if they are different at at least one point. Let D be distribution differentiation and let $D^{-1}g$ denote a primitive distribution of $g \in \mathcal{D}'(\mathbf{R})$. If for some integer j $D^j g \in \mathcal{P}^0$ then g is said to be in $\mathcal{P}^j$. Outside distribution theory we agree that all functions of $\mathcal{P}^1$ are right continuous and that all functions of $\mathcal{P}^j$, $j > 1$, are continuous. In the same way if for some integer j $D^j g \in \mathcal{B}^0$ then g is said to be in $\mathcal{B}^j$. Outside distribution theory we agree that all functions of $\mathcal{B}^j$, $j > 0$, are continuous. Further if $f \in \mathcal{B}^j$ then we fix a certain $g \in \mathcal{B}^0$ such that $D^j f = g$. When we write $f \in \mathcal{B}^j$ then we mean the pair (f, g) . By convention we mean (f, f) when $f \in \mathcal{B}^0$. Let $f_1 \in \mathcal{B}^j$ with $D^j f_1 = g_1$. Let $j > 0$. Then $f = f_1$ in $\mathcal{B}^j$ if $(f, g) = (f_1, g_1)$ pointwise. If $j < 0$ then $f = f_1$ in $\mathcal{B}^j$ if $f = f_1$ in $\mathcal{D}'(\mathbf{R})$ and $g - g_1$ is a polynomial of at most order $-1 - j$.

We define multiplication of elements in $\mathcal{B}^j$ and $\mathcal{P}^j$.

Definition 1.2. Let j be an integer, let $a \in \mathcal{P}^{-j}$ and let $f \in \mathcal{B}^j$. We choose $b \in \mathcal{P}^0$ and $g \in \mathcal{B}^0$ such that $D^j b = a$ and $D^j f = g$. If $j \geq 0$ then the distribution fa at $\phi \in \mathcal{D}(\mathbf{R})$ is defined as

$$(1.1) \quad \langle fa, \phi \rangle = (-1)^j \int D^j(f\phi)db.$$

If $j < 0$ then

$$(1.2) \quad \langle fa, \phi \rangle = (-1)^j \int g d(D^{-j}(\phi a)).$$

Definition 1.2 gives

Proposition 1.3. *Let j be an integer. If $j \geq 0$ then $\mathcal{P}^{-j}$ is a $\mathcal{B}^j$ module. If $j > 0$ then $\mathcal{B}^{-j}$ is a $\mathcal{P}^j$ module.*

Remark. We regard the modules of Proposition 1.3 as two-sided to make the book-keeping easier. We also notice that one has $\mathcal{P}^{j+1} \subset \mathcal{B}^j$ if for $f \in \mathcal{P}^{j+1}$ one agrees that in $(f, g) \in \mathcal{B}^j$, $D^j f = g \in \mathcal{P}^1$, i.e. g is unique and right continuous. In the same way for $(f, g) \in \mathcal{B}^j$, $j \geq 0$, g is unique in $\mathcal{P}^0$. It is unique modulo a polynomial of at most order $-j-1$ if $j < 0$. Thus we get $\mathcal{B}^j \subset \mathcal{P}^j$ for all j . Proposition 1.3 shows that $\mathcal{P}^j$ is a $\mathcal{P}^{-j+1}$ module, $j \leq 0$, and that $\mathcal{P}^{-j+1}$ is a $\mathcal{P}^j$ module, $j > 0$. In the following the ambiguity with the space $\mathcal{B}^j$ is removed if one replaces it by the $\mathcal{P}^{j+1}$ space.

