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A Connection Problem for the Generalized Hypergeometric Equation

By

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§ 1. Introduction

The first author has been developing the theory on systems of linear ordinary differential equations of the form

$$(tI-B)dx/dt = Ax. \tag{E}$$

Here t is a complex variable, x is a complex column vector, A and B are constant square matrices and I is the identity matrix ([4]). It is known that hypergeometric equations such as Gauss' hypergeometric equation, Jordan-Pochhammer equation, the generalized hypergeometric equation and Kummer's confluent hypergeometric equation can be reduced to the systems of the form (E). We note that system (E) has a remarkable property that if $x(t)$ is a solution of (E) then $y(t)=dx(t)/dt$ is a solution of $(tI-B)dy/dt=(A-I)y$. In this paper, we make a study in the large on the generalized hypergeometric equation by reducing it to a system of the form (E).

The generalized hypergeometric function ${}_nF_{n-1}(\alpha_1,\ldots,\alpha_n;\beta_1,\ldots,\beta_{n-1};t)$ is defined by the power series

$$\sum_{m=0}^{\infty} \frac{(\alpha_1, m)\cdots\cdots(\alpha_n, m)}{(\beta_1, m)\cdots(\beta_{n-1}, m)(1, m)} t^m \tag{1.1}$$

convergent in $|t|<1$. Here (a, m) denotes the factorial function

$$(a, m) = a(a+1)\cdots(a+m-1) = \Gamma(a+m)/\Gamma(a). \tag{1.2}$$

In case $n=2$, it is the well known Gauss' hypergeometric function. Denote by δ the operator td/dt. Then we see that $u(t)={}_nF_{n-1}(\alpha_1,\ldots,\alpha_n;\beta_1,\ldots,\beta_{n-1};t)$ satisfies the n-th order linear differential equation

$$[(\delta+\alpha_n)\cdots(\delta+\alpha_1)-t^{-1}\delta(\delta+\beta_{n-1}-1)\cdots(\delta+\beta_1-1)]u = 0, \tag{1.3}$$

which is called the generalized hypergeometric differential equation.

Set

$$(1.4)\qquad x^1(t)=u(t),\ x^2(t)=(\delta+\beta_1-1)x^1(t),\ldots,\ x^n(t)=(\delta+\beta_{n-1}-1)x^{n-1}(t)$$

and denote by $x(t)$ the column vector ${}^t(x^1(t),\ldots,x^n(t))$. Then it can be verified that $x(t)$ satisfies a system of the form (E) where $A=(a_{jk})_{1\leqslant j,k\leqslant n}$ is a matrix given by

$$(1.5)\qquad a_{jj}=1-\beta_j,\ a_{j,j+1}=1,\ a_{jk}=0\quad(1\leqslant j\leqslant n-1,\ k\neq j,\ j+1)$$

$$a_{nk}=-\sum_{h_1+\cdots+h_n=k-1,\,0\leqslant h_\ell\leqslant 1}\ \prod_{\ell=1}^{n}(\alpha_\ell-\beta_{1+h_1+\cdots+h_\ell}+1)^{1-h_\ell}\quad(1\leqslant k\leqslant n-1)$$

$$a_{nn}=-\sum_{\ell=1}^{n-1}(1-\beta_\ell)-\sum_{\ell=1}^{n}\alpha_\ell$$

and

$$(1.6)\qquad B=\operatorname{diag}(0,\ldots,0,1).$$

We see that the eigenvalues of the minor matrix $(a_{jk})_{1\leqslant j,k\leqslant n-1}$ of A are $1-\beta_1,\ldots,1-\beta_{n-1}$ and those of A are $-\alpha_1,\ldots,-\alpha_n$. Now assume that $\beta_j\neq\beta_k$ for all $1\leqslant j,k\leqslant n-1$ with $j\neq k$ and make the linear transformation $y=(P\oplus 1)x$, $P=(p_{jk})_{1\leqslant j,k\leqslant n-1}$ where

$$(1.7)\qquad p_{jk}=0\quad(j>k),\quad p_{jj}=1$$

$$p_{jk}=-1/[\textstyle\prod_{\ell=j}^{k-1}(\beta_\ell-\beta_k)]\quad(j<k).$$

Then we obtain a system of the form (E) in which B is unchanged and A is changed to

$$(1.8)\qquad A=\begin{pmatrix} a_1 & & 0 & c_1\\ & \ddots & \ddots & \vdots\\ & 0 & a_{n-1} & c_{n-1}\\ b_1 & \cdots & b_{n-1} & a_n\end{pmatrix}$$

where

$$(1.9)\qquad a_j=1-\beta_j\ (1\leqslant j\leqslant n-1),\ a_n=-\sum_{\ell=1}^{n-1}(1-\beta_\ell)-\sum_{\ell=1}^{n}\alpha_\ell$$

$$(b_1,\ldots,b_{n-1})=(a_{n1},\ldots,a_{n,n-1})P,\ {}^t(c_1,\ldots,c_{n-1})=P^{-1}\cdot{}^t(0,\ldots,0,1).$$