Proposition 1.4. *Let $j > 0$ be an integer, let $a \in \mathcal{P}^{-j}$, $f \in \mathcal{B}^j$ and let $b \in \mathcal{P}^0$ be such that $D^j b = a$. Then*

$$(1.3) \quad D^{-1}(fa) = D^{-1}(fD^j b) = fD^{j-1}b - D^{-1}((Df)D^{j-1}b) + \text{constant}.$$

Proposition 1.5. *Let $j > 0$ be an integer, let $a \in \mathcal{P}^j$, $f \in \mathcal{B}^{-j}$ and let $g \in \mathcal{B}^0$ be such that $D^j g = f$. Then*

$$(1.4) \quad D^{-1}(af) = D^{-1}(aD^j g) = aD^{j-1}g - D^{-1}((Da)D^{j-1}g) + \text{constant}.$$

These propositions are proved in [3] and the proof is not repeated here. Induction then gives as in [3]

Proposition 1.6. *Let j , b , and f be as in Proposition 1.4. Then*

$$(1.5) \quad D^{-j}(fD^j b) = \sum_{k=0}^j (-1)^k \binom{j}{k} D^{-k}((D^k f)b) + p,$$

where p is a polynomial of at most degree $j-1$.

Proposition 1.7. *Let j , a and g be as in Proposition 1.5. Then*

$$(1.6) \quad D^{-j}(aD^j g) = \sum_{k=0}^j (-1)^k \binom{j}{k} D^{-k}(gD^k a) + p,$$

where p is a polynomial of at most degree $j-1$.

We want to solve the one-sided Cauchy problem with initial values given at $x=0$ for the equation

$$(1.7) \quad u^{(n)} + a_{n-1}u^{(n-1)} + \dots + a_0 u = f.$$

Here we assume that $a_j \in \mathcal{P}^{k(j)}$ and $f \in \mathcal{P}^{k'}(\mathcal{B}^{k'-1})$ with some integers $k(j)$, $0 \leq j < n$, and k' . We seek the solution u in $\mathcal{P}^{k'+n}(\mathcal{B}^{k'+n-1})$. The integers $k(j)$ and k' are chosen so that $a_j u^{(j)} \in \mathcal{P}^{k'}$ when we use Definition 1.2. As the simplest example we may choose $k' = k(j) = 0$, $0 \leq j < n$. Then (1.7) is a measure differential equation of order n . The two-sided Cauchy problem for such equations are treated in Persson [4, Theorem 2.1]. We include the one-sided part of that theorem in our theorems below but we do not repeat the proof. It is obvious that if $u \in \mathcal{P}^2$ then $D^2 u(0)$ is not necessarily defined. To avoid these difficulties we assume that the support of f together with those of the coefficients is in $x \geq 0$. For $x < 0$ (1.7) is solved by $u = \sum_{k=0}^{n-1} c_k x^k / k!$ for $c = (c_0, \dots, c_{n-1}) \in \mathbb{C}^n$. Here c gives the initial values. If one lets $w = u - \sum_{k=0}^{n-1} c_k x^k / k!$ then (1.7) is equivalent to

$$(1.8) \quad w^{(n)} + a_{n-1} w^{(n-1)} + \dots + a_0 w = f_0,$$

where $f_0 = f - \sum_{j=0}^{n-1} a_j (\sum_{k=j}^{n-1} c_k x^{k-j} / (k-j)!)$. Let $g \in \mathcal{P}^j(\mathcal{B}^j)$ with support in $x \geq 0$. From now on we let $D^{-1}g$ denote the primitive element of g with support in $x \geq 0$.

The Propositions 1.6 and 1.7 are true in distribution sense. We take them as a model for the formulation of the Cauchy problem for (1.7).

Theorem 1.8. Let m, l and n be integers such that $n = 2l - 1$, $1 \leq m \leq l$. Let $a_j \in \mathcal{P}^{j+2-l-m}$, $j > 2m - 2$, and let $a_j \in \mathcal{P}^{m-l}$, $0 \leq j \leq 2m - 2$, and let $f \in \mathcal{P}^{m-l}(\mathcal{B}^{m-l-1})$. Let

$$(1.9) \quad a_{n-1}(\{x\}) \neq -1, \quad x \geq 0 \quad \text{if} \quad m = l,$$

and let a_j , $0 \leq j < n$, and f have their support in $x \geq 0$. Choose $b_j \in \mathcal{P}^0$ such that $D^{l-m} b_j = a_j$, $0 \leq j \leq 2m - 2$, and $D^{l+m-2-j} b_j = a_j$, $2m - 1 \leq j \leq m + 1 - 2$. Then there is a unique $u \in \mathcal{P}^{l+m-1}(\mathcal{B}^{l+m-2})$ fulfilling

$$(1.10) \quad u + \sum_{j=0}^{2m-2} \sum_{k=0}^{l-m} (-1)^k \binom{l-m}{k} D^{1-l-m-k} ((D^{j+k} u) b_j) \\ + \sum_{j=2m-1}^{m+l-2} \sum_{k=0}^{m+l-2} (-1)^k \binom{l+m-j-2}{k} D^{m-l-j-k-1} ((D^{j+k} u) b_j) \\ + \sum_{j=m+l-1}^{2l-2} \sum_{k=0}^{j+2-l-m} (-1)^k \binom{j+2-l-m}{k} D^{j+3-3l-m-k} ((D^k a_j) D^{l+m-2} u) \\ = D^{1-2l} f + \sum_{k=0}^{2l-2} c_k x^k / k!.$$

Here the last two double sums of the left member are assumed to be void if $l = m$.

Theorem 1.9. *Let m, l be as in Theorem 1.8 and let $n=2l$. Let $a_j \in \mathcal{P}^{m-l}$, $0 \leq j \leq 2m-1$, let $a_j \in \mathcal{P}^{j+1-l-m}$, $j \geq 2m$, and let $f \in \mathcal{P}^{m-l}(\mathcal{B}^{m-l-1})$. Let f and a_j , $0 \leq j < n$, have their support in $x \geq 0$. Let (1.9) be fulfilled. Choose $b_j \in \mathcal{P}^0$ such that $D^{l-m}b_j = a_j$, $0 \leq j < 2m$, and $D^{l+m-j-1}b_j = a_j$, $2m \leq j < m+1$. Then to each choice $c = (c_0, \dots, c_{n-1}) \in \mathbb{C}^n$ there is a unique $u \in \mathcal{P}^{l+m}(\mathcal{B}^{l+m-1})$ fulfilling*

$$\begin{aligned}
 (1.11) \quad u &+ \sum_{j=0}^{2m-1} \sum_{k=0}^{l-m} (-1)^k \binom{l-m}{k} D^{-l-m-k} ((D^{j+k}u)b_j) \\
 &+ \sum_{j=2m}^{m+l-1} \sum_{k=0}^{l+m-j-1} (-1)^k \binom{l+m-j-1}{k} D^{m-l-j-k-1} ((D^{j+k}u)b_j) \\
 &+ \sum_{j=m+l}^{2l-1} \sum_{k=0}^{j+1-l-m} (-1)^k \binom{j+1-l-m}{k} D^{j+1-3l-m-k} ((D^k a_j) D^{l+m-1}u) \\
 &= D^{-2l}f + \sum_{k=0}^{2l-1} c_k x^k / k!.
 \end{aligned}$$

Here the last two double sums of the left member are assumed to be void if $l=m$.

In Section 2 we sketch the proof of the theorems. In Section 3 there are some short comments.