Here the inverse matrix $P^{-1}=(q_{jk})_{1\leqslant j,k\leqslant n-1}$ is given by

$$(1.10)\qquad q_{jk}=0\quad(j>k),\quad q_{jj}=1$$

$$q_{jk}=1/[\textstyle\prod_{\ell=j+1}^{k}(\beta_\ell-\beta_j)]\quad(j<k).$$

Thus we have seen that the generalized hypergeometric equation (1.3) with $\beta_j\neq\beta_k$ $(j\neq k)$ is transformed to a system (E) with (1.6) and (1.8). Conversely, by con-

sidering a transformation $y=\mathrm{diag}(s_1,\ldots, s_n)\cdot x$ $(s_1,\ldots, s_n\in \boldsymbol{C}-\{0\})$ in addition to (1.7), (1.9) and (1.10), we can verify that every system (E) with (1.6), (1.8), $a_j\neq a_k$ $(1\leqslant j, k\leqslant n-1, j\neq k)$ and $b_1\cdots b_{n-1}c_1\cdots c_{n-1}\neq 0$ is derived from (1.3) for some values of α_j's and β_j's. Therefore we study system (E) with (1.6) and (1.8) in the following of this paper.

Denote by $\rho_1,\ldots, \rho_n$ the eigenvalues of A. We see that (E) is of Fuchsian type with regular singular points at $t=0$, 1, ∞ of which the characteristic exponents are

$$\begin{aligned} a_1,\ldots\ldots\ldots, a_{n-1}, 0 \quad & \text{at} \quad t=0 \\ 0,\ldots\ldots\ldots, 0, a_n \quad & \text{at} \quad t=1 \\ -\rho_1,\ldots\ldots\ldots, -\rho_n \quad & \text{at} \quad t=\infty. \end{aligned}$$

We note that $a_j=1-\beta_j$ $(1\leqslant j\leqslant n-1)$, $a_n=-\sum_{\ell=1}^{n-1}(1-\beta_\ell)-\sum_{\ell=1}^{n}\alpha_\ell$, $\rho_j=-\alpha_j$ $(1\leqslant j\leqslant n)$ for the system transformed from (1.3).

In §2, we give some preliminary propositions on the structures of local solutions of (E) with (1.6) and (1.8). We denote by $x_1,\ldots, x_{n-1}$ the normalized solutions at $t=0$ of which the characteristic exponents are $a_1,\ldots, a_{n-1}$ and by x_n the one at $t=1$ of which the characteristic exponent is a_n. Then, in §3, we prove the identity which gives the value of $\det(x_1,\ldots, x_n)$ explicitly in terms of a_j's and ρ_j's by using the above-mentioned property of (E). We call the identity a generalized Gauss' formula because the well known formula $\lim_{t\to 1}F(\alpha,\beta,\gamma;t)=\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)/[\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)]$ $(0<t<1)$ is easily derived from it. We remark that the identity plays an essential role in our study. In §4, we obtain the explicit expressions of a set of generators of the monodromy group of (E) in terms of a_j's and ρ_j's. We take $X=(x_1,\ldots, x_n)$ as a fundamental matrix where $x_1,\ldots, x_n$ are the solutions mentioned above. Here the linear independence of $x_1,\ldots, x_n$ is not at all a trivial fact, but it can be verified by the above identity. Then each generator of the group with repsect to X is written in terms of not only a_j's and ρ_j's but also the constants ξ_j's and η_j's, $1\leqslant j\leqslant n-1$, where ξ_j's (or η_j's) are the connection coefficients between X and a fundamental matrix at $t=0$ (or at $t=1$). We see that the generators of the group with respect to $X\cdot\mathrm{diag}(\xi_1,\ldots, \xi_{n-1}, 1)$ are expressed by a_j's, ρ_j's and $\xi_j\eta_j$'s. In order to compute the values of $\xi_j\eta_j$'s, we use the trivial relation among the three generators of $\pi_1(\boldsymbol{P}^1(\boldsymbol{C})-\{0, 1, \infty\})$ as Riemann did in computing the group of Gauss' hypergeometric equation ([6]). In §5, we compute the values of ξ_j's and η_j's in terms of a_j's, ρ_j's and b_j's by using also the above identity.

We note that the group of the generalized hypergeometric equation was first computed by A. H. M. Levelt ([3]) by a method completely different from ours and a connection problem for the same equation in the general setting was initiated

by Y. Kawabata ([2]). We all were involved in the problem since then. We took two different ways of approach but to get the same result which follows.

The notation used in this paper is as follows. D_0 and D_1 denote the domains defined by

$$D_0 = \{t \in \boldsymbol{C} \mid |t|<1\}, \quad D_1 = \{t \in \boldsymbol{C} \mid |t-1|<1\}$$

and $D=D_0 \cap D_1$. ε_j $(1 \leqslant j \leqslant n)$ is the column vector of which the k-th component is δ_{jk}, δ_{jk} being Kronecker's delta. For a column vector x, we denote by x^k the k-th component of x. O denotes Landau's symbol. We abbreviate $\prod_{k=1}^{n}$, $\sum_{k=1}^{n}$, $\prod_{k=1}^{n-1}$ and $\sum_{k=1}^{n-1}$ to $\prod_k$, $\sum_k$, $\prod_k'$ and $\sum_k'$ respectively.

§2. Local solutions

In this section, we shall give some propositions concerning the local properties of

$$(\mathrm{E})_\mu \qquad (tI-B)dx/dt = (A+\mu I)x$$

at $t=0$, 1. Here A and B are the matrices given by (1.8) and (1.6) and μ is a complex parameter. Remark that every formal solution conveges because $(\mathrm{E})_\mu$ is of Fuchsian type.

Proposition 1 (Singular solutions at $t=0$). *Given $1 \leqslant j \leqslant n-1$, suppose that $a_j \neq -1-\mu$, $-2-\mu,\ldots, a_j-a_k \neq -1$, $-2,\ldots$ for every k with $1 \leqslant k \leqslant n-1$, $k \neq j$. Then we have a unique solution $x_j(t, \mu)$ of $(\mathrm{E})_\mu$ of the form*

$$x_j(t, \mu) = t^{a_j+\mu} \sum_{m \geqslant 0} g_j(m, \mu) t^m \tag{2.1}$$

convergent in D_0 with

$$g_j(0, \mu) = \varepsilon_j. \tag{2.2}$$

The coefficients $g_j^k(m, \mu)$ are determined recursively by

$$\begin{aligned} &(a_j-a_k+m)g_j^k(m, \mu) = c_k g_j^k(m, \mu), \quad 1 \leqslant k \leqslant n-1, \quad m \geqslant 0 \\ &(a_j+\mu+m+1)g_j^n(m+1, \mu) = -\sum_k' b_k g_j^k(m, \mu) \\ &\qquad\qquad + (a_j-a_n+m)g_j^n(m, \mu), \quad m \geqslant 0. \end{aligned} \tag{2.3}$$

Proposition 2 (A singular solution at $t=1$). *Suppose $a_n \neq -1-\mu, -2-\mu,\ldots$ Then we have a unique solution $x_n(t, \mu)$ of $(\mathrm{E})_\mu$ of the form*

$$x_n(t, \mu) = (t-1)^{a_n+\mu} \sum_{m \geqslant 0} g_n(m, \mu)(t-1)^m \tag{2.4}$$

convergent in D_1 satisfying

$$g_n(0, \mu) = \varepsilon_n. \tag{2.5}$$

The coefficients $g_n^k(m)$ *are determined recursively by*

$$\begin{aligned} mg_n^n(m, \mu) &= \sum_k' b_k g_n^k(m, \mu), \qquad m \geqslant 0 \\ (a_n+\mu+m+1)g_n^k(m+1, \mu) &= -(a_n-a_k+m)g_n^k(m, \mu) \\ &\quad + c_k g_n^n(m, \mu), \quad 1 \leqslant k \leqslant n-1, \quad m \geqslant 0. \end{aligned} \tag{2.6}$$

For holomorphic solutions of (E) at $t=0$ and $t=1$, we have the following propositions.

Proposition 3 (A holomorphic solution at $t=0$). *Under the assumption* $a_k \neq 0, 1, \ldots$ *for every* $1 \leqslant k \leqslant n-1$, *there exists a unique solution* $x_n^*(t)$ *of* (E) *of the form*

$$x_n^*(t) = \sum_{m \geqslant 0} g_n^*(m) t^m \tag{2.7}$$

convergent in D_0 *with*

$$g_n^{*n}(0) = 1. \tag{2.8}$$

The coefficients $g_n^{*k}(m)$ *are determined by*

$$\begin{aligned} (m-a_k)g_n^{*k}(m) &= c_k g_n^{*n}(m), \qquad 1 \leqslant k \leqslant n-1, \quad m \geqslant 0 \\ (m+1)g_n^{*n}(m+1) &= -\sum_k' b_k g_n^{*k}(m) + (m-a_n)g_n^{*n}(m), \quad m \geqslant 0. \end{aligned} \tag{2.9}$$

Proposition 4 (Holomorphic solutions at $t=1$). *Under the assumption* $a_n \neq 0, 1, \ldots$, *there exist linearly independent* $n-1$ *solutions of* (E) *holomorphic at* $t=1$.

§ 3. Generalized Gauss' formula

We first remark

Proposition 5 (Fuchs' relation).

$$\sum_k a_k = \sum_k \rho_k. \tag{3.1}$$

Denote by $x_j(t)$ $(1 \leqslant j \leqslant n)$ the solution $x_j(t, 0)$ of (E) defined by (2.1), (2.2) or (2.4), (2.5). Then we obtain

Theorem 1. *Suppose* $a_k - a_\ell \neq \pm 1, \pm 2, \ldots$ *for all* $1 \leqslant k, \ell \leqslant n-1$ *with* $k \neq \ell$, $a_k \neq 0, \pm 1, \ldots$, $\rho_k \neq -1, -2, \ldots$ *for all* $1 \leqslant k \leqslant n$. *Then we have the identity*

$$\begin{aligned} &\det(x_1(t), \ldots, x_n(t)) \\ &\quad = t^{\sum_k' a_k}(t-1)^{a_n} \prod_k [\Gamma(a_k+1)/\Gamma(\rho_k+1)] \end{aligned} \tag{3.2}$$

for $t \in D$. *Here the branches of* $x_1(t),\dots, x_n(t)$ *in* D *are determined by* $\arg t=0$, $\arg(t-1)=\pi$ *for* $0<t<1$.