2. A sketch of a proof

The case $m=l$ is already covered by [4, Theorem 2.1]. We sketch the proof of Theorem 1.8 when $m < l$. We let

$$(2.1) \quad w = u - \sum_{k=0}^{n-1} c_k x^k / k!$$

which implies that $\text{supp } w$ is in $x \geq 0$. Then we let

$$(2.2) \quad v = D^{l+m-2}w.$$

It follows from (1.10) that

$$\begin{aligned}
 (2.3) \quad v &+ \sum_{j=0}^{2m-2} \sum_{k=0}^{l-m} (-1)^k \binom{l-m}{k} D^{-k-1} ((D^{j+k+2-l-m}v)b_j) \\
 &+ \sum_{j=2m-1}^{m+l-2} \sum_{k=0}^{m+l-j-2} (-1)^k \binom{l+m-2-j}{k} D^{2m-j-k-3} ((D^{j+2+k-l-m}v)b_j) \\
 &+ \sum_{j=m+l-1}^{2l-2} \sum_{k=0}^{j+2-l-m} (-1)^k \binom{j+2-l-m}{k} D^{j+1-2l-k} ((D^k a_j) D^{l+m-2}v) \\
 &= h,
 \end{aligned}$$

with

$$(2.4) \quad h = D^{-l+m-1}f - D^{-l+m-1}\left(\sum_{j=0}^{2l-2} \left(\sum_{k=j}^{2l-2} a_j c_k x^{k-j}/(k-j)!\right)\right) \\ + \sum_{k=l+m-2}^{2l-2} c_k x^{k-l-m+2}/(k-l-m+2)!.$$

Then one solves (2.3) by successive approximations just as is done in Persson [3]. We do not repeat the proof here. The resulting v is unique in $\mathcal{P}^1(\mathcal{B}^0)$. Then (2.2) and (2.1) give that u solves (1.10) and is unique in $\mathcal{P}^{l+m}(\mathcal{B}^{l+m-1})$.

In the proof of Theorem 1.9 one lets

$$(2.5) \quad v = D^{m+l-1}w.$$

Then the proof follows the lines of the proof of Theorem 1.8.

3. Comments

To exemplify we write down (1.10) in detail when $l=2$, $m=1$ and $n=2l-1=3$. We differentiate (1.10) once and get

$$(3.1) \quad Du + D^{-1}(ub_0) - D^{-2}((Du)b_0) + D^{-2}((Du)b_1) + D^{-1}(a_2 Du) \\ - D^{-2}((Da_2)Du) = D^{-2}f + c_1 + c_2 x.$$

Or with $c_0 = u(0)$

$$(3.2) \quad Du + \int_0^x \left(c_0 + \int_0^t Du(s)ds \right) db_0(t) - \int_0^x \left(\int_0^t Du(s)db_0(s) \right) dt \\ + \int_0^x \left(\int_0^t Du(s)db_1(s) \right) dt + \int_0^x a_2(t)Du(t)dt - \int_0^x \left(\int_0^t Du(s)d(Da_2)(s) \right) dt \\ = D^{-2}f + c_1 + c_2 x.$$

Here $c_1 = Du(0^-)$ and $c_2 = D^2u(0^-)$.

We emphasize that the integrals of (3.2) are taken over the closed intervals $[0, x]$ and $[0, t]$. In the case of measure differential equations, $m=l$, this differs from [4] where the integrals are taken over the intervals $(0, x]$. However as long as (1.9) is fulfilled one can always go from the one case to the other one.

If $f=0$ then one sees that the solution space of (1.7) has dimension n . So one way imagine that in some case it could be possible to solve boundary value problems for distribution differential equations which are not measure differential equations. For boundary value problems for such equations see Persson [1] and [2].

Maybe it should be pointed out explicitly that the function u of Theorem 1.8

or Theorem 1.9 solves (1.7) in distribution sense too.

Let $y > 0$ and let $x > y$. An examination of (1.10) and (1.11) shows that we can redefine D^{-1} in the left members as integration from y^+ to x . Then one gets a right member of the type

$$D^{-n}f + \sum_{k=0}^{n-1} c_k(x-y)^k/k!, \quad x > y,$$

with new constants c_k . We get Cauchy data at y although some higher derivatives of u are not necessarily pointwise defined.

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(Ricevita la 4-an de septembro, 1985)