Proof. Let D' be a simply connected compact subset of D. We have only to show (3.2) in D'.

For every integer $\mu \geqslant 0$, let $x_1(t, \mu),\dots, x_n(t, \mu)$ be the unique solutions of $(\mathrm{E})_\mu$ given by Propositions 1 and 2. Set

$$X(t, \mu) = (x_1(t, \mu),\dots, x_n(t, \mu)),\ w(t, \mu) = \det X(t, \mu).$$

We shall first obtain a relation between $w(t, \mu)$ and $w(t, 0)$.

Let μ be a positive integer. Then $y_j(t, \mu)=dx_j(t, \mu)/dt$, $1 \leqslant j \leqslant n$, are solutions of $(\mathrm{E})_{\mu-1}$ with

$$y_j(t, \mu) = t^{a_j+\mu-1}[(a_j+\mu)\varepsilon_j + O(t)], \qquad 1 \leqslant j \leqslant n-1$$

$$y_n(t, \mu) = (t-1)^{a_n+\mu-1}[(a_n+\mu)\varepsilon_n + O(t-1)].$$

Hence, by the uniqueness in Propositions 1 and 2, we have $y_j(t, \mu)=(a_j+\mu)x_j(t, \mu-1)$ $(1 \leqslant j \leqslant n)$, which implies

$$(t-B)X(t, \mu-1)\,\mathrm{diag}\,(a_1+\mu,\dots, a_n+\mu) = (A+\mu)X(t, \mu).$$

From this, we have a relation between $w(t, \mu)$ and $w(t, \mu-1)$ and then

$$w(t, \mu) = w(t, 0)t^{\mu(n-1)}(t-1)^\mu \textstyle\prod_k [\Gamma(\rho_k+1)/\Gamma(\rho_k+\mu+1)] \times \prod_k [\Gamma(a_k+\mu+1)/\Gamma(a_k+1)]. \tag{3.3}$$

We shall next investigate the asymptotic behabior of $w(t, \mu)$ as $\mu \to \infty$, $\mu>0$. Set

$$g_j(m, \mu) = h_j(m, \mu)(a_j+1, m)/(a_j+\mu+1, m), \qquad 1 \leqslant j \leqslant n,$$

where (a, m) is defined by (1.2). Then each $h_j(m, \mu)$ $(1 \leqslant j \leqslant n)$ satisfies the equation (2.3) or (2.6) with $\mu=0$ and $h_j(0, \mu)=\varepsilon_j$. Therefore, from the uniqueness property for (2.3) or (2.6), it follows $h_j(m, \mu)=g_j(m)$ $(=g_j(m, 0))$. Hence we have

$$g_j(m, \mu) = g_j(m)(a_j+1, m)/(a_j+\mu+1, m), \qquad 1 \leqslant j \leqslant n. \tag{3.4}$$

Now we estimate $\sum_{m \geqslant 1} g_j^k(m, \mu)t^m$ $(1 \leqslant j \leqslant n-1)$ and $\sum_{m \geqslant 1} g_n^k(m, \mu)(t-1)^m$. We can verify

$$|(a_j+1, m)/(a_j+\mu+1, m)| \leqslant 2(|a_j|+1)\mu^{-1}, \qquad 1 \leqslant j \leqslant n \tag{3.5}$$

for all $m \geqslant 1$, $\mu \geqslant 2(|a_j|+1)$. Then it follows

(3.6) $|\sum_{m\geqslant 1} g_j^k(m,\mu)t^m| \leqslant M\mu^{-1}\sum_{m\geqslant 1}|g_j^k(m)|\,|t|^m, \quad 1\leqslant j\leqslant n-1, \quad 1\leqslant k\leqslant n$

$|\sum_{m\geqslant 1} g_n^k(m,\mu)(t-1)^m| \leqslant M\mu^{-1}\sum_{m\geqslant 1}|g_n^k(m)|\,|t-1|^m, \quad 1\leqslant k\leqslant n$

for $t\in D$, $\mu\geqslant M=\max_{1\leqslant j\leqslant n}\{2(|a_j|+1)\}$. We see that the right hand sides of (3.6) are of order $O(1/\mu)$ for $t\in D'$, because D' is a compact subset of D. Therefore, it holds

$$x_j(t,\mu) = t^{a_j+\mu}[\varepsilon_j+O(1/\mu)], \qquad 1\leqslant j\leqslant n-1$$
$$x_n(t,\mu) = (t-1)^{a_n+\mu}[\varepsilon_n+O(1/\mu)]$$

for $t\in D'$, which implies

(3.7) $$w(t,\mu) = t^{\Sigma_k' a_k+\mu(n-1)}(t-1)^{a_n+\mu}[1+O(1/\mu)].$$

By combing (3.3) and (3.7), we have

(3.8) $$w(t,0) = t^{\Sigma_k' a_k}(t-1)^{a_n}[1+O(1/\mu)]$$
$$\times \prod_k [\Gamma(a_k+1)/\Gamma(a_k+\mu+1)]\prod_k [\Gamma(\rho_k+\mu+1)/\Gamma(\rho_k+1)]$$

for $t\in D'$, $\mu\geqslant M$. Take the limit $\mu\to\infty$ in (3.8). Then, by (3.1) and Stirling's formula, we obtain (3.2). Thus we have proved Theorem 1.

As an immediate consequence of the theorem, we have

Corollary (Gauss' formula). *Assume that* γ, $\gamma-\alpha-\beta\neq 0$, $\pm 1,\ldots$, $\gamma-\alpha$, $\gamma-\beta\neq 0$, $-1,\ldots$ *and* $\mathrm{Re}\,(\gamma-\alpha-\beta)>0$. *Then we have*

(3.9) $$\lim_{t\to 1,0<t<1} F(\alpha,\beta,\gamma;t) = \Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)/[\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)].$$

Proof. By definition, $F(\alpha,\beta,\gamma;t)={}_2F_1(\alpha,\beta;\gamma;t)$. Consider (E) with $B=\mathrm{diag}\,(0,1)$, $A=(a_{jk})$ where $a_{11}=a_1=\gamma-1$, $a_{22}=a_2=\gamma-\alpha-\beta-1$, $a_{21}=b_1=-\beta$, $a_{12}=c_1=\alpha$. The eigenvalues of A are $\gamma-\alpha-1$, $\gamma-\beta-1$. Then the assumptions in Theorem 1 are satisfied. Let $x_1(t)=t^{\gamma-1}\sum_{m\geqslant 0} g_1(m)t^m$ $(g_1(0)=\varepsilon_1)$ and $x_2(t)=(t-1)^{\gamma-\alpha-\beta-1}\sum_{m\geqslant 0} g_2(m)(t-1)^m (g_2(0)=\varepsilon_2)$ be solutions of our system. Then, from Theorem 1, it follows

$$\det(x_1(t), x_2(t)) = c\, t^{\gamma-1}(t-1)^{\gamma-\alpha-\beta-1}$$

where c is the constant on the right hand side of (3.9). We see that $g_1^1(m)=(\alpha,m)(\beta,m)/[(\gamma,m)(1,m)]$, that is, $x_1^1(t)=t^{\gamma-1}F(\alpha,\beta,\gamma;t)$. Let η be a constant and $x_1^*(t)$ a solution holomorphic at $t=1$ such that

$$x_1(t) = x_1^*(t) + \eta x_2(t), \qquad t\in D.$$

Then, we have

$$\det(x_1^*(t),\ x_2(t)) = c\ t^{\gamma-1}(t-1)^{\gamma-\alpha-\beta-1}. \tag{3.10}$$

Since $x_2^1(t)=(t-1)^{\gamma-\alpha-\beta}O(1)$, $x_2^2(t)=(t-1)^{\gamma-\alpha-\beta-1}[1+O(t-1)]$ as $t\to 1$ in D, we see by (3.10) that $x_1^{*1}(t)=c+O(t-1)$ as $t\to 1$ in D. Hence we have

$$t^{\gamma-1}F(\alpha,\ \beta,\ \gamma;\ t) = c + O(t-1) + (t-1)^{\gamma-\alpha-\beta}O(1)$$

as $t\to 1$ in D, which proves (3.9).

§ 4. Monodromy group

In this section, we shall compute the monodromy group of (E) with (1.6) and (1.8). First we note

Lemma 1. *The determinant of the matrix* (1.8) *is given by*

$$\det A = \prod_k a_k - \sum_k' b_k c_k \prod_{\ell\neq k}' a_\ell . \tag{4.1}$$

Now fix a base point $t=t_0$ in D. Let ℓ_0 (or ℓ_1) be a closed curve in $\boldsymbol{C}-\{0, 1\}$ from the base point to itself, encircling the point $t=0$ (or $t=1$) once in the positive direction. Let X be a fundamental matrix of (E) at $t=t_0$. If we continue X analytically along ℓ_j $(j=0, 1)$, then we have another fundamental matrix which is expressed as XM_j where $M_j\in GL(n, \boldsymbol{C})$. The matrix M_0 (or M_1) is called the circuit matrix of X around $t=0$ (or $t=1$) and the group generated by M_0 and M_1 is called the monodromy group of (E) with respect to X.

Set

$$e_k = e^{2\pi i a_k},\quad f_k = e^{2\pi i \rho_k},\qquad 1\leqslant k\leqslant n \tag{4.2}$$

where $i=\sqrt{-1}$, then we have

Theorem 2. *Assume that none of* a_k $(1\leqslant k\leqslant n)$, a_k-a_ℓ $(1\leqslant k, \ell\leqslant n-1, k\neq\ell)$, $a_k-\rho_\ell$ $(1\leqslant k\leqslant n-1,\ 1\leqslant\ell\leqslant n)$ *are integers and none of* ρ_k $(1\leqslant k\leqslant n)$ *are nonnegative integers. Then the monodromy group of* (E) *with* (1.6) *and* (1.8) *with respect to a certain fundamental matrix is generated by*

$$\begin{pmatrix} e_1 & & & \\ & \ddots & & \\ & & e_{n-1} & \\ 0 & & & 1 \end{pmatrix}\begin{pmatrix} e_1-1 & & & \\ & \ddots & & \\ & & e_{n-1}-1 & \\ & & & 1 \end{pmatrix},\quad \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ d_1(e_n-1)\cdots d_{n-1}(e_n-1) & & & e_n \end{pmatrix} \tag{4.3}$$

where

$$(4.4)\quad d_j = -\prod_k (f_k - e_j)/[e_j(e_j-1)(e_n-1)\prod'_{k\neq j}(e_k-e_j)]$$
$$= -\prod_k \sin\pi(\rho_k - a_j)/[\sin\pi a_j \sin\pi a_n \prod'_{k\neq j}\sin\pi(a_k-a_j)].$$

Proof. Let $x_1(t),\ldots, x_n(t)$ be the solutions stated in Theorem 1. Then, by our assumptions and by Theorem 1, $X(t)=(x_1(t),\ldots, x_n(t))$ is a fundamental matrix of (E), of which the circuit matrices will be first investigated.

Express x_n as a linear combination of the independent solutions $x_1,\ldots, x_{n-1}$, x_n^* at $t=0$ as

$$(4.5)\qquad x_n(t) = \sum'_k \xi_k x_k(t) + \xi_n x_n^*(t), \qquad t\in D$$

where $\xi_1,\ldots, \xi_n$ are constants and x_n^* is the holomorphic solution at $t=0$ given by Proposition 3. Next express each x_j $(1\leqslant j\leqslant n-1)$ by independent solutions at $t=1$ as

$$(4.6)\qquad x_j(t) = x_j^*(t) + \eta_j x_n(t), \qquad t\in D$$

where η_j is a constant and x_j^* is a holomorphic solution at $t=1$. Set

$$X_0(t) = (x_1,\ldots, x_{n-1}, \xi_n x_n^*)(t), \quad X_1(t) = (x_1^*,\ldots, x_{n-1}^*, x_n)(t),$$

then (4.5) and (4.6) are written as

$$X(t) = X_0(t)C_0, \quad X(t) = X_1(t)C_1$$

where

$$C_0 = \begin{pmatrix} 1 & & & \xi_1 \\ & \ddots & 0 & \vdots \\ & & 1 & \xi_{n-1} \\ 0 & & & 1 \end{pmatrix}, \quad C_1 = \begin{pmatrix} 1 & & & \\ & \ddots & & 0 \\ & & 1 & \\ 0 & & & \\ \eta_1 & \cdots\cdots & \eta_{n-1} & 1 \end{pmatrix}$$

Remark that both $X_0(t)$ and $X_1(t)$ are fundamental matrices of (E).

From the local expressions of $x_1,\ldots, x_{n-1}$, x_n^* at $t=0$, it follows that the circuit matrix E_0 of $X_0(t)$ around $t=0$ is given by

$$E_0 = \operatorname{diag}(e_1,\ldots, e_{n-1}, 1).$$

Similarly, the circuit matrix E_1 of $X_1(t)$ around $t=1$ is

$$E_1 = \operatorname{diag}(1,\ldots, 1, e_n).$$

Therefore, the circuit matrices $M_0=C_0^{-1}E_0C_0$, $M_1=C_1^{-1}E_1C_1$ of $X(t)$ around $t=0$ and $t=1$ are

$$M_0 = \begin{pmatrix} e_1 & & & \xi_1(e_1-1) \\ & \ddots & & \vdots \\ & & 0 & \\ 0 & & e_{n-1} & \xi_{n-1}(e_{n-1}-1) \\ & & & 1 \end{pmatrix}, \quad M_1 = \begin{pmatrix} 1 & & & 0 \\ & \ddots & & \\ & & 0 & \\ & & & 1 \\ \eta_1(e_n-1)\cdots\eta_{n-1}(e_n-1) & & & e_n \end{pmatrix}$$

We shall next determine the values of

$$d_j = \xi_j\eta_j, \qquad 1 \leqslant j \leqslant n-1. \tag{4.7}$$

We see that M_0M_1 is the inverse circuit matrix of $X(t)$ around $t=\infty$, and hence the eigenvalues of $(M_0M_1)^{-1}$ are the characteristic multipliers of (E) around $t=\infty$. On the other hand, since $\rho_1,\ldots,\rho_n$ are eigenvalues of A, the characteristic multipliers of (E) are $f_1^{-1}=\exp(-i2\pi\rho_1),\ldots,f_n^{-1}=\exp(-i2\pi\rho_n)$. Therefore, we have the identity in λ

$$\det(M_0M_1-\lambda I) = \prod_k (f_k-\lambda). \tag{4.8}$$

By direct computation, we see that the components r_{jk} of M_0M_1 are

$$r_{jk} = \delta_{jk}e_j + \xi_j\eta_k(e_j-1)(e_n-1) \quad (1\leqslant j,\ k\leqslant n-1) \tag{4.9}$$

$$r_{jn} = \xi_j(e_j-1)e_n\ (1\leqslant j\leqslant n-1),\ r_{nk} = \eta_k(e_n-1)\ (1\leqslant k\leqslant n-1),\ r_{nn} = e_n,$$

and hence

$$\det(M_0M_1-\lambda) = \begin{vmatrix} e_1-\lambda & & & \xi_1(e_1-1)\lambda \\ & \ddots & & \vdots \\ & & 0 & \\ 0 & & e_{n-1}-\lambda & \xi_{n-1}(e_{n-1}-1)\lambda \\ \eta_1(e_n-1)\cdots\cdots\eta_{n-1}(e_n-1) & & & e_n-\lambda \end{vmatrix}. \tag{4.10}$$

Therefore, by (4.7) and (4.8) and Lemma 1, we obtain

$$\prod_k (e_k-\lambda) - (e_n-1)[\sum_k{}' d_k(e_k-1)\prod_{\ell\neq k}{}' (e_\ell-\lambda)]\lambda = \prod_k (f_k-\lambda). \tag{4.11}$$

By putting the special value $\lambda=e_j$ in (4.11), we see that the value of d_j is given by the first equation of (4.4). The second equation of (4.4) follows from (3.1) and $(e^{i\pi\theta}-e^{-i\pi\theta})/(2i)=\sin\pi\theta$. We note that $d_j=\xi_j\eta_j\neq 0$ $(1\leqslant j\leqslant n-1)$ under our assumptions.

Finally we see that the circuit matrices of $X(t)\mathrm{diag}(\xi_1,\ldots,\xi_{n-1},1)$ are of the

form given in the theorem. Thus we have proved Theorem 2. The values of $\xi_1,\ldots,\xi_{n-1}$ will be determined in the next section.

§5. Connection coefficients

We shall compute the values of $\xi_1,\ldots,\xi_n$, $\eta_1,\ldots,\eta_{n-1}$ appearing in (4.5) and (4.6) under the assumptions in Theorem 2.

5.1. The value of ξ_n. From (4.5), it follows that

$$\det(x_1(t),\ldots,x_n(t)) = \xi_n \det(x_1(t),\ldots,x_{n-1}(t),\, x_n^*(t)) \tag{5.1}$$

for $t\in D$. By the local expressions of $x_1,\ldots,x_{n-1}$, x_n^* near $t=0$, we have

have

$$\det(x_1(t),\ldots,x_{n-1}(t),\, x_n^*(t)) = t^{\sum_k a_k}[1+O(t)] \tag{5.2}$$

as $t\to 0$ in D. Therefore, by taking the limit $t\to 0$ in (5.1), we obtain

$$\xi_n = e^{i\pi a_n}\prod_k [\Gamma(a_k+1)/\Gamma(\rho_k+1)] \tag{5.3}$$

by means of Theorem 1 and (5.2).

5.2. The values of ξ_j, η_j, $1\leqslant j\leqslant n-1$. We first investigate the n-th component of $x_n^*(t)=\sum_{m\geqslant 0} g_n^*(m)t^m$ $(g_n^{*n}(0)=1)$ given in Proposition 3. By (2.9), we have

$$g_n^{*n}(m+1) = \{u(m)/[(m+1)\prod_k'(m-a_k)]\}g_n^{*n}(m), \qquad m\geqslant 0,$$

$$u(m) = \prod_k(m-a_k) - \sum_k' b_k c_k \prod_{l\neq k}'(m-a_l).$$

Since the characteristic equation of A is

$$\prod_k(a_k-\rho) - \sum_k' b_k c_k \prod_{l\neq k}'(a_l-\rho) = 0$$

by Lemma 1, $u(m)$ is equal to $\prod_k(m-\rho_k)$. Therefore it follows that $g_n^{*n}(m)=\prod_k(-\rho_k,\, m)/[(1,\, m)\prod_k'(-a_k,\, m)]$, namely,

$$x_n^{*n}(t) = {}_nF_{n-1}(-\rho_1,\ldots,-\rho_n;\, -a_1,\ldots,-a_{n-1};\, t). \tag{5.4}$$

Express x_n^* as a linear combination of $x_1^*,\ldots,x_{n-1}^*$, x_n^* as

$$x_n^*(t) = \eta_n x_n(t) + \sum_k' \zeta_k x_k^*(t), \qquad t\in D.$$

On the other hand, from (4.5) and (4.6), we have

$$x_n^*(t) = [(1-\sum_k' d_k)/\xi_n]x_n(t) - \sum_k'(\xi_k/\xi_n)x_k^*(t),$$

which implies

$$\eta_n = (1-\sum{}'_k d_k)/\xi_n . \tag{5.5}$$

Now define the polynomial $v(\lambda)$ by

$$v(\lambda) = \prod_k [(f_k-\lambda)/(e_k-\lambda)] .$$

Then, by (4.11), we have

$$v(\lambda) = 1 - \lambda[(e_n-1)/(e_n-\lambda)] \prod{}'_k d_k[(e_k-1)/(e_k-\lambda)] . \tag{5.6}$$

By putting $\lambda=1$ in (5.6), we have

$$1 - \sum{}'_k d_k = v(1) = \prod_k (f_k-1)/(e_k-1) = \prod_k \sin \pi\rho_k/\sin \pi a_k \tag{5.7}$$

by (3.1). From (5.3), (5.5) and (5.7), it follows

$$\eta_n = e^{-i\pi a_n} \prod_k \Gamma(-a_k)/\Gamma(-\rho_k) .$$

Thus we have obtained

Lemma 2. *Under the assumptions in Theorem* 2, *we have*

$$\begin{aligned} &{}_nF_{n-1}(-\rho_1,\ldots, -\rho_n; -a_1,\ldots, -a_{n-1}; t) \\ &\quad = e^{-i\pi a_n} \prod_k [\Gamma(-a_k)/\Gamma(-\rho_k)] (t-1)^{a_n}[1+O(t-1)] + O(1) \end{aligned} \tag{5.8}$$

as $t\to 1$ *in* D *where* $a_n = \sum_k \rho_k - \sum{}'_k a_k$.

Next we shall calculate η_j $(1\leqslant j\leqslant n-1)$ by comparing the n-th components on both sides of (4.6). By the similar way as above, we see

$$\begin{aligned} x_j^{\eta}(t) = &- [b_j/(a_j+1)]t^{a_j+1}{}_nF_{n-1}(a_j-\rho_1+1,\ldots, a_j-\rho_n+1; \\ &a_j - a_1 + 1,\ldots, a_j + 2,\ldots, a_j - a_{n-1} + 1; t) . \end{aligned} \tag{5.9}$$

Therefore, by Lemma 2, we have

$$\eta_j = - b_j e^{-i\pi a_n}\Gamma(-a_n)\Gamma(a_j+1)\prod{}'_{k\neq j} \Gamma(a_j-a_k+1)/\prod_k \Gamma(a_j-\rho_k+1) . \tag{5.10}$$

The values of ξ_j, $1\leqslant j\leqslant n-1$, are obtained by (4.4), (4.7) and (5.10):

$$\xi_j = b_j^{-1} e^{i\pi a_n}\Gamma(a_n+1)\Gamma(-a_j)\prod{}'_{k\neq j} \Gamma(a_k-a_j)/\prod_k \Gamma(\rho_k-a_j) . \tag{5.11}$$

Thus we have proved

Theorem 3. *Under the same assumptions as in Theorem* 2, *the values of* $\xi_1,\ldots, \xi_n, \eta_1,\ldots, \eta_{n-1}$ *are given by* (5.11), (5.3) *and* (5.10).

Remark. As the values of ξ_j's and η_j's have been determined, the expression

of the connection matrix C_∞ between $X=(x_1,\ldots, x_n)$ and the fundamental matrix $X_\infty=(x_{n+1},\ldots, x_{2n})$ at $t=\infty$ defined by $X(t)=X_\infty(t)C_\infty$ $(t\in D)$ can be easily obtained. Here $x_{n+j}(t)=t^{\rho_j}\sum_{m\geqslant 0} g_{n+j}(m)t^{-m}$ $(g^n_{n+j}(0)=1)$ and the branch of X_∞ in D is defined in an arbitrary but fixed way. Indeed, we can determine C_∞ by the relation $C_\infty M_0M_1C_\infty^{-1}=\mathrm{diag}\,(f_1,\ldots,f_n)$ (M_0M_1 is the matrix given by (4.9)) and by Lemma 2.

References

[1] Erdélyi, A., Higher Transcendental Functions, Vol. 1, McGraw-Hill, 1953.

[2] Kawabata, Y., Connection problem for an n-th order differential equation of Fuchsian type with three regular singular points II, Tsudajuku Daigaku Kiyo, **10** (1978), 45–55, [in Japanese].

[3] Levelt, A. H. M., Hypergeometric functions III, Indag. Math., **23** (1961), 386–403.

[4] Okubo, K., Connection problems for systems of linear differential equations, Lect. Notes in Math., **243**, Springer, 238–248.

[5] Okubo, K. and Takano, K., Generalized hypergeometric functions, preprint, 1981.

[6] Riemann, B., Beiträge zur Theorie der durch die Gauss'sche Reihe $F(\alpha, \beta, \gamma, x)$ darstellbaren Functionen, The Collected Works of Bernhard Riemann, Dover.

[7] Sasai, T., On a monodromy group and irreducibility conditions of a fourth order Fuchsian differential system of Okubo type, J. Reine Angew. Math., 299/300 (1978), 38–50.

[8] ———, The monodromy group and the reducibility conditions of the one dimensional section of Appell's hypergeometric equation for $F_3(\alpha, \alpha', \beta, \beta', \gamma; x, y)$, Proc. Japan Acad., **54** (1978), 59–61.

